Matryoshka arithmetic

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Motivation: Dynamical systems

- $u_0 \in \mathbb{A}^m$, $u_{k+1} = f(u_k)$ with $f : \mathbb{A}^m \to \mathbb{A}^m$ any kind of function.
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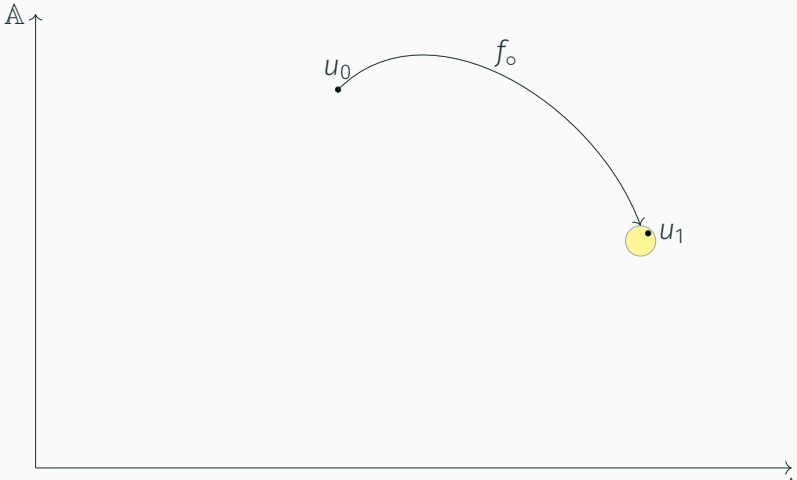

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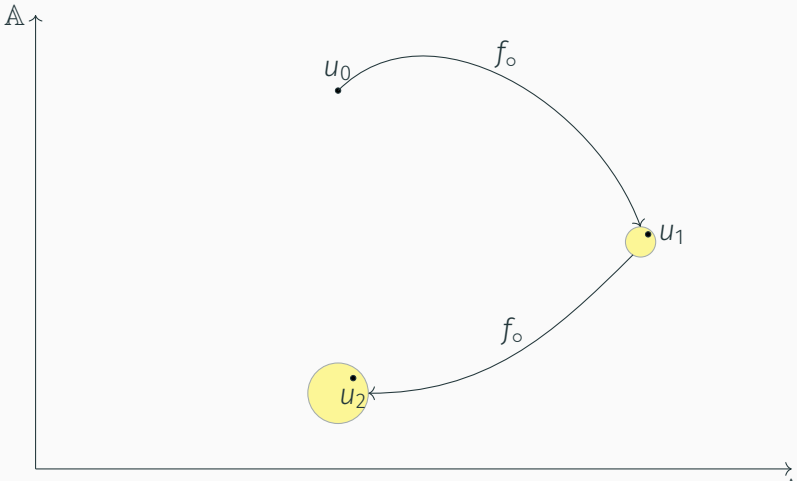

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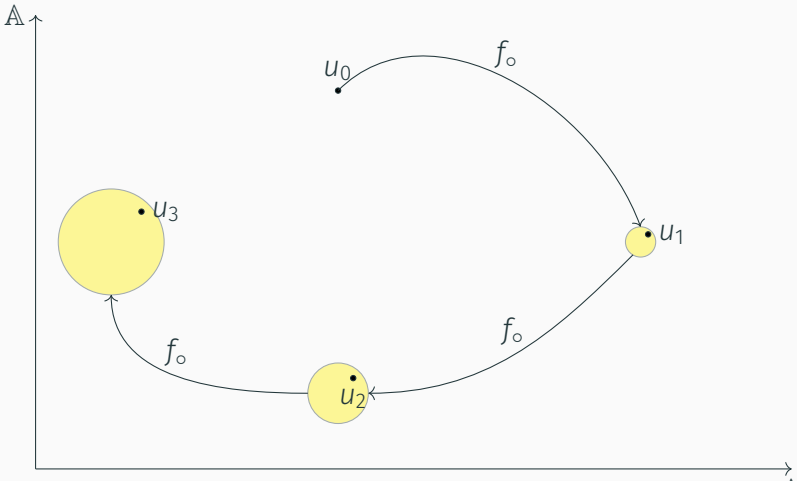

