Computational Geometry and Topology
Géométrie et topologie algorithmiques

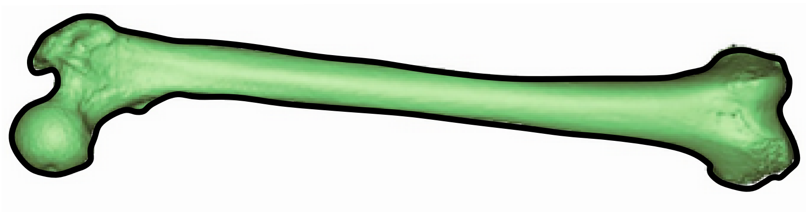
Steve OUDOT *Pooran MEMARI*

Acknowledgments of Involved Colleagues:

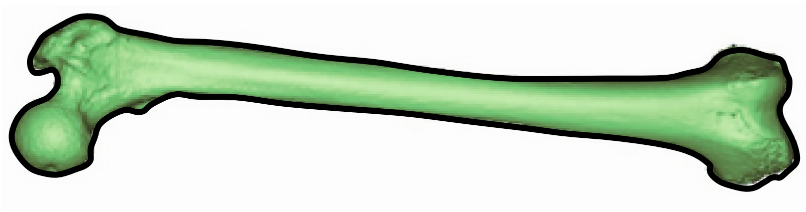
Jean Daniel Boissonnat, Frederic Chazal, Marc Glisse,
As well as Olivier Devillers and Luca Castelli

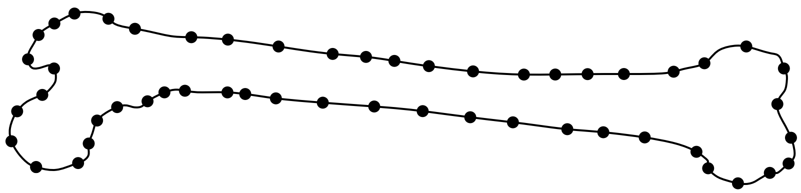
This lecture: slides courtesy of
Luca Castelli and Steve Oudot

Sampling and reconstruction issues



Sampling and reconstruction: goals and issues





Assume the sample is dense enough, the recontruction should be *homeomorphic* and (geometrically) close to the original shape

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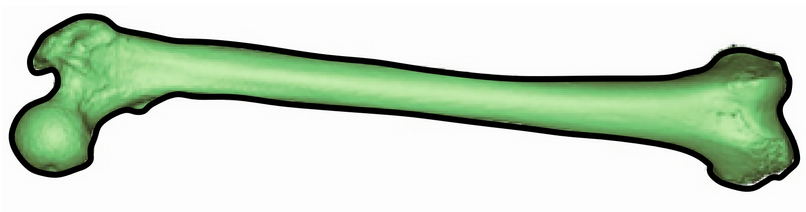

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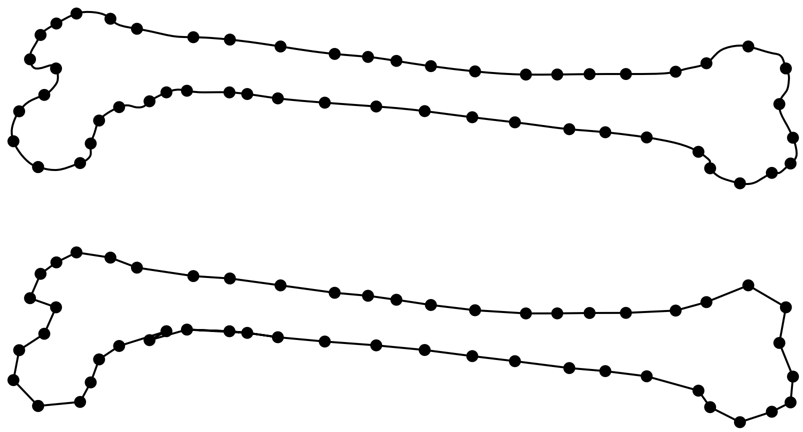

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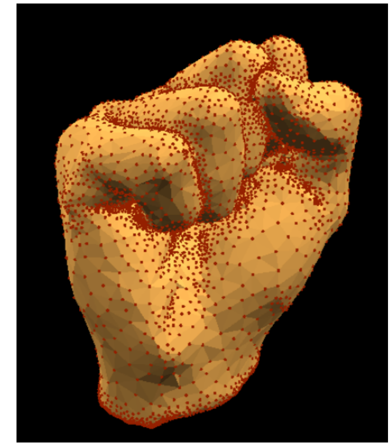

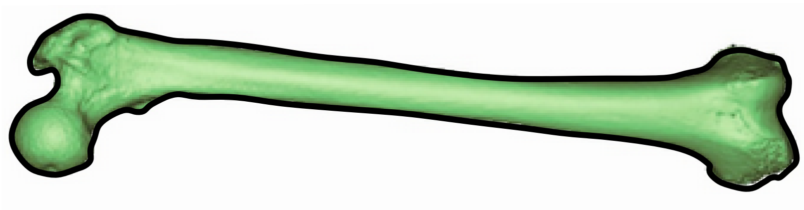

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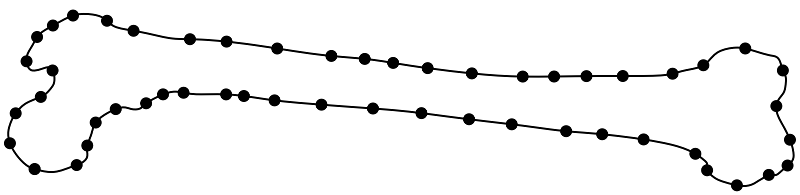

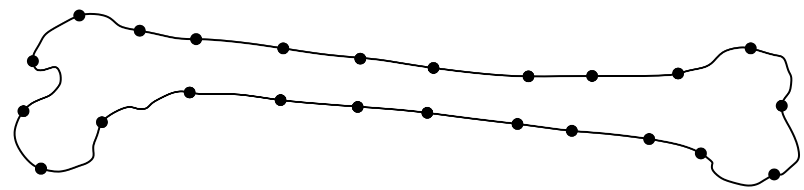

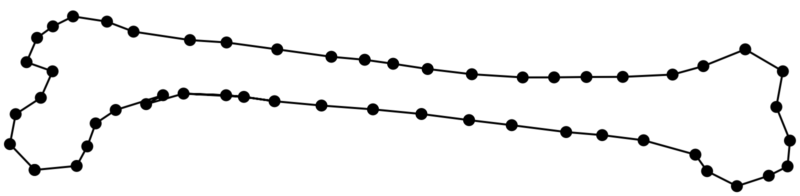

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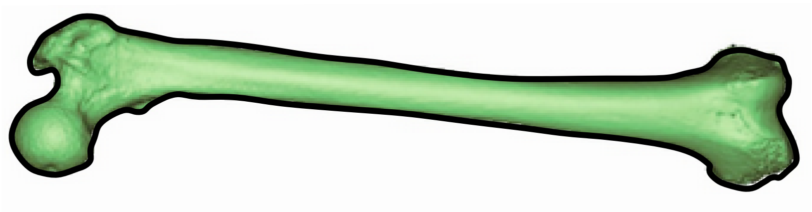

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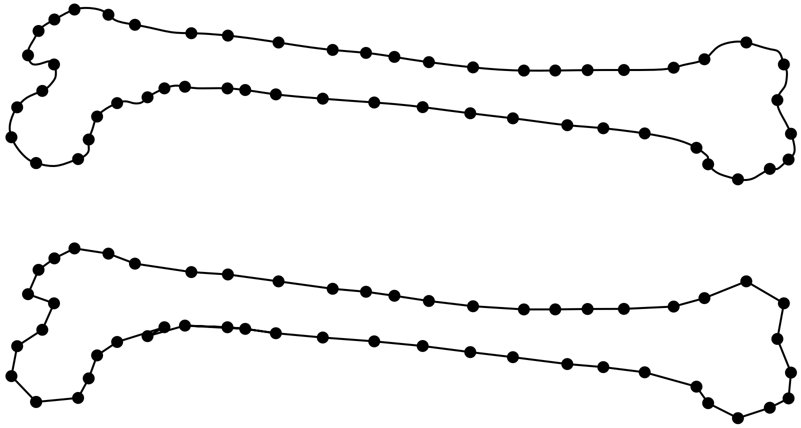
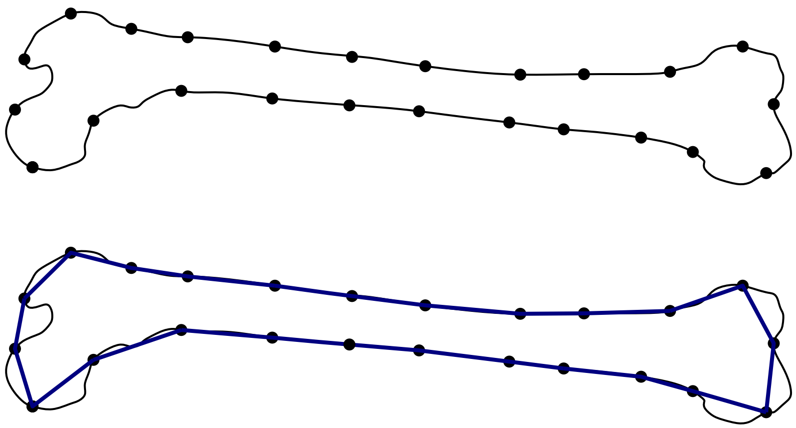
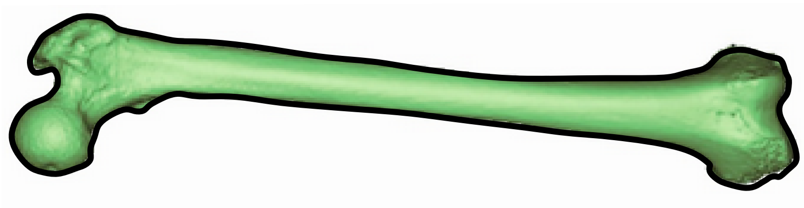




The sampling should be dense where curvature is high

Sampling and reconstruction: goals and issues



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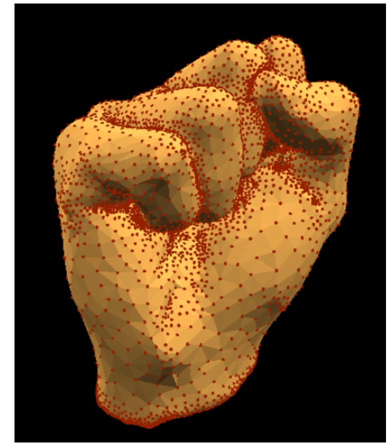

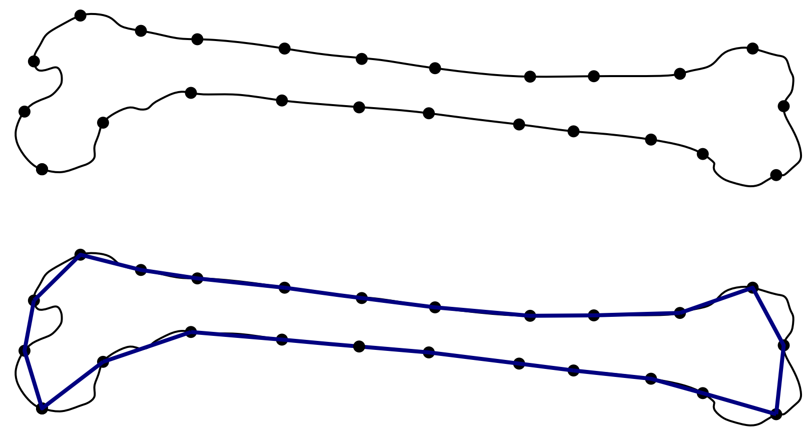

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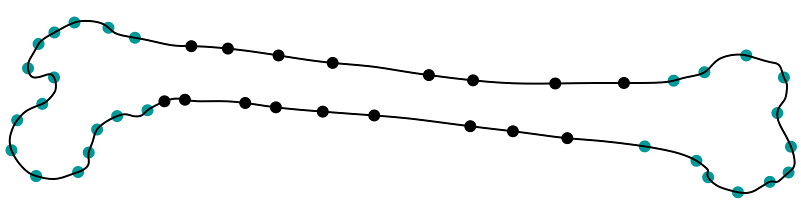

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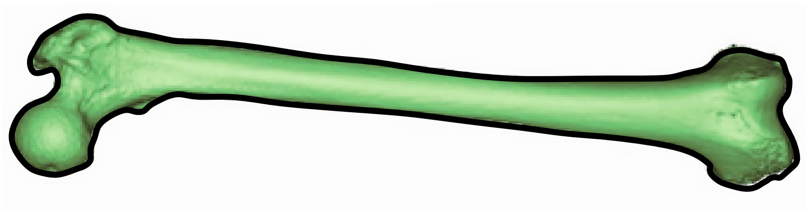

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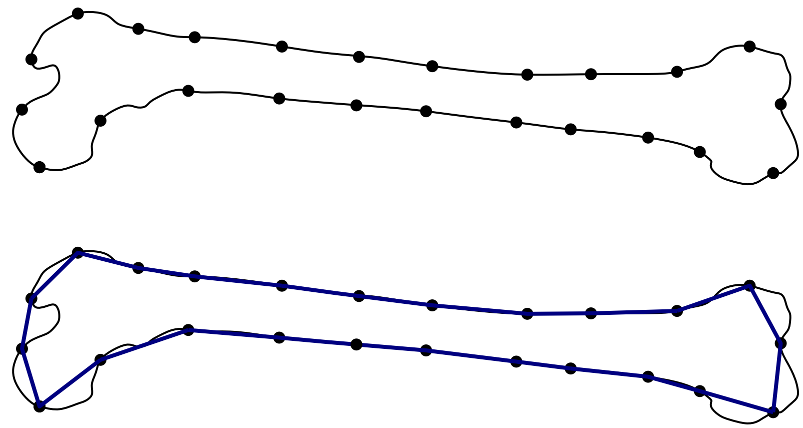

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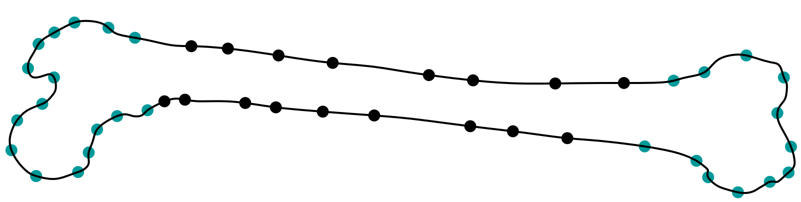
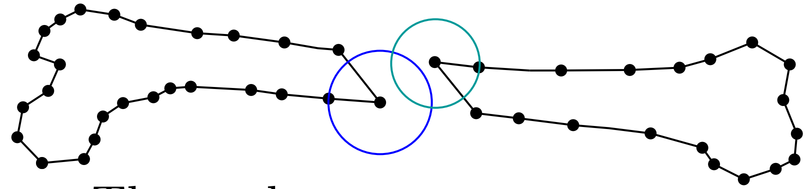




The only curvature parameter does not suffice

Various reconstruction techniques

Delaunay-based

- Crust / Power Crust
- Cocone
- Gabriel / α-shape / β-skeleton
- flow complex

Implicitization

- Local polynomial fitting
- Natural Neighbors (Voronoi-based)
- Radial Basis Functions

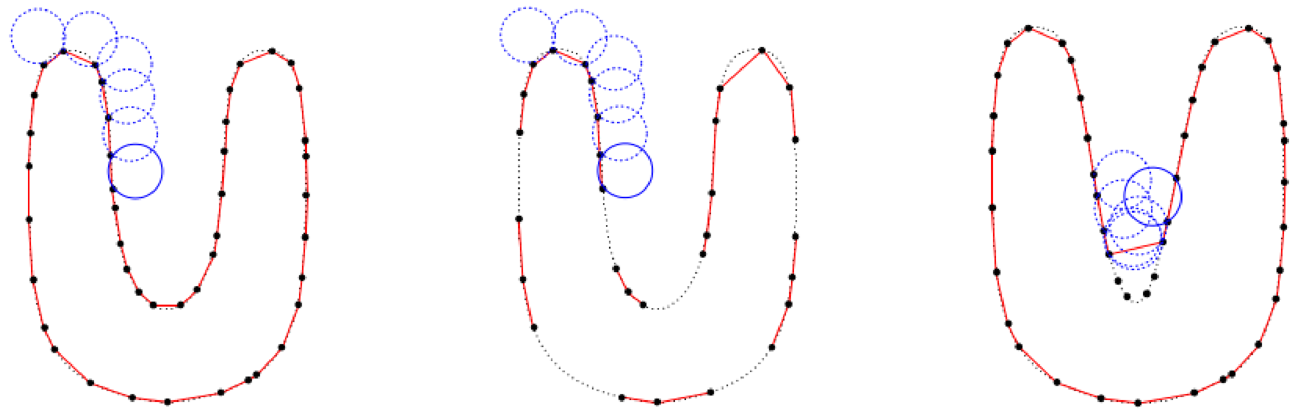
Projection operators

- Moving Least Squares
- Extremal surfaces

For arbitrary dimensions and co-dimensions

- Unions of balls / nerves
- Witness Complex

Various reconstruction techniques


Ball Pivoting (Bernardini et al.)

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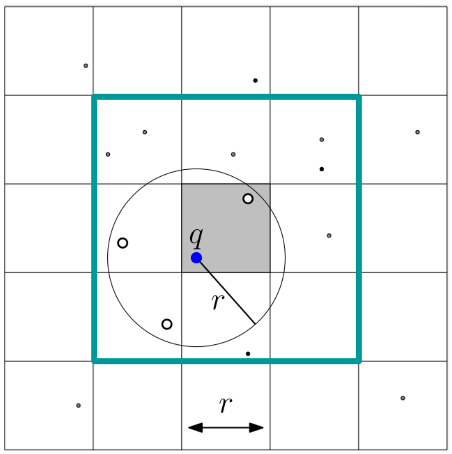

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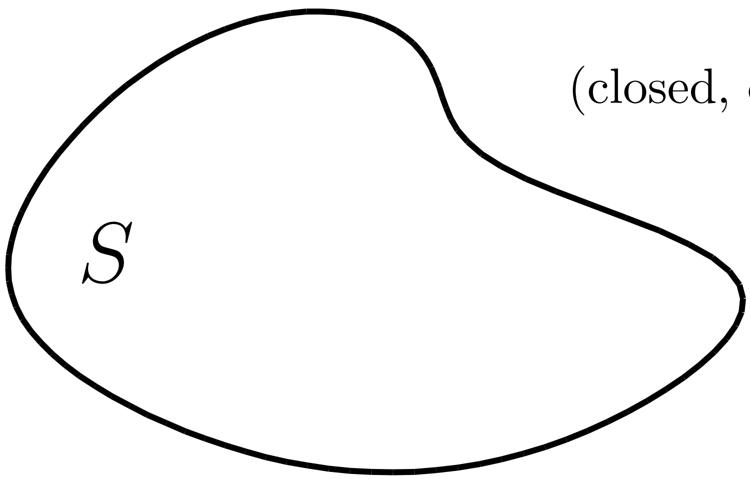
General assumptions

we assume S is a *smooth curve*

(closed, compact, twice differentiable 1-manifold without boundary)

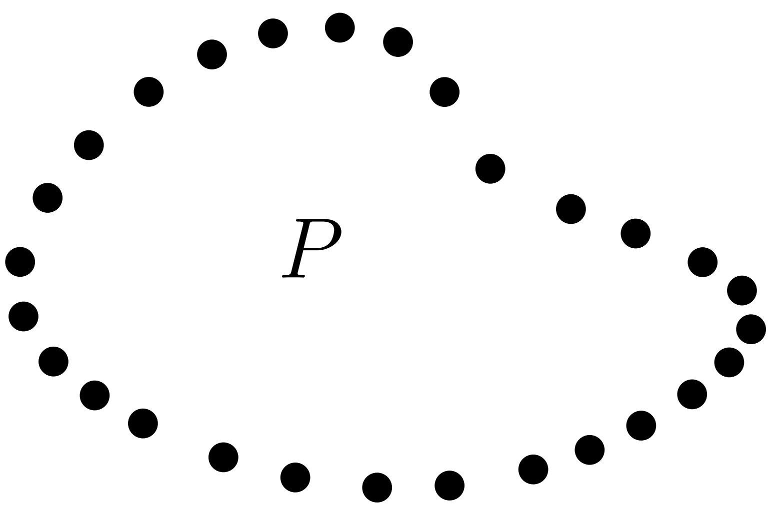


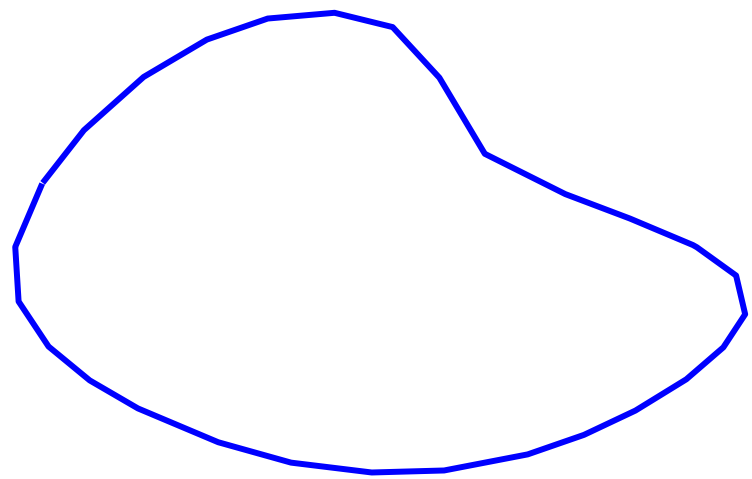
S can have multiple connected components

S cannot have: endpoints, branches, self-intersections

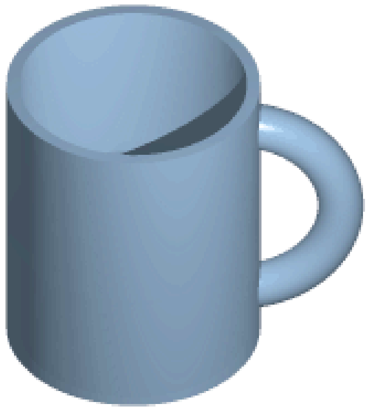
Goal: compute a *polygonal reconstruction* of S from P

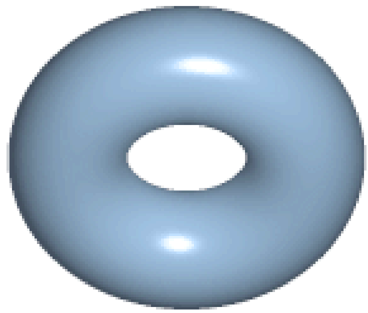
(a graph connecting consecutive points of P on S)

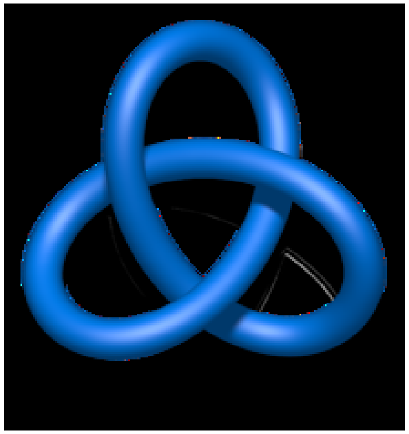




Quality of the reconstruction: topological equivalences





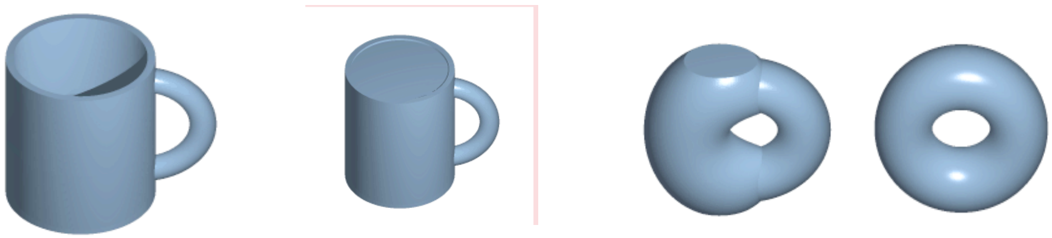


These three surfaces are *homeomorphic* (they all have genus 1)

There exists a continuous bijection between surfaces, whose inverse is also continuous

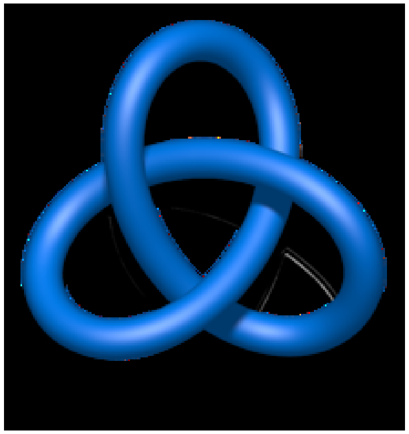
homeomorphisms are weak equivalences (they do not take into account the ambient space)

isotopic surfaces (unknotted torus)



There exists a family of deformations which continuously transform the surfaces

knotted torus



Simplicial complexes

abstract simplicial complex K (set of simplices)

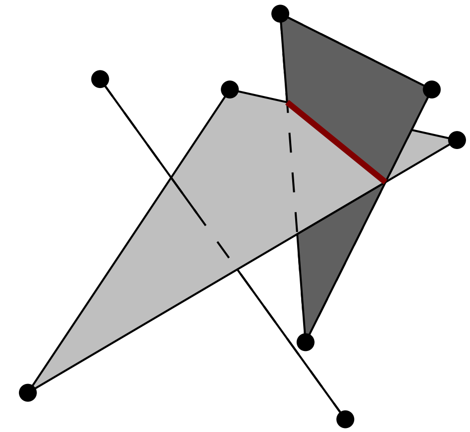
$$V = \{v_0, v_1, \ldots, v_{n-1}\}$$
$$E = \{\{i,j\}, \{k,l\}, \ldots\}$$
$$F = \{\{i,j,k\}, \{j,i,l\}, \ldots\}$$

inclusion property:

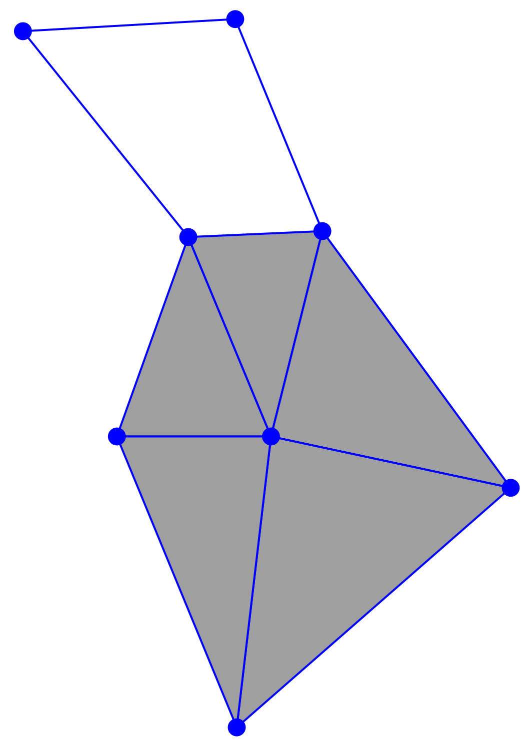
$$\rho \in K \text{ and } \sigma \subset \rho \longrightarrow \sigma \in K$$

intersection property:

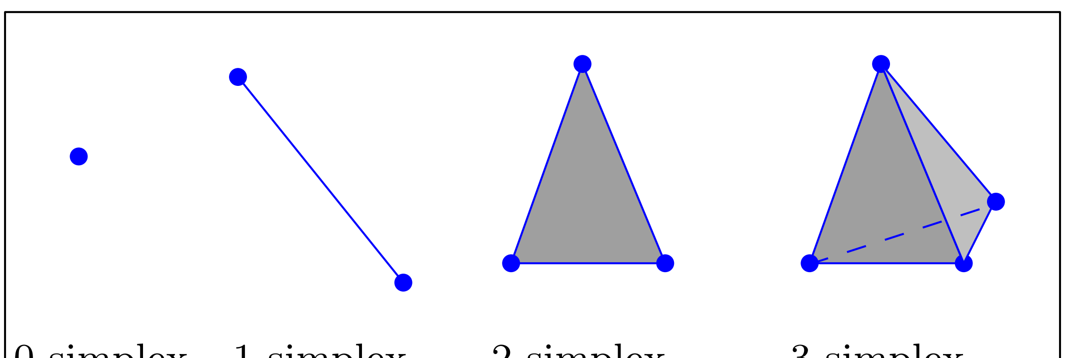
given two simplices σ_1, σ_2 of K, the intersection $\sigma_1 \cup \sigma_2$ is a face of both



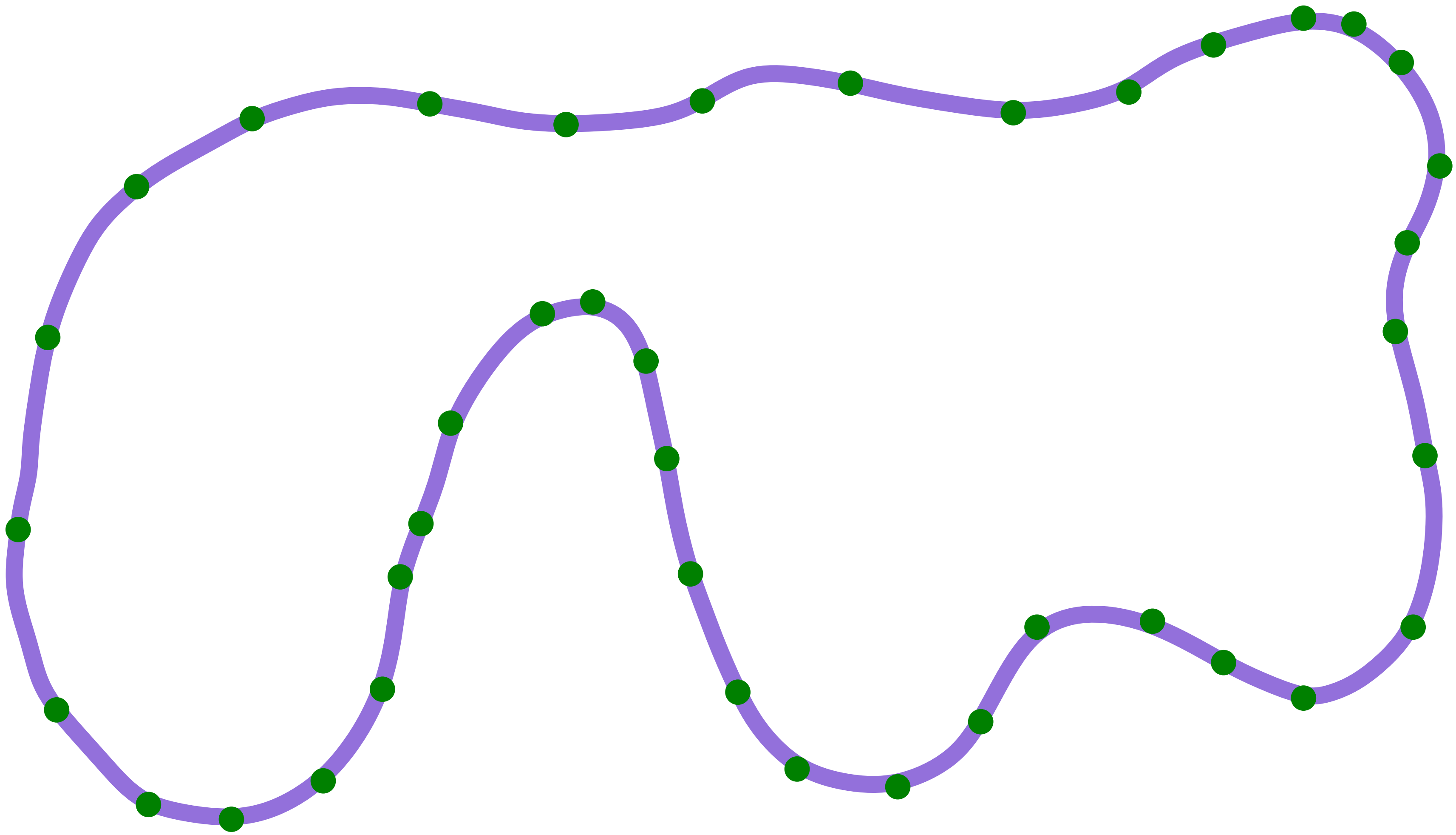
not valid simplicial complex



valid simplicial complex



What Delaunay has to do with reconstruction



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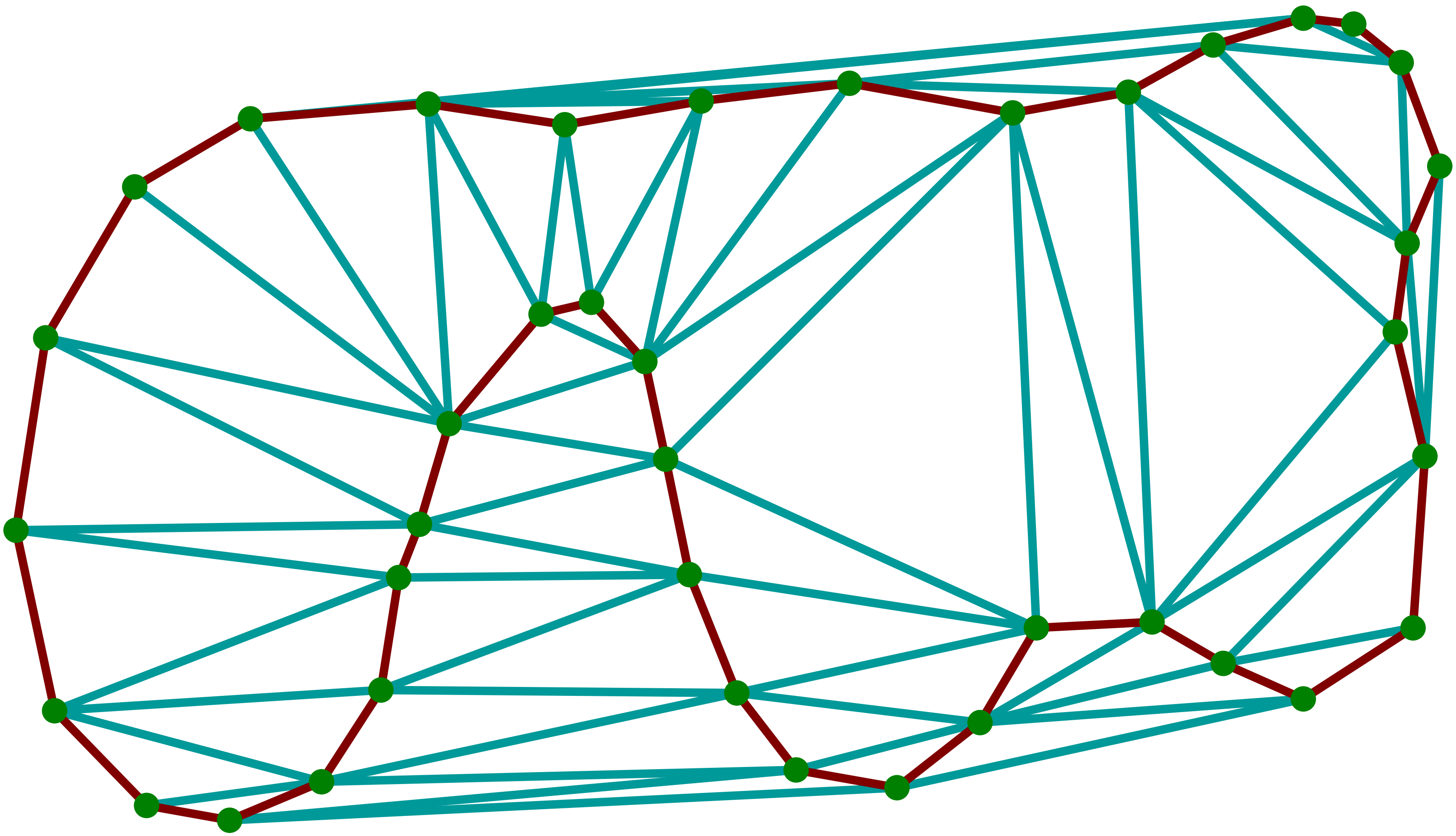

→ a faithful approximation of the curve appears as a subcomplex of the Delaunay

→ this should hold whenever the point cloud is sufficiently densely sampled along the curve

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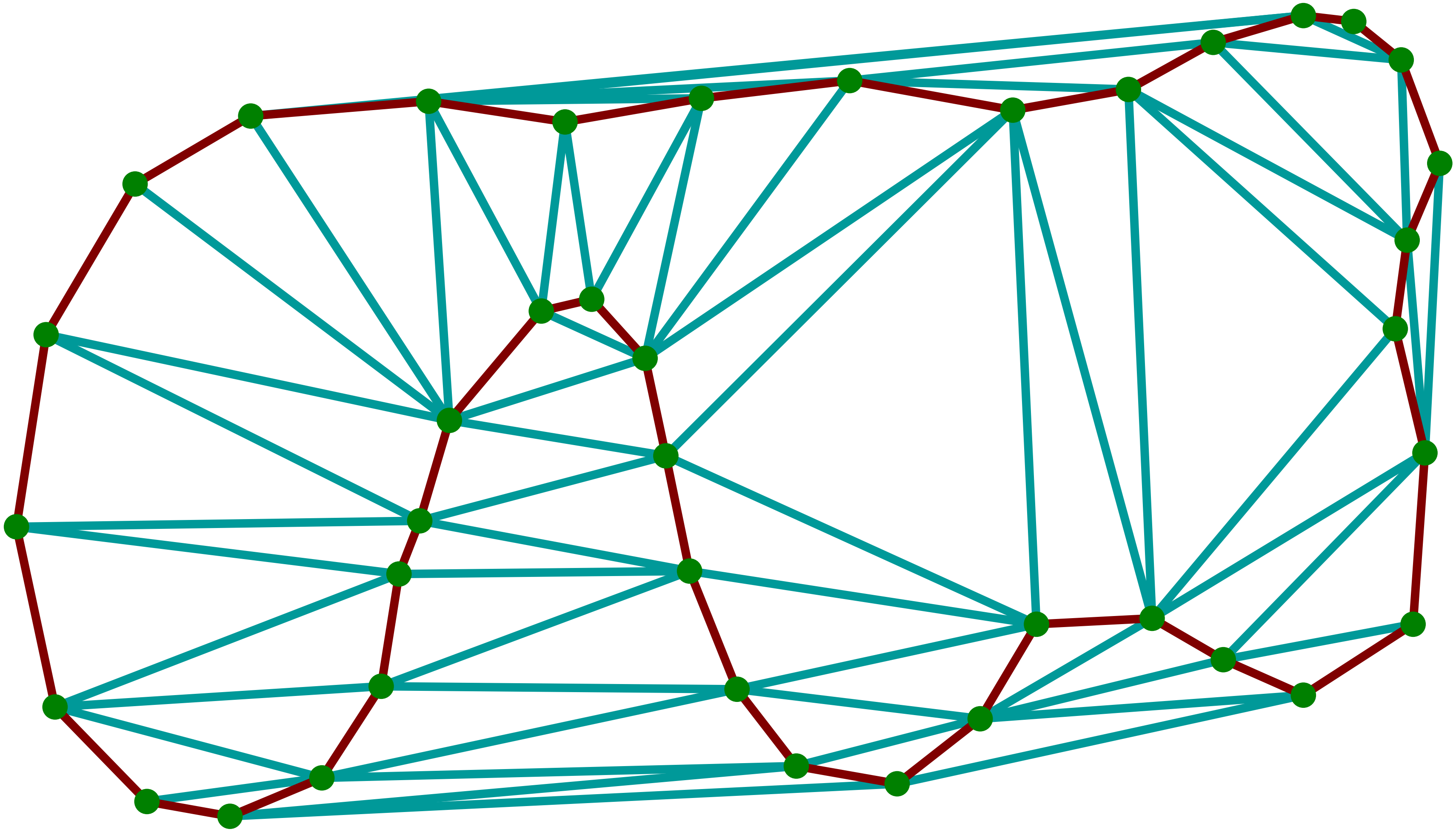

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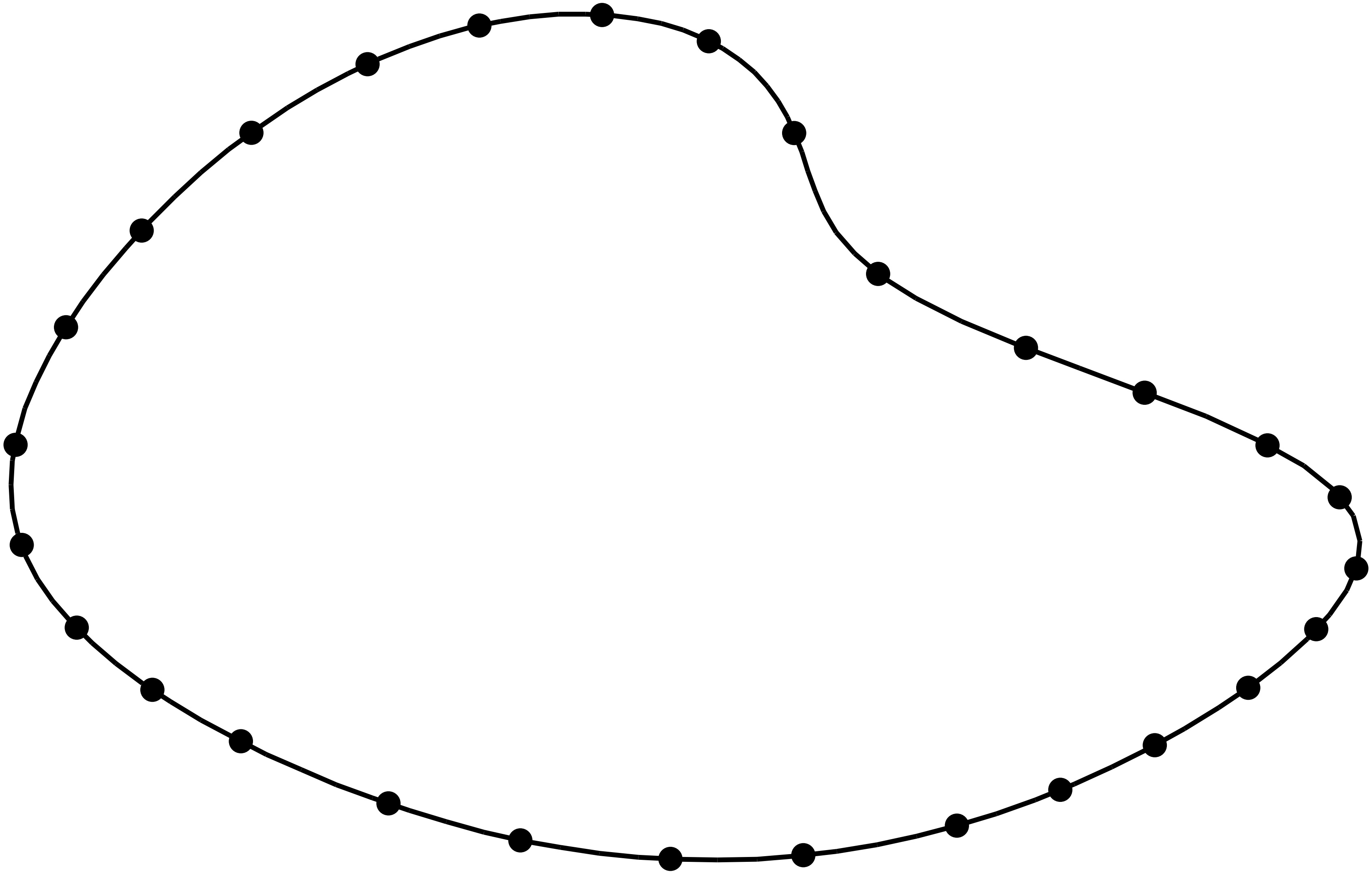
Q What is this *good* subcomplex? Can it be defined in some canonical way?