Speed of certified computations

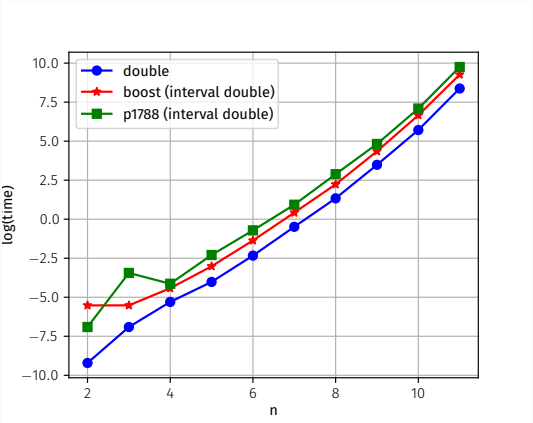

Figure 1: Time to compute: $\det(A) = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$ in millisecond with double and interval of double. See boost: Brönnimann, Melquiond, and Pion 2006, p1788:Nehmeier 2014

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- Given a numerical program, that uses basic arithmetic operations, how accurate is the result? Can we bound rounding errors?

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- Given a numerical program, that uses basic arithmetic operations, how accurate is the result? Can we bound rounding errors?
- Can we accelerate existing ball arithmetic ?

Ball arithmetic

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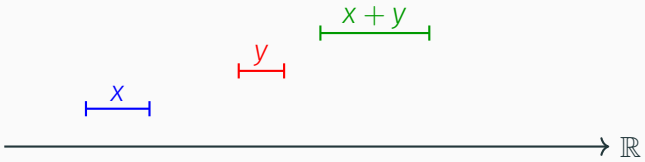

Ball arithmetic

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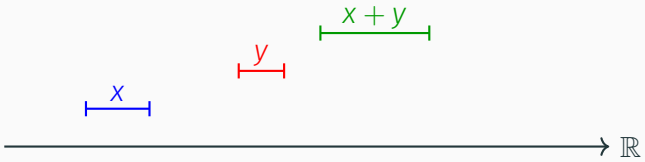

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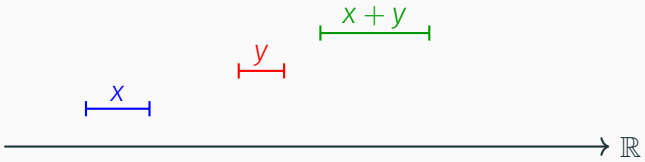

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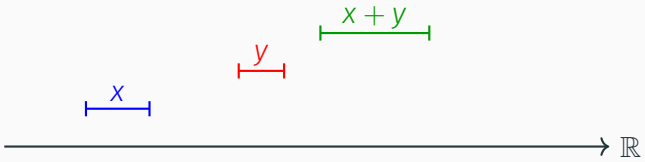

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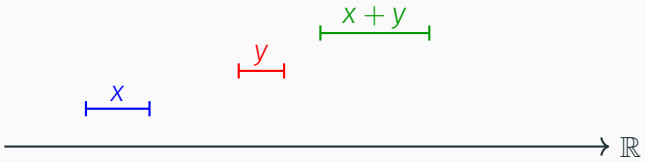

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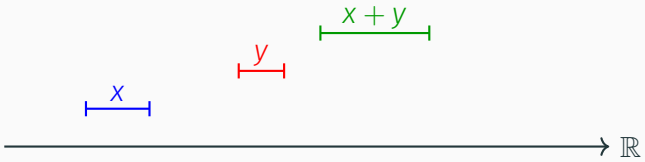

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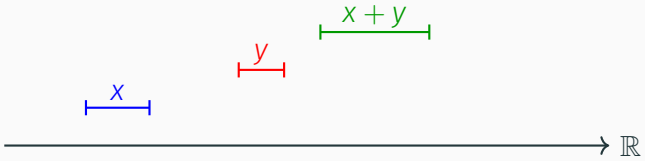
- Inclusion principle: for all input sets, the resulting set encloses the image of the input sets.



Ball arithmetic

- Ball lift of $f : \mathbb{A}^m \to \mathbb{A}^n$ is a function $f : \mathcal{B}(\mathbb{A}, \mathbb{R})^m \to \mathcal{B}(\mathbb{A}, \mathbb{R})^n$ that satisfies the inclusion principle:

$$\forall x \in \mathbb{A}^m, \forall \mathbf{x} \in \mathcal{B}(\mathbb{A}, \mathbb{R})^m, (\mathbf{x} \circ\!\!- x \implies f(\mathbf{x}) \circ\!\!- f(x))$$

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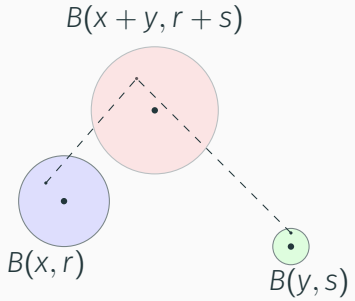

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- $\mathbb{A}_p$ machine numbers.

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- Assume $\epsilon \in \mathbb{R} \cap 2^{\mathbb{Z}}$ with $\epsilon \leq 1/16$ such that for all operations $* \in \{+, -, \cdot\}$ and all machine numbers $a, b \in \mathbb{A}_p$:

$$|a * b - a *_\circ b| \leq \epsilon |a *_\circ b|$$
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- If $\mathbb{A} = \mathbb{R}$ then $\epsilon = 2^{-p}$. If $\mathbb{A} = \mathbb{C}$ then $\epsilon = 4 \cdot 2^{-p}$.

Floating point arithmetic

- Implementation of ball arithmetic using floating point arithmetic:

$$\mathcal{B}(a,r) \pm \mathcal{B}(b,s) = \mathcal{B}(a \pm_\circ b, \uparrow [r + s + \epsilon|(a +_\circ b)|]$$
$$\mathcal{B}(a,r) \cdot \mathcal{B}(b,s) = \mathcal{B}(a \cdot_\circ b, \uparrow [(|a| + r) \cdot s + |b| \cdot r + \epsilon|a \cdot_\circ b|])$$

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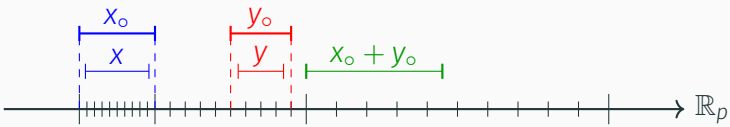

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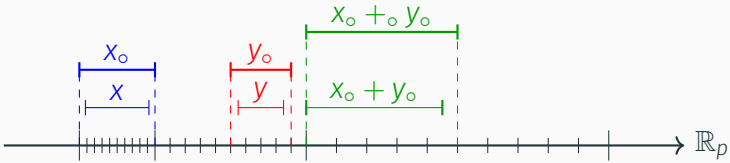


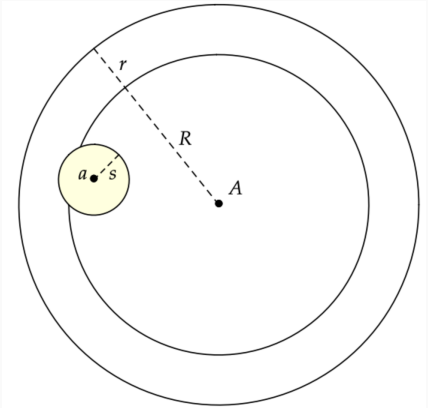
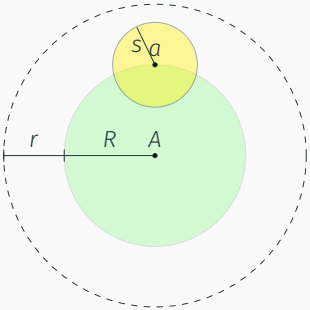


Figure 2: On the left matryoshki. On the right a matryoshka.

Matryoshki

- A matryoshka is a generalized ball for which the center is itself a ball.