Restricted Delaunay triangulation



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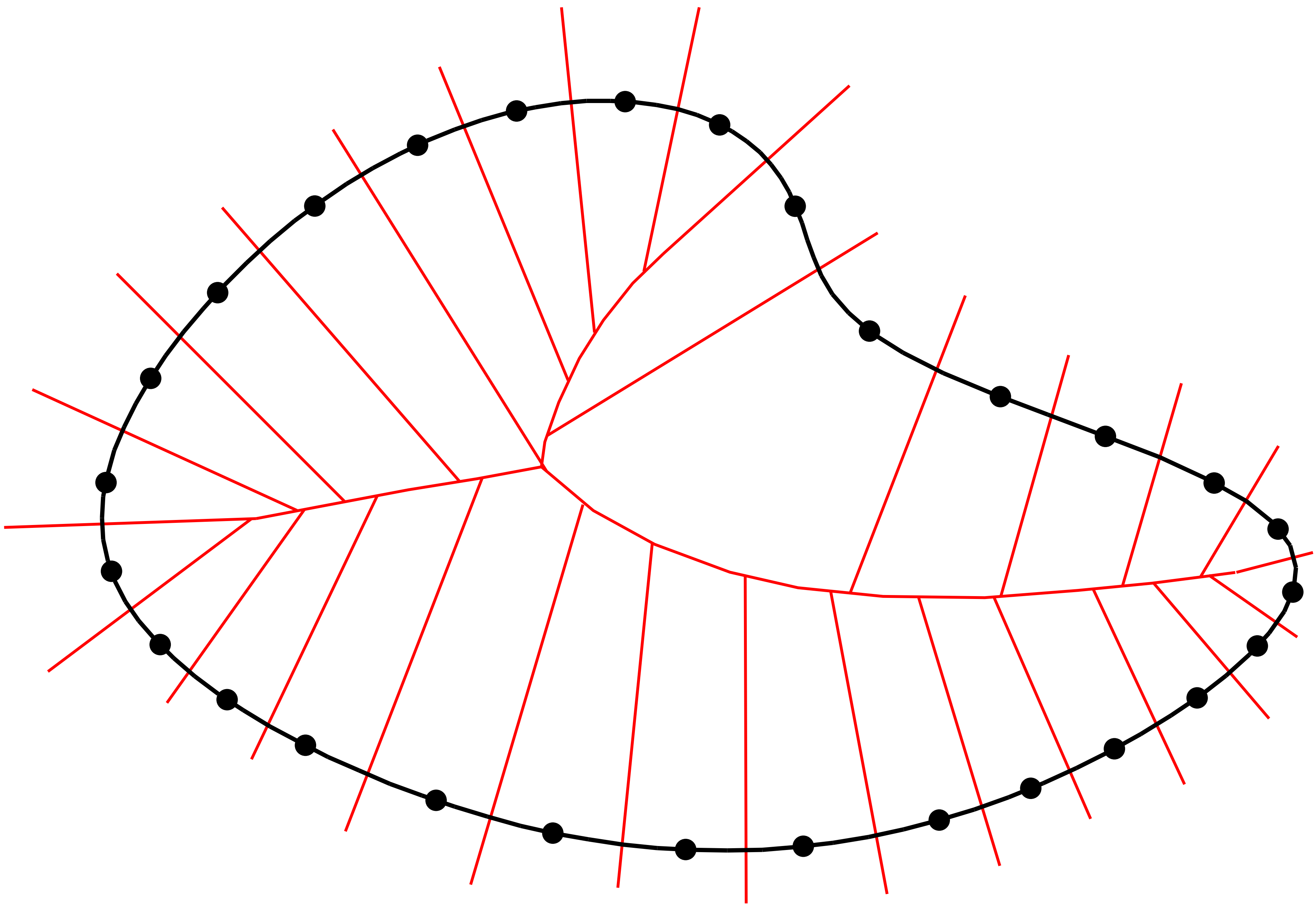
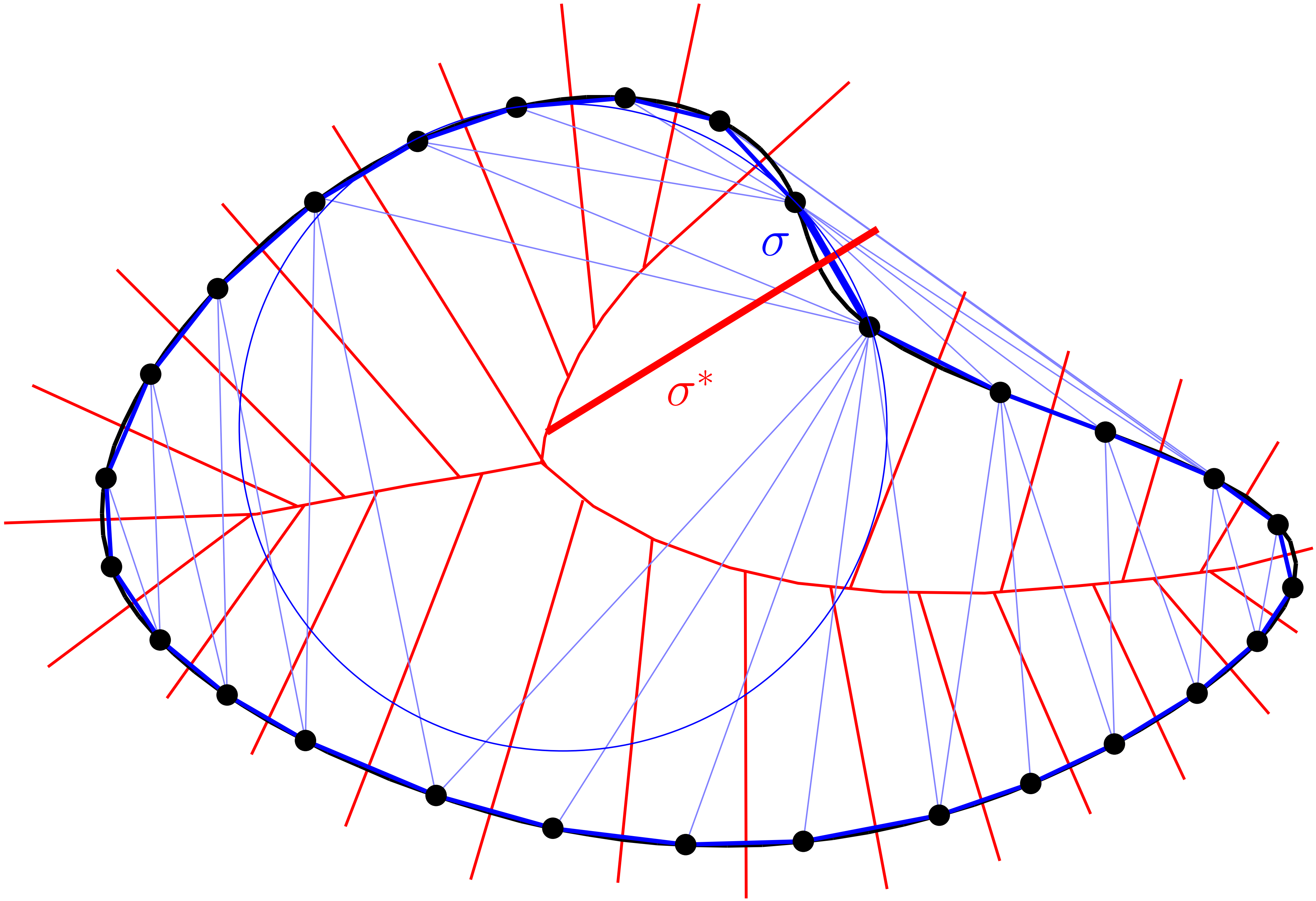


Restricted Delaunay triangulation

Def: $\mathrm{Del}_{\mathcal{S}}(P) := \{\sigma \in \mathrm{Del}(P) \mid \sigma^* \cap \mathcal{S} \neq \emptyset\}$



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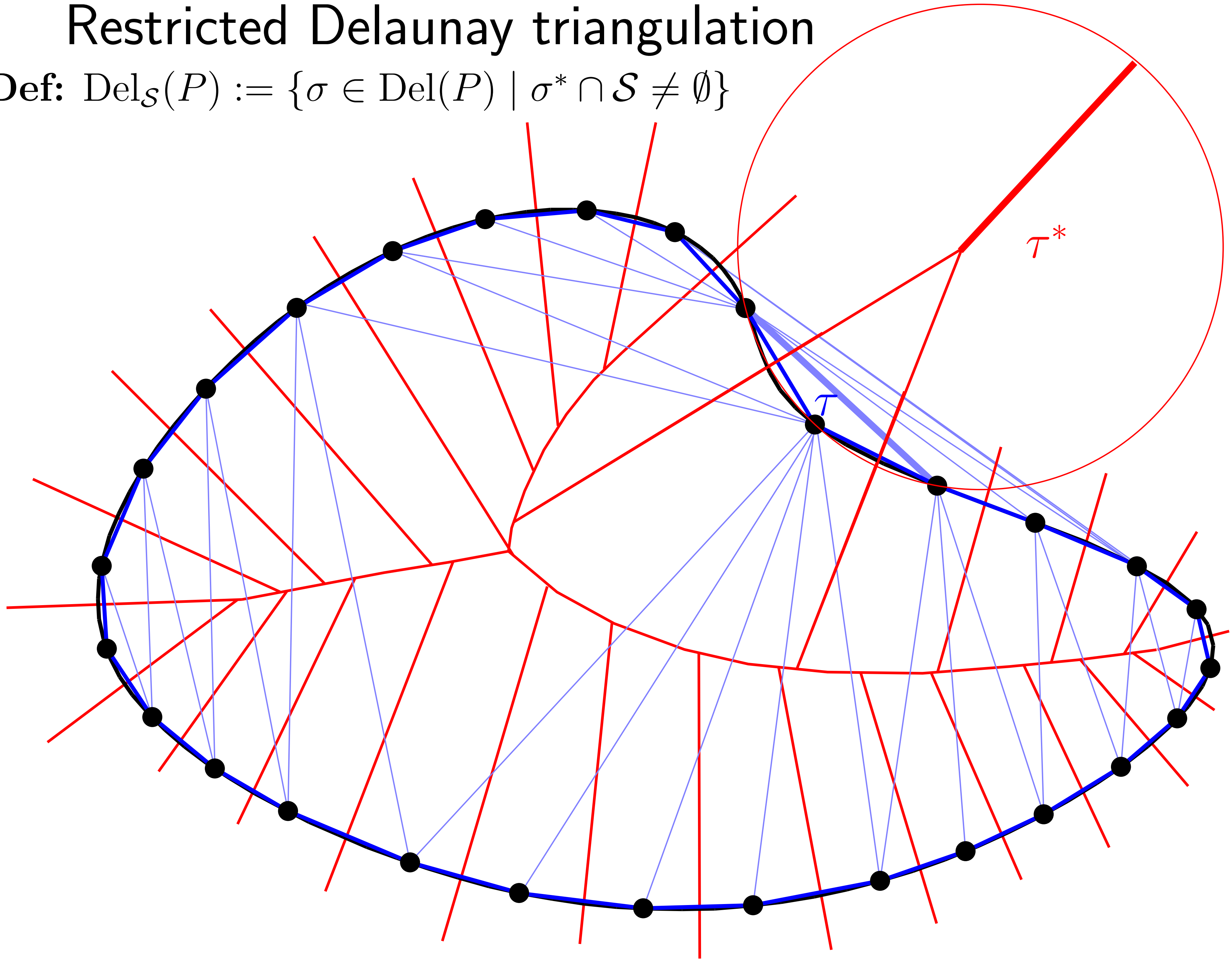
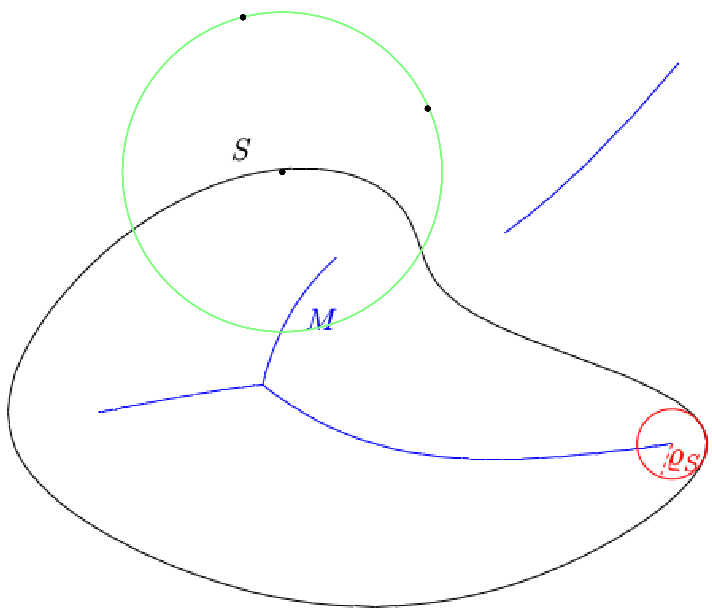
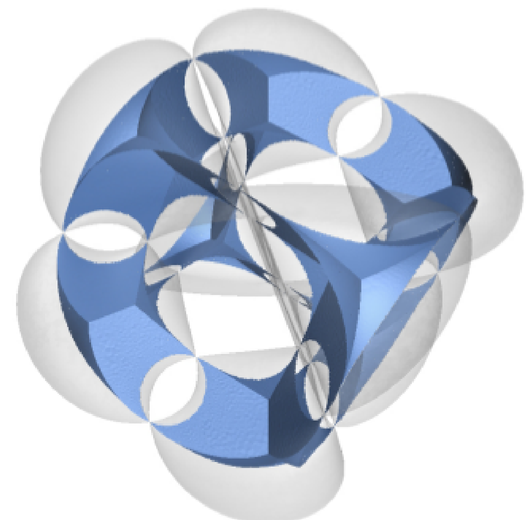


Medial axis

Def: $M_{\mathcal{S}}$ is the closure of the set of points of $\mathbb{R}^d$ that have ≥ 2 nearest neighbors on $\mathcal{S}$.

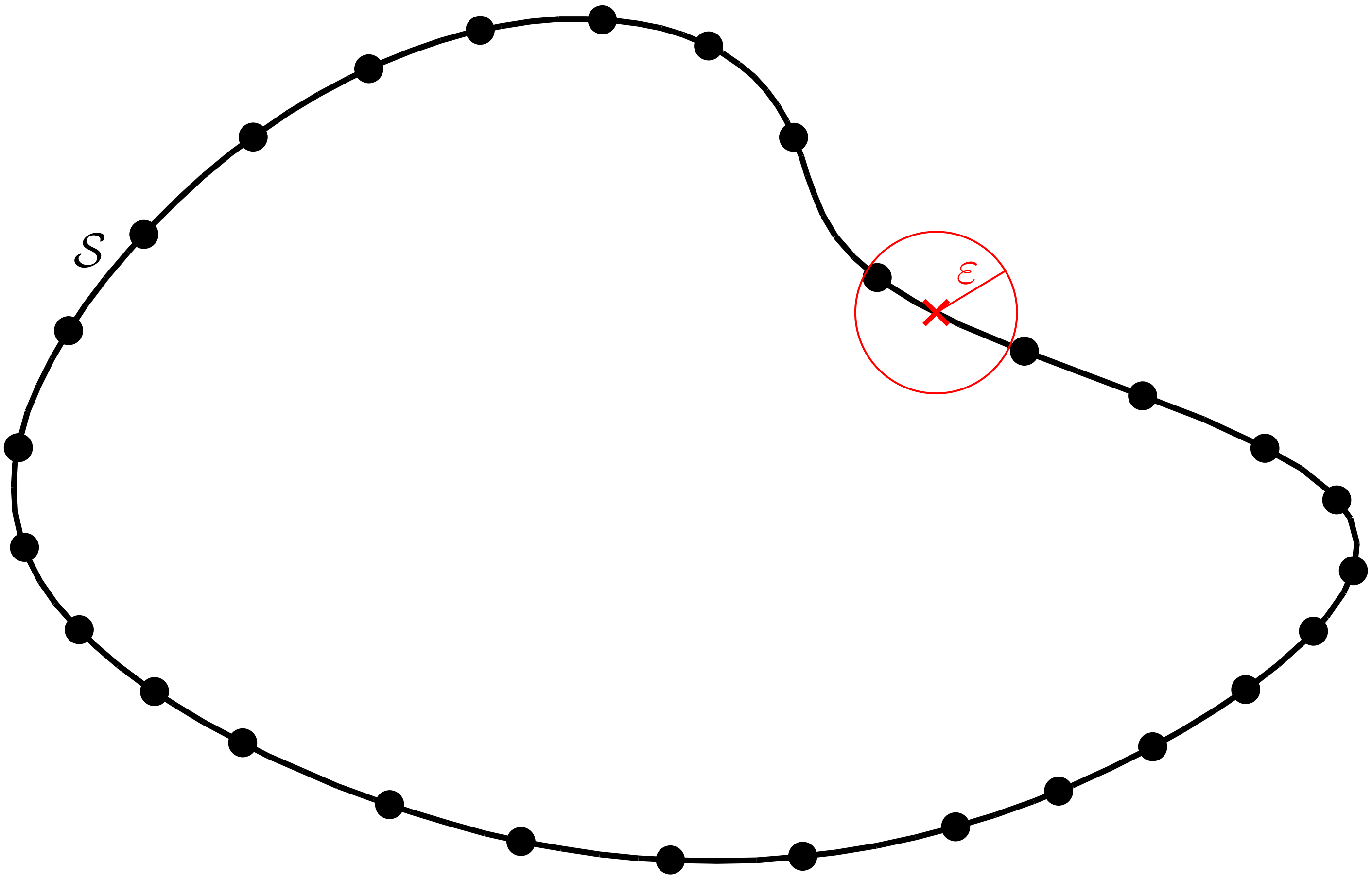
Def (equivalent): locus of centers of maximal inscribed circles (spheres)





ε-samples

Def: P is an ε-sample of $\mathcal{S}$ if $\forall x \in \mathcal{S}$, $\min\{\|x - p\| \mid p \in P\} \leq \varepsilon$.



Shapes with positive reach [Federer 1958]

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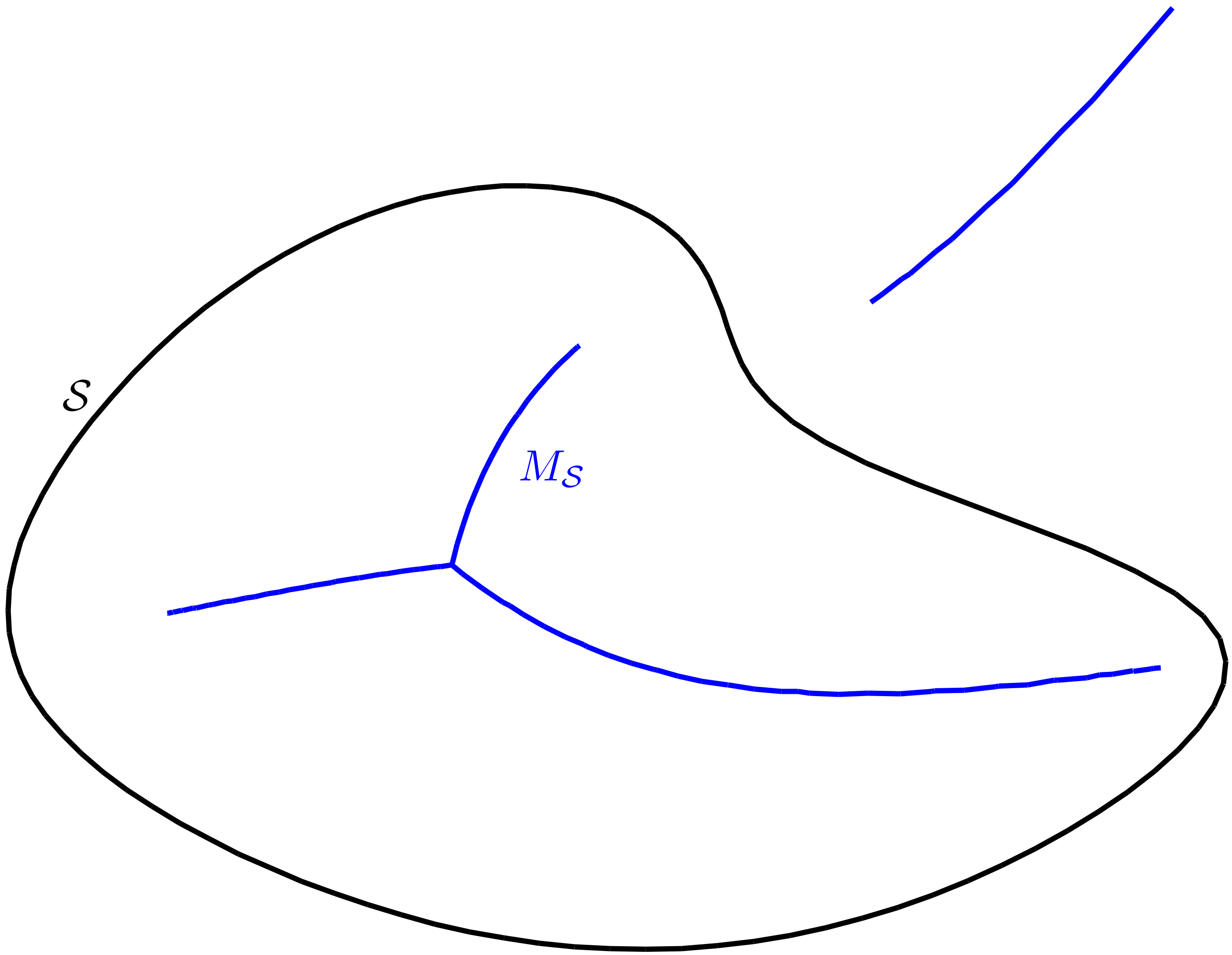

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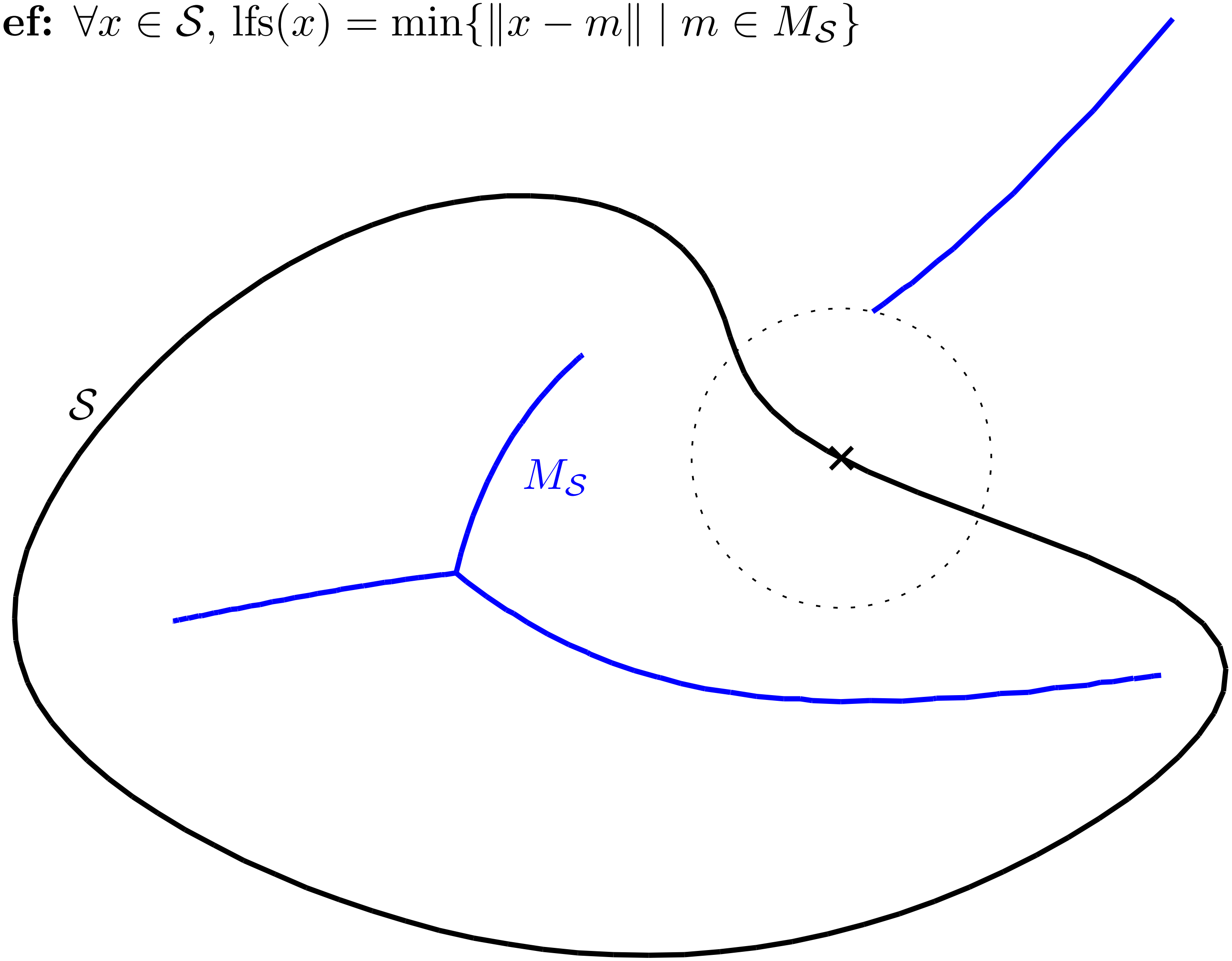

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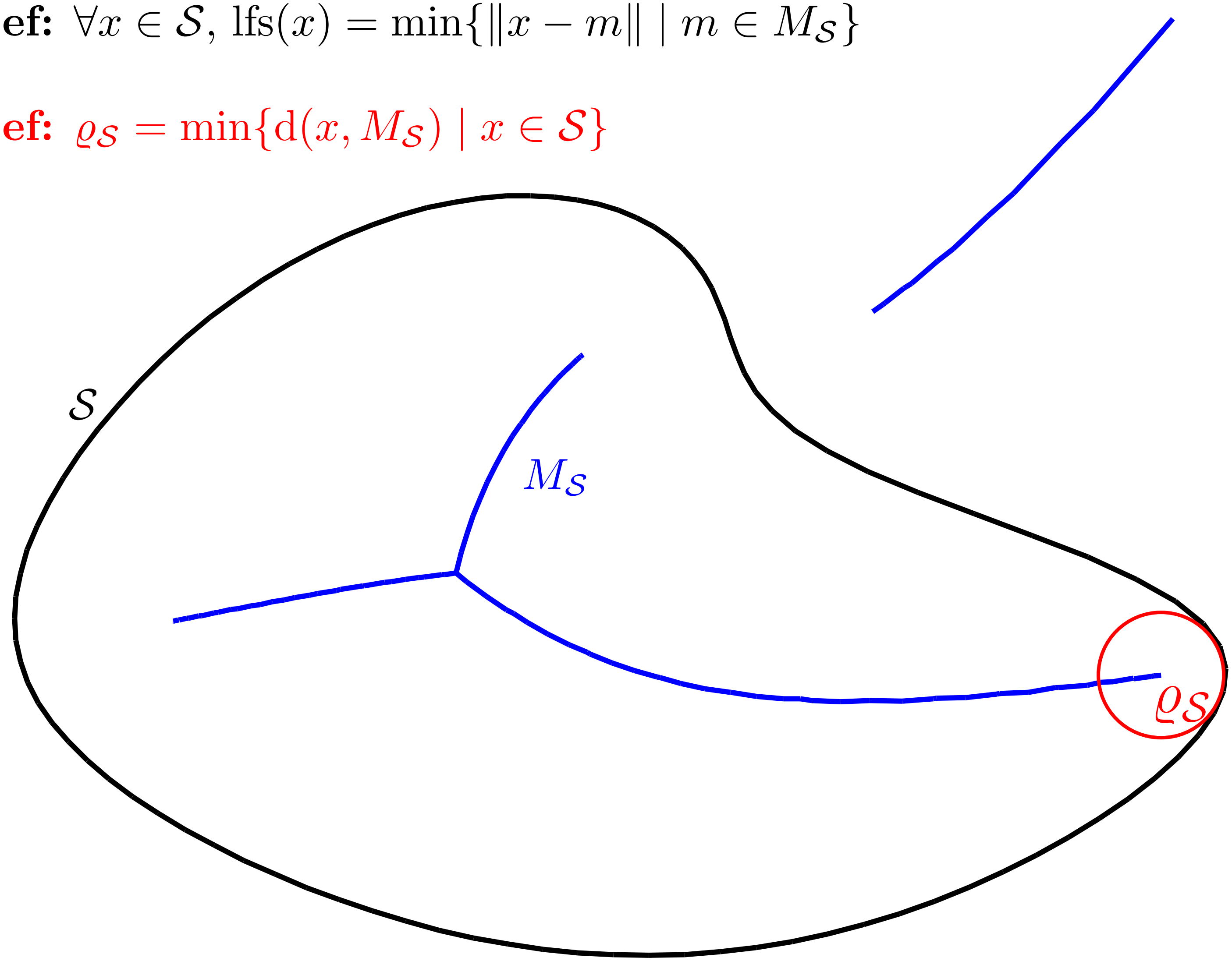

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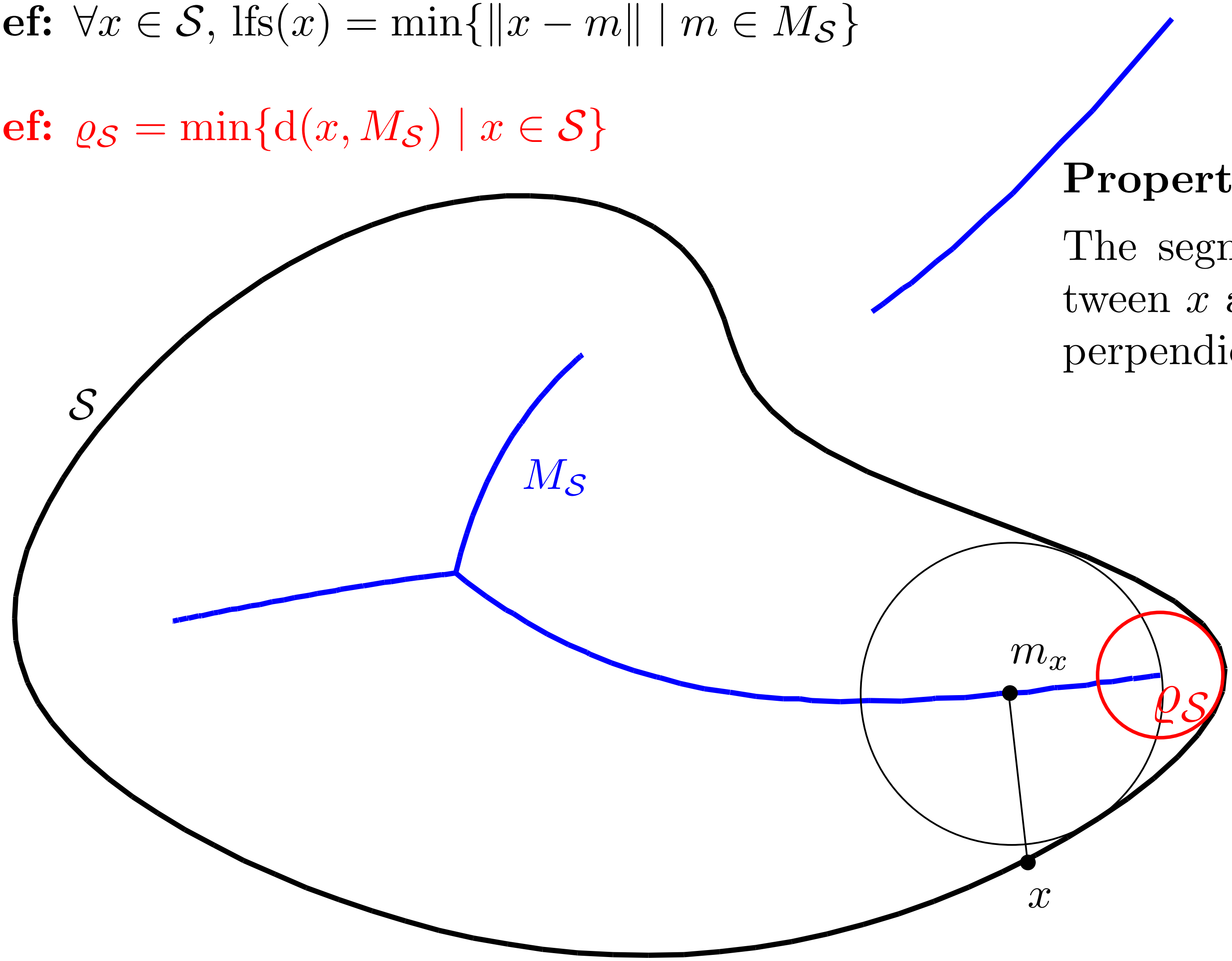

Property 1:

The segment of length $lfs(x)$ between x and the medial axis $M_{\mathcal{S}}$ is perpendicular to $M_{\mathcal{S}}$

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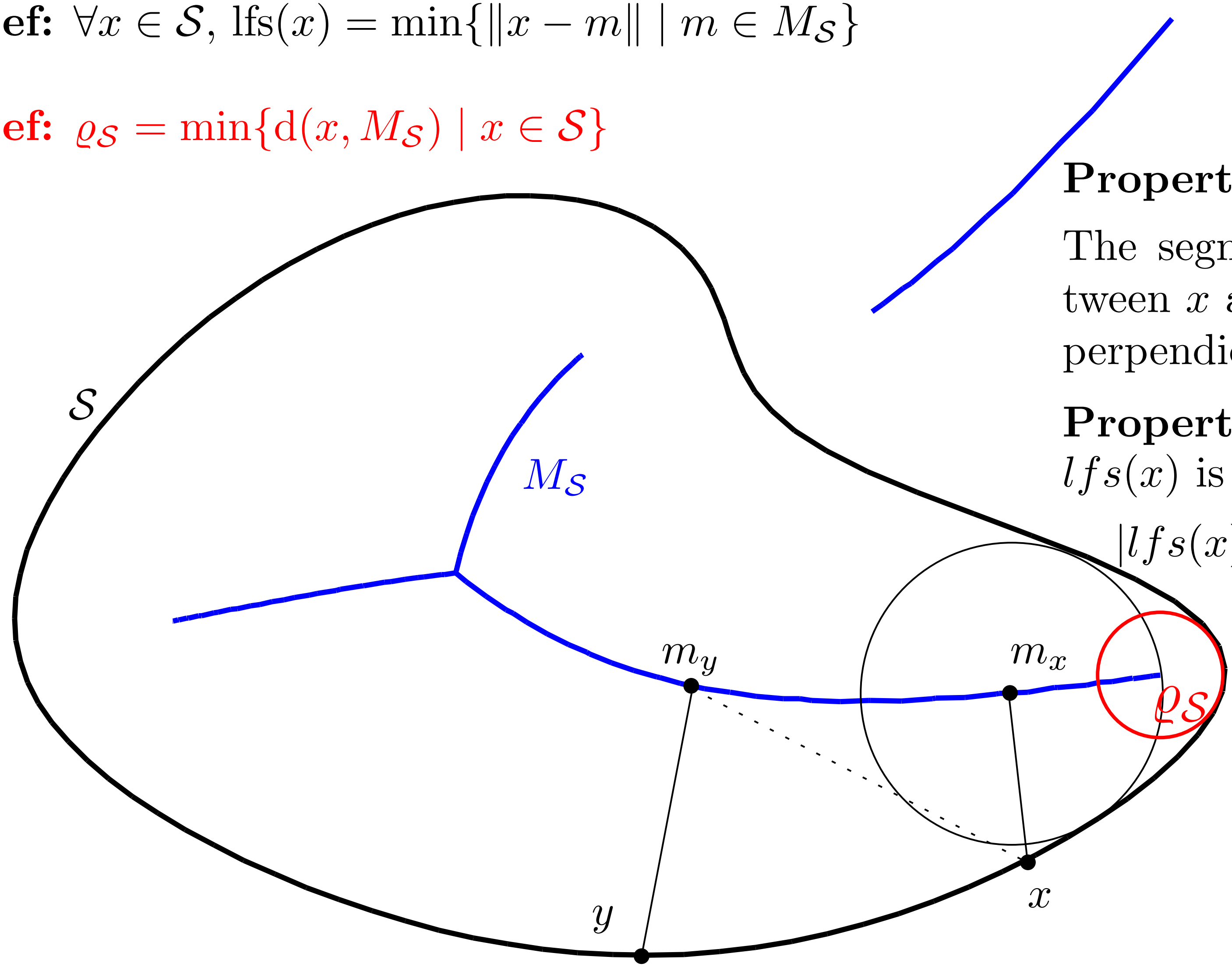

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Property 2:
$lfs(x)$ is Lipschitz continuous

$$|lfs(x) - lfs(y)| \leq d(m_x, m_y)$$

$$lfs(x) := d(m_x, x) \leq d(x, m_y) \leq d(x, y) + d(y, m_y) = d(x, y) + lfs(y)$$

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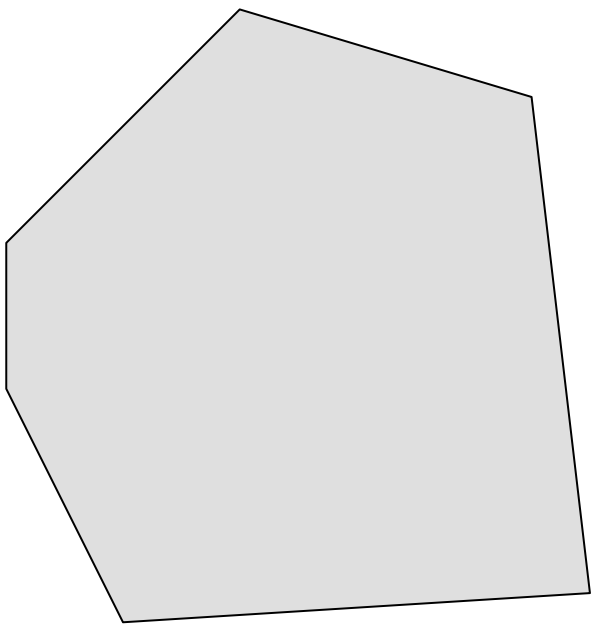