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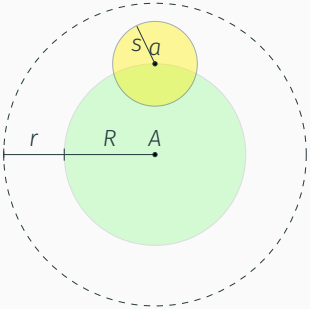
- A matryoshka is a generalized ball for which the center is itself a ball.
- Let $A \in \mathbb{A}, R, r \in \mathbb{R}^{\geq}$, a matryoshka $\boldsymbol{A} := \mathcal{B}(A, R, r)$



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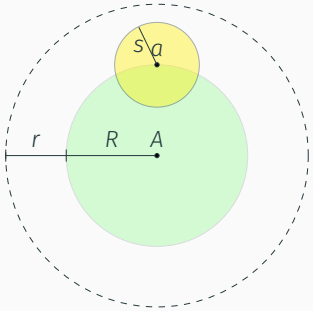
- Let $\boldsymbol{a} := \mathcal{B}(a, s)$,

$$\boldsymbol{A} \circ\!\!- \boldsymbol{a} \iff \mathcal{B}(A, R) \circ\!\!- a \text{ and } s \leqslant r.$$



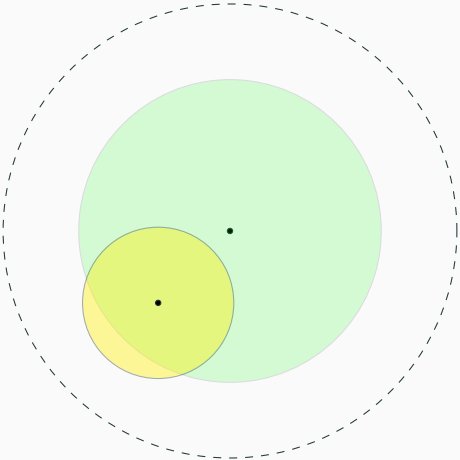


Figure 3: A matryoshka $\mathcal{B}(A, R, r)$ encloses a ball.

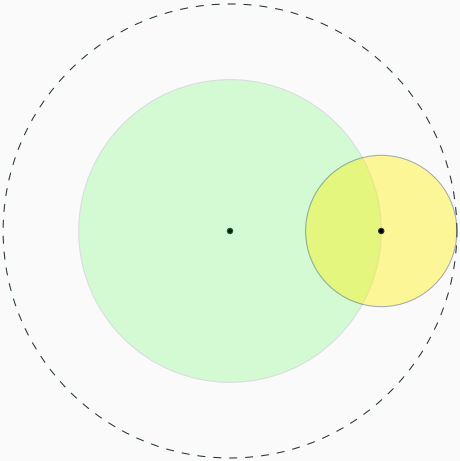


Figure 4: Example of a ball enclosed by the matryoshka $\mathcal{B}(A, R, r)$

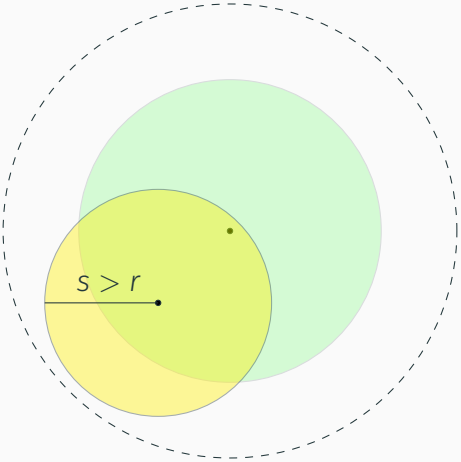


Figure 5: Example of a ball not enclosed by the matryoshka $\mathcal{B}(A, R, r)$.

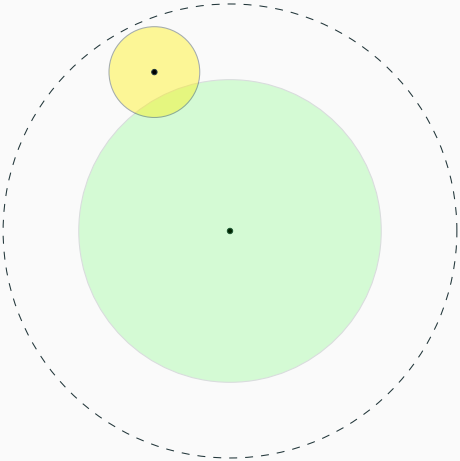
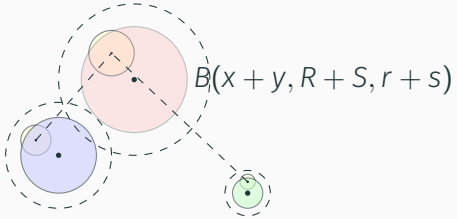


Figure 6: Example of a ball not enclosed by the matryoshka $\mathcal{B}(A, R, r)$.

Matryoshka lift

- Matryoshka lift of a function $f : \mathbb{A}^m \to \mathbb{A}^n$ is a function $f : \mathcal{B}(\mathbb{A}, \mathbb{R}, \mathbb{R})^m \to \mathcal{B}(\mathbb{A}, \mathbb{R}, \mathbb{R})^n$ that satisfies the inclusion principle

$$A \circ\!\!- a \quad \Longrightarrow \quad f(A) \circ\!\!- f(a)$$



Matryoshka arithmetic

Let $|\mathcal{B}(a,r)|_{\mathcal{B}(\mathbb{A},\mathbb{R})} = |a| + r$. Ring operations admit lifts:

$$\mathcal{B}(a,r) \pm \mathcal{B}(b,s) = \mathcal{B}(a \pm b, r+s)$$
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- For all normed algebra, $(\mathbb{A}, |\cdot|)$, $\mathbb{G} = (\mathcal{B}(\mathbb{A},\mathbb{R}), |\cdot|_{\mathcal{B}(\mathbb{A},\mathbb{R})})$ formally implements the structure of a normed algebra.

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- For all normed algebra, $(\mathbb{A}, |\cdot|)$, $\mathbb{G} = (\mathcal{B}(\mathbb{A},\mathbb{R}), |\cdot|_{\mathcal{B}(\mathbb{A},\mathbb{R})})$ formally implements the structure of a normed algebra.
- Matryoshka can be seen as balls over the formal normed algebra $\mathcal{B}(\mathbb{A},\mathbb{R})$.

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- Let $|\mathcal{B}(a,r)|_{\mathcal{B}(\mathbb{A},\mathbb{R},\mathbb{R})} = |a|_{\mathcal{B}(\mathbb{A},\mathbb{R})} + r$. Consequently, $(\mathcal{B}(\mathbb{A},\mathbb{R},\mathbb{R}), |\cdot|_{\mathcal{B}(\mathbb{A},\mathbb{R},\mathbb{R})})$ formally implements the structure of a normed algebra.

Matryoshki

$\mathcal{B}(a, r_1, r_2, ..., r_k)$ can be seen as a ball whose center is itself a ball whose center... is itself a ball, where the dots repeat the sentence $k - 1$ times.

Figure 7: Matryoshki

- Let $\boldsymbol{a} = \mathcal{B}(a, r), \boldsymbol{b} = \mathcal{B}(b, s)$ be two balls, define $\boldsymbol{a} -_{\text{vec}} \boldsymbol{b} = \mathcal{B}(a - b, r - s)$.
- Let $\epsilon \in \mathbb{R} \cap 2^{\mathbb{Z}}$ with $\epsilon \leq 1/16$. Corresponding last bit error for balls:

$$|\boldsymbol{a} * \boldsymbol{b} -_{\text{vec}} \boldsymbol{a} *_\circ \boldsymbol{b}| \leq \epsilon |\boldsymbol{a} *_\circ \boldsymbol{b}|$$

- Implementation in computer needs to take care of the rounding errors:

$$\mathcal{B}(\boldsymbol{a}, r) \pm \mathcal{B}(\boldsymbol{b}, s) = \mathcal{B}(\boldsymbol{a} \pm_\circ \boldsymbol{b}, \uparrow [r + s + \bar{\epsilon}_\circ(\boldsymbol{a} \pm \boldsymbol{b})])$$
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Straight Line Programs

All simple functions built of arithmetic operations $(+, -, \cdot)$ can be computed using straight line programs.

For instance
$f : (a_1, a_2) \mapsto 5a_1a_2 + a_1$ can be computed by Γ.