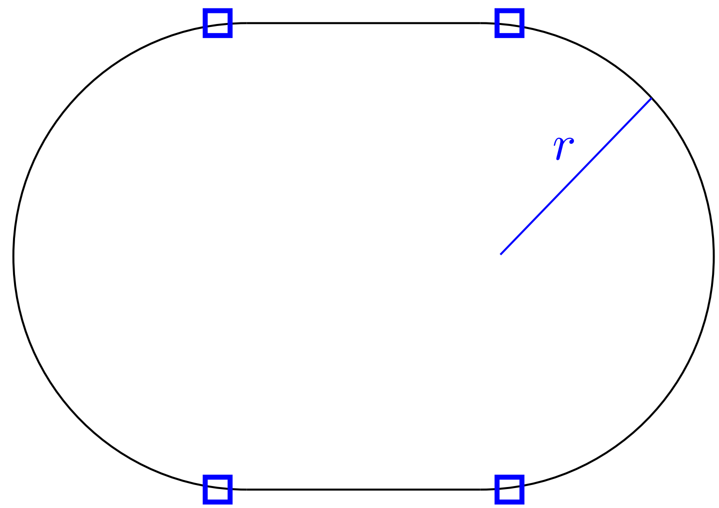

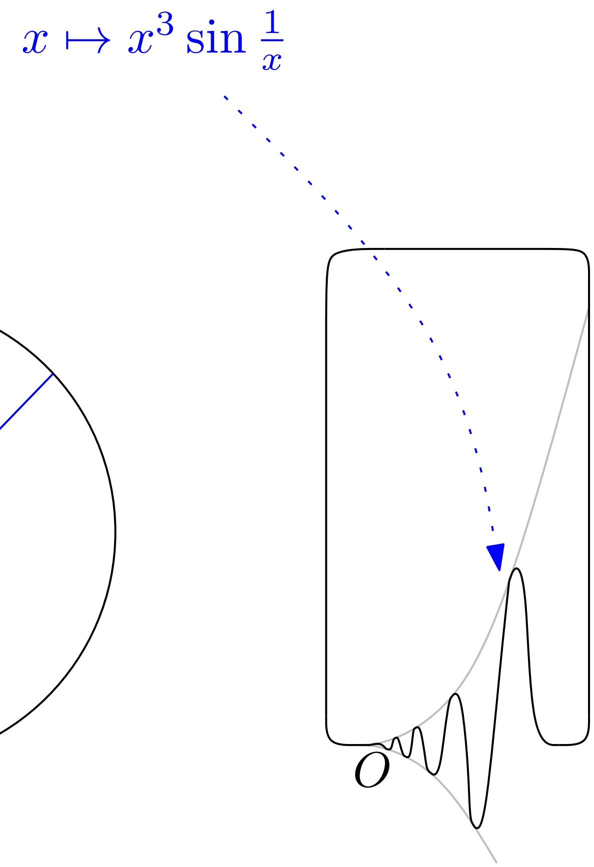

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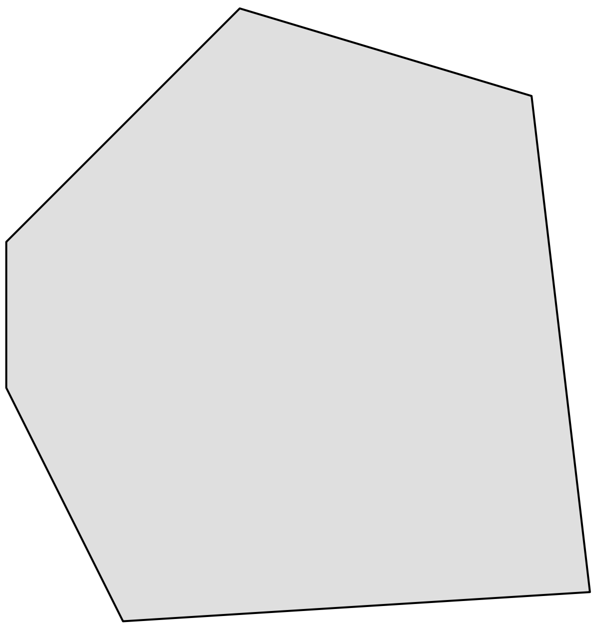
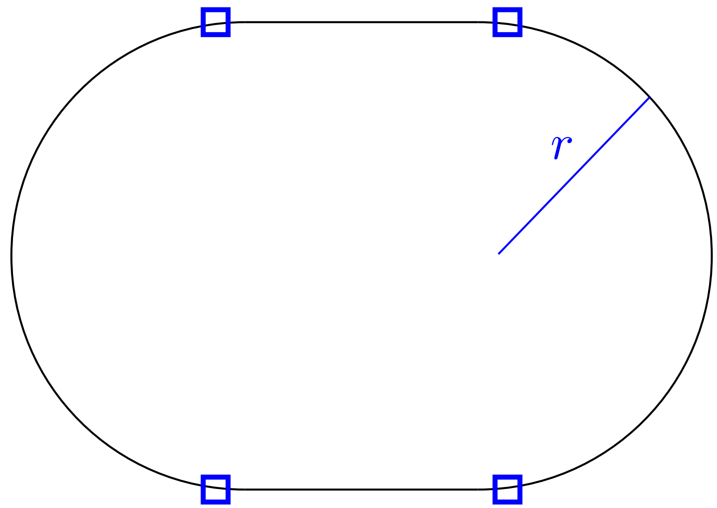


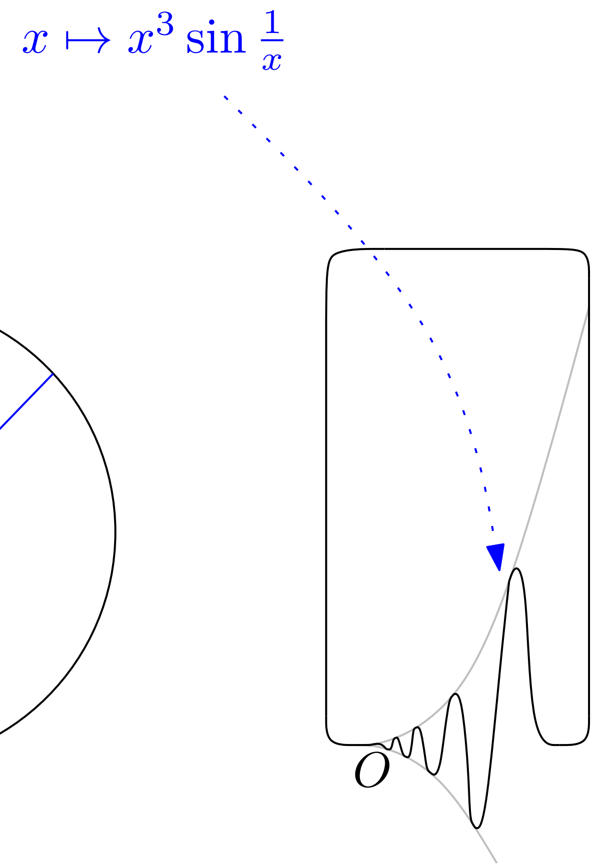
$\varrho_{\mathcal{S}} = +\infty$

(convex)



$\varrho_{\mathcal{S}} = r$

($C^{1,1}$ but not C^2)



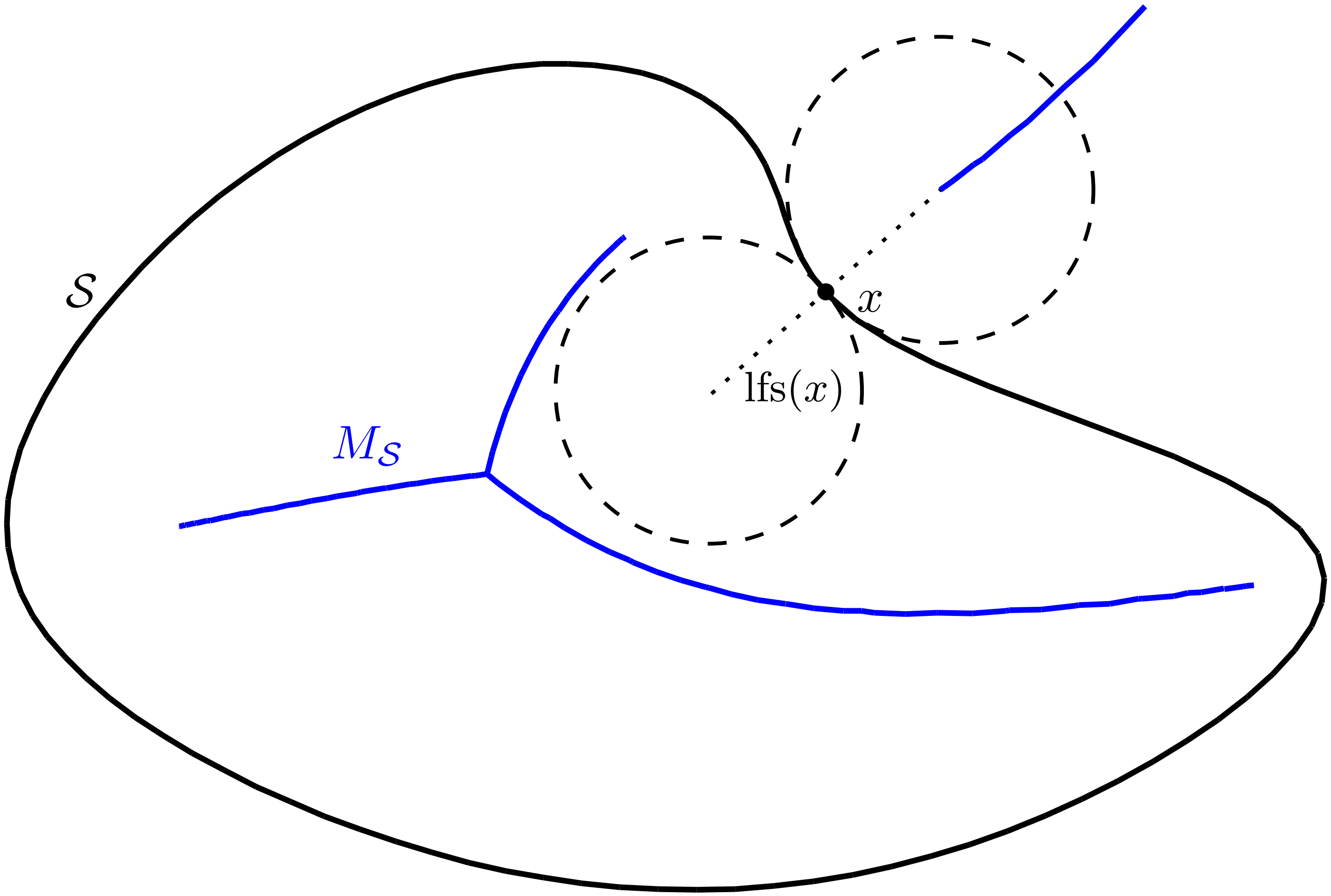
$\varrho_{\mathcal{S}} = 0$

(C^1 but not $C^{1,1}$)

Shapes with positive reach (Cont'd)

$\rightarrow$ Fundamental properties: (see [Federer 1958])

Tangent Ball Lemma: $\forall x \in \mathcal{S}$, $\forall c \in n_x\mathcal{S}$, $\|x - c\| < \mathrm{lfs}(x) \Rightarrow B(c, \|x - c\|) \cap \mathcal{S} = \emptyset$.



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$\rightarrow$ Fundamental properties: (see [Federer 1958])

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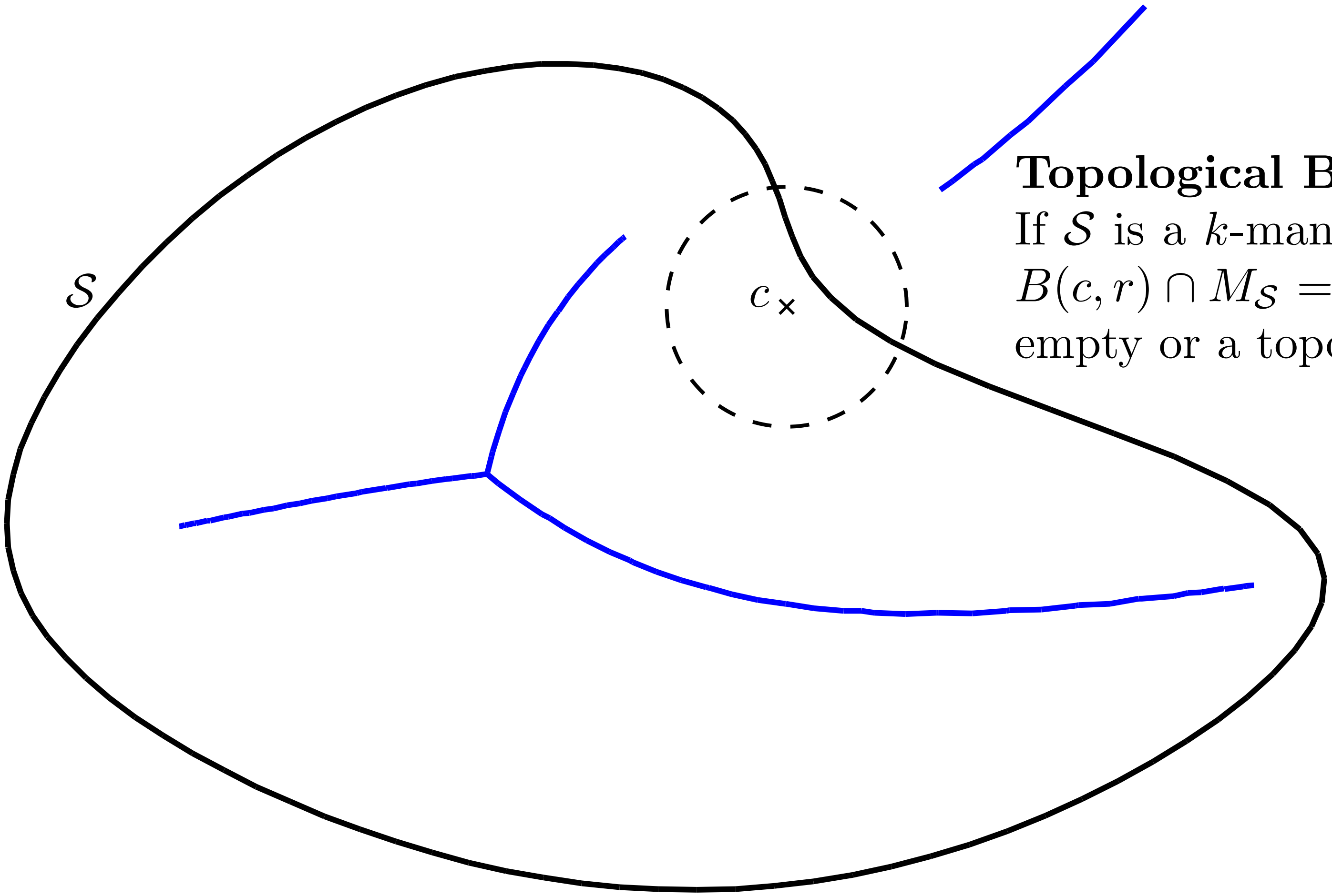
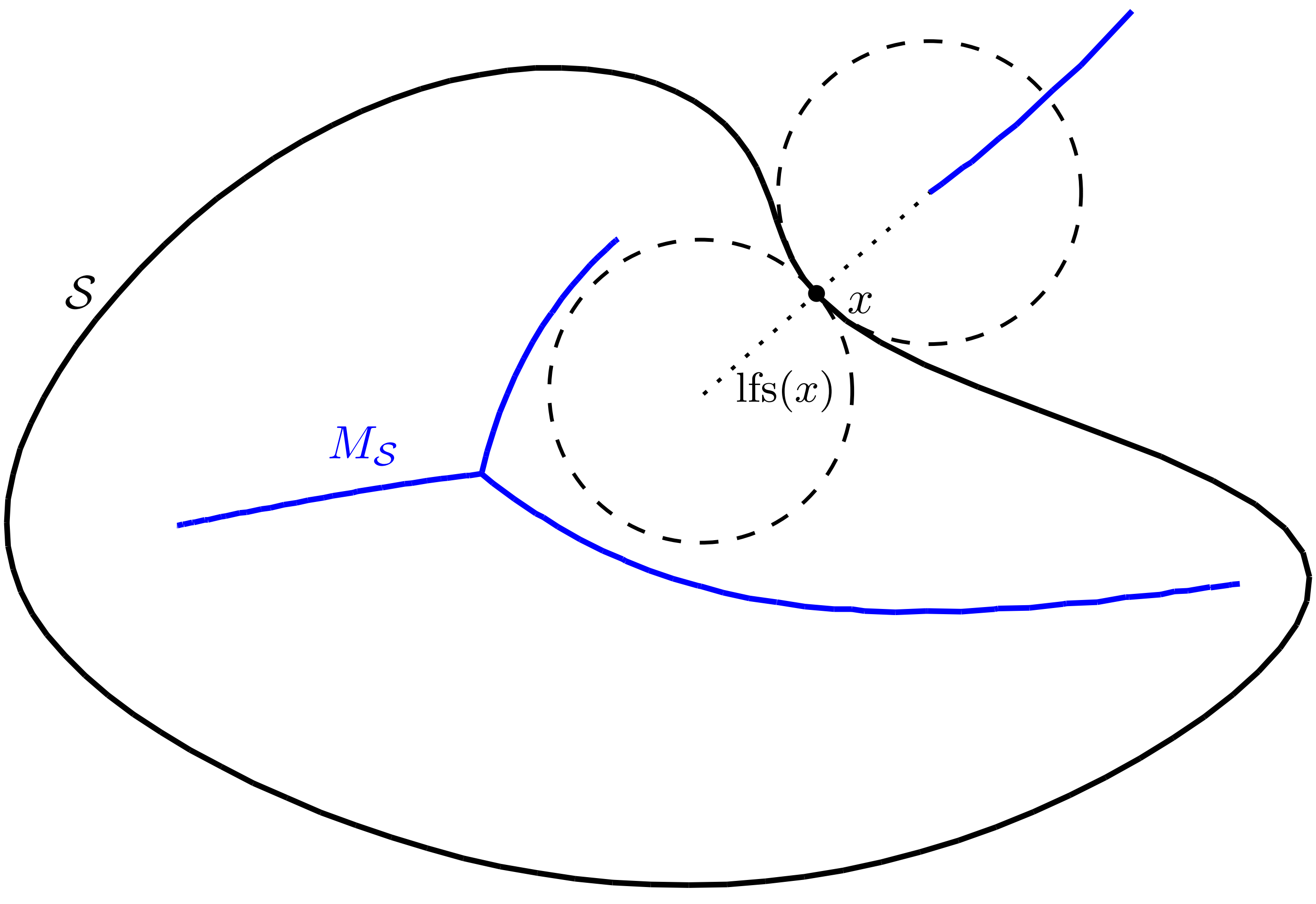


Topological Ball Lemma:
If $\mathcal{S}$ is a k-manifold, then $\forall B(c, r)$ s.t. $B(c, r) \cap M_{\mathcal{S}} = \emptyset$, $B(c, r) \cap \mathcal{S}$ is either empty or a topological k-ball.

A first (topological) proof

Tangent Ball Lemma: $\forall x \in \mathcal{S}$, $\forall c \in n_x\mathcal{S}$, $\|x - c\| < \mathrm{lfs}(x) \Rightarrow B(c, \|x - c\|) \cap \mathcal{S} = \emptyset$.

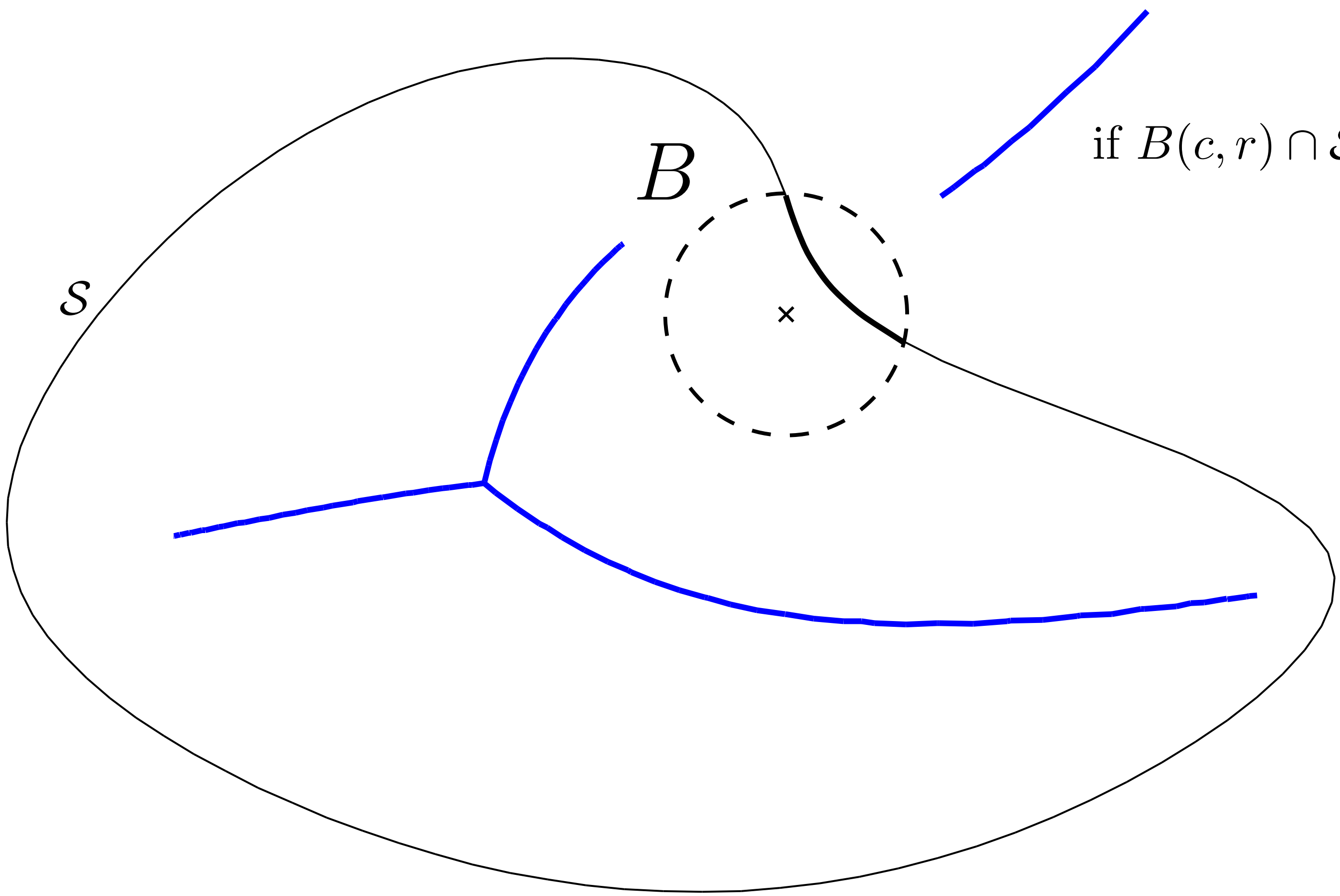


Second (topological) proof

Topological Ball Lemma (for curves)
If $\mathcal{S}$ is a smooth curve, given a disk $B(c,r)$ (intersecting $\mathcal{S}$ in at least two points)

either $B(c,r) \cap \mathcal{S}$ is a topological 1-ball (arc of curve).

or $B(c,r) \cap M_{\mathcal{S}} \neq \emptyset$ (the intersection has several connected components)



if $B(c,r) \cap \mathcal{S}$ is a topological we are done

Second (topological) proof

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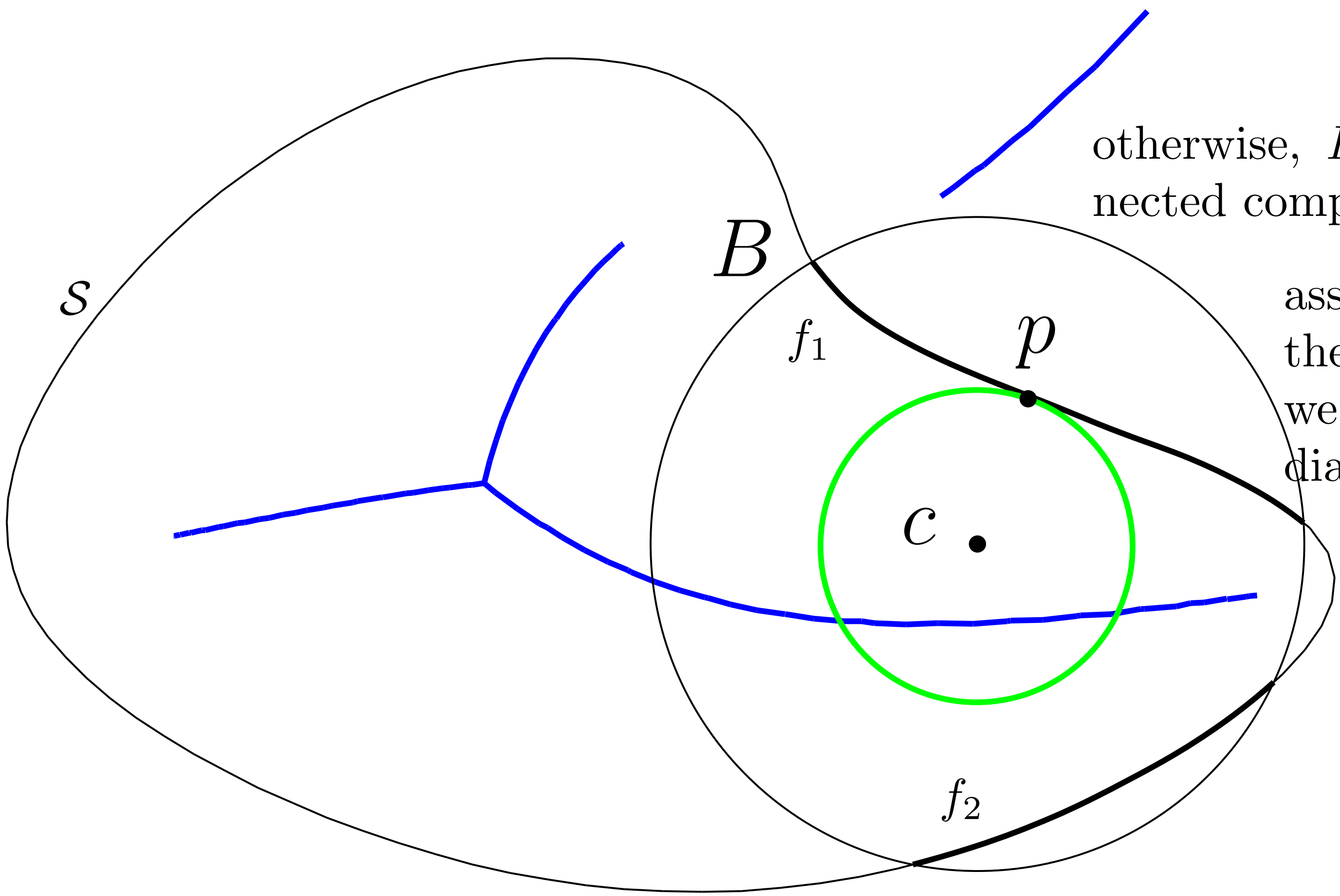

otherwise, $B(c,r) \cap \mathcal{S}$ has at least 2 connected components f_1 and f_2

assume that the closest point p to the center c is unique (otherwise, we are done, by definition of medial axis)

Second (topological) proof

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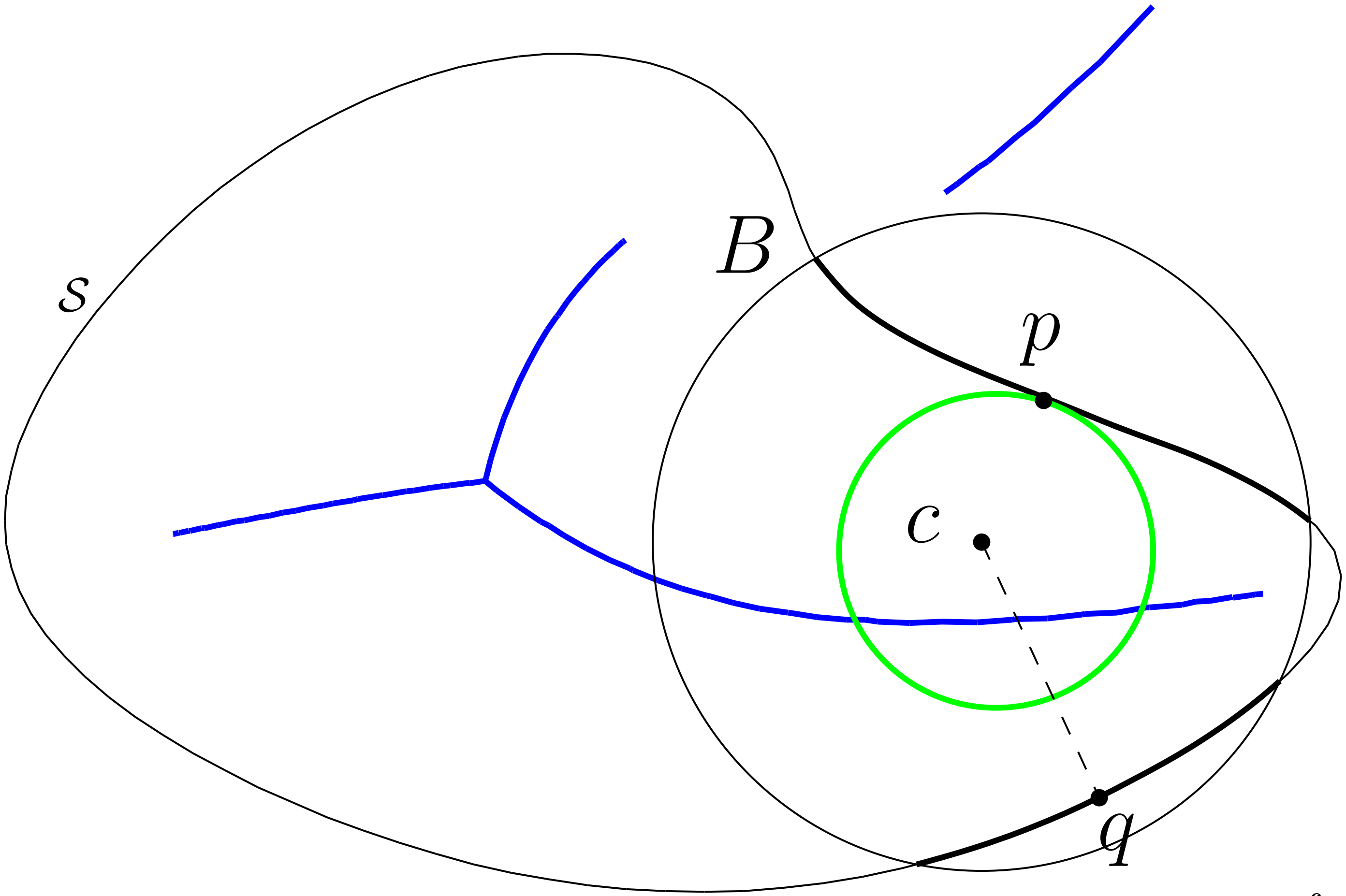

$q \in f_2$ closest point

Second (topological) proof

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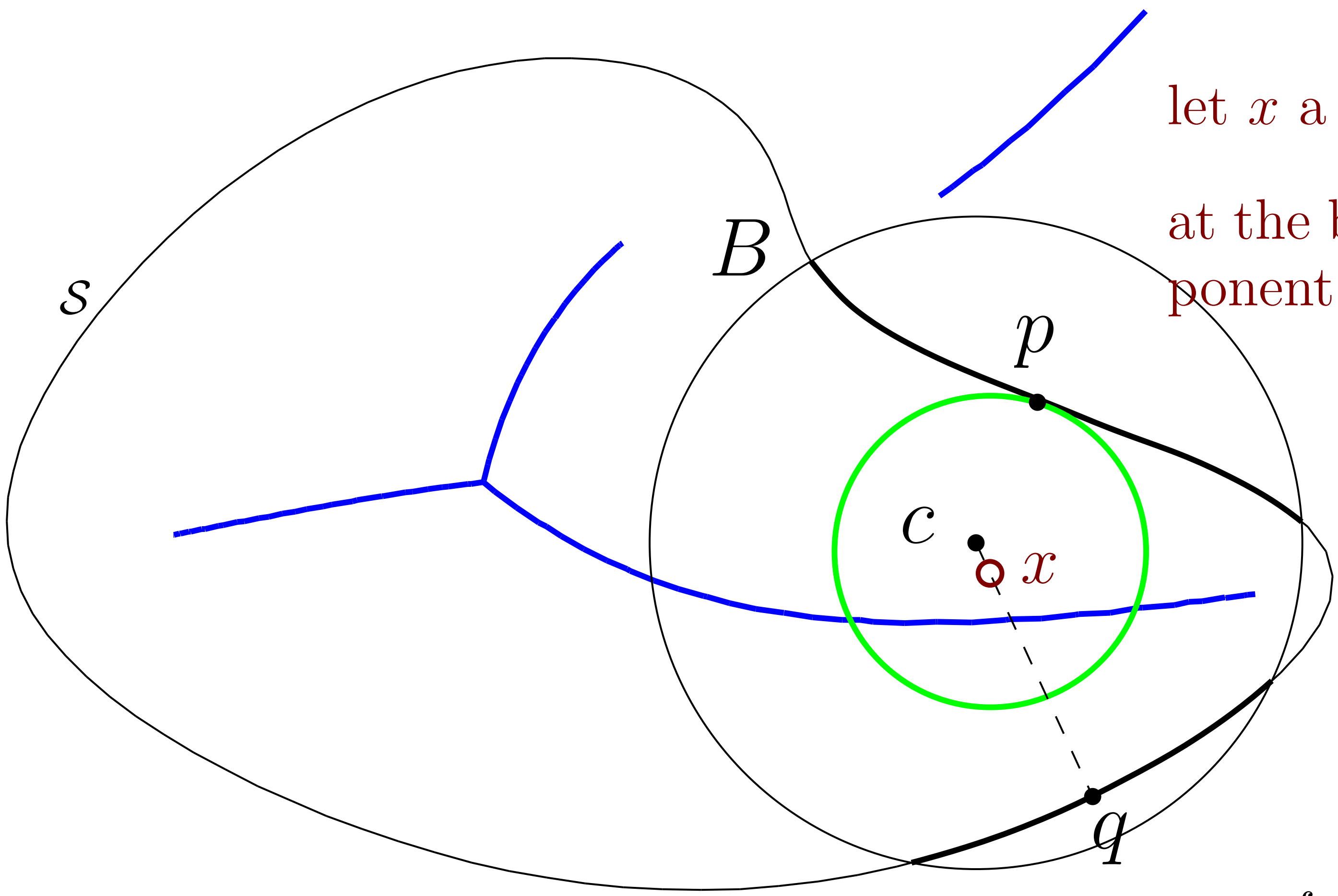

let x a point moving from c to q

at the beginning the closest component of x is f_1

$q \in f_2$ closest point

Second (topological) proof

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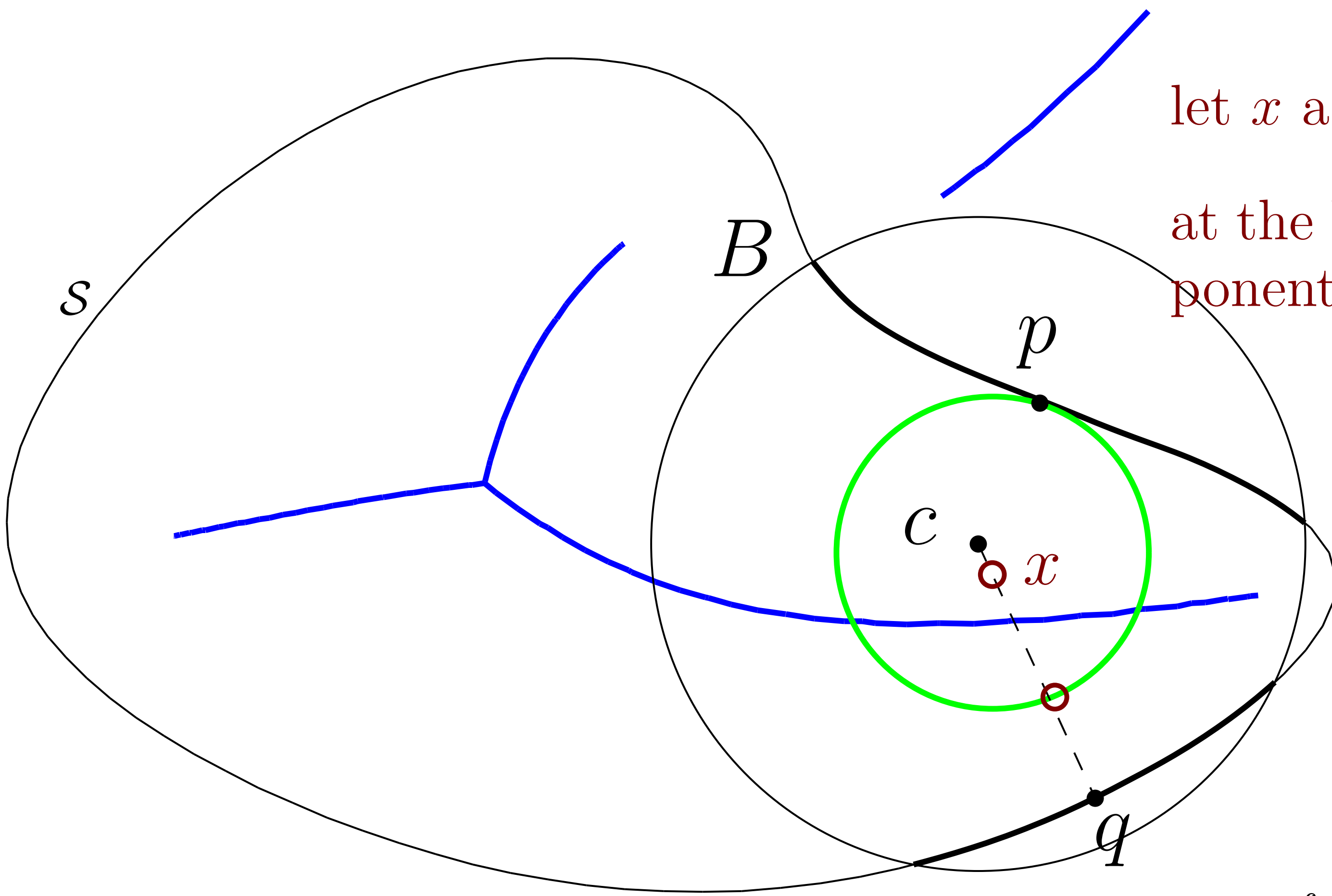

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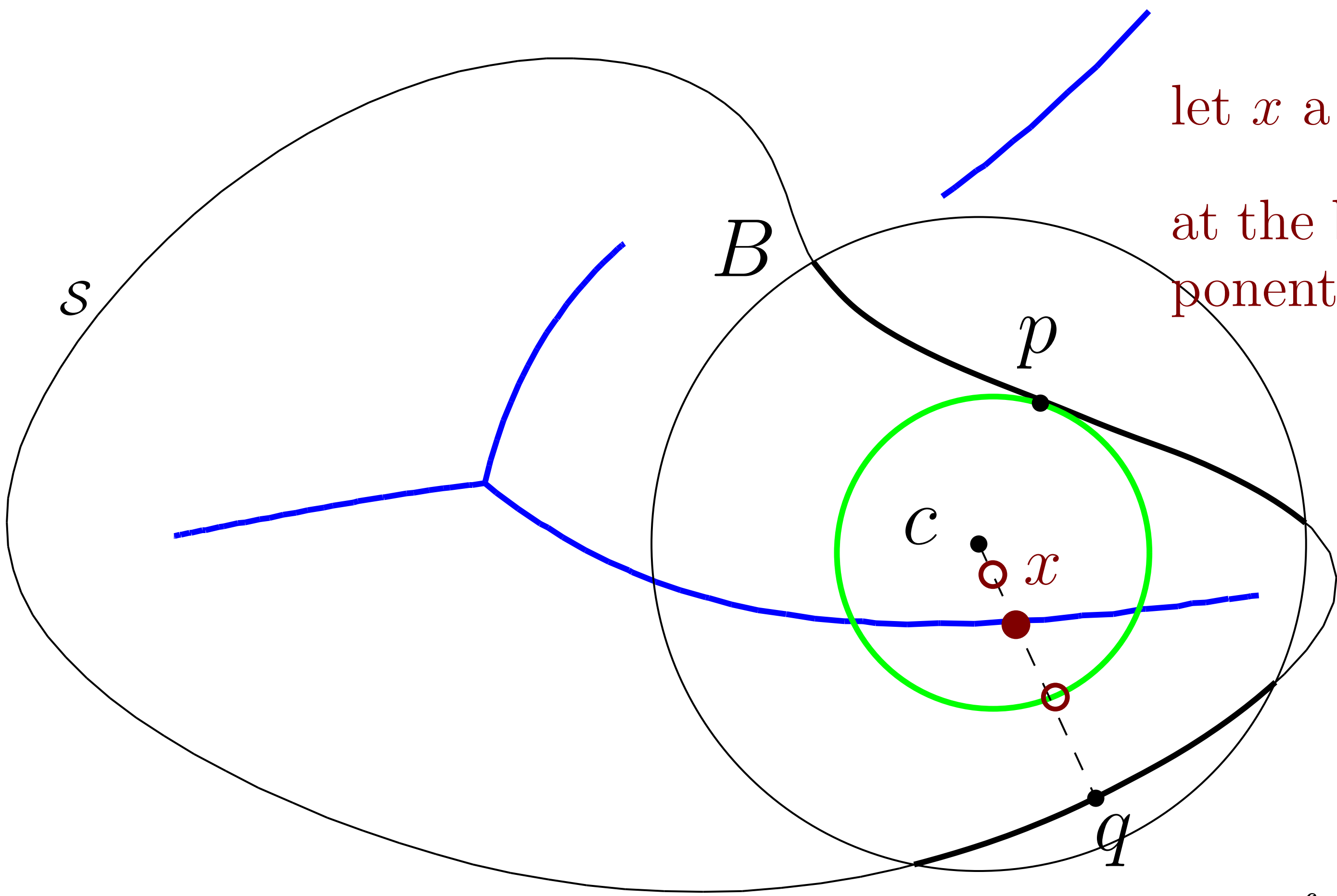


let x a point moving from c to q

at the beginning the closest component of x is f_1

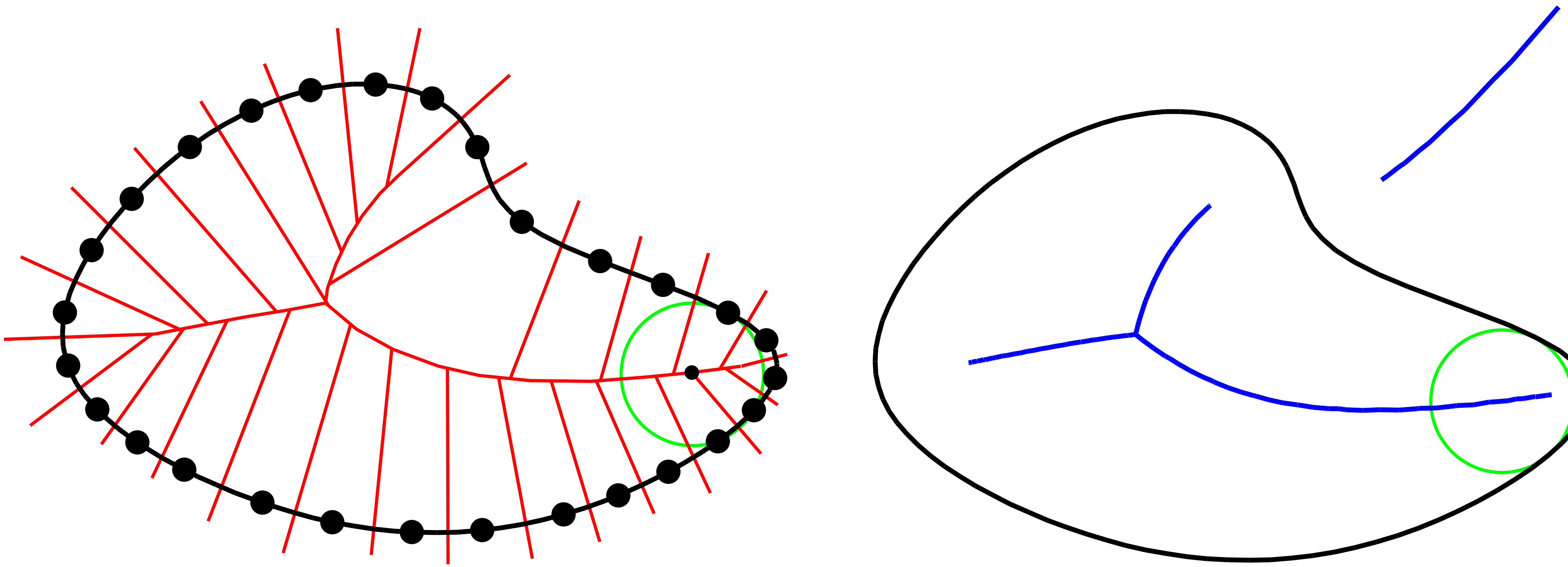
at the end the closest component to x is f_2

there is a point $x \in M_{\mathcal{S}}$ with two closest neighbors on distinc components

$q \in f_2$ closest point

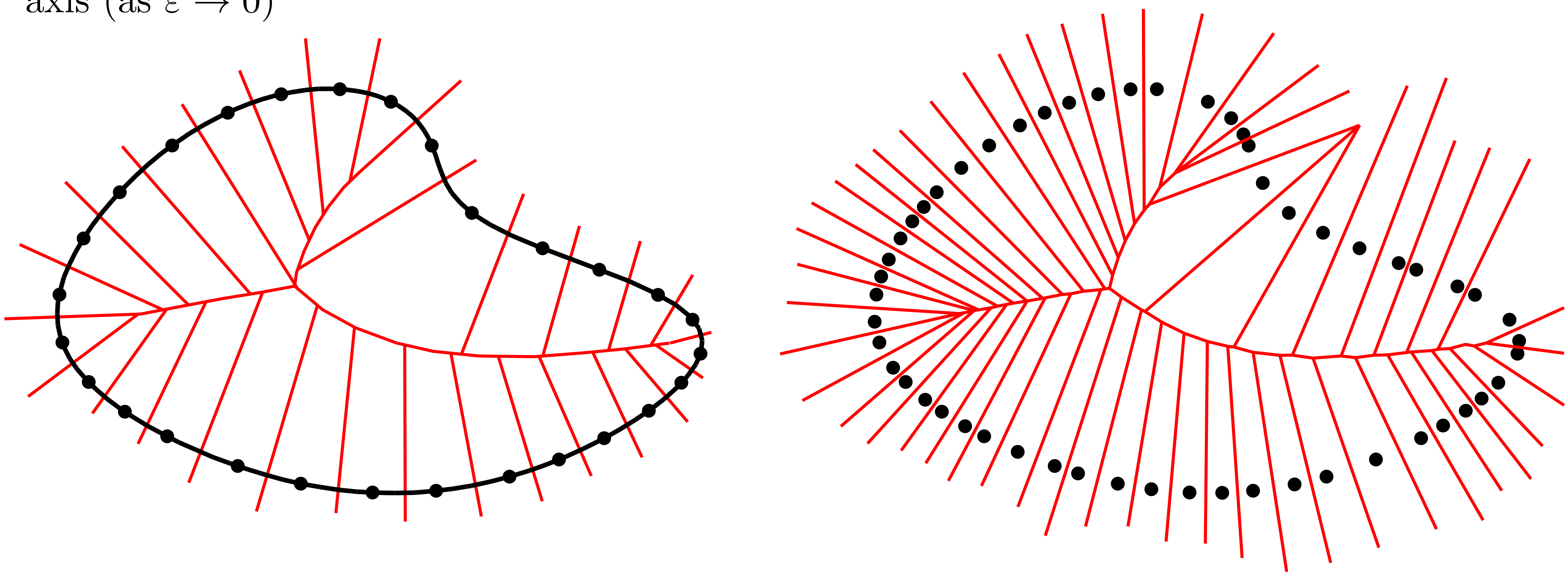
Medial axis and Voronoi vertices

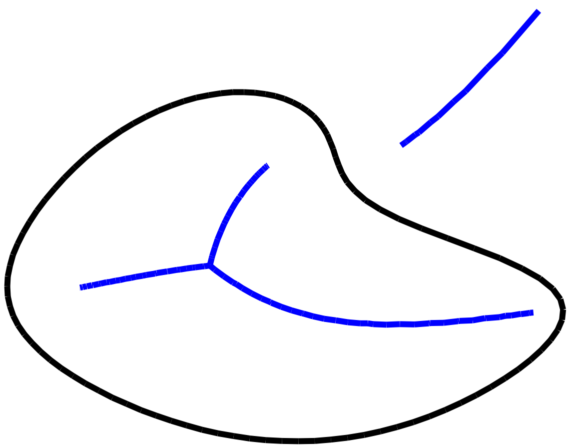
Property (for curves in the plane)
Given a set of points $P \subset \mathcal{S}$, any Voronoi disk B (maximal empty disk centered at a Voronoi vertex) must intersect the medial axis $M_{\mathcal{S}}$



Medial axis and Voronoi vertices

Remark (for curves in the plane)
If P is an ε-sample of $\mathcal{S}$ (smooth), then all Voronoi vertices converge to the medial axis (as $\varepsilon \to 0$)





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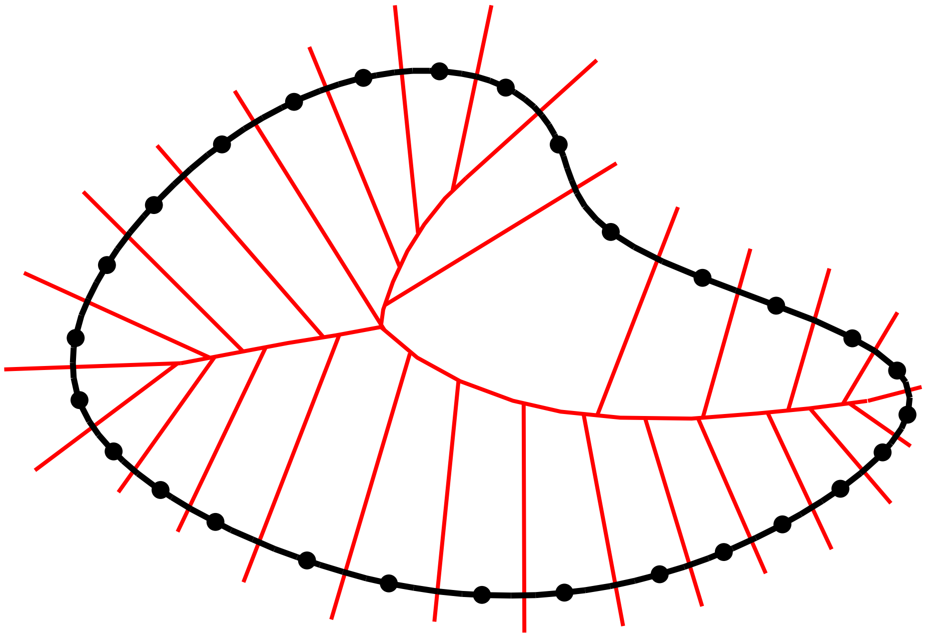
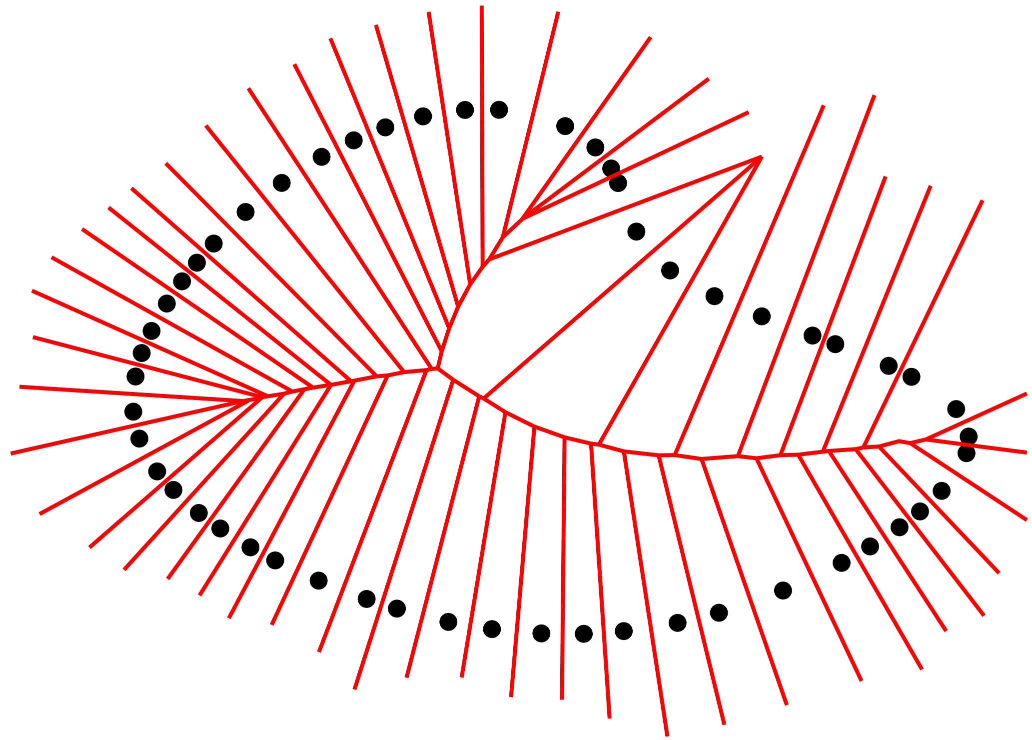
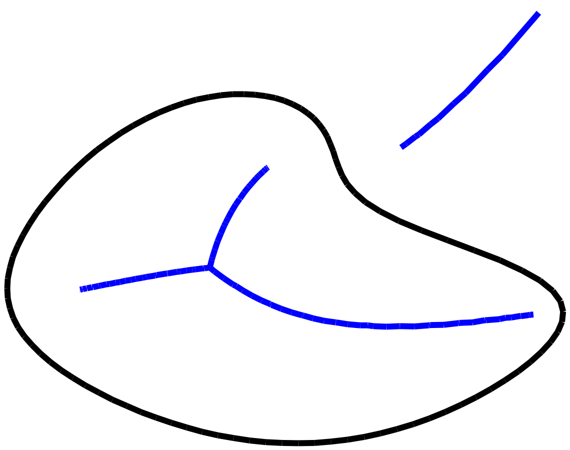
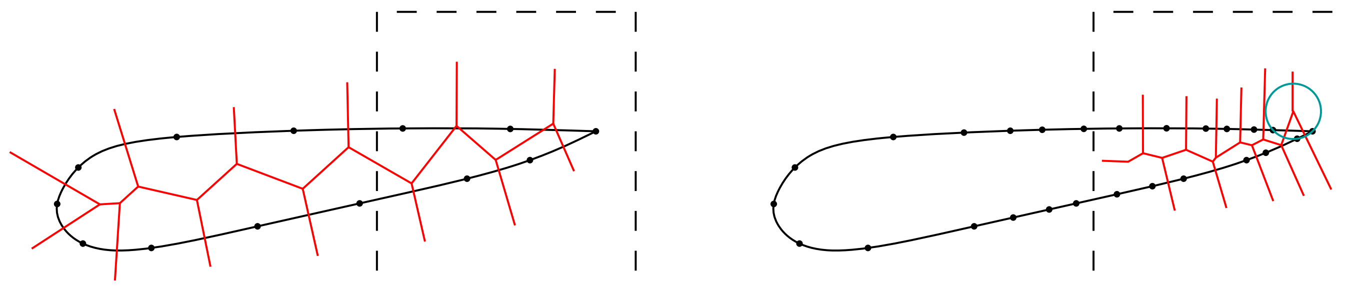






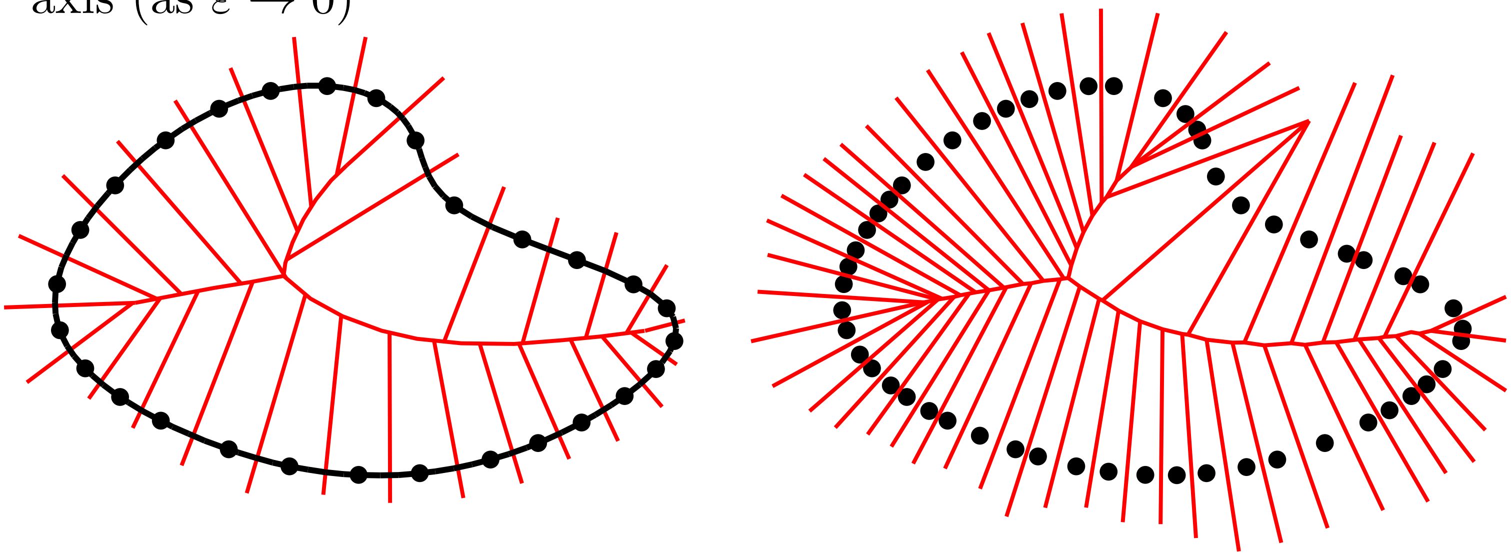


when $\mathcal{S}$ is not smooth some Voronoi vertices may not converge to the medial axis (as $\varepsilon \to 0$)

Medial axis and Voronoi vertices

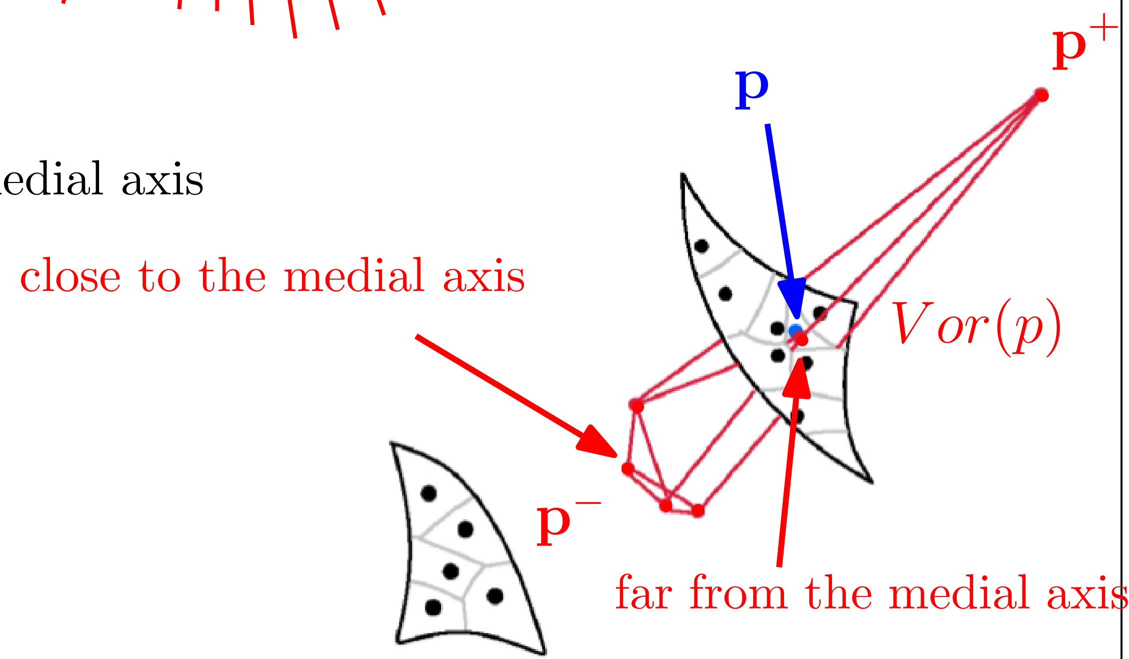
Remark (for curves in the plane)

If P is an ε-sample of $\mathcal{S}$ (smooth), then all Voronoi vertices converge to the medial axis (as $\varepsilon \to 0$)



Remark (surfaces in 3D)

Not all Voronoi vertices are close to the medial axis



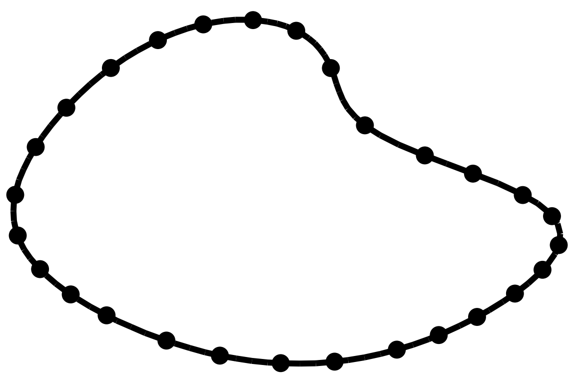
Approximation power of the restricted Delaunay

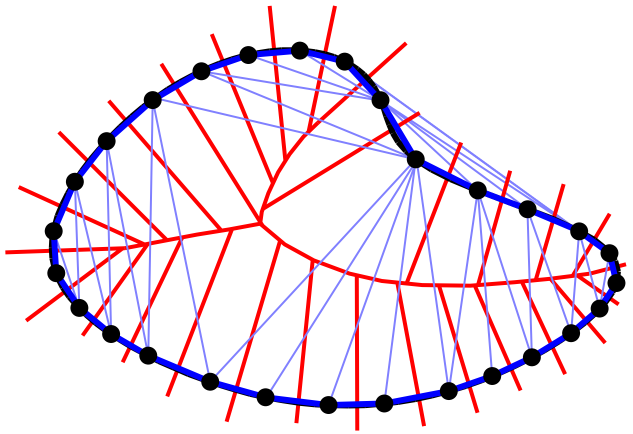
1. the underlying shape $\mathcal{S}$ is a closed curve or surface with positive *reach* $\varrho_\mathcal{S}$

2. the point cloud P is an ε-*sample* of $\mathcal{S}$ with $\varepsilon \in O(\varrho_\mathcal{S})$.

Theorem: [Amenta et al. 1998-99]
If $\mathcal{S}$ is a curve or surface with positive *reach*, and if P is an ε-*sample* of $\mathcal{S}$ with $\varepsilon < \varrho_\mathcal{S}$ (curve) or $\varepsilon < 0.1\varrho_\mathcal{S}$ (surface), then:

- $\mathrm{Del}_\mathcal{S}(P)$ is homeomorphic to $\mathcal{S}$,
- $\mathrm{d_H}(\mathrm{Del}_\mathcal{S}(P), \mathcal{S}) \in O(\varepsilon^2)$, $\quad$ $\mathcal{S}$ and $Del_S(P)$ are close
- $\forall f \in \mathrm{Del}_\mathcal{S}(P)$, $\forall v \in f$, $\angle n_f n_v \mathcal{S} \in O(\varepsilon)$, $\quad$ the angles between consectuive points on $Del_S(P)$ are flat
- $\cdots$ (similar areas, curvature estimation, etc.)

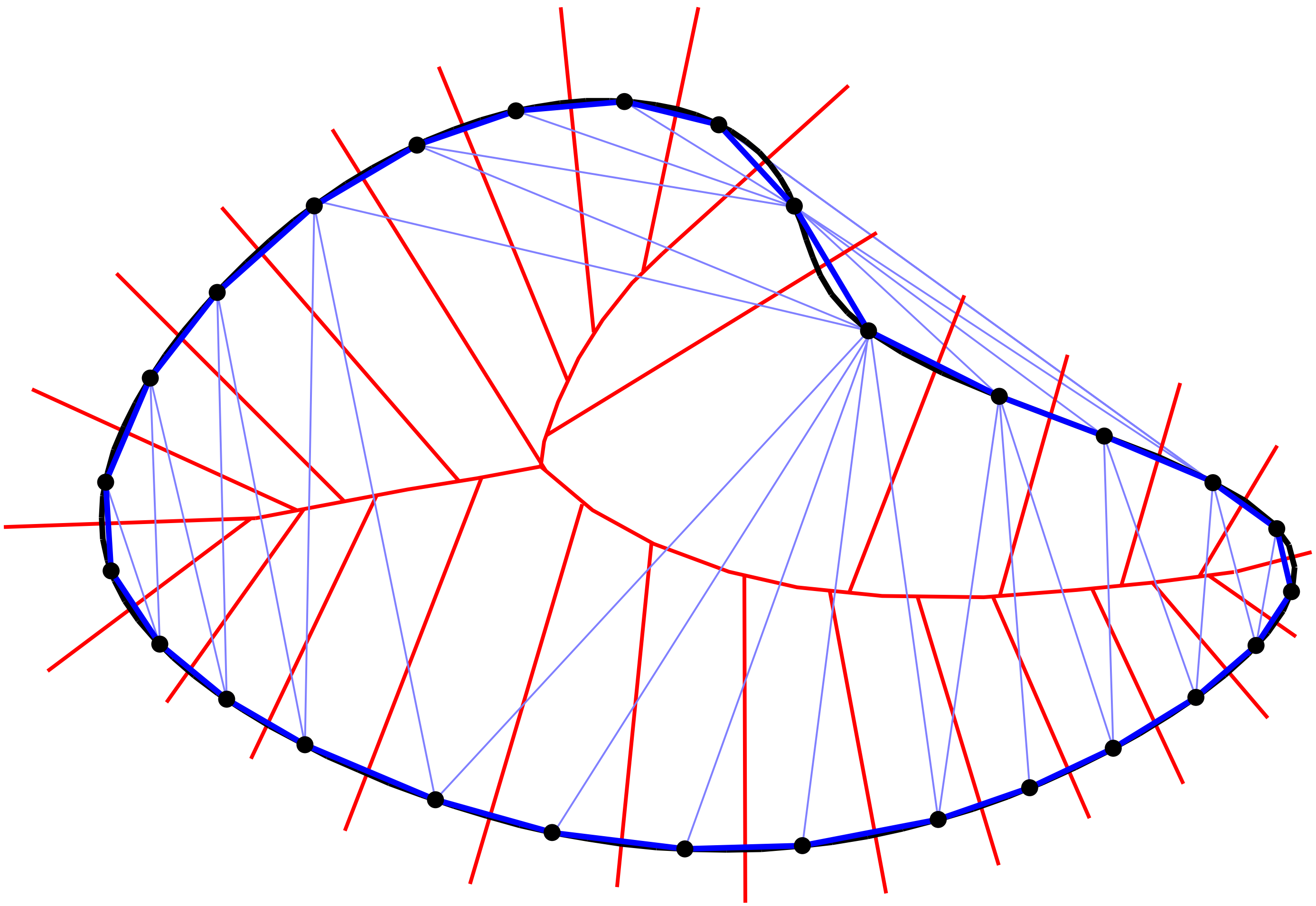




Approximation power of the restricted Delaunay

Proof for curves:

show that every edge of $\mathrm{Del}_{\mathcal{S}}(P)$ connects consecutive points of P along $\mathcal{S}$, and vice-versa



Approximation power of the restricted Delaunay

Proof for curves:

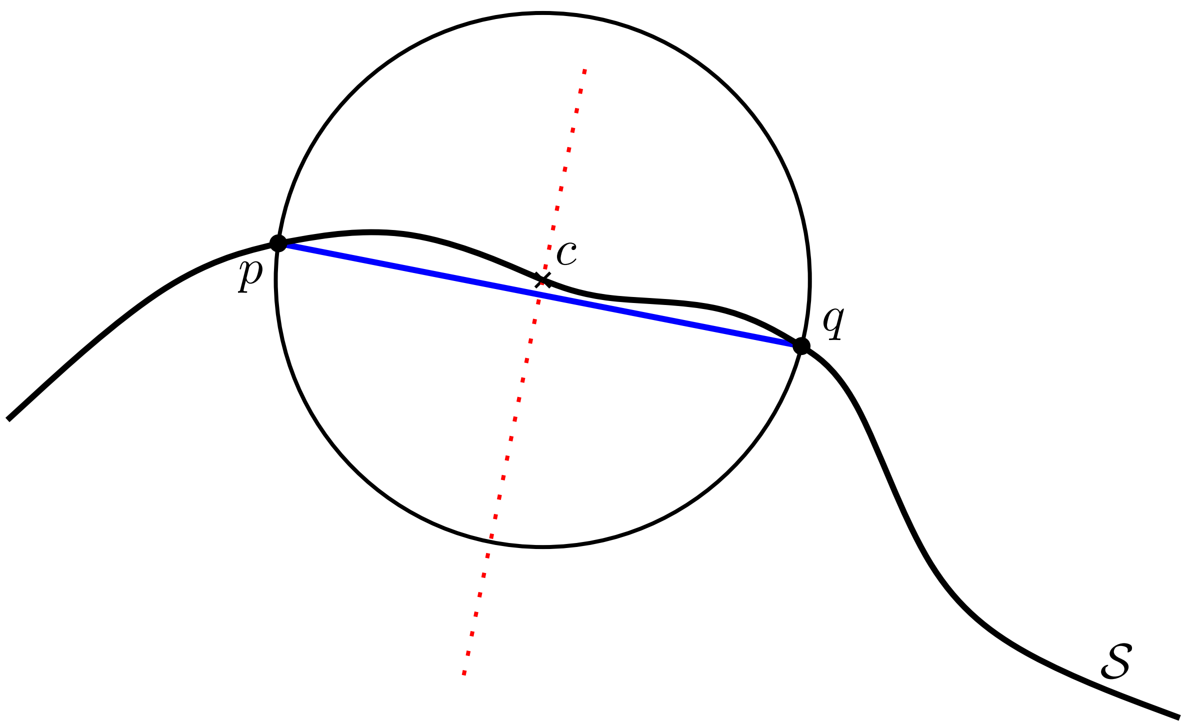
show that every edge of $\mathrm{Del}_{\mathcal{S}}(P)$ connects consecutive points of P along $\mathcal{S}$, and vice-versa

$\rightarrow$ Assume $(p,q) \in Del_{\mathcal{S}}(P)$.

Let $c \in (p,q)^* \cap \mathcal{S}$.

$r = \|c-p\| = \|c-q\| = \mathrm{d}(c,P) \leq \varepsilon < \varrho_{\mathcal{S}} \leq \mathrm{lfs}(c)$

$\Rightarrow B(c,r) \cap \mathcal{S}$ is a topological arc



$B(c,r)$ is a Voronoi disk (empty of points in its interior)

Approximation power of the restricted Delaunay

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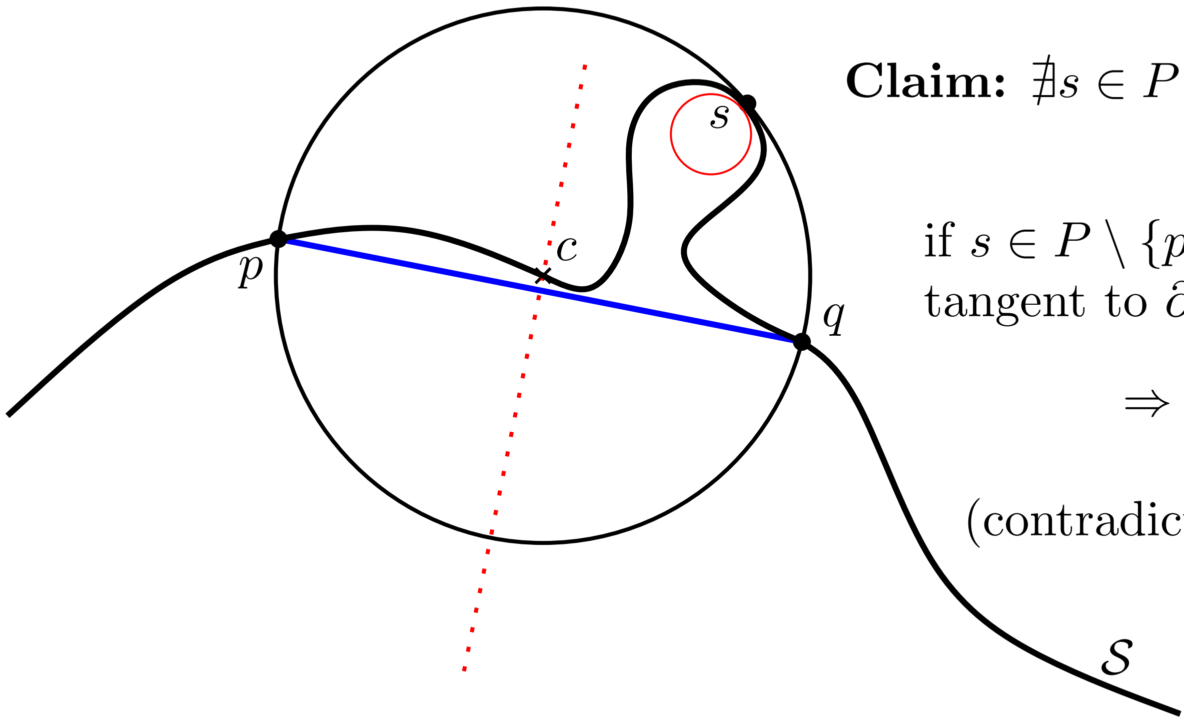


Claim: $\nexists s \in P \setminus \{p, q\}$ on the arc on between p and q

if $s \in P \setminus \{p, q\}$ belongs to this arc, then the arc is tangent to $\partial B(c, r)$ in p, q or s (say s)

$\Rightarrow \mathrm{d}(c, P) = r = \|c - s\| \geq \mathrm{lfs}(s) > \varepsilon.$

(contradiction with the hypothesis of the theorem)

by the **Tangent ball lemma** B should not intersect $\mathcal{S}$ (its radius r is too small)

$\square$

$B(c, r)$ is a Voronoi disk (empty of points in its interior)

Approximation power of the restricted Delaunay

Proof for curves:

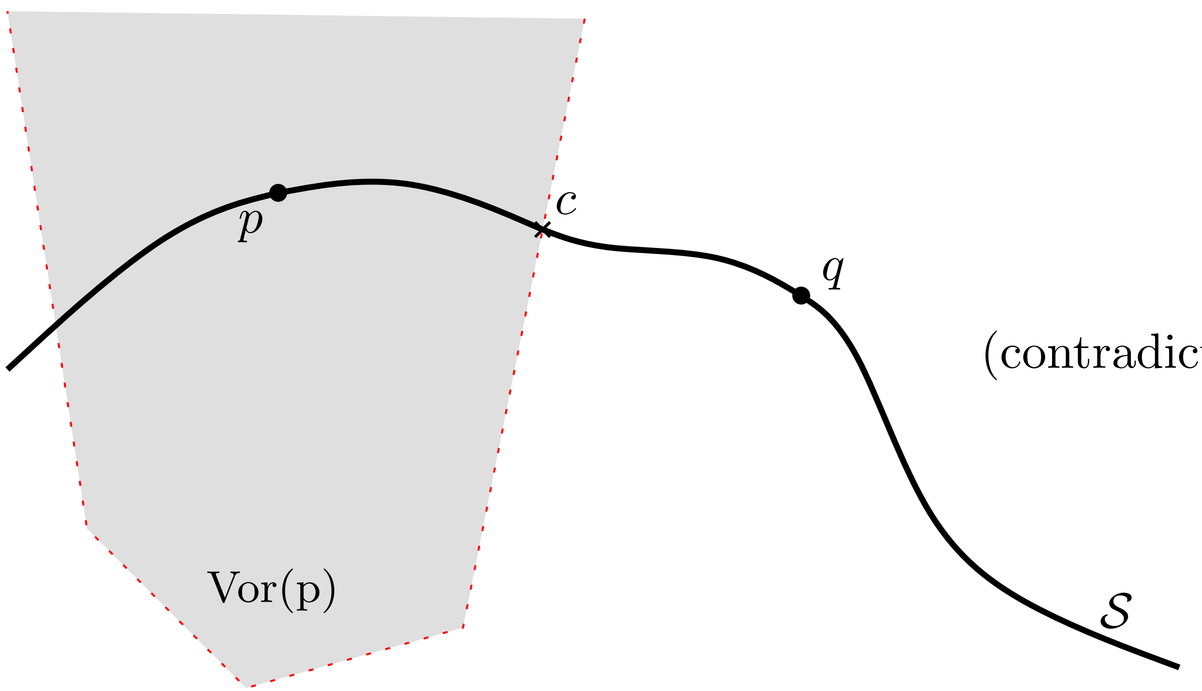
show that every edge of $\mathrm{Del}_{\mathcal{S}}(P)$ connects consecutive points of P along $\mathcal{S}$, and vice-versa

$\leftarrow$ Assume p and q are consecutive on $\mathcal{S}$

Let $c \in \mathrm{arc}_{\mathcal{S}}(pq) \cap \partial p^*$.

if $c \in \partial q^*$, we are done

otherwise $c \in (p, s)^*$ for some $s \in P \setminus \{p\}$



(contradiction)

Approximation power of the restricted Delaunay

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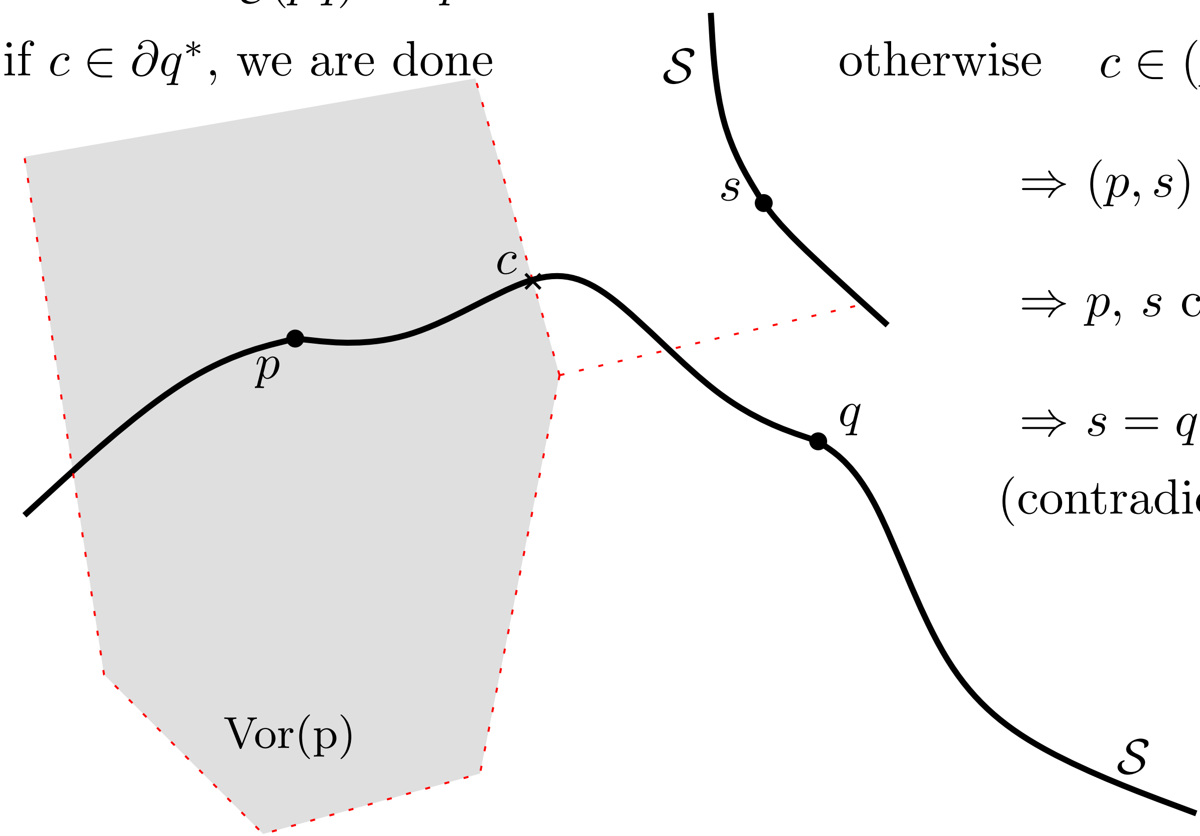


otherwise $\quad c \in (p, s)^*$ for some $s \in P \setminus \{p\}$

$\Rightarrow (p, s) \in \mathrm{Del}_{\mathcal{S}}(P)$

$\Rightarrow p$, s consecutive along $\mathcal{S}$, with $c \in \mathrm{arc}_{\mathcal{S}}(ps)$

(by previous part of the proof)