Straight-line program Γ:

$$\begin{array}{lllll} \Gamma_1 & \equiv & X_1 & := & I_1 \times I_2 \\ \Gamma_2 & \equiv & X_2 & := & X_1 \times 5 \\ \Gamma_3 & \equiv & X_3 & := & X_2 + I_1 \end{array}$$

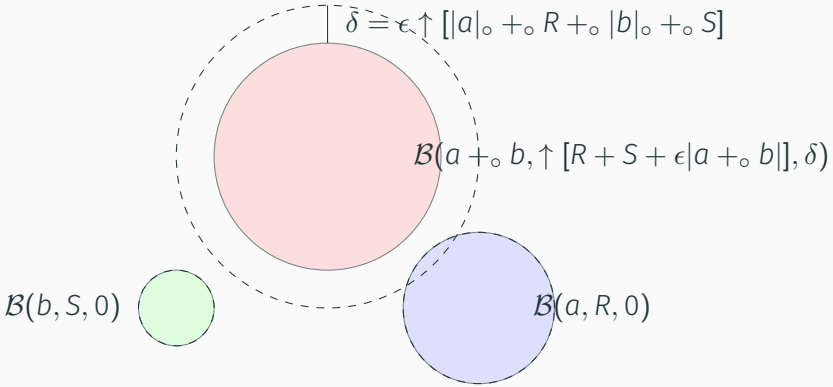

Figure 8: Sum of two particular matryoshka

Proposition 1

$f : \mathbb{A}^m \to \mathbb{A}^n$ any straight line program.
Ball ansatz: $A = (A_1, \ldots, A_m) \in \mathcal{B}(\mathbb{A}_p, \mathbb{R}_p)^m$
Matryoshka $\mathcal{B}(A, 0) := (\mathcal{B}(A_1, 0), \ldots, \mathcal{B}(A_m, 0))$
Ball lift: $\boldsymbol{f}$, Matryoshka lift: $\boldsymbol{F}$
Assume $\boldsymbol{F}(\mathcal{B}(A, 0)) = (\mathcal{B}(C_1, E_1), \ldots, \mathcal{B}(C_n, E_n))$.

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Ball ansatz: $\boldsymbol{A} = (\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m) \in \mathcal{B}(\mathbb{A}_p, \mathbb{R}_p)^m$
Matryoshka $\mathcal{B}(\boldsymbol{A}, 0) := (\mathcal{B}(\boldsymbol{A}_1, 0), \ldots, \mathcal{B}(\boldsymbol{A}_m, 0))$
Ball lift: $\boldsymbol{f}$, Matryoshka lift: $\boldsymbol{F}$
Assume $\boldsymbol{F}(\mathcal{B}(\boldsymbol{A}, 0)) = (\mathcal{B}(\boldsymbol{C}_1, E_1), \ldots, \mathcal{B}(\boldsymbol{C}_n, E_n))$.

Then for all $a \in \mathbb{A}_p^m$
where $\boldsymbol{A} \circ\!\!- a$, we have

$$|f_{\circ,i}(a) - f_i(a)| \leqslant E_i.$$

Static rounding errors: proof

- Ball lift property: $\mathcal{B}(a, 0) \circ\!\!- a$ then $\boldsymbol{f}(\mathcal{B}(a, 0)) \circ\!\!- f(a)$.

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- Then $r_i \leq E_i$.
- Then $|f_\circ(a) - f(a)| \leq r_i \leq E_i$.

Efficient ball lifts

Compute bounds $(B_{i,j})_{1\leq i\leq m, 1\leq j\leq n}$ on the Jacobian matrix:

$$\left\| \frac{\partial f_i}{\partial x_j} \right\|_{\boldsymbol{A}} := \sup_{A \multimap a} \left| \frac{\partial f_i}{\partial x_j}(a) \right| \leqslant B_{i,j}.$$

Evaluate a ball lift of the Jacobian of f at $\boldsymbol{A}$, which yields a matrix $\boldsymbol{J} \in \mathcal{B}(\mathbb{A}_p, \mathbb{R}_p)^{n\cdot m}$, after which let $B_{i,j} := |\boldsymbol{J}_{i,j}|$.