$\Rightarrow s = q$

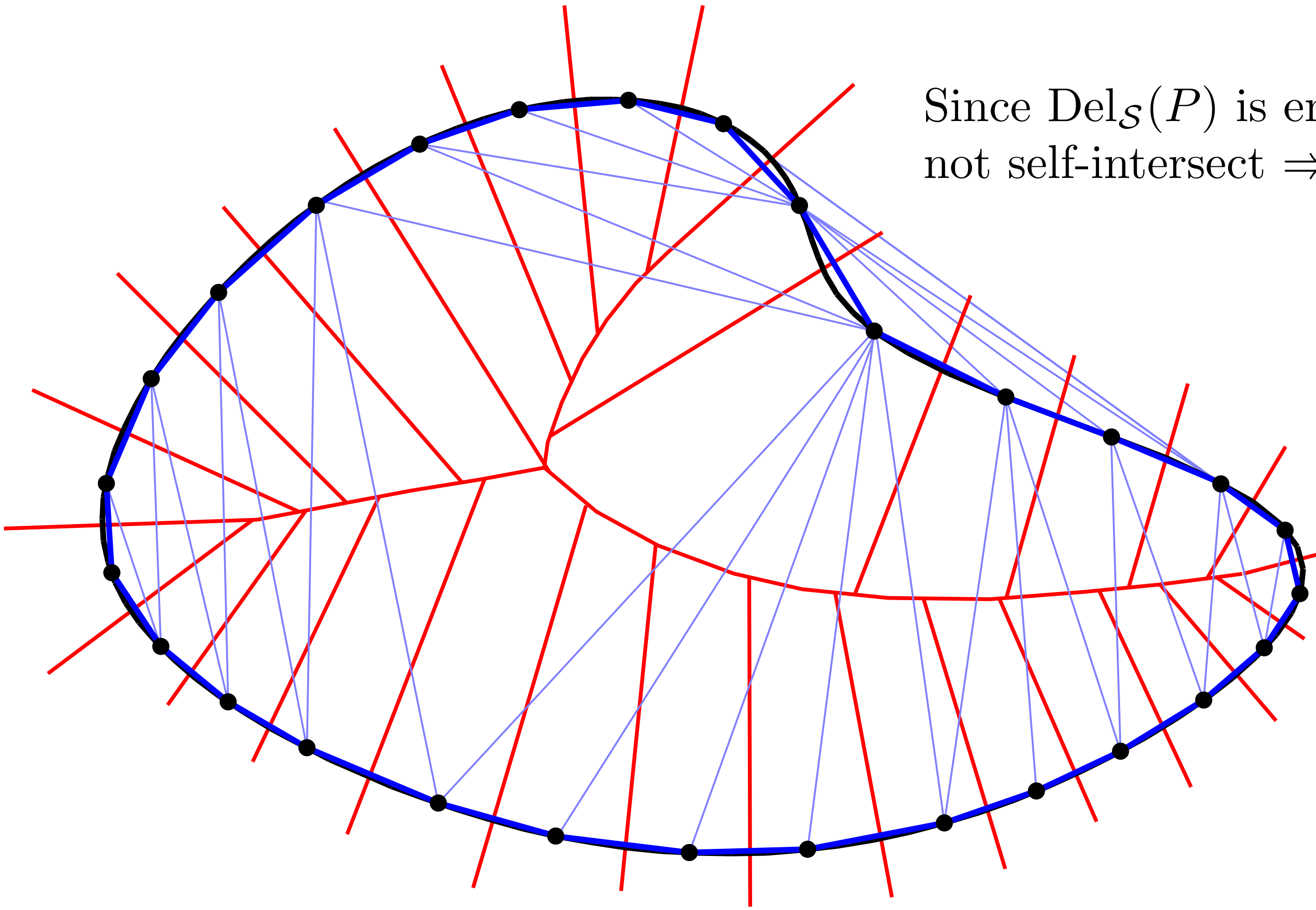
(contradiction)

Approximation power of the restricted Delaunay

Proof for curves:

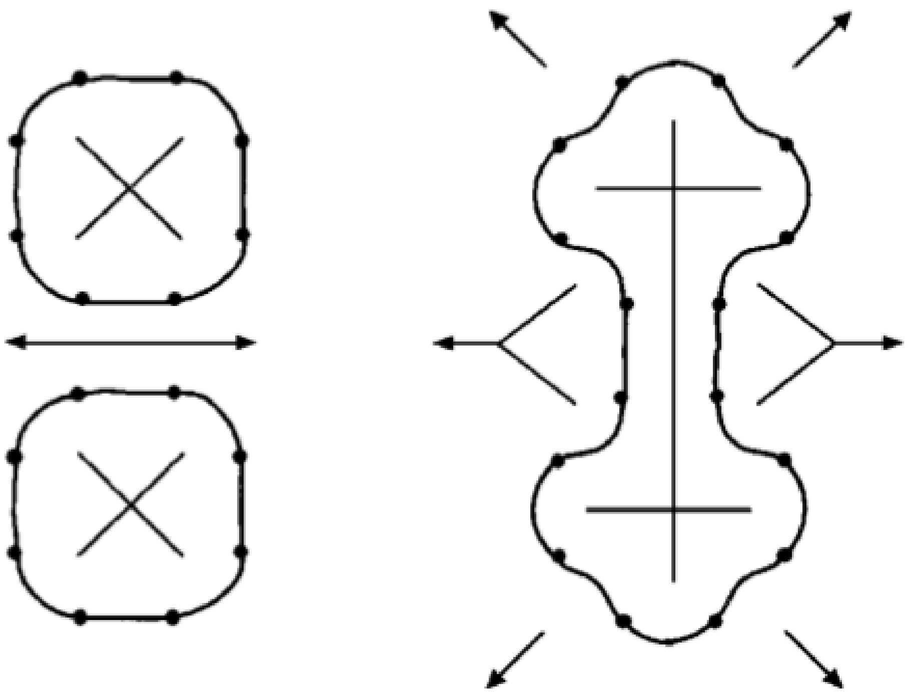
show that every edge of $\mathrm{Del}_{\mathcal{S}}(P)$ connects consecutive points of P along $\mathcal{S}$, and vice-versa

$\Rightarrow$ $\mathrm{Del}_{\mathcal{S}}(P)$ is homeomorphic to $\mathcal{S}$ between each pair of consecutive points of P



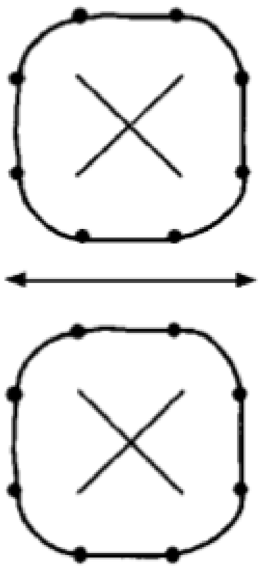
Since $\mathrm{Del}_{\mathcal{S}}(P)$ is embedded in $\mathrm{Del}(P)$, it does not self-intersect $\Rightarrow$ global homeomorphism

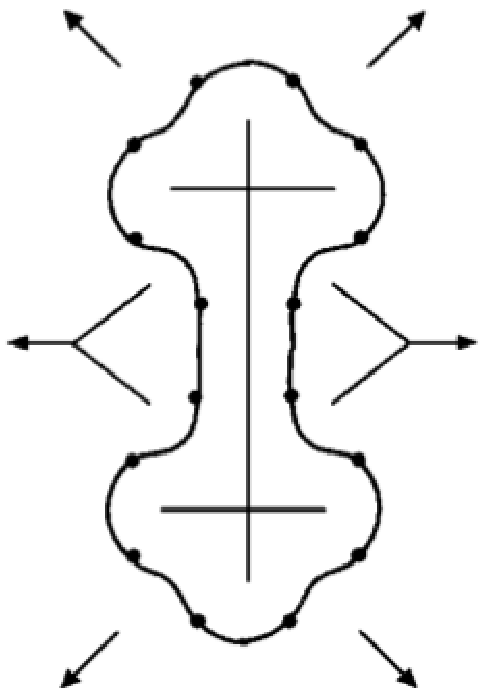
Approximation power of the restricted Delaunay

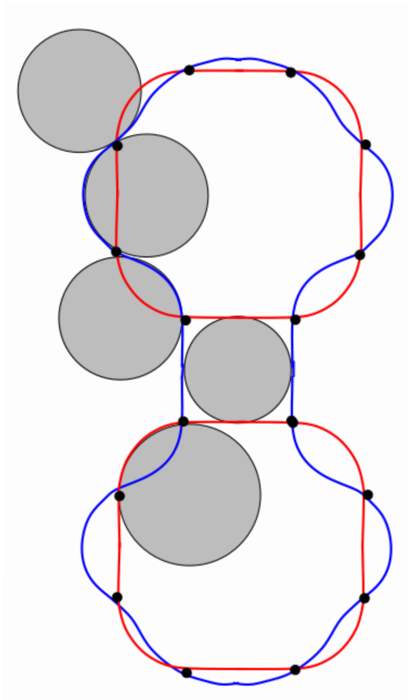


The polygonal reconstruction may no be unique

Approximation power of the restricted Delaunay







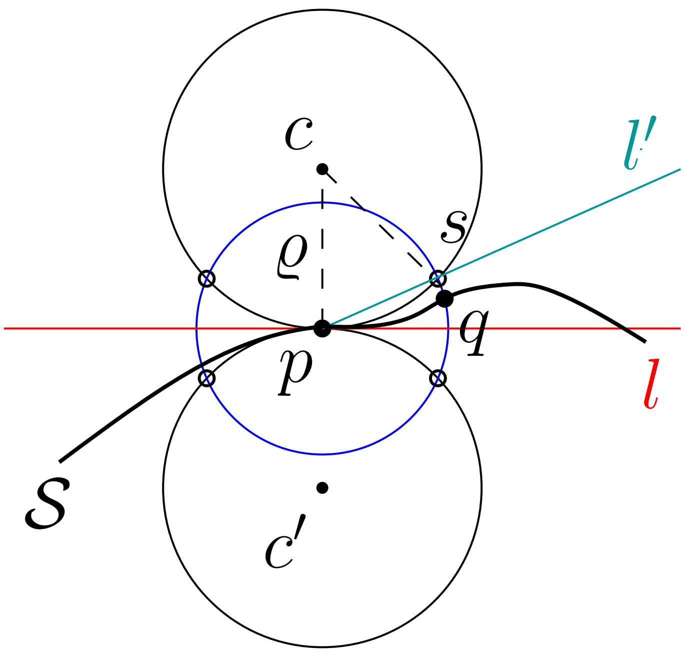
The polygonal reconstruction may no be unique

(if P is not an ε-sample)

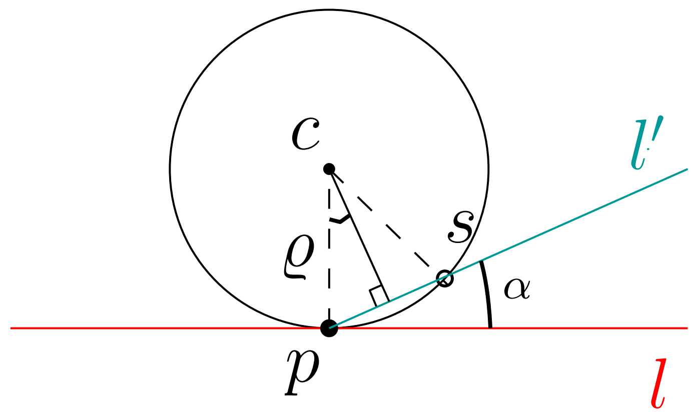
Flatness of the reconstruction

Property:
Let $\mathcal{S}$ be a curve with positive *reach* ϱ, and let p, q two points on $\mathcal{S}$. If $d(p,q) < 2\varrho$, then the angle between (p,q) and the tangent line l is at most $\arcsin \frac{d(p,q)}{2\varrho}$

$$\angle(pq,l) \leq \angle(l,l') := \alpha$$

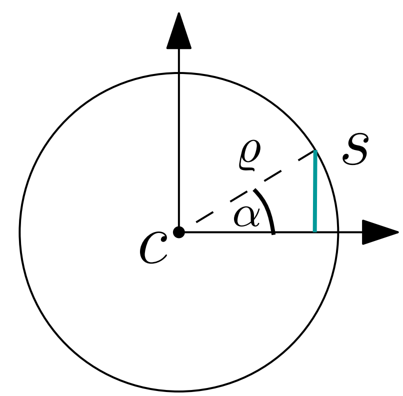


$$\alpha = 2\angle(pcs)$$



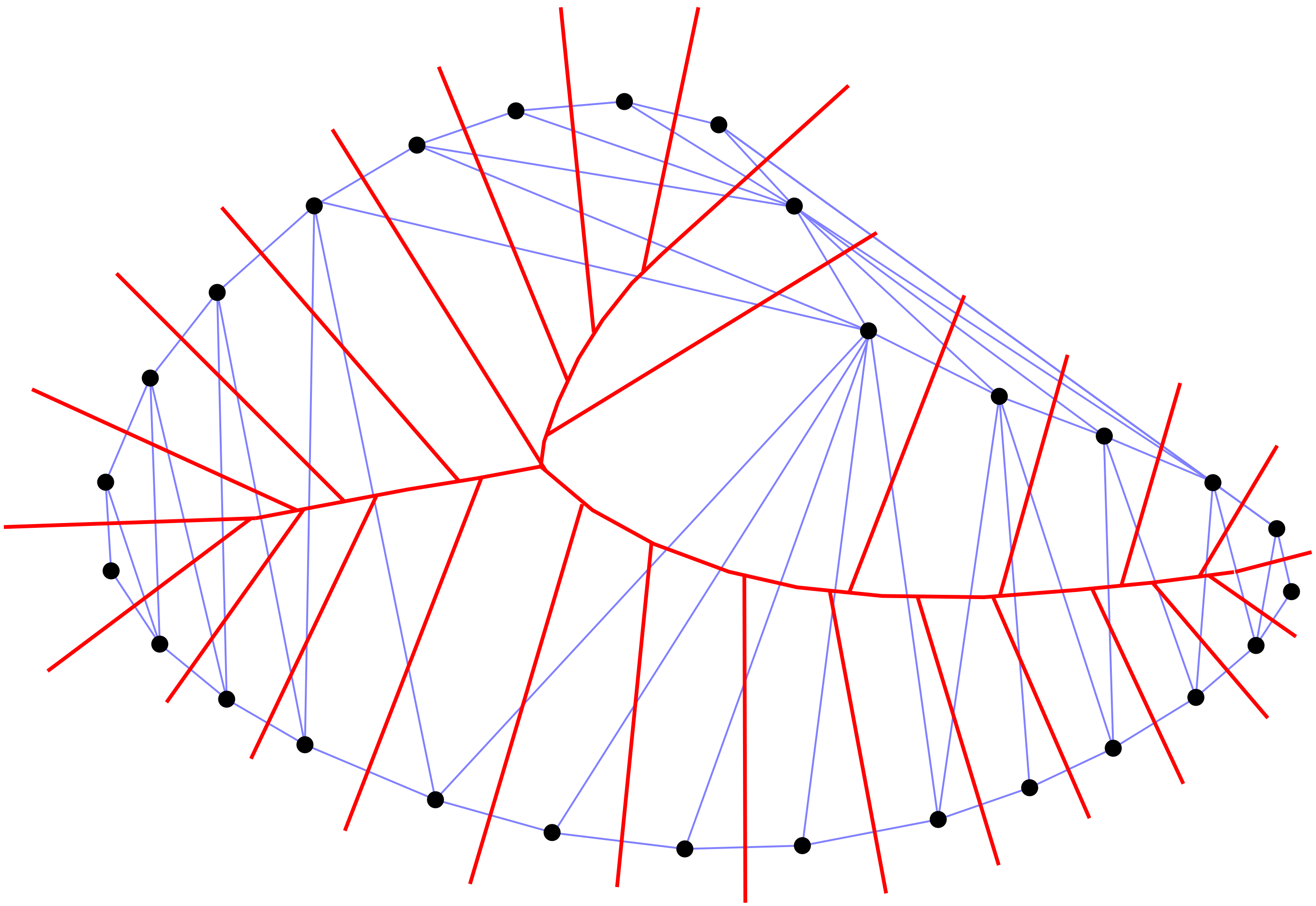
$$\tfrac{1}{2}d(p,s) = \varrho \sin \alpha$$

$$\alpha = \arcsin \frac{d(p,s)}{2\varrho}$$



Computing the restricted Delaunay

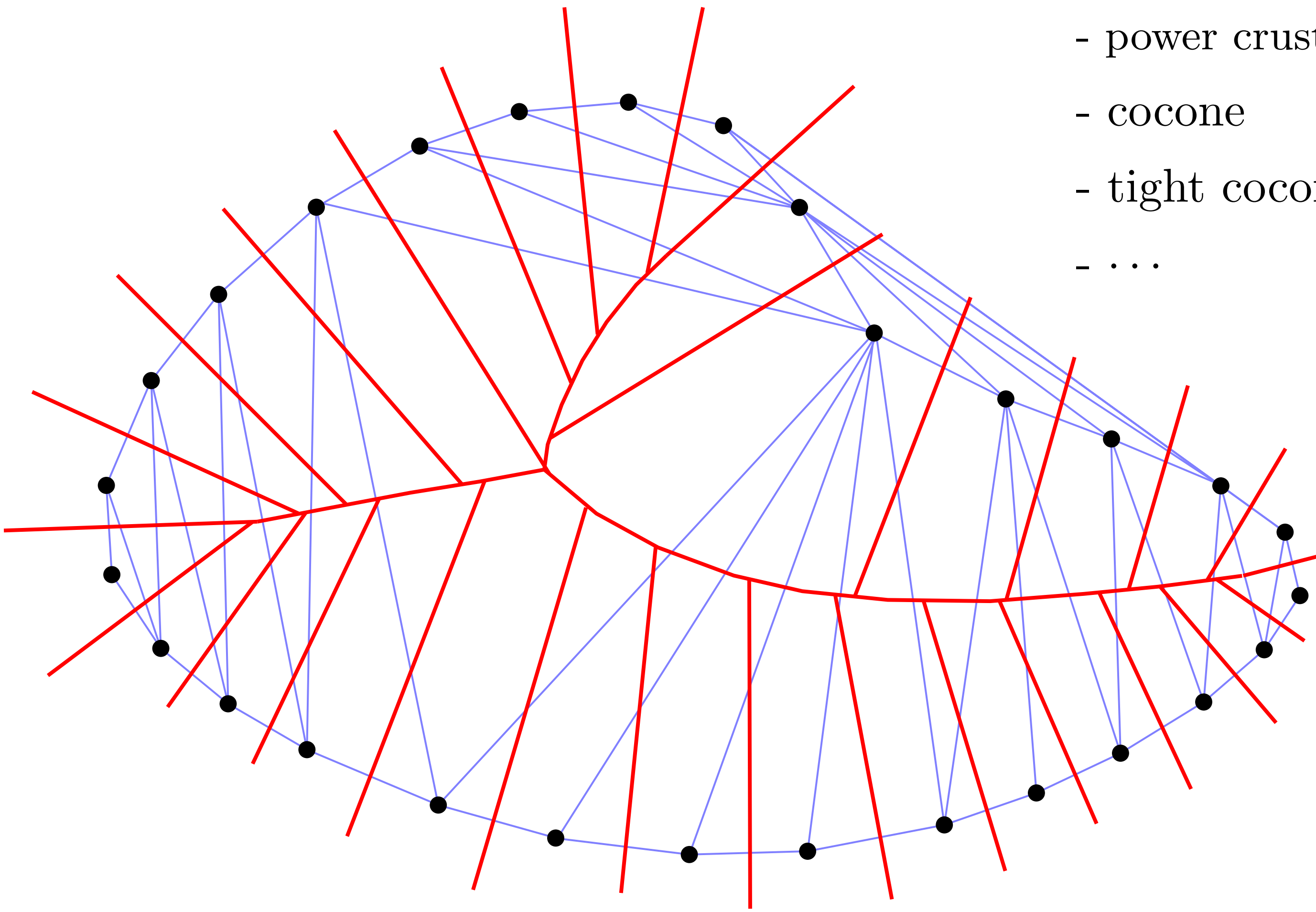
Q How to compute $\mathrm{Del}_{\mathcal{S}}(P)$ when $\mathcal{S}$ is unknown?



Computing the restricted Delaunay

Q How to compute $\mathrm{Del}_{\mathcal{S}}(P)$ when $\mathcal{S}$ is unknown?

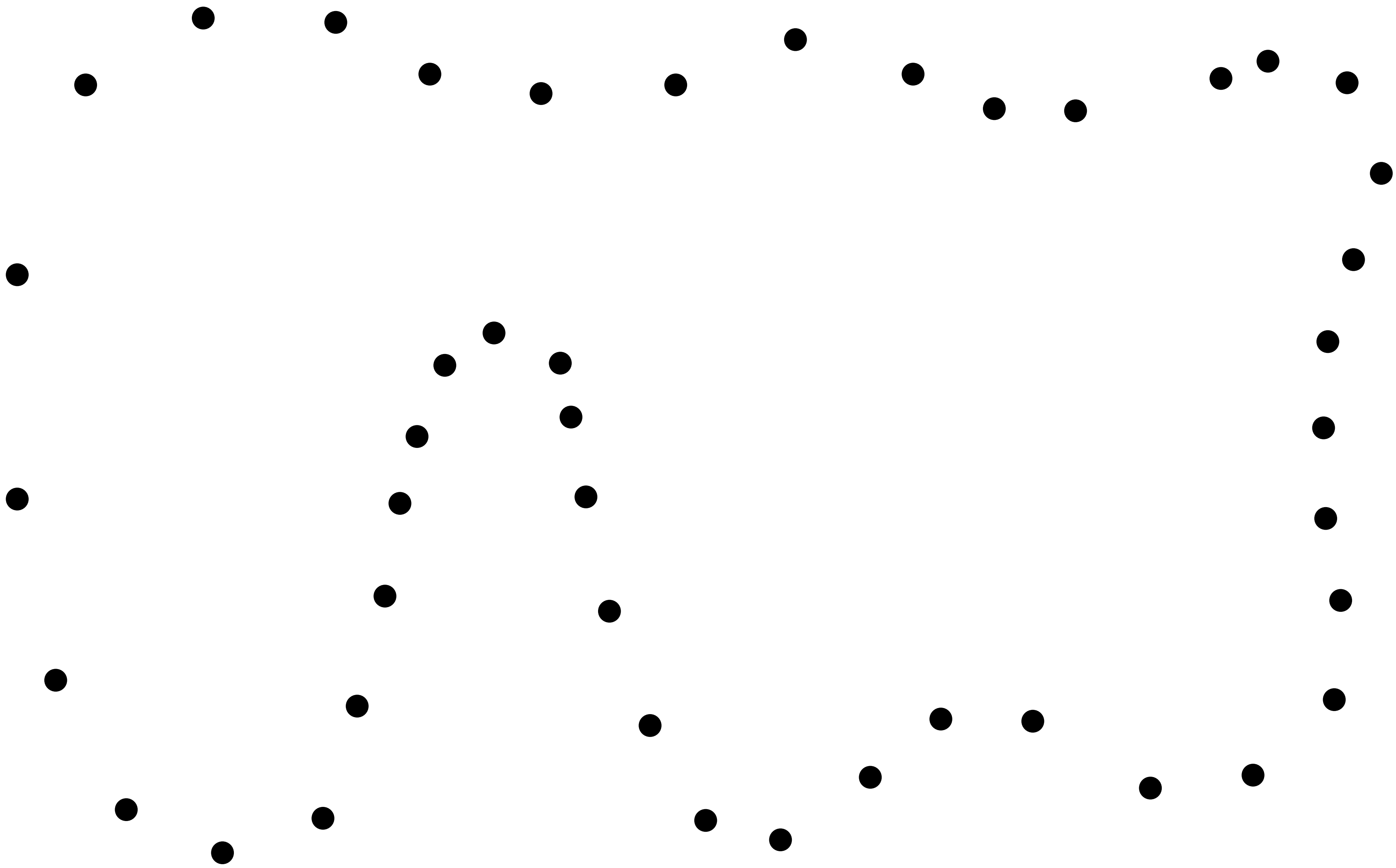
$\rightarrow$ a whole family of algorithms use various Delaunay extraction criteria:



- crust
- power crust
- cocone
- tight cocone
- $\cdots$

Crust algorithm

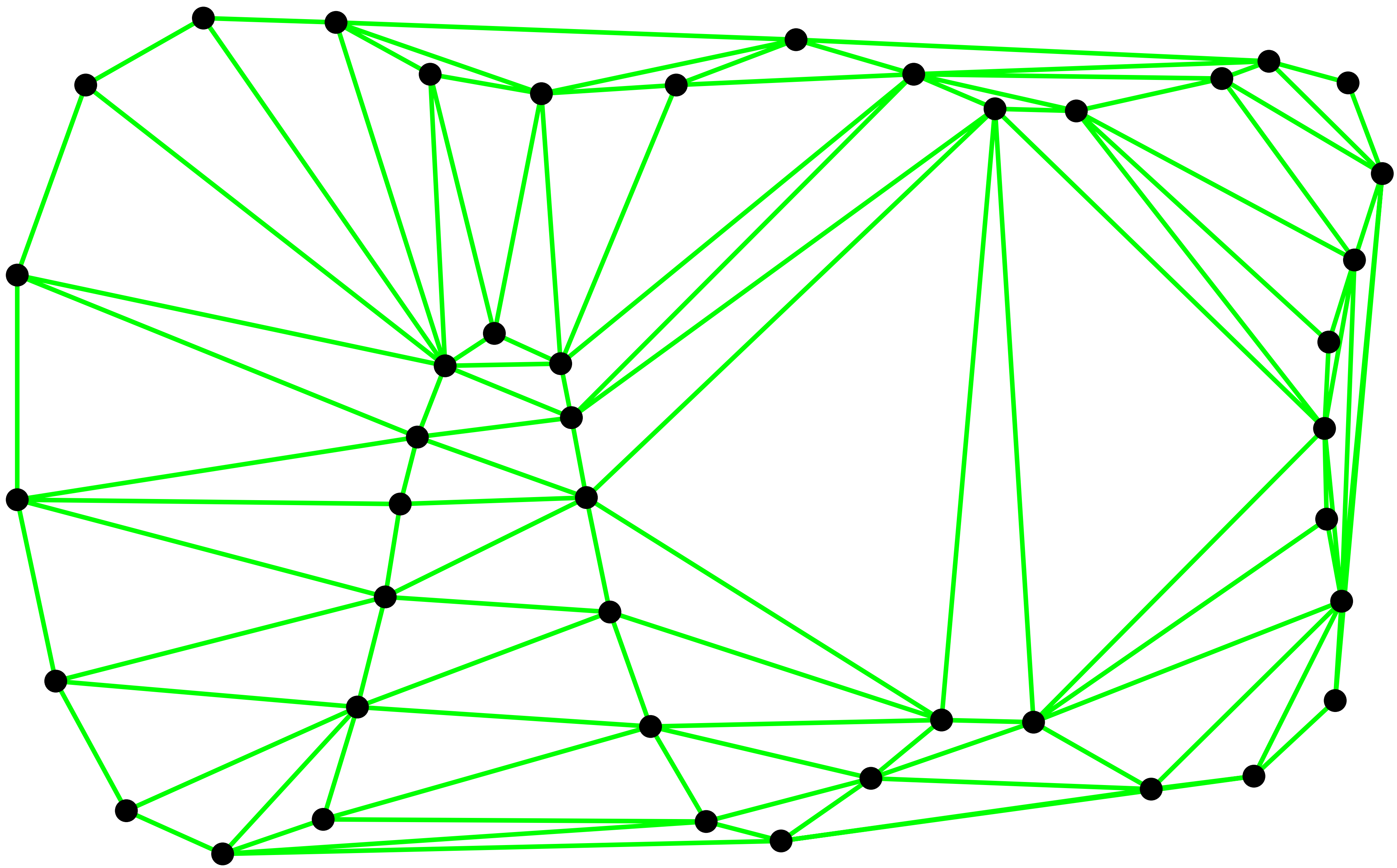
[Amenta et al. 1997-98]



Crust algorithm

[Amenta et al. 1997-98]

1. Compute Delaunay triangulation of P



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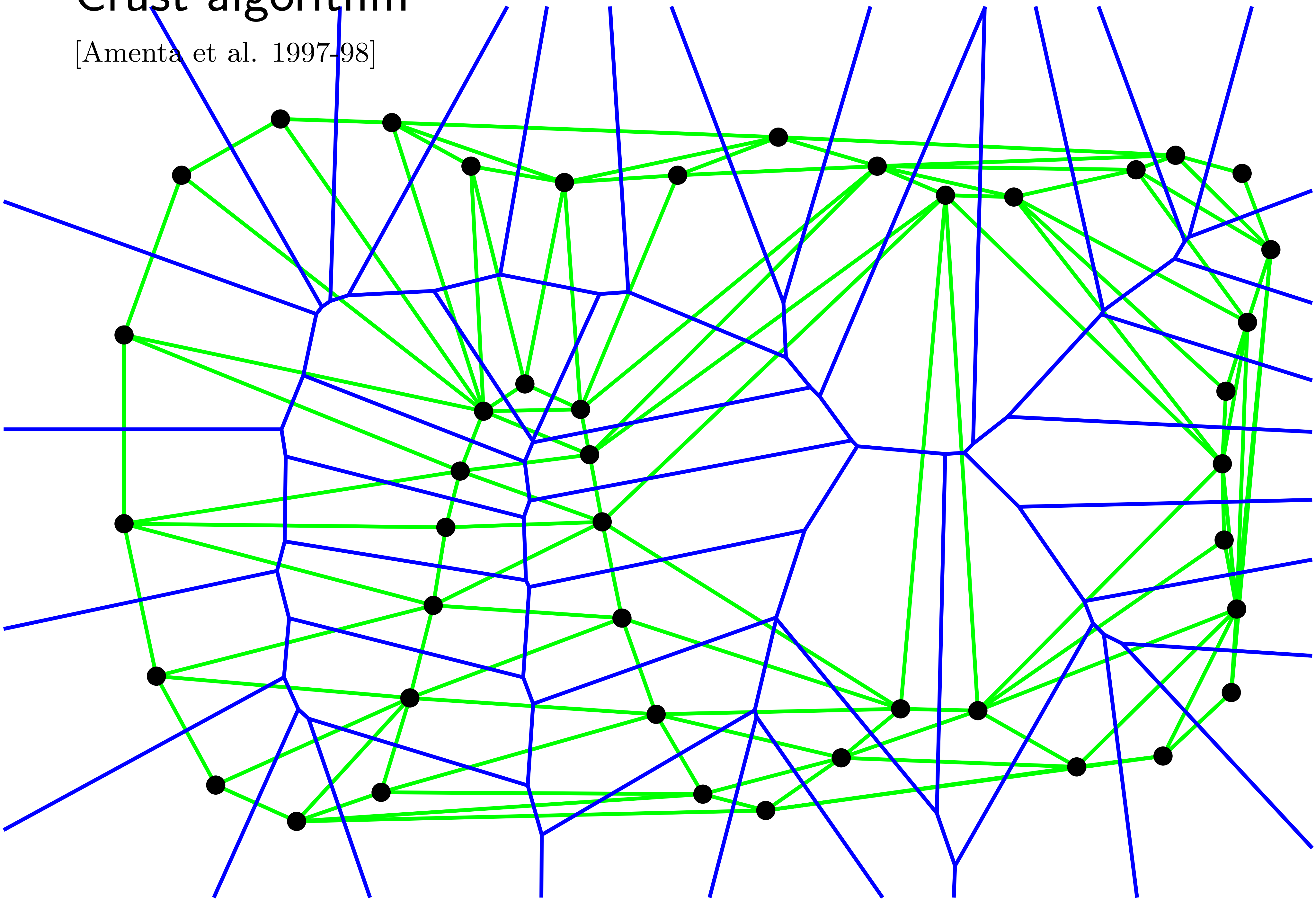
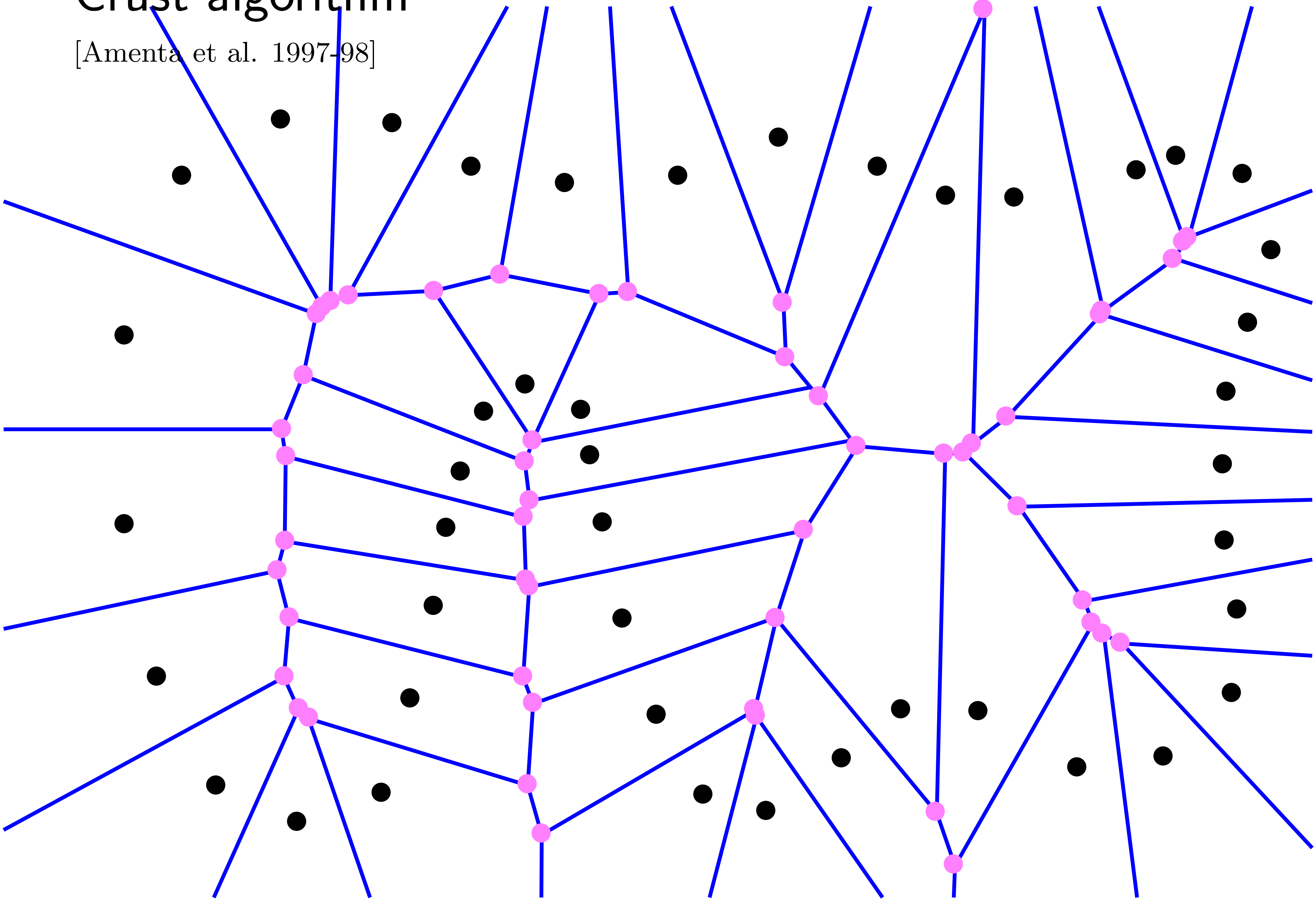


Crust algorithm

[Amenta et al. 1997-98]

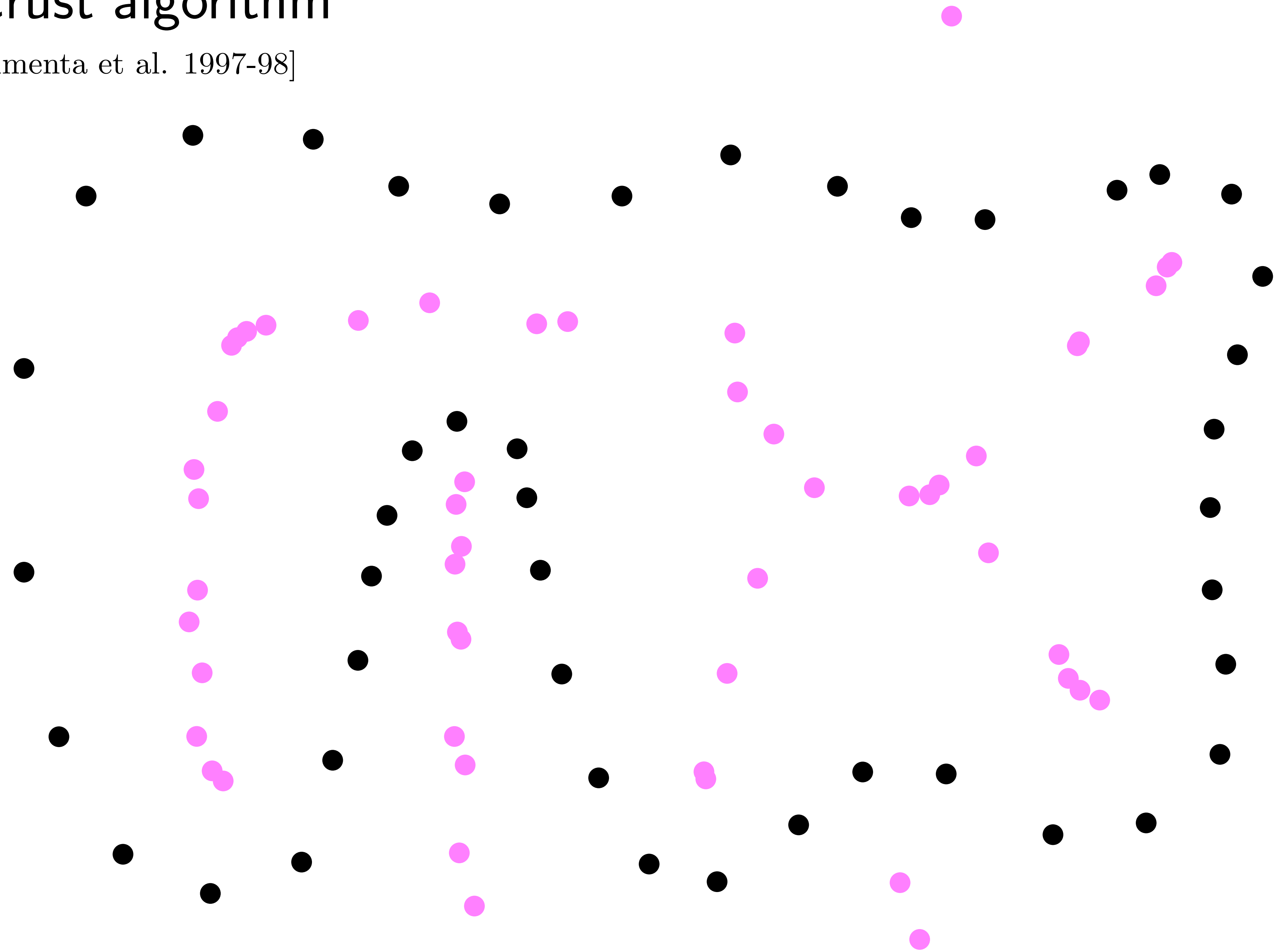
2. Compute *poles* (furthest Voronoi vertices)



Crust algorithm

[Amenta et al. 1997-98]

3. Add poles to the set of vertices



Crust algorithm

[Amenta et al. 1997-98]

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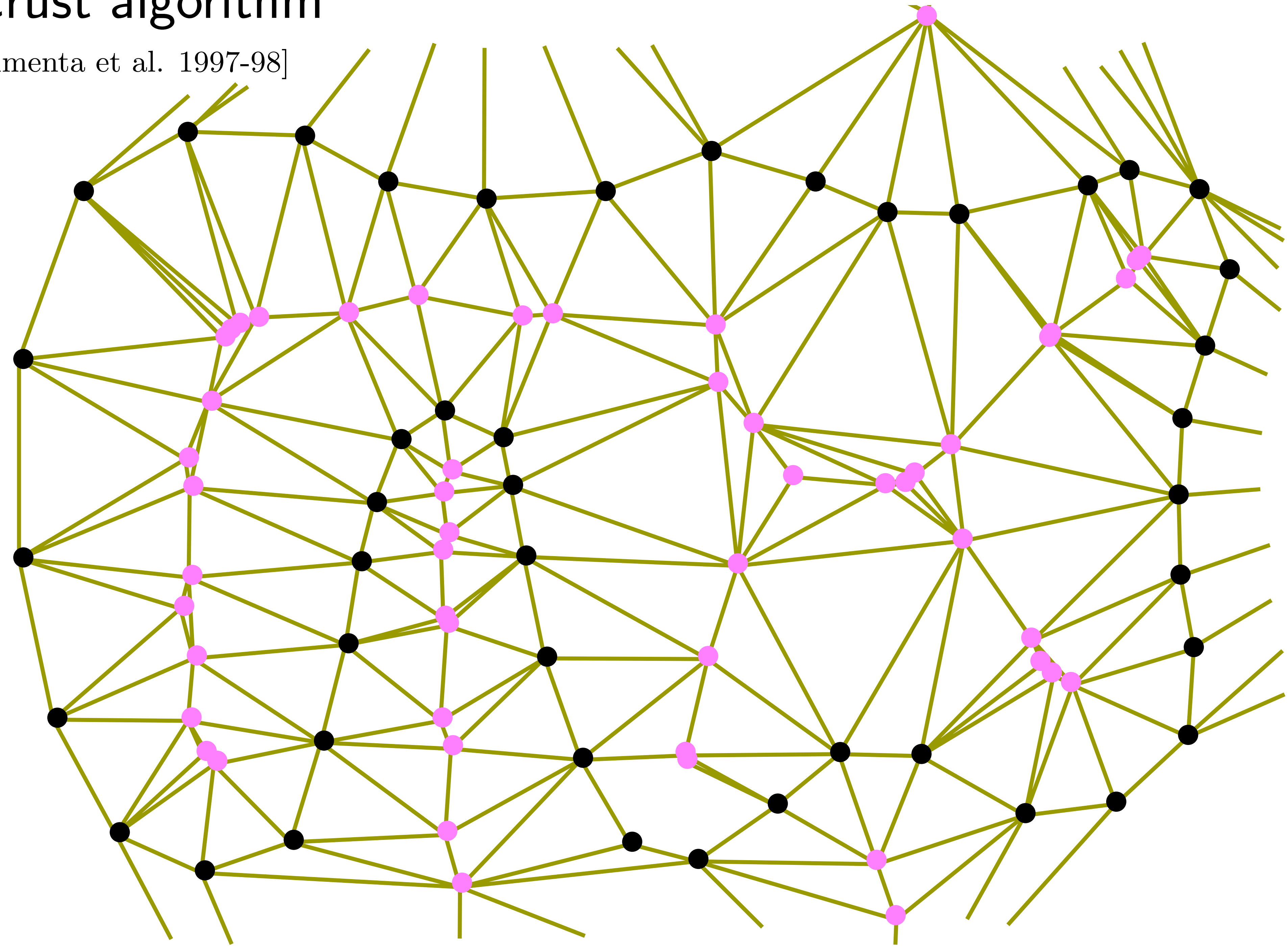
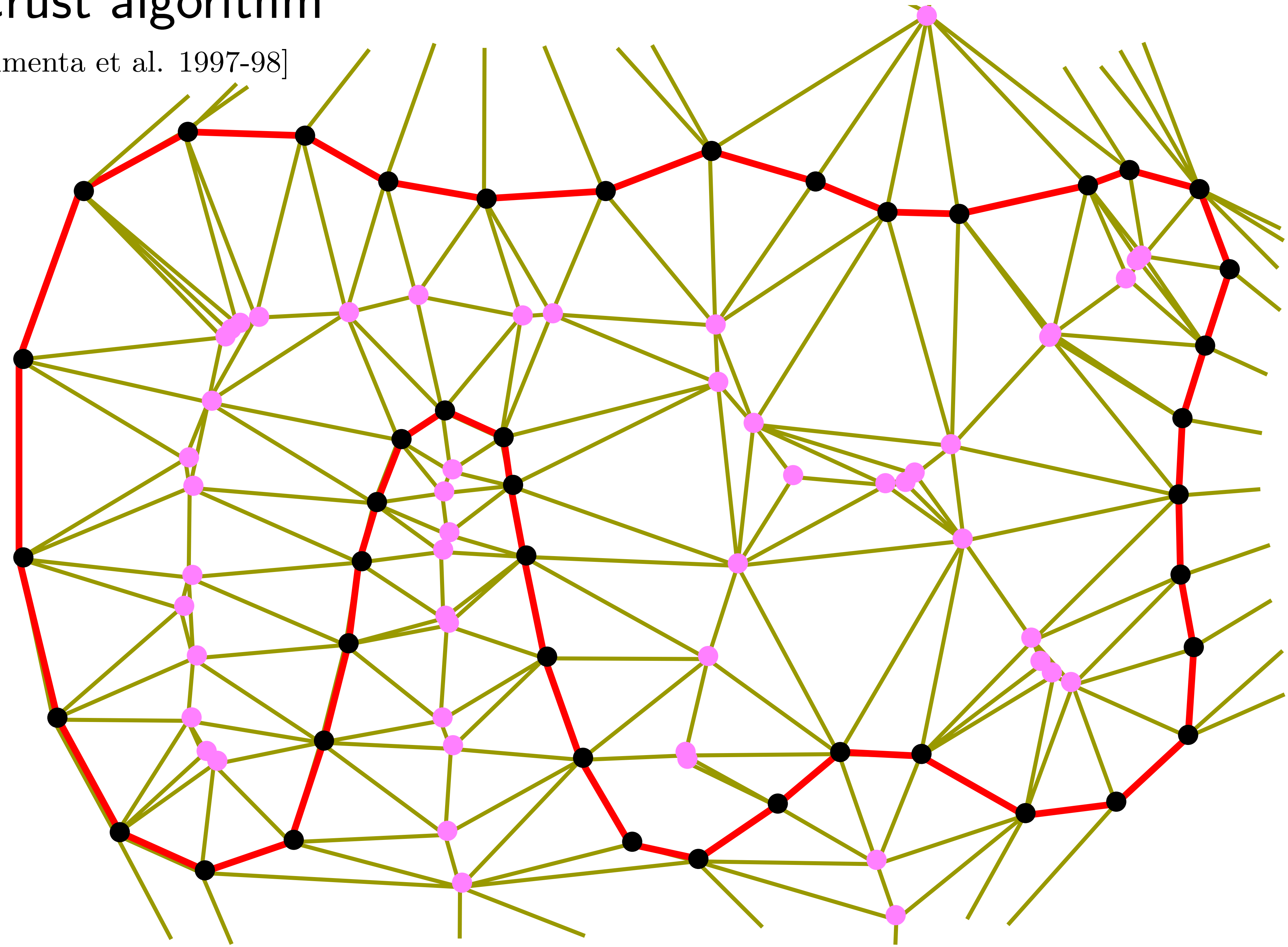


Crust algorithm

[Amenta et al. 1997-98]

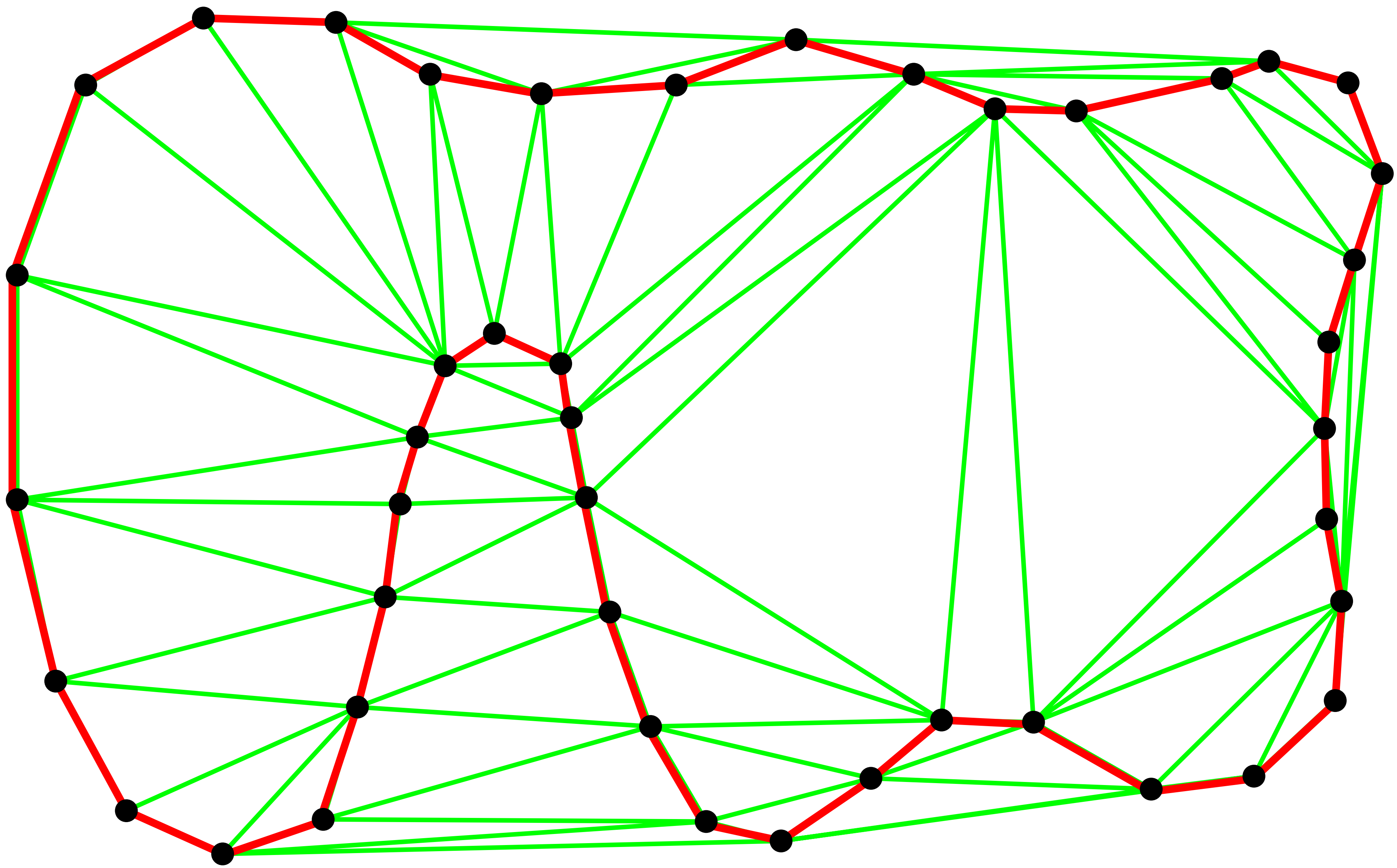
4. Keep Delaunay simplices whose vertices are in P



Crust algorithm

$\rightarrow$ in 2-d, crust $= \mathrm{Del}_{\mathcal{S}}(P) \approx \mathcal{S}$

[Amenta et al. 1997-98]



Crust algorithm

[Amenta et al. 1997-98]

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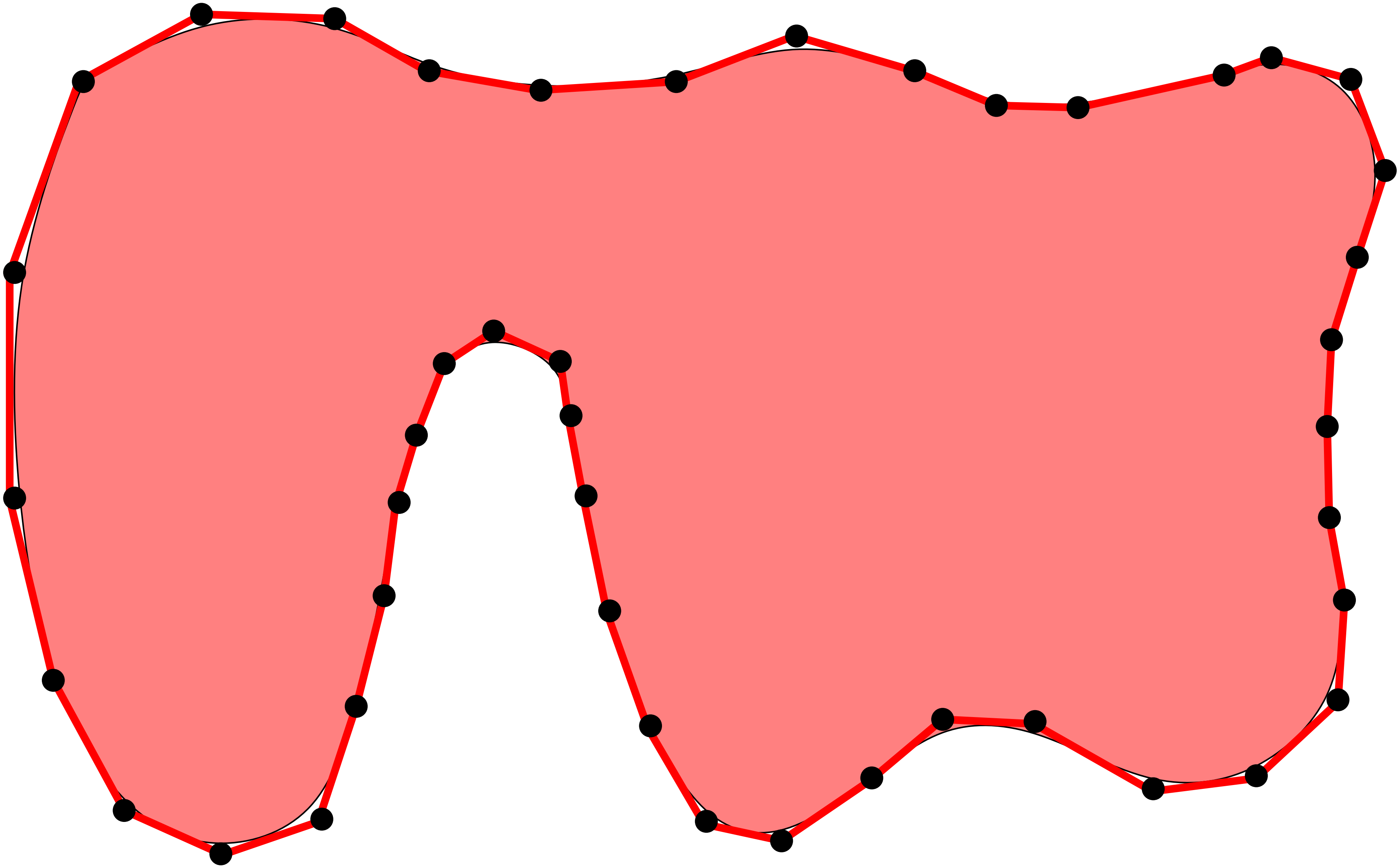
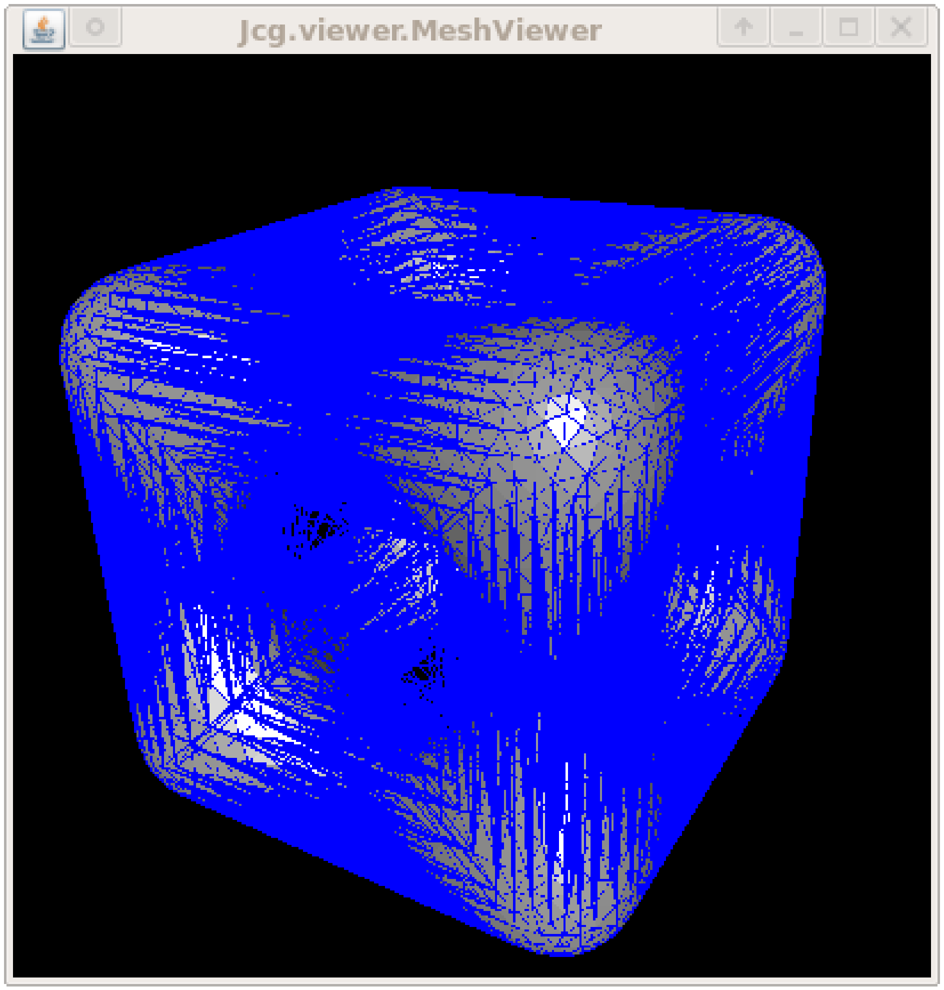


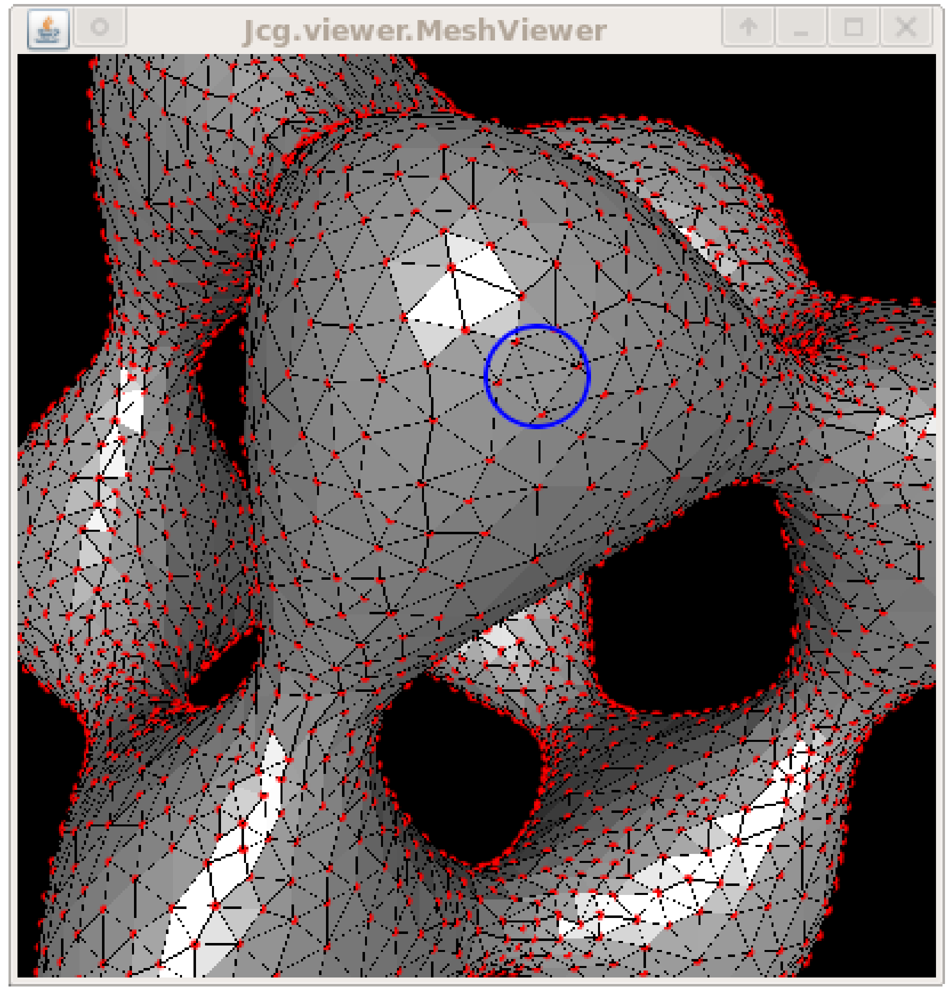
Crust algorithm

[Amenta et al. 1997-98]

$\rightarrow$ in 2-d, crust $= \mathrm{Del}_{\mathcal{S}}(P) \approx \mathcal{S}$

$\rightarrow$ in 3-d, crust $\supseteq \mathrm{Del}_{\mathcal{S}}(P) \approx \mathcal{S}$





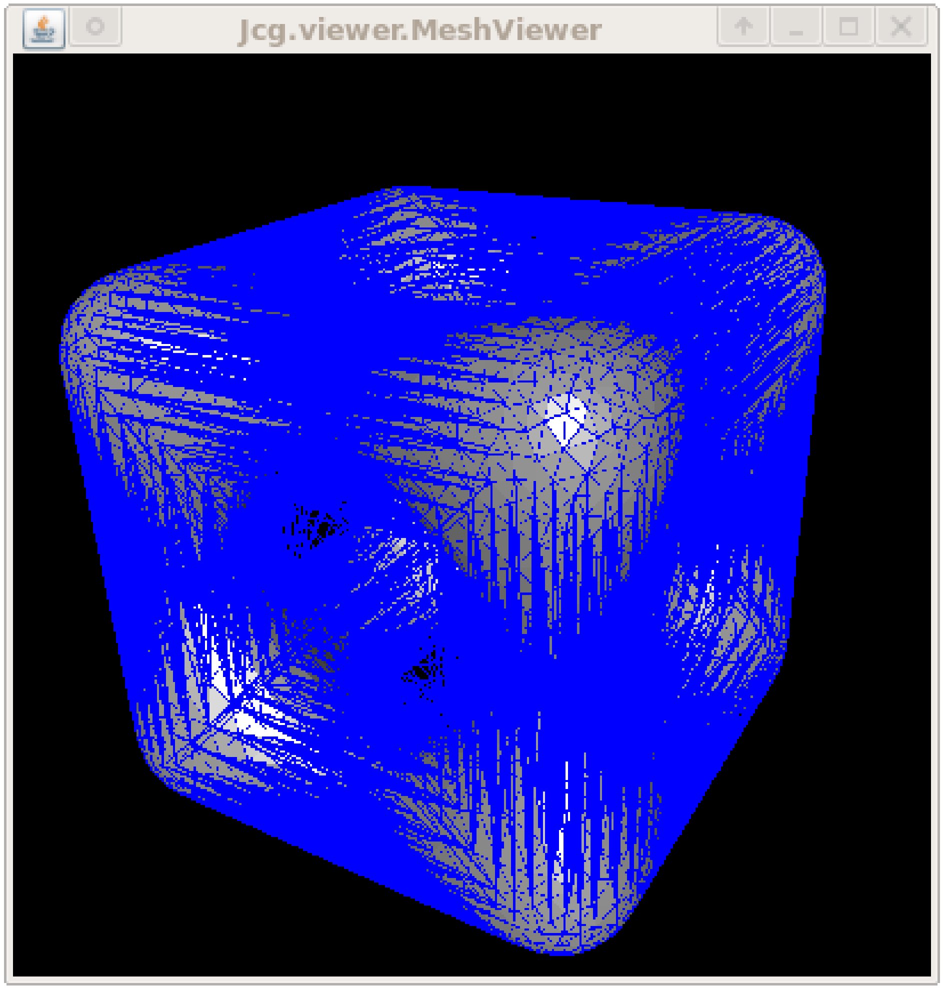
Crust algorithm

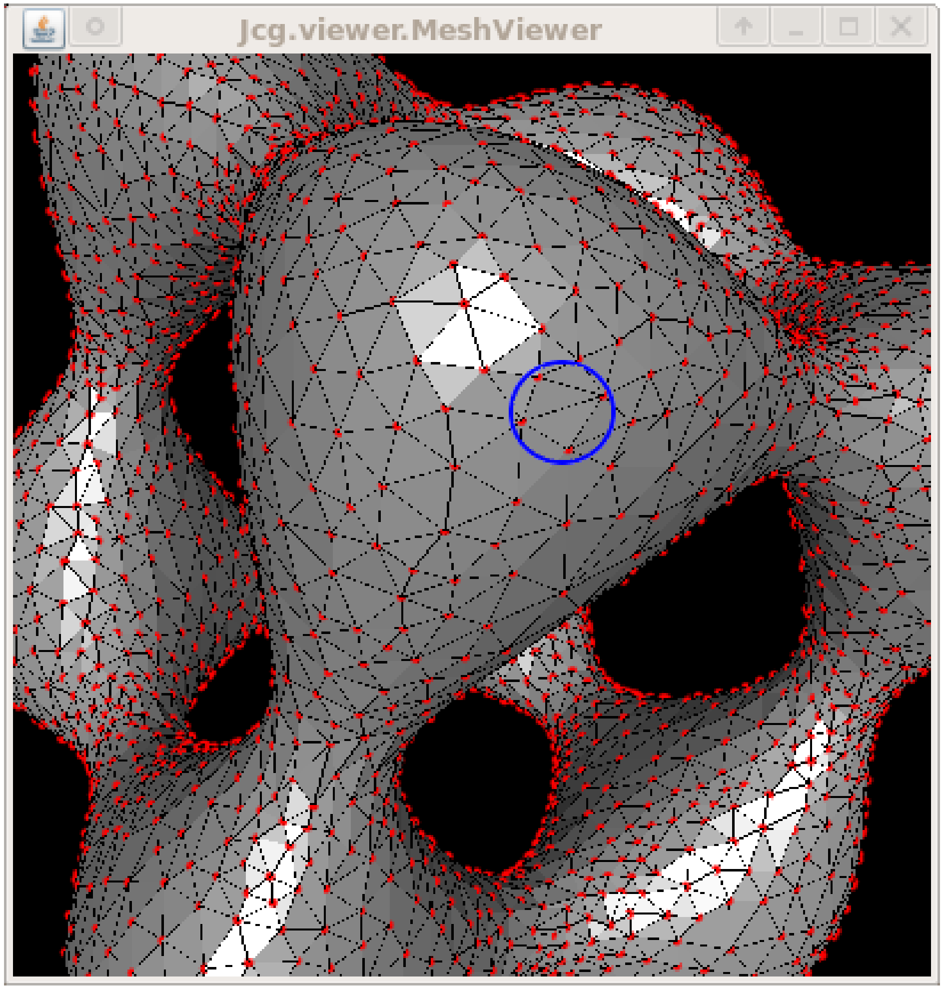
[Amenta et al. 1997-98]

$\rightarrow$ in 2-d, crust $= \mathrm{Del}_{\mathcal{S}}(P) \approx \mathcal{S}$

$\rightarrow$ in 3-d, crust $\supseteq \mathrm{Del}_{\mathcal{S}}(P) \approx \mathcal{S}$

$\Rightarrow$ manifold extraction step in post-processing





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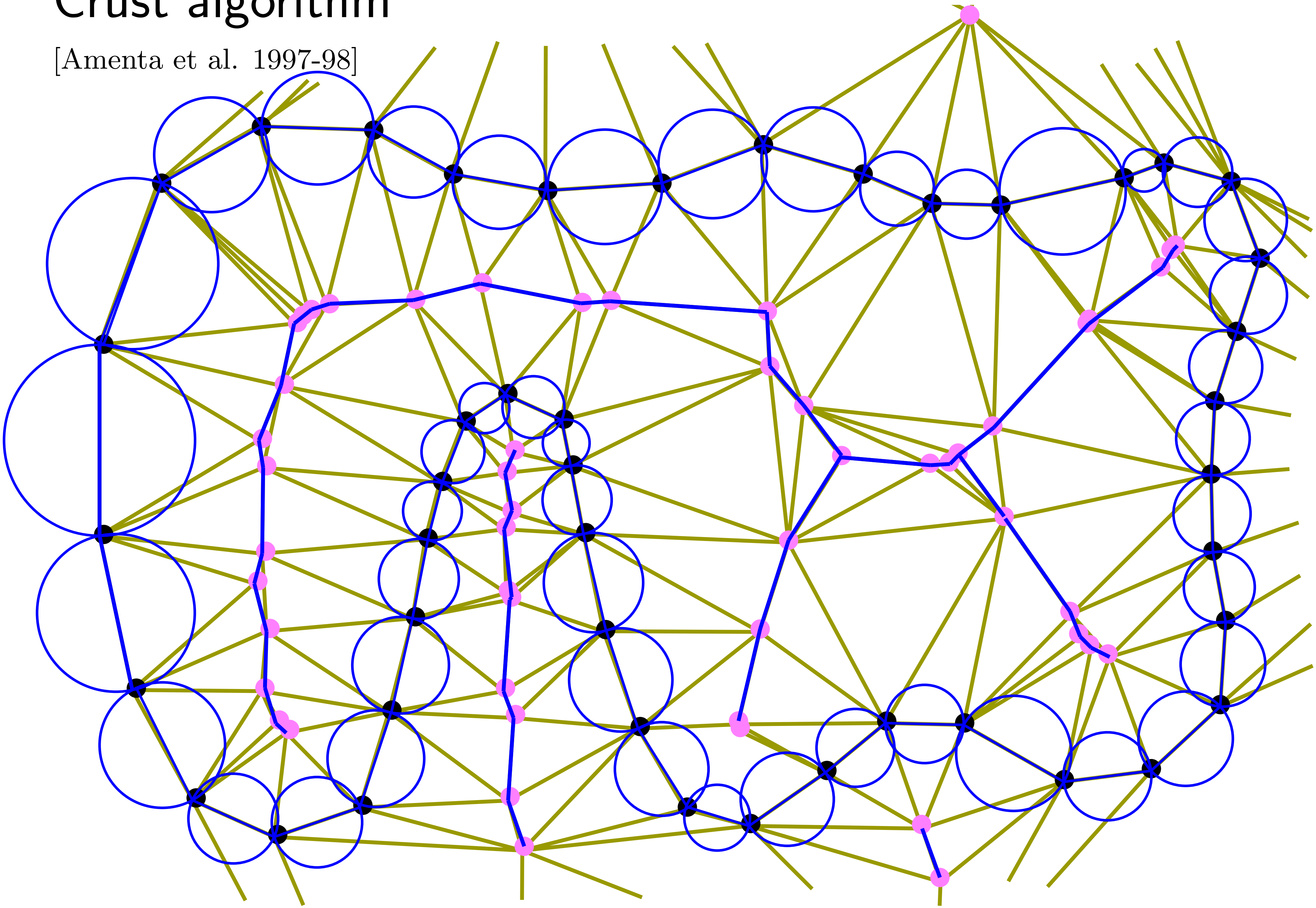

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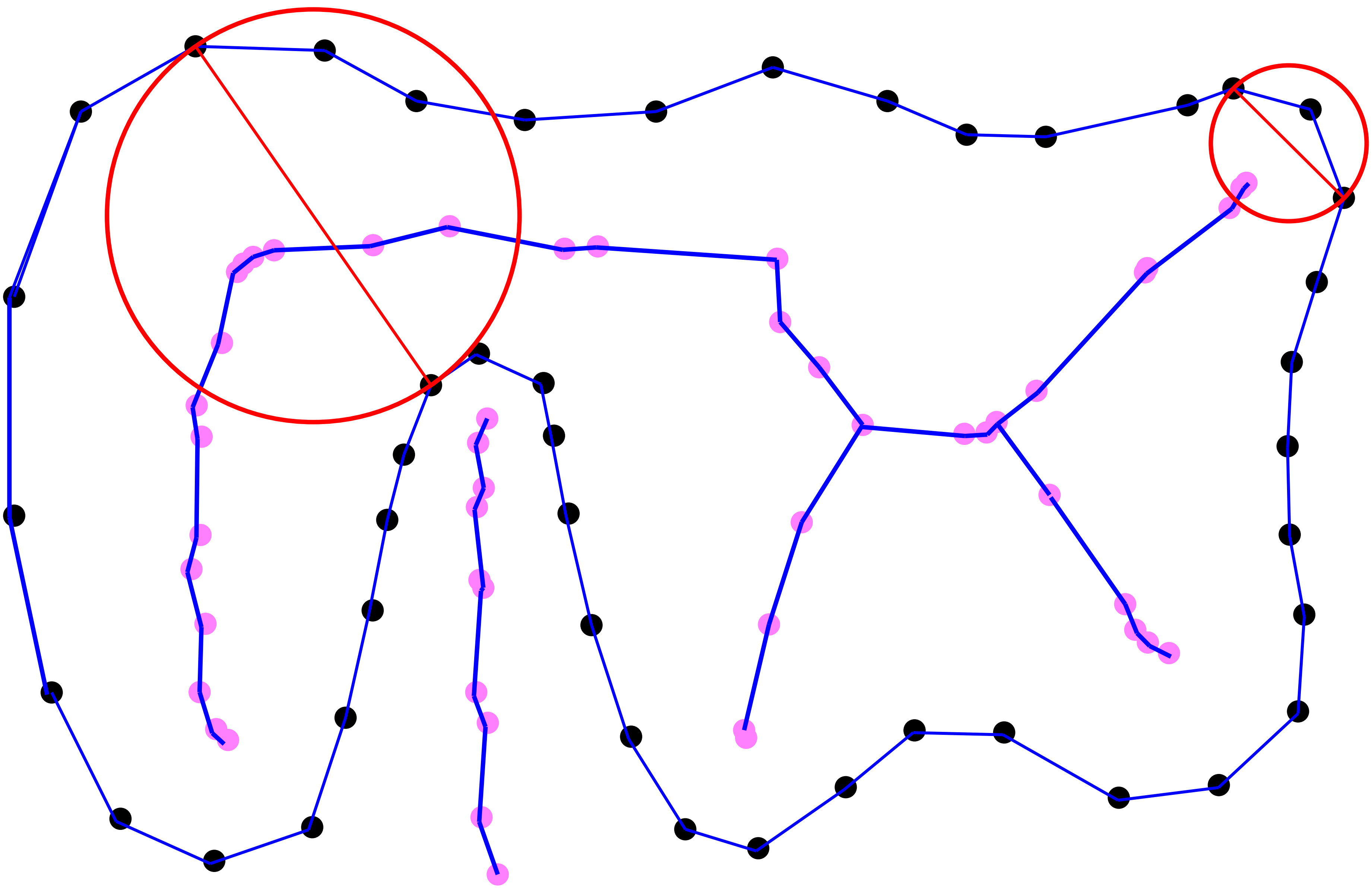
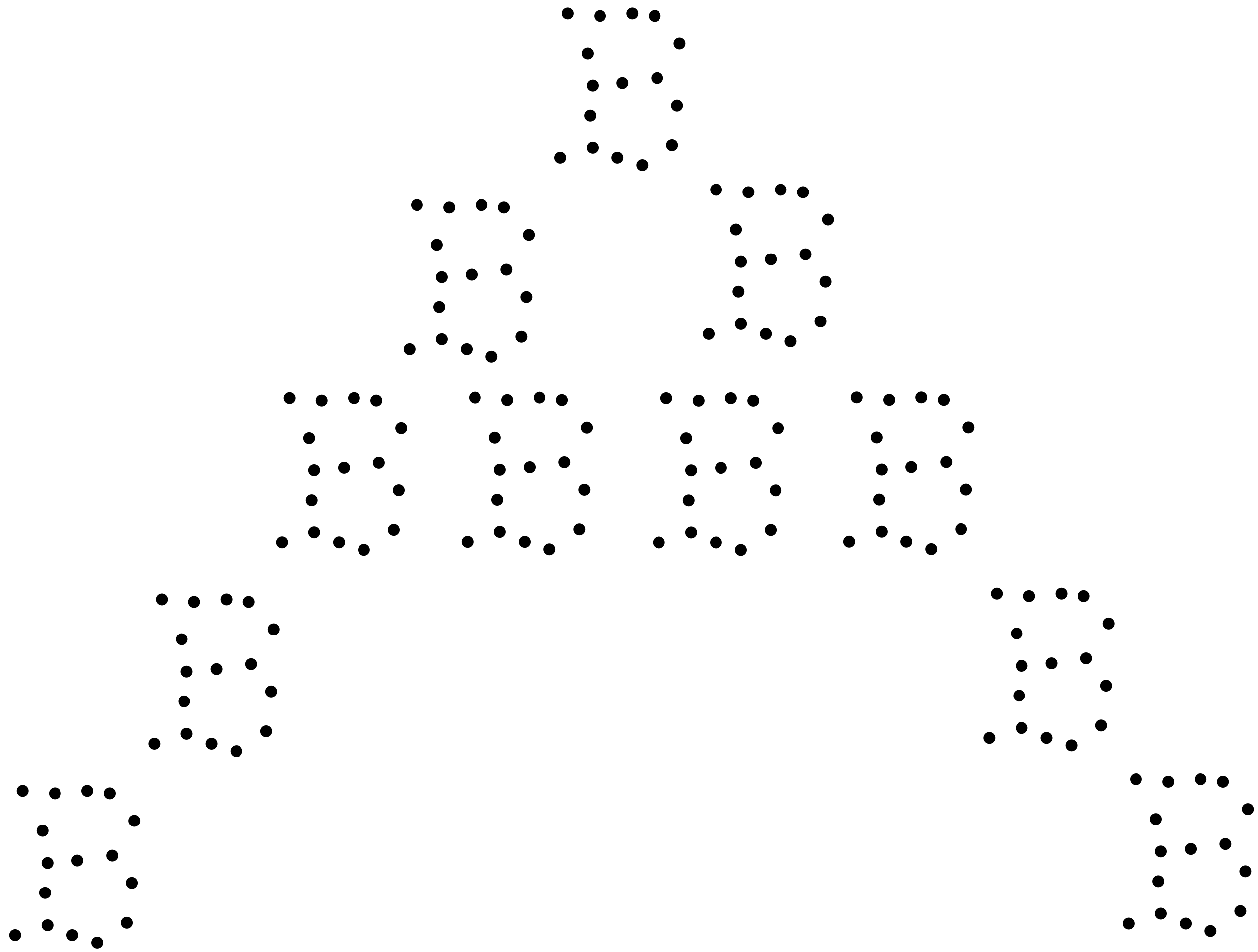


Back to the reconstruction paradigm

Q What do you see?

Why?