Proposition 2

$f : \mathbb{A}^m \to \mathbb{A}^n$ any straight line program.
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Then for all $\boldsymbol{a} = \mathcal{B}(a, r) \in \mathcal{B}(\mathbb{A}_p, \mathbb{R}_p)^m$ such that $\boldsymbol{a}_1 \subseteq \boldsymbol{A}_1, \ldots, \boldsymbol{a}_m \subseteq \boldsymbol{A}_m$

$$\boldsymbol{f}_*(\boldsymbol{a}) := \mathcal{B}\left(f_\circ(a), E + Br\right).$$

Efficient ball lifts: rounding

Proposition 3
Let $\lg m = \lceil \log_2(m) \rceil$, $\epsilon := 2^{-p}$, $\eta := 2^{E_{\min}-p+1}$, and ball ansatz: $\boldsymbol{A} = (\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m)$.
Assume that:

- $(\lg m)^2 < \epsilon^{-1}$
- $\boldsymbol{F}(\mathcal{B}(\boldsymbol{A}, 0)) = (\mathcal{B}(\boldsymbol{C}_1, E_1), \ldots, \mathcal{B}(\boldsymbol{C}_n, E_n)$

Then for all $\boldsymbol{a} = \mathcal{B}(a, r) \in \mathcal{B}(\mathbb{A}_p, \mathbb{R}_p)^m$ such that $\boldsymbol{a}_1 \subseteq \boldsymbol{A}_1, \ldots, \boldsymbol{a}_m \subseteq \boldsymbol{A}_m$,

$$\boldsymbol{f}_*(\boldsymbol{a}) := \mathcal{B}\left(f_\circ(a), \circ[(E + Br)(1 + (\lg m + 8)\epsilon) + (m+1)\eta]\right).$$

defines a ball lift of $f_{|\boldsymbol{A}}$.

Application to polynomial evaluation: Homogenization

- $P \in \mathbb{K}[x_1, \ldots, x_m]$
- $P^{\text{hom}} \in \mathbb{K}(x_1, \ldots, x_m, t)$ such that all its monomials are of same degree and $P^{\text{hom}}(x_1, \ldots, x_m, 1) = P(x_1, \ldots, x_m)$.

Ex: $P = XY + Z + X^2ZY$, then $P^{\text{hom}} = XYT^2 + ZT^3 + X^2ZY$.

$$\lambda^d P^{\text{hom}}(x_1, x_2, \ldots, x_m, t) = P^{\text{hom}}(\lambda x_1, \lambda x_2, \ldots, \lambda x_m, \lambda t)$$

Application to polynomial evaluation: Projective bounds

For all $x \in \mathbb{K}^m$, let $\lambda := \max_{i=1,\dots,m}(|x_i|, 1)$. Then $(x, 1)/\lambda \in \mathcal{B}(0, 1)$. Apply Proposition 3 with $f = P^{\text{hom}}$ and domain $A = \mathcal{B}(0, 1)$.

$$x \xrightarrow{\quad\lambda\quad} x^{\text{hom}} \xrightarrow{\quad P^{\text{hom}}\quad} P^{\text{hom}}(x^{\text{hom}}) \xrightarrow{\quad\lambda^d\quad} P(x)$$

Figure 9: Scheme for certifying a polynomial evaluation

Benchmark

slp	prefp	prelip	preball	fp	lip	ball
test1	21.782	510.672	382	0.011	0.012	0.030
det 2	20.381	444.718	149	0.010	0.013	0.513
det 3	30.147	800.342	183	0.010	0.031	0.054
det 4	32.157	1676.93	304	0.013	0.030	0.109
det 5	55.217	4435.34	442	0.027	0.031	0.729
det 6	97.460	12342.2	723	0.068	0.066	0.333
det 7	297.25	39931.9	1463	0.219	0.225	0.928
det 8	491.46	102470	2794	0.369	0.374	6.797
det 9	1128.5	264722	5961	0.830	1.301	10.64
det 10	2365.3	687691	10907	2.504	5.126	17.70
det 11	4717.3	1791649	21818	5.601	11.79	37.01

Table 1: Time to compute: $\det(A) = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$ in microsecond

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