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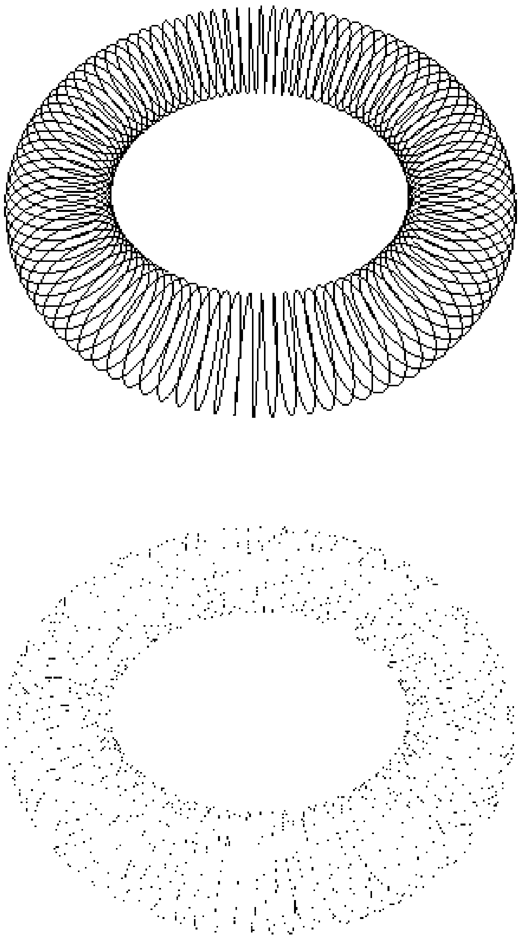
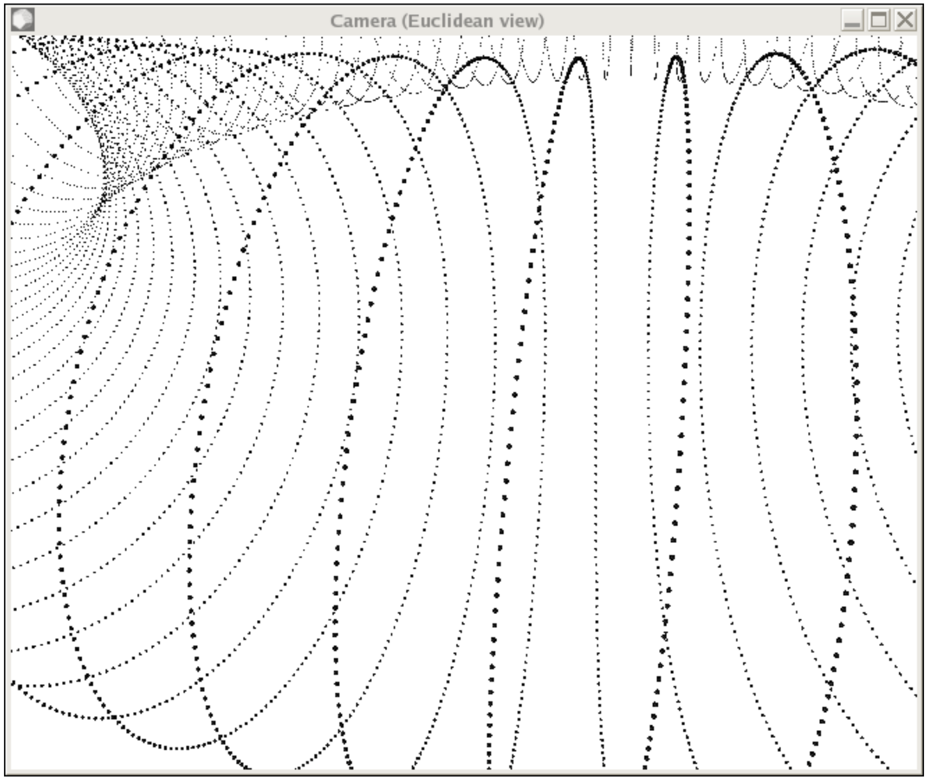




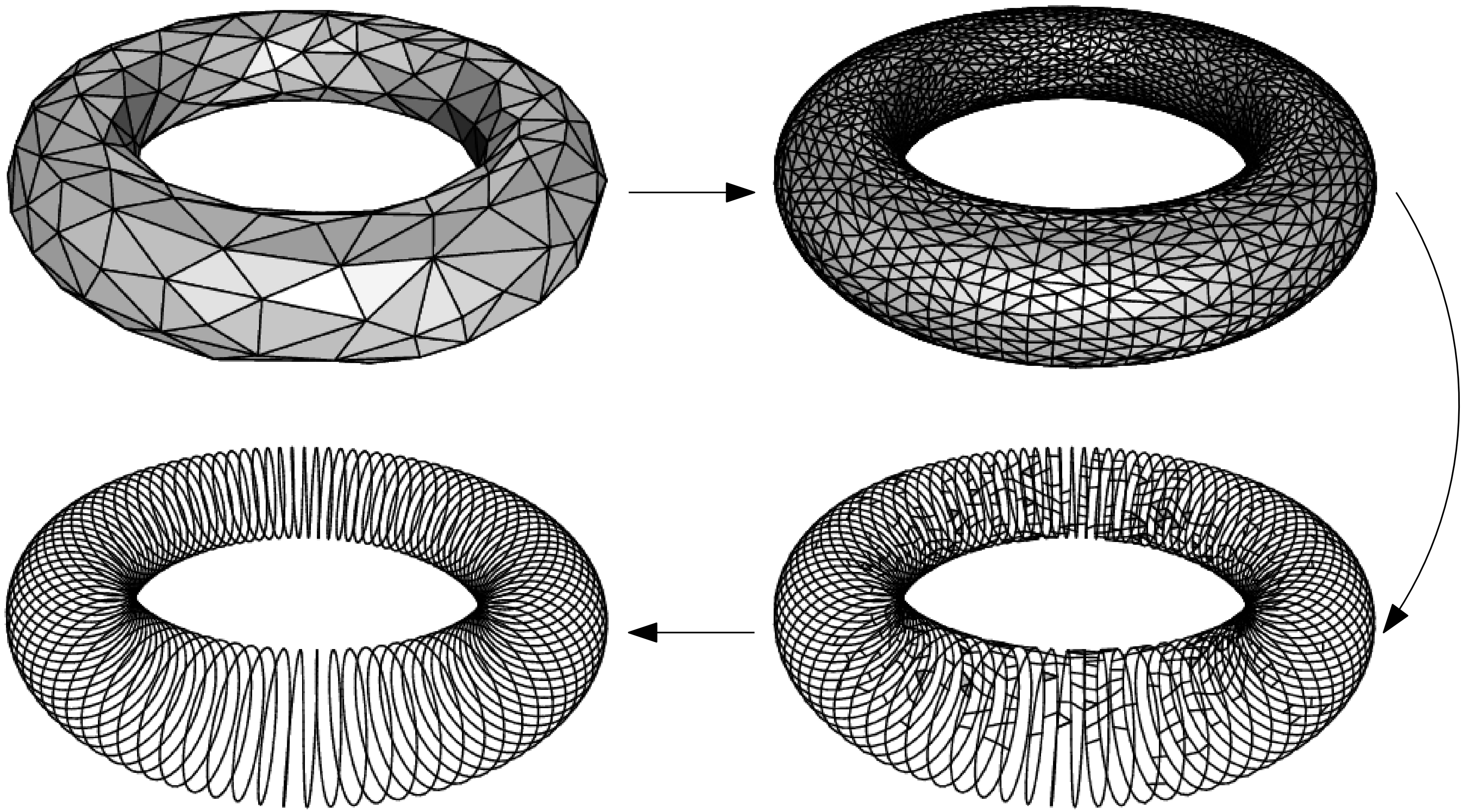
Back to the reconstruction paradigm

$\rightarrow$ **When the dimensionality of the data is unknown or there is noise, the reconstruction result depends on the scale at which the data is looked at.**

$\rightarrow$ need for multi-scale reconstruction techniques

Multi-scale approach in a nutshell

$\rightarrow$ build a one-parameter family of complexes approximating the input at various scales



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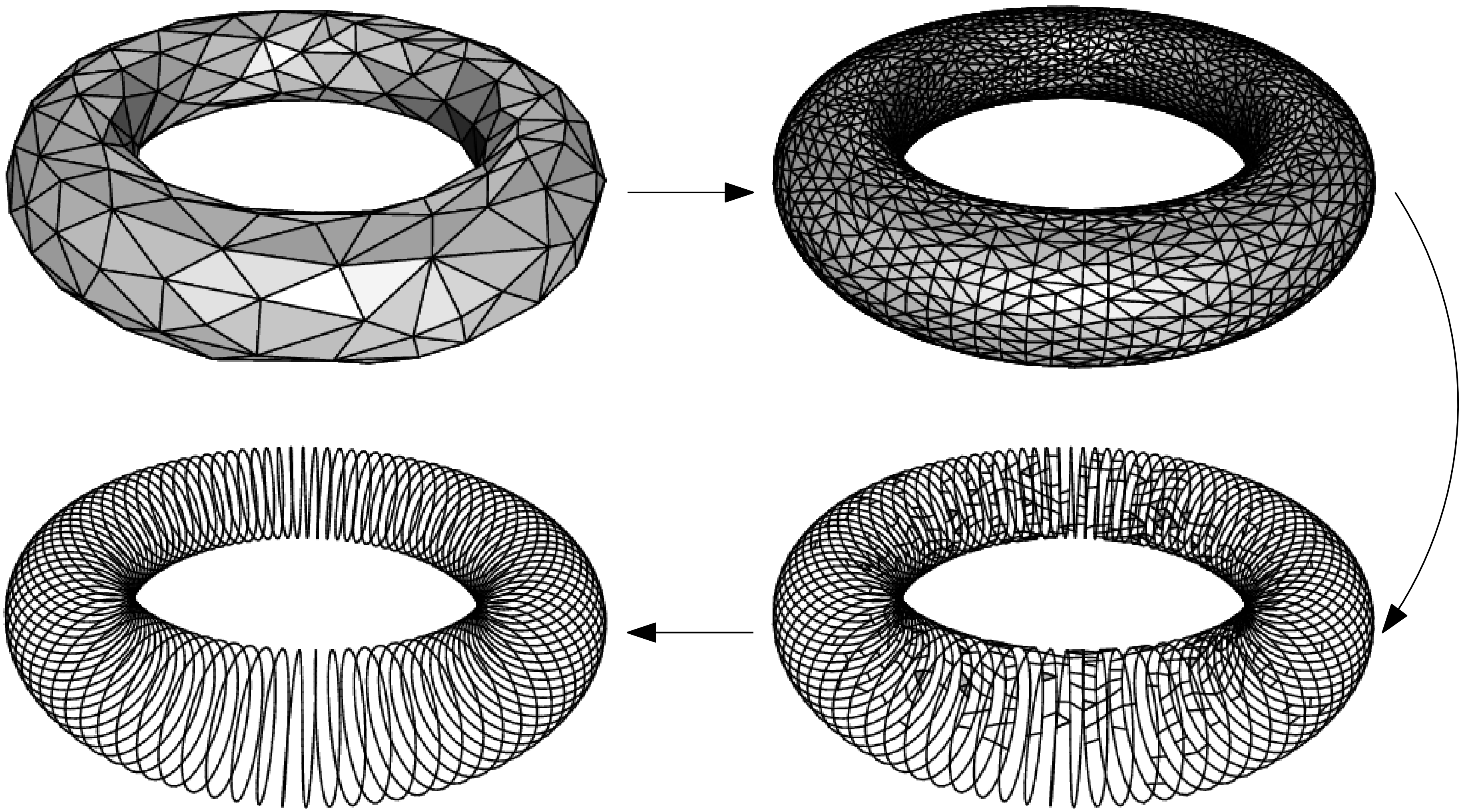
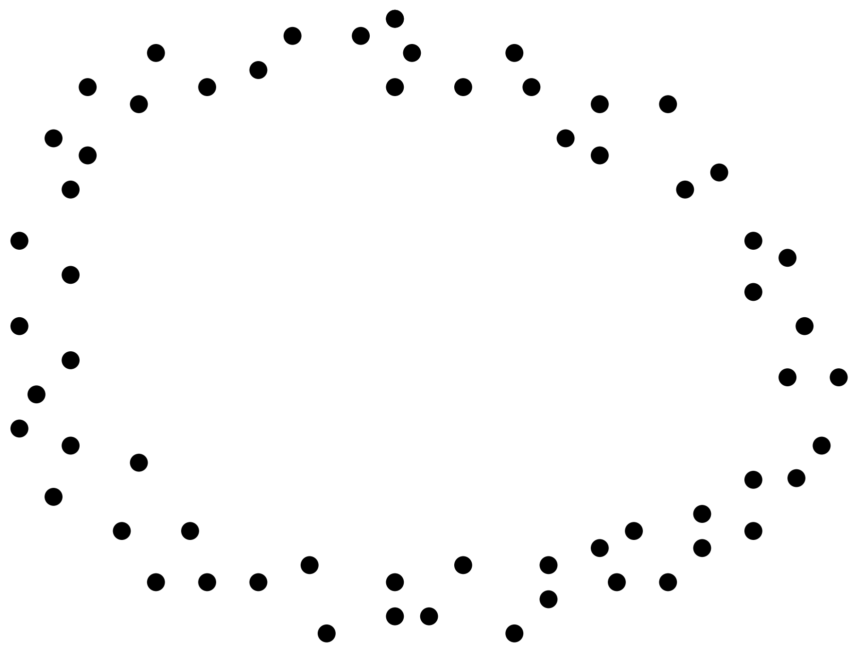


→ connections with manifold learning and topological persistence

Multi-scale algorithm [Guibas, Oudot 2007]

Input: a finite point set $W \subset \mathbb{R}^n$

$\rightarrow$ resample W iteratively, and maintain a simplicial complex:



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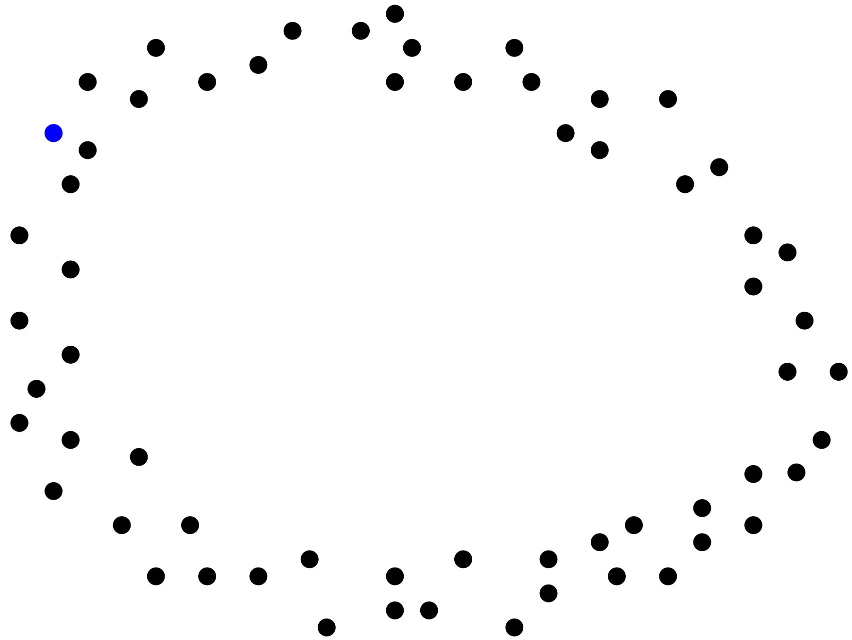

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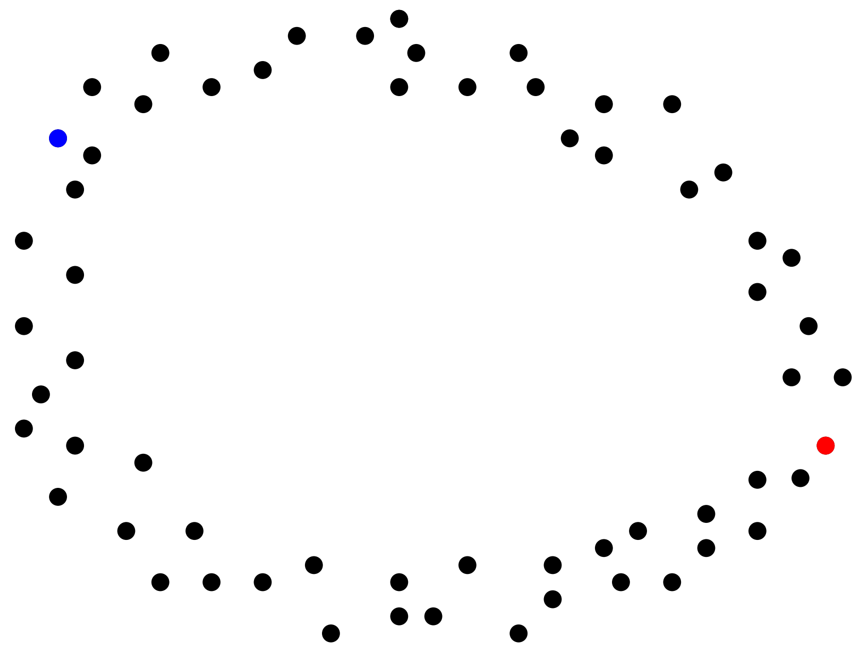

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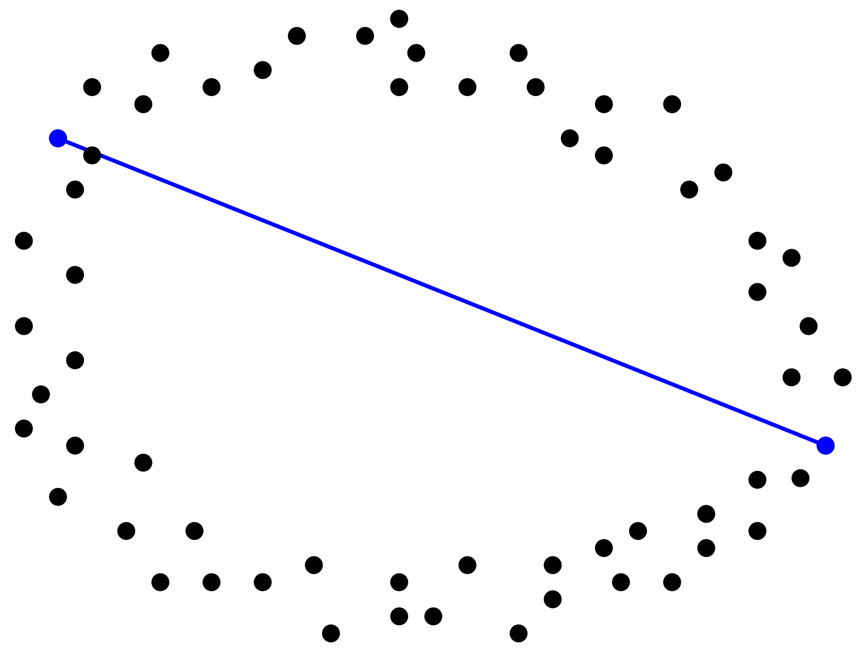

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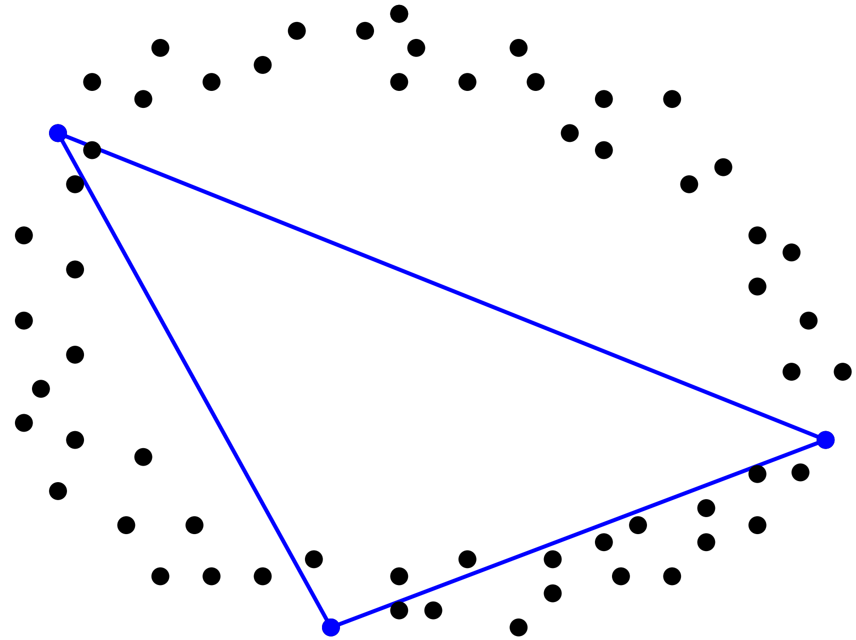

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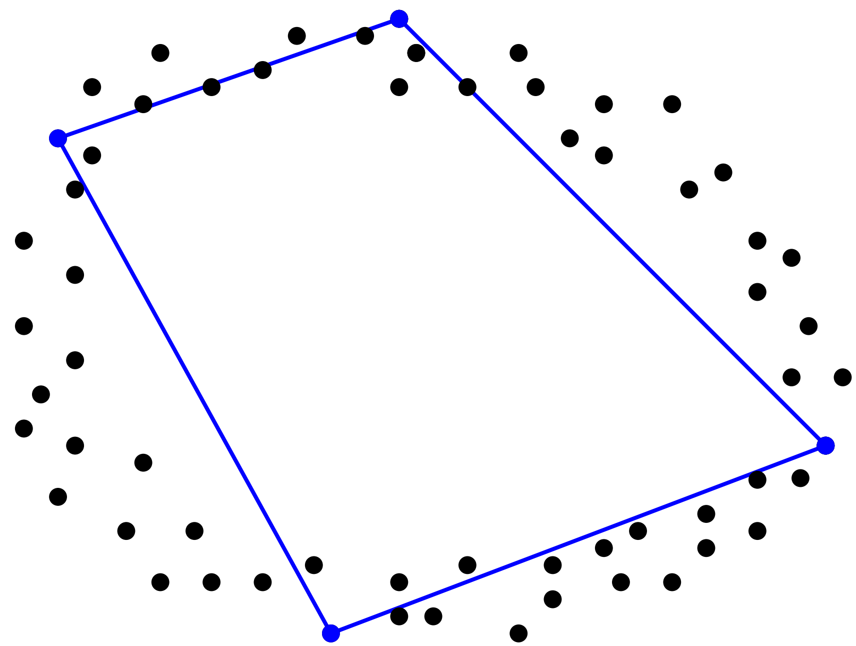
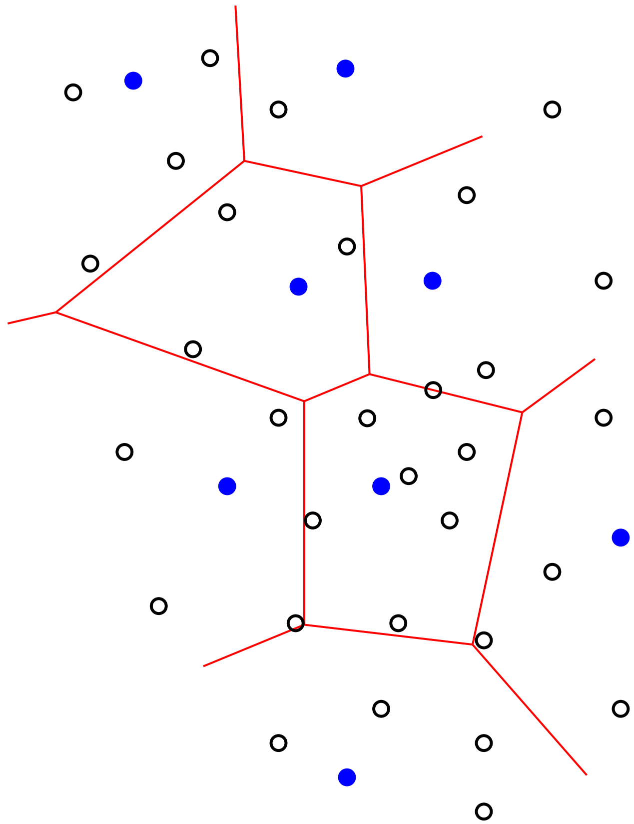
Output: the sequence of simplicial complexes



The simplicial complex to maintain

$\rightarrow$ maintain the **witness complex** $C^W(L)$ [de Silva 2003]:

Let $L \subseteq \mathbb{R}^d$ (landmarks) s.t. $|L| < +\infty$ and $W \subseteq \mathbb{R}^d$ (witnesses)



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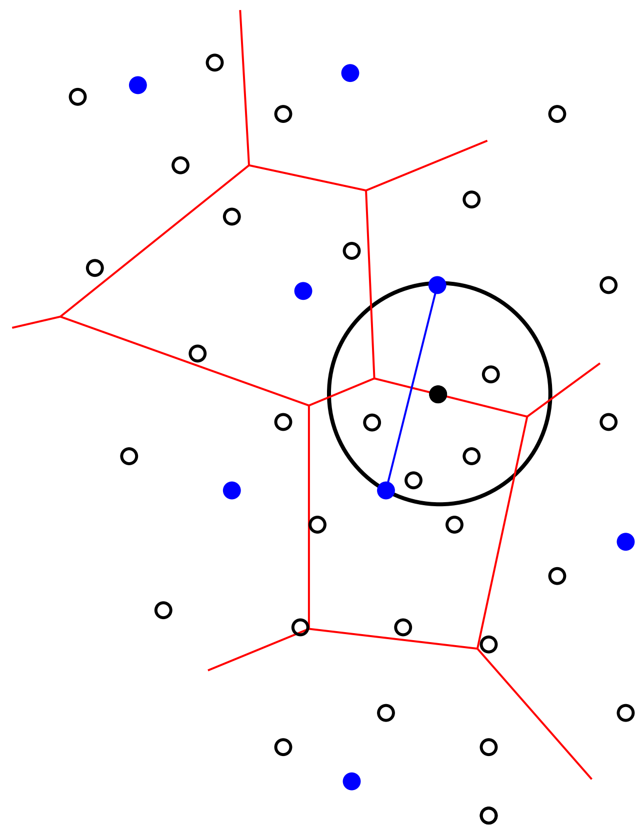

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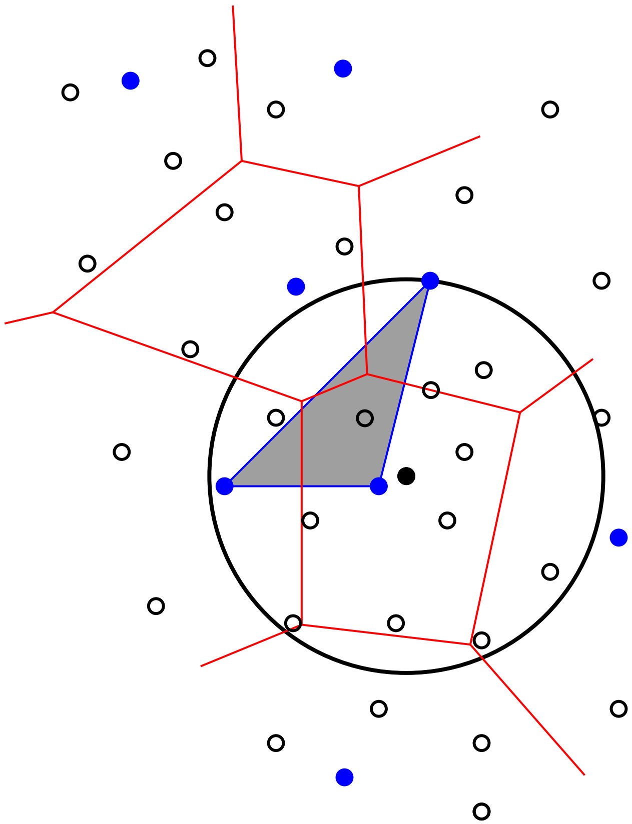

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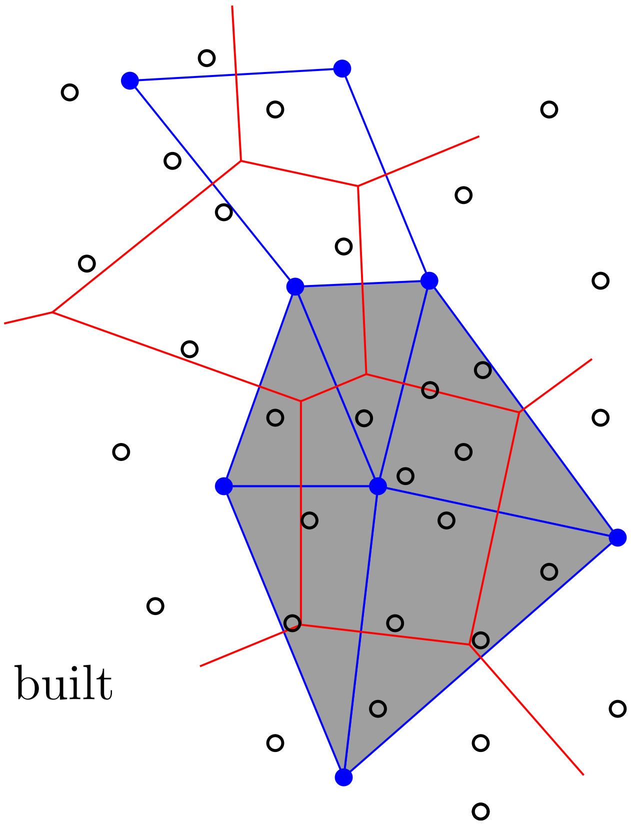

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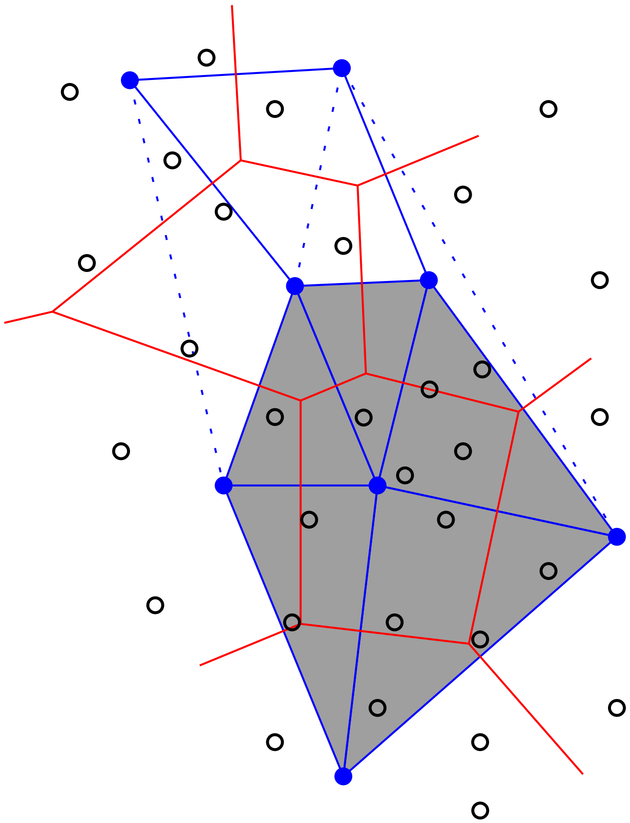
Def. $\mathrm{C}^W(L)$ is the largest abstract simplicial complex built over L, whose faces are weakly witnessed by points of W.

The witness complex (properties)

Thm. 1 [de Silva 2003] $\forall W, L, \forall \sigma \in \mathrm{C}^W(L), \exists c \in \mathbb{R}^d$ that strongly witnesses σ.

$\Rightarrow \mathrm{C}^W(L)$ is a subcomplex of $\mathrm{Del}(L)$

$\Rightarrow \mathrm{C}^W(L)$ is embedded in $\mathbb{R}^d$



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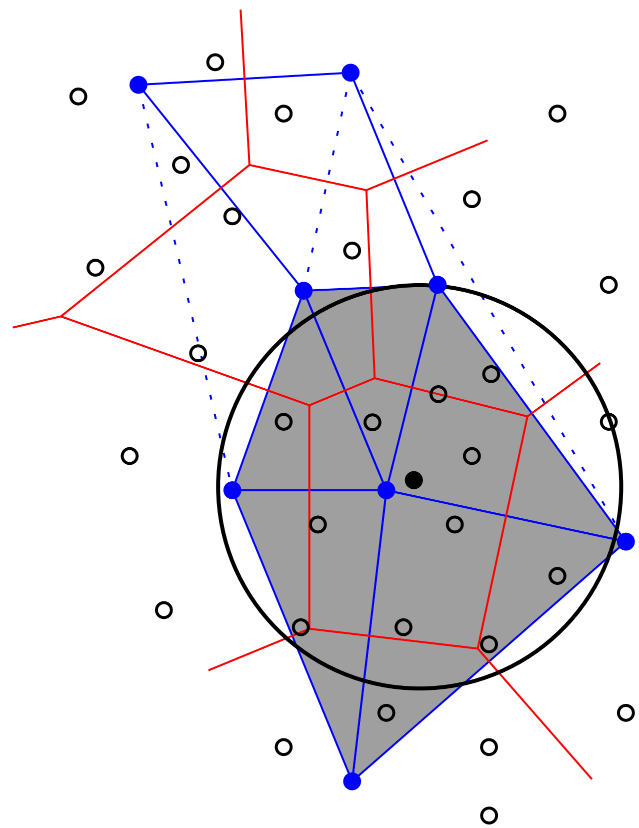

Thm. 2 [de Silva, Carlsson 2004]
- The size of $\mathrm{C}^W(L)$ is $O(d|W|)$
- The time to compute $\mathrm{C}^W(L)$ is $O(d|W||L|)$

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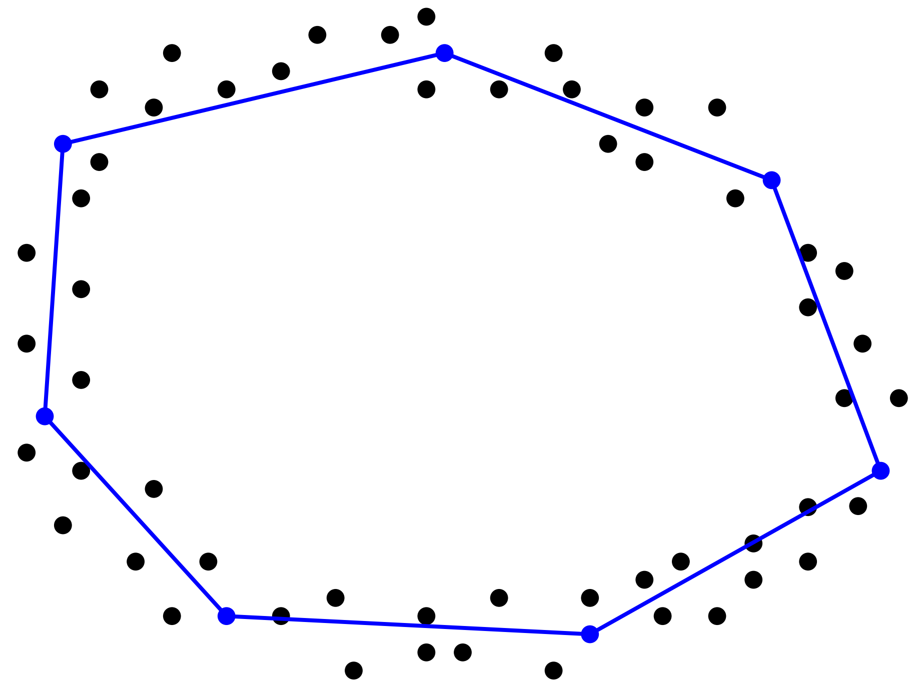

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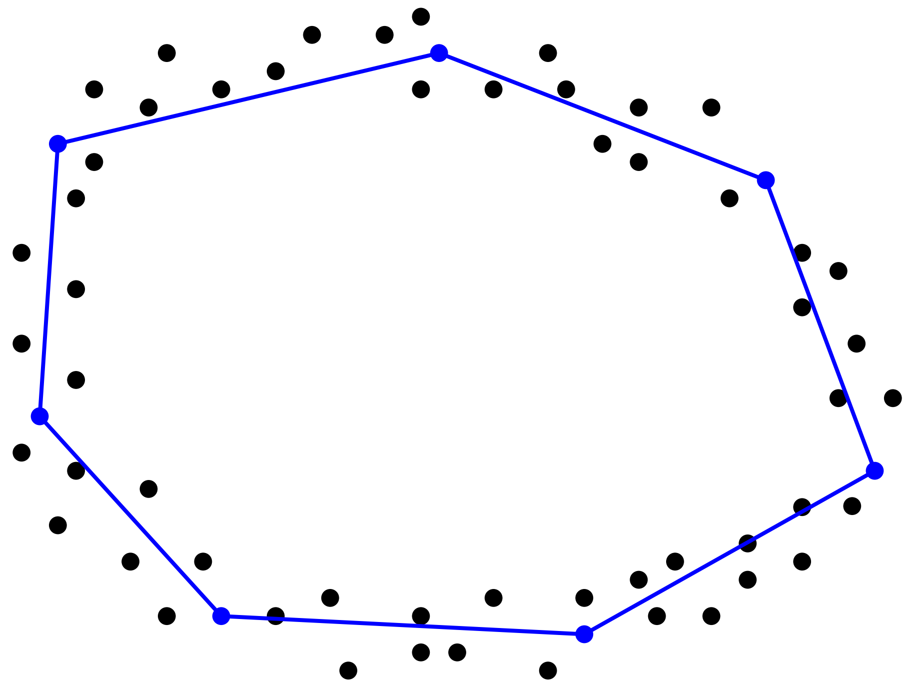
Thm. 3 [Guibas, Oudot 2007]
[Attali, Edelsbrunner, Mileyko 2007]
Under some conditions, $\mathrm{C}^W(L) = \mathrm{Del}_{\mathcal{S}}(L) \approx \mathcal{S}$

The witness complex (properties)

$\rightarrow$ connection with reconstruction:

- $W \subset \mathbb{R}^d$ is given as input
- $L \subseteq W$ is generated
- underlying manifold $\mathcal{S}$ unknown
- only distance comparisons

$\Rightarrow$ algorithm is applicable in any metric space



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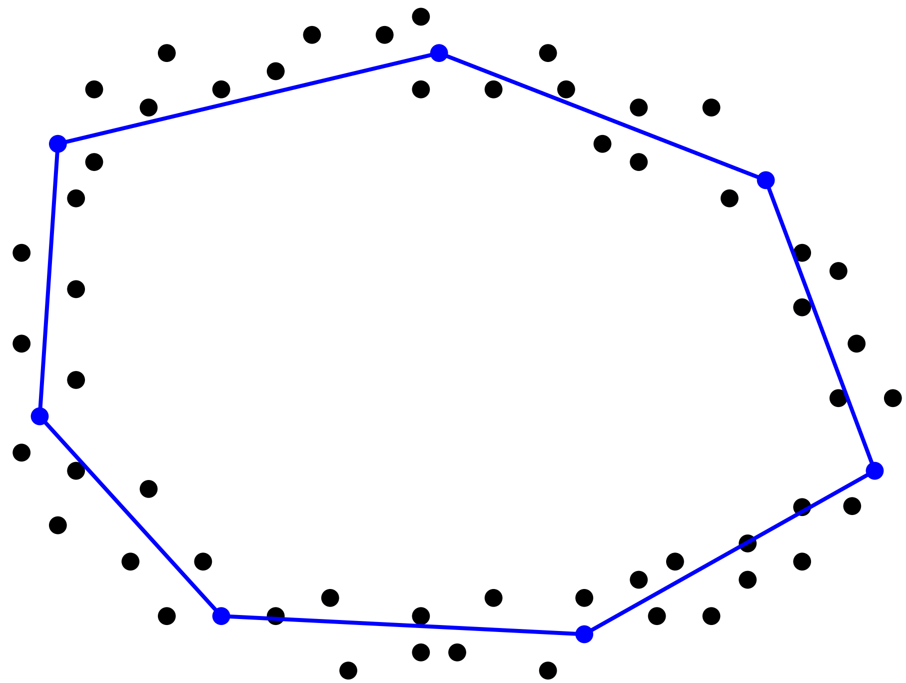
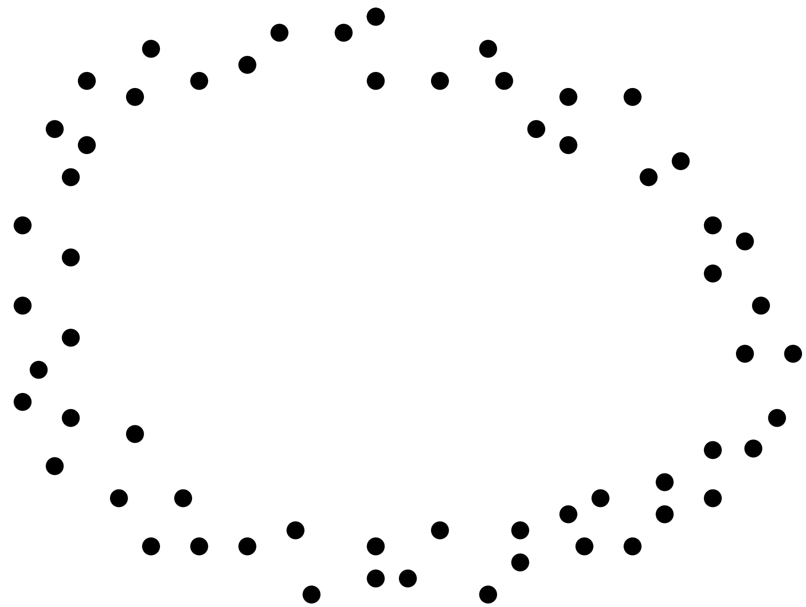


• In $\mathbb{R}^d$, $\mathrm{C}^W(L)$ can be maintained by updating, for each witness w, the list of $d+1$ nearest landmarks of w.

$$\Rightarrow \begin{array}{lcl} \text{space} & \leq & O\left(d|W|\right) \\ \text{time} & \leq & O\left(d|W|^2\right) \end{array}$$

The full algorithm

Input: a finite point set $W \subset \mathbb{R}^d$.



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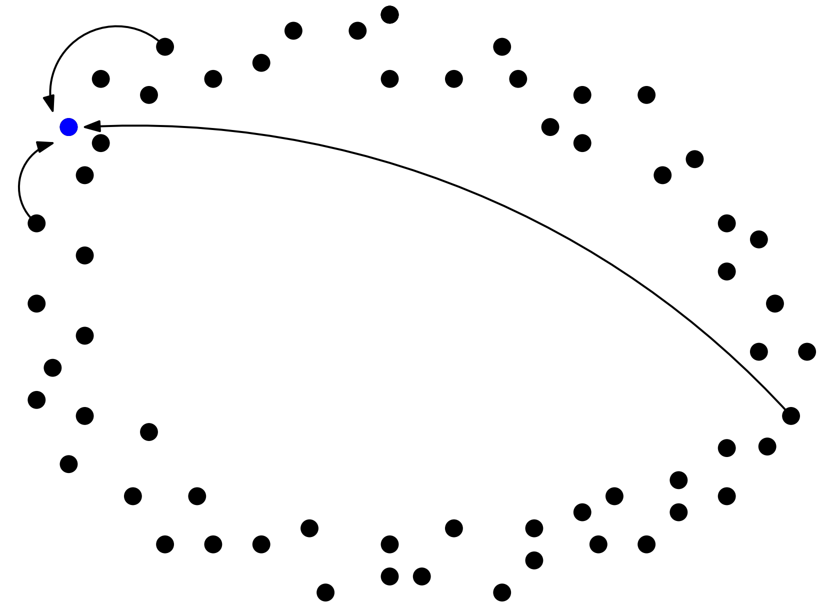
Init: $L := \{p\}$; construct lists of nearest landmarks; $\mathrm{C}^W(L) = \{[p]\}$;

Invariant: $\forall w \in W$, the list of $d+1$ nearest landmarks of w is maintained throughout the process.



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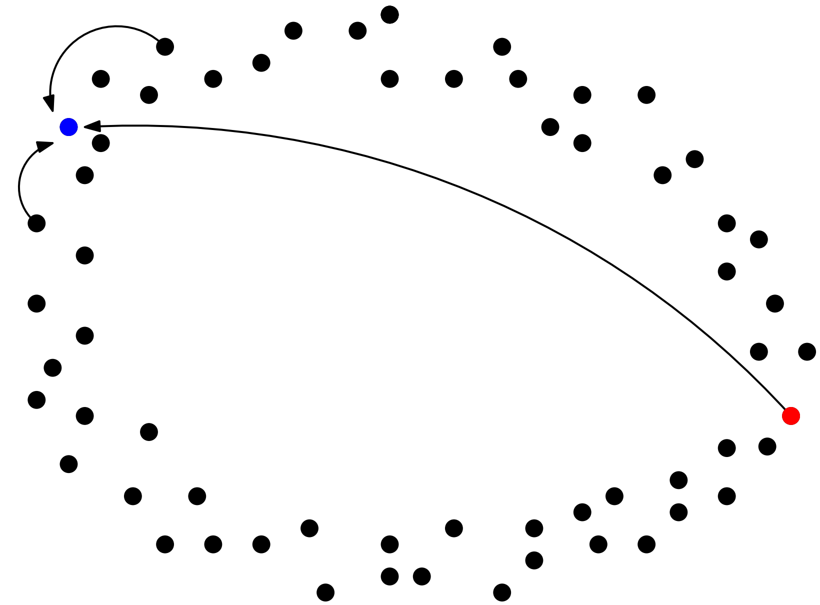

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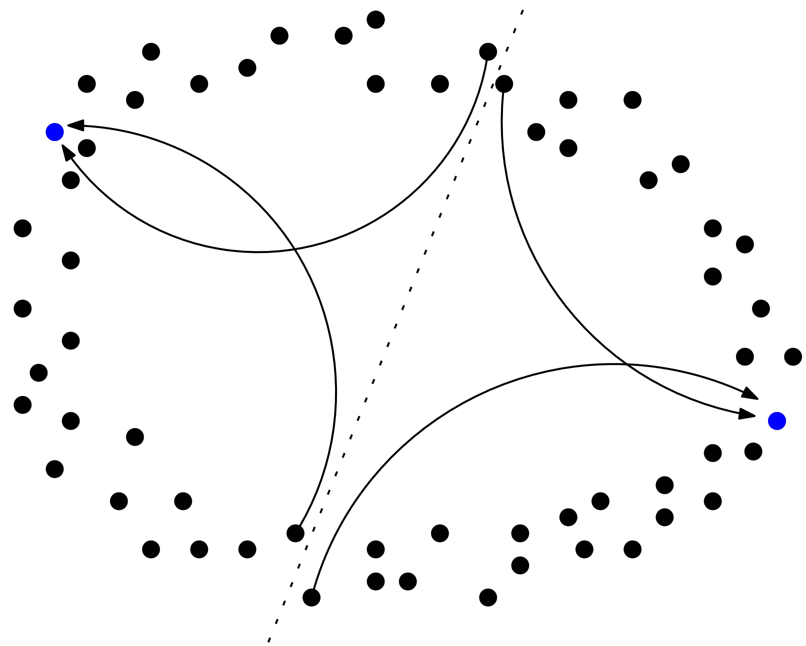

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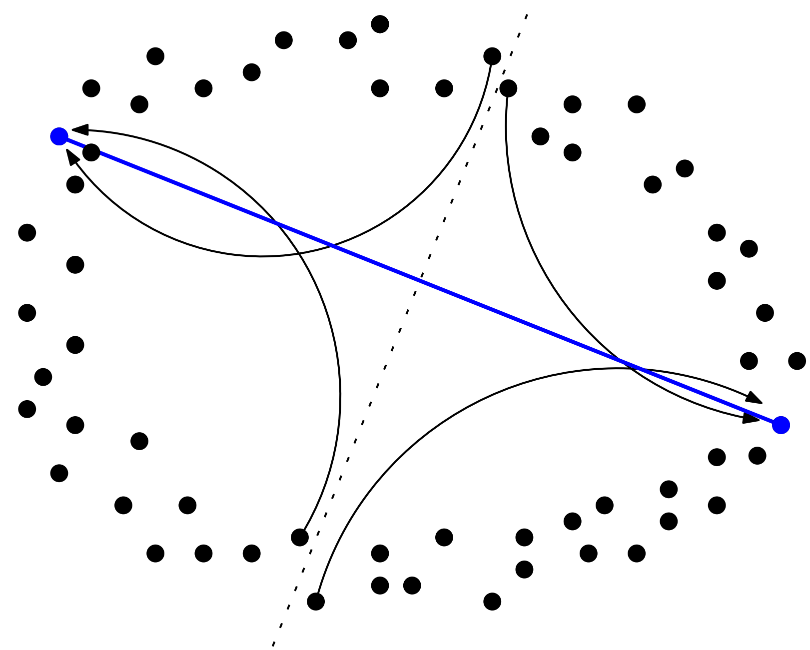

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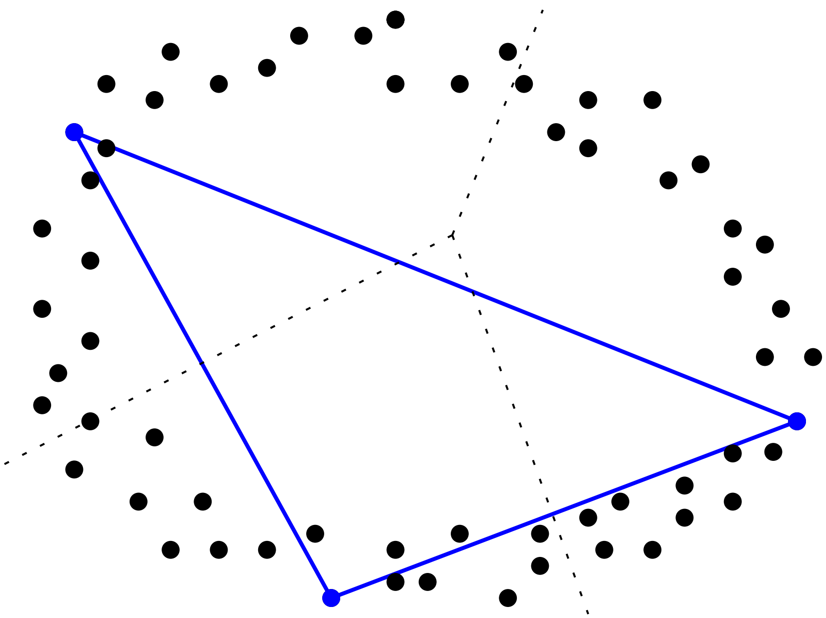

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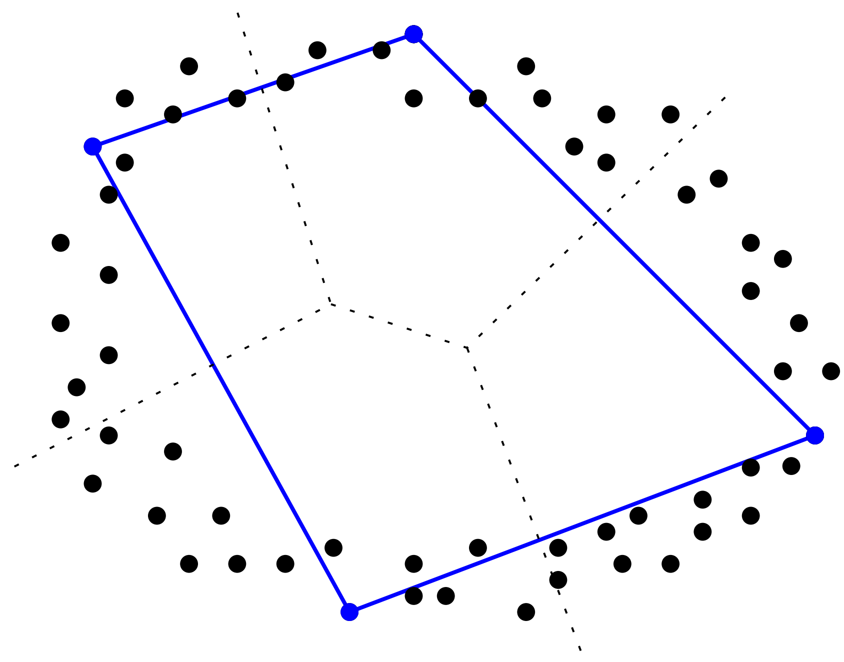

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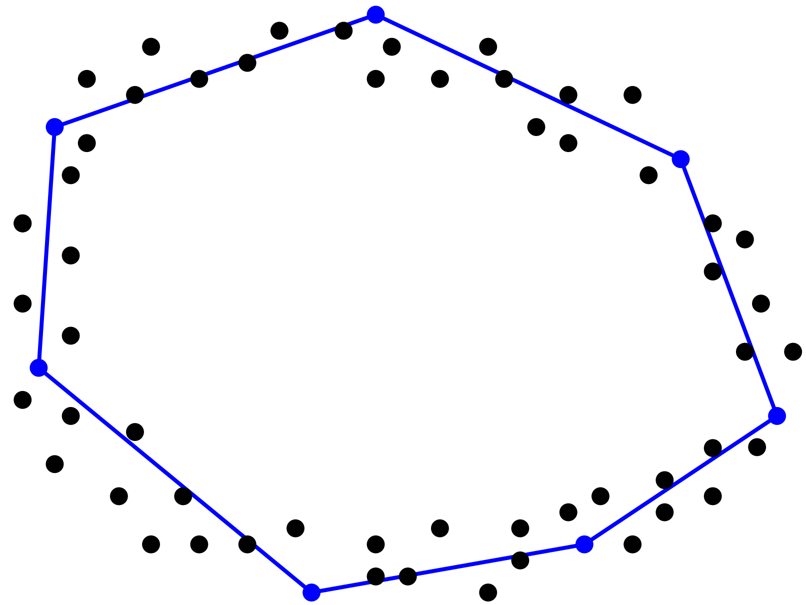
Output: the sequence of complexes $\mathrm{C}^W(L)$

Theoretical guarantees

$\rightarrow$ case of curves:

Conjecture [Carlsson, de Silva 2004]: $\mathrm{C}^W(L)$ coincides with $\mathrm{Del}_{\mathcal{S}}(L)$...



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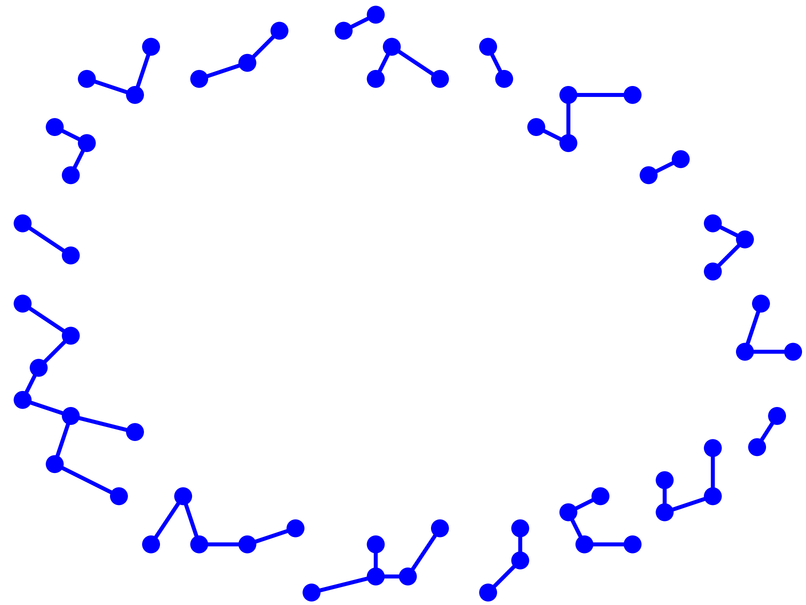

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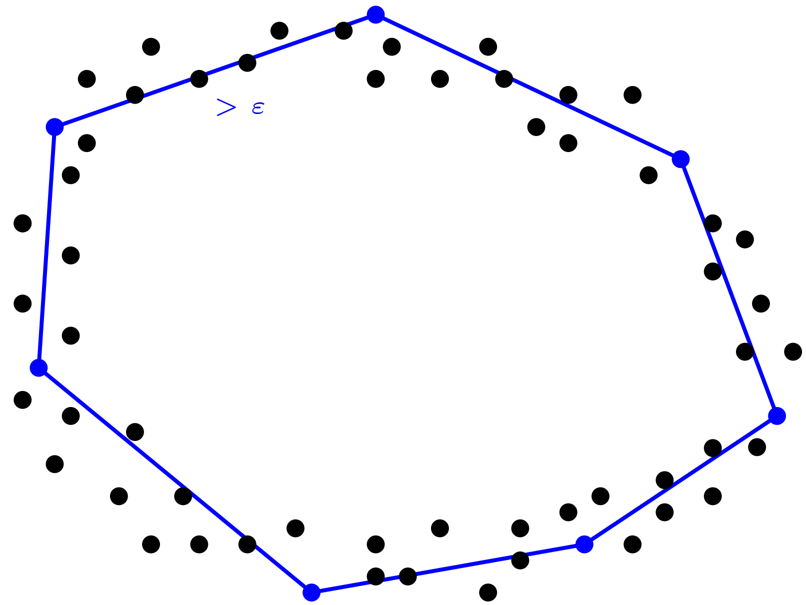
Thm. 3 If $\mathcal{S}$ is a closed curve with positive reach, $W \subset \mathbb{R}^d$ s.t. $\mathrm{d_H}(W, \mathcal{S}) \leq \delta$, $L \subseteq W$ ε-sparse ε-sample of W with $\delta \ll \varepsilon \ll \varrho_{\mathcal{S}}$, then $\mathrm{C}^W(L) = \mathrm{Del}_{\mathcal{S}}(L) \approx \mathcal{S}$.



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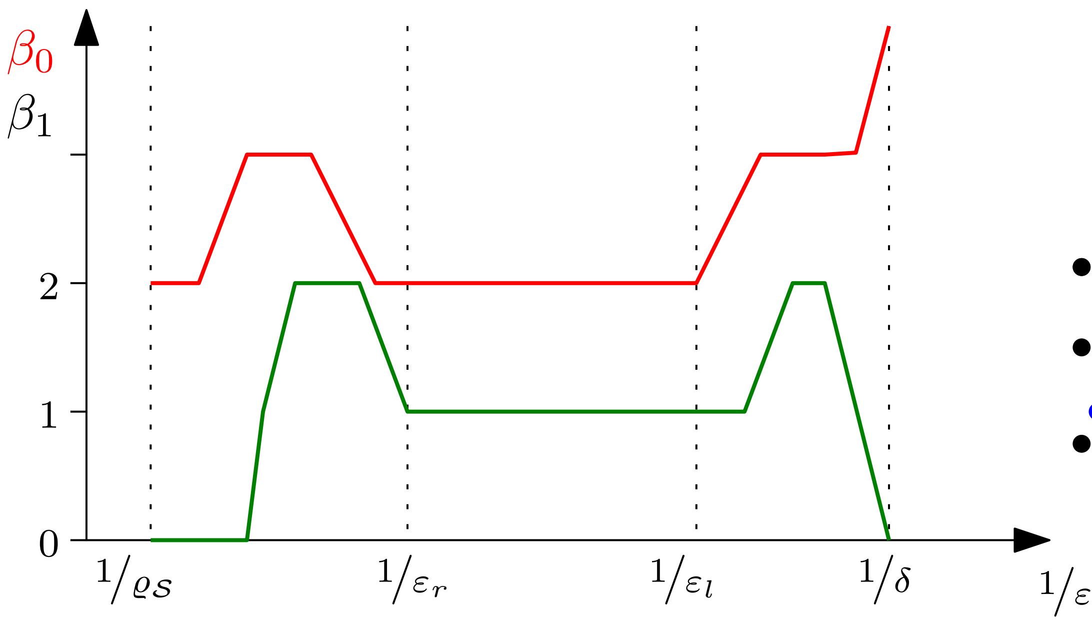

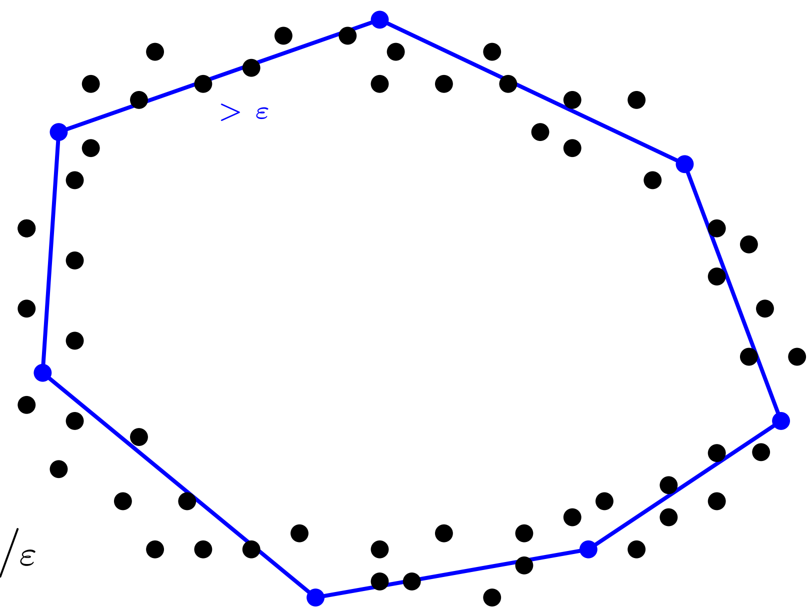

$\rightarrow$ There is a plateau in the diagram of Betti numbers of $\mathrm{C}^W(L)$.

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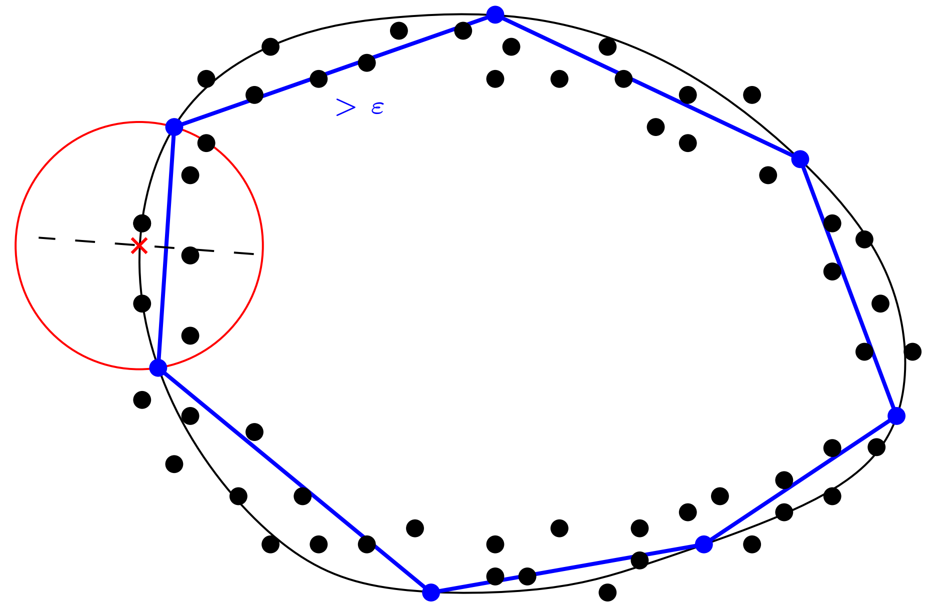
- $\mathrm{Del}_{\mathcal{S}}(L) \subseteq \mathrm{C}^W(L)$



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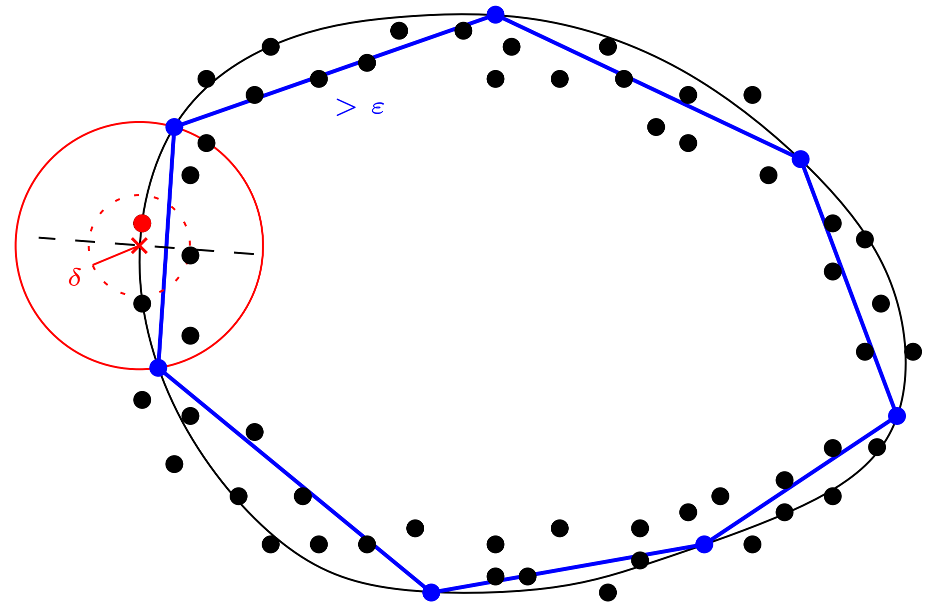

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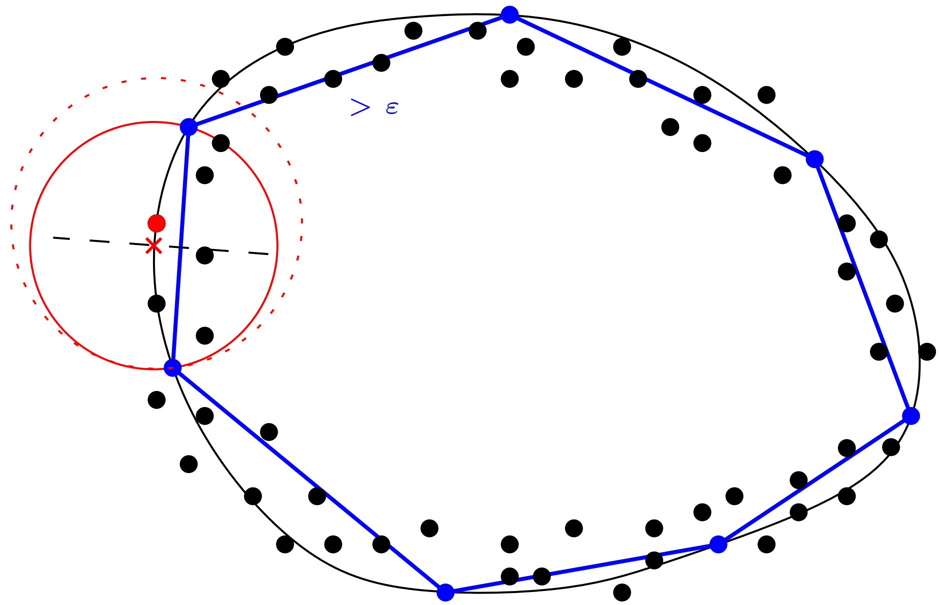
Thm. 3 If $\mathcal{S}$ is a closed curve with positive reach, $W \subset \mathbb{R}^d$ s.t. $\mathrm{d_H}(W, \mathcal{S}) \leq \delta$, $L \subseteq W$ ε-sparse ε-sample of W with $\delta \ll \varepsilon \ll \varrho_{\mathcal{S}}$, then $\mathrm{C}^W(L) = \mathrm{Del}_{\mathcal{S}}(L) \approx \mathcal{S}$.

- $\mathrm{Del}_{\mathcal{S}}(L) \subseteq \mathrm{C}^W(L)$



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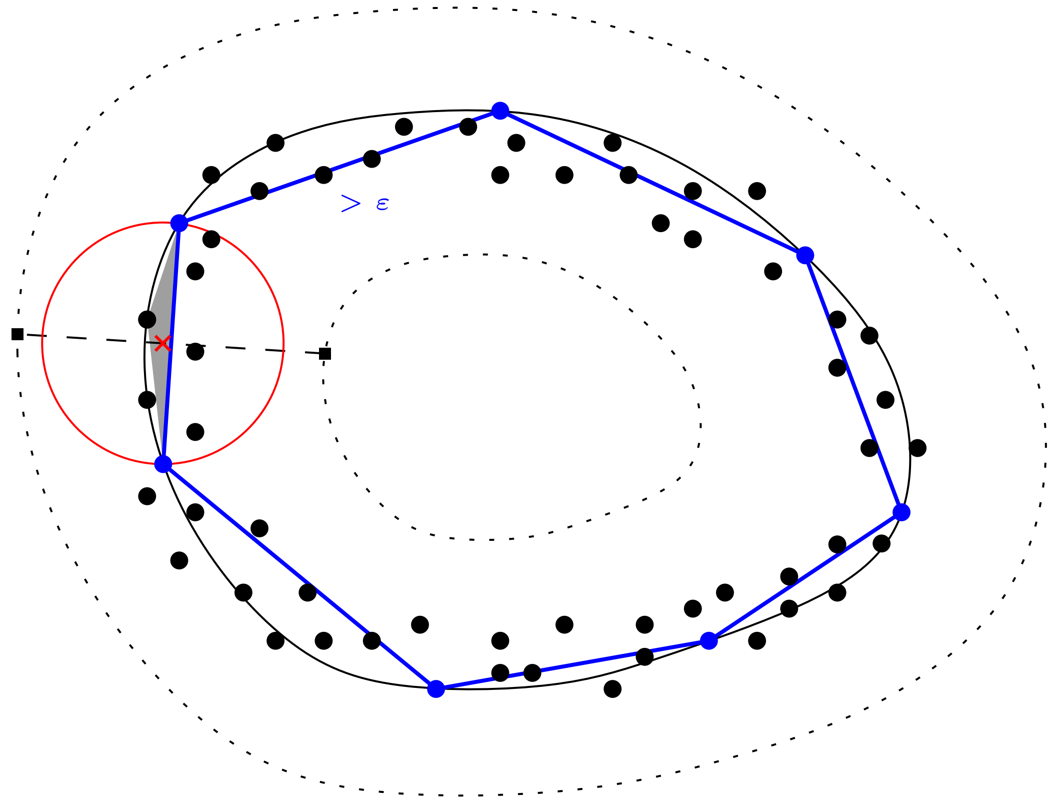

Theoretical guarantees

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Thm [Attali, Edelsbrunner, Mileyko]
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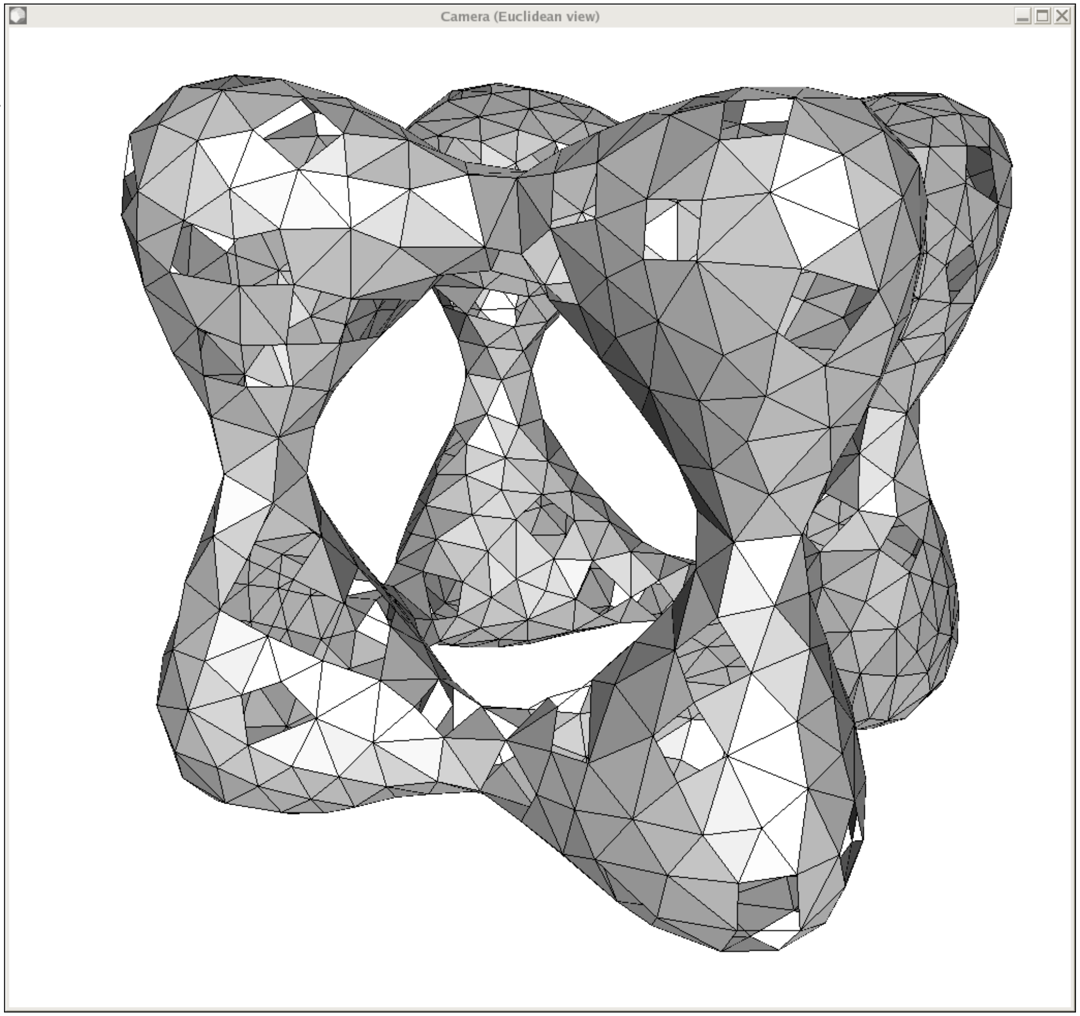

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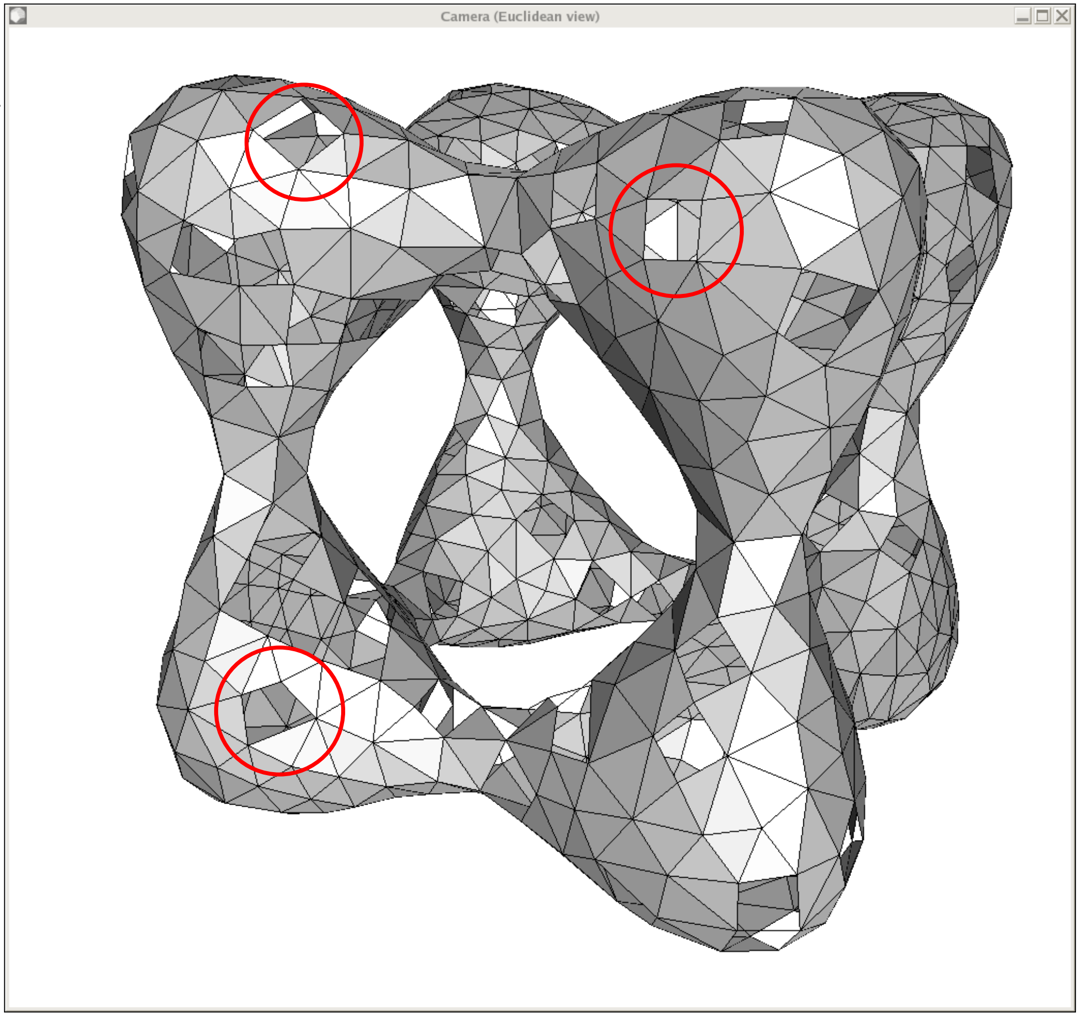

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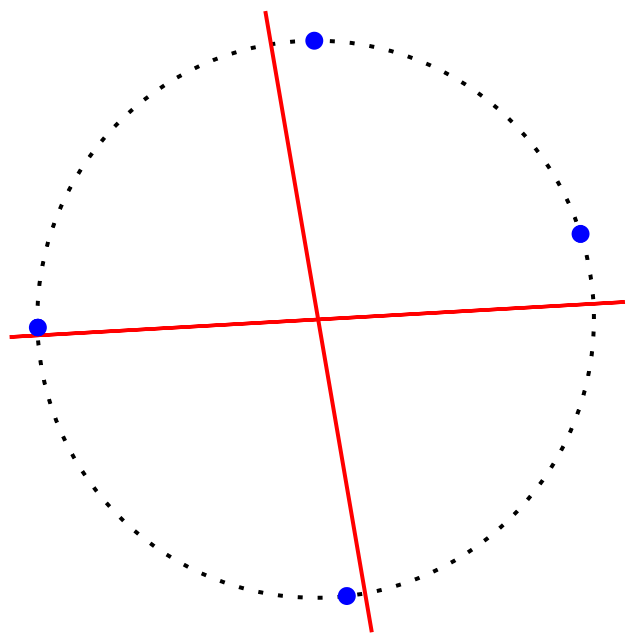
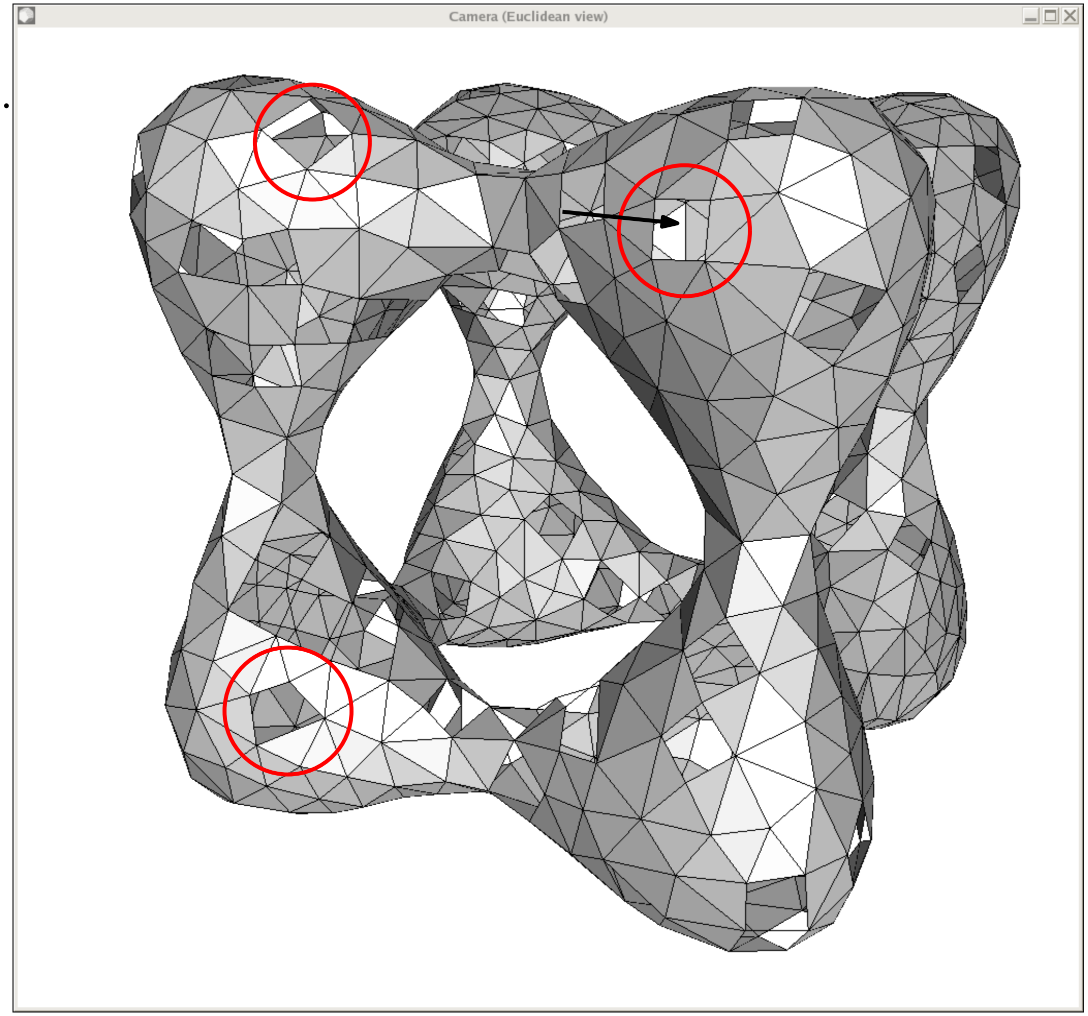


order-2 Voronoi diagram



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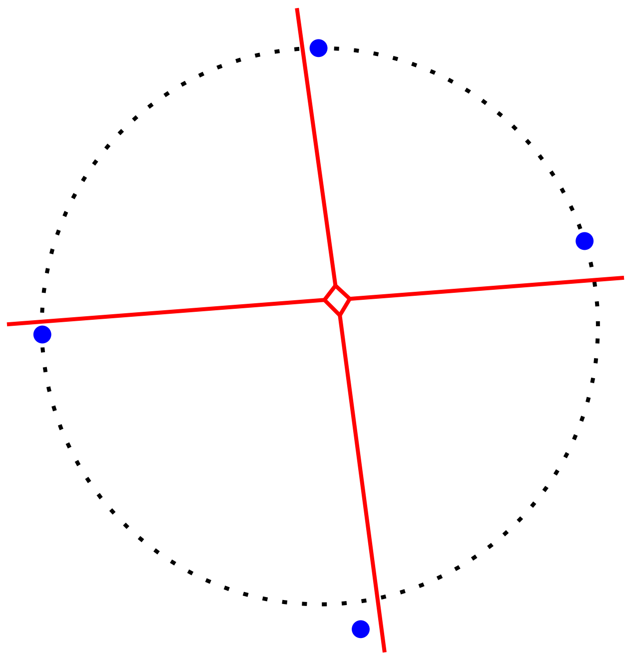

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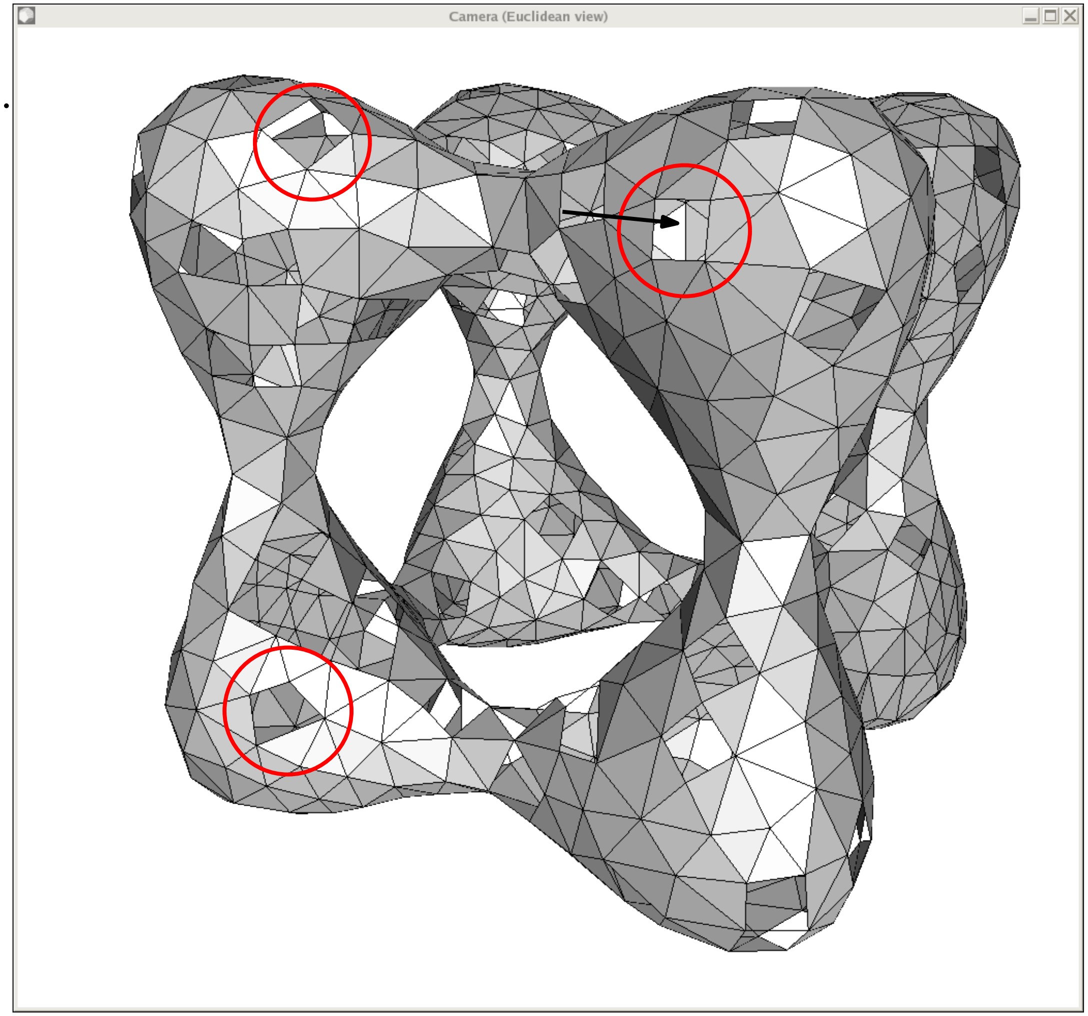

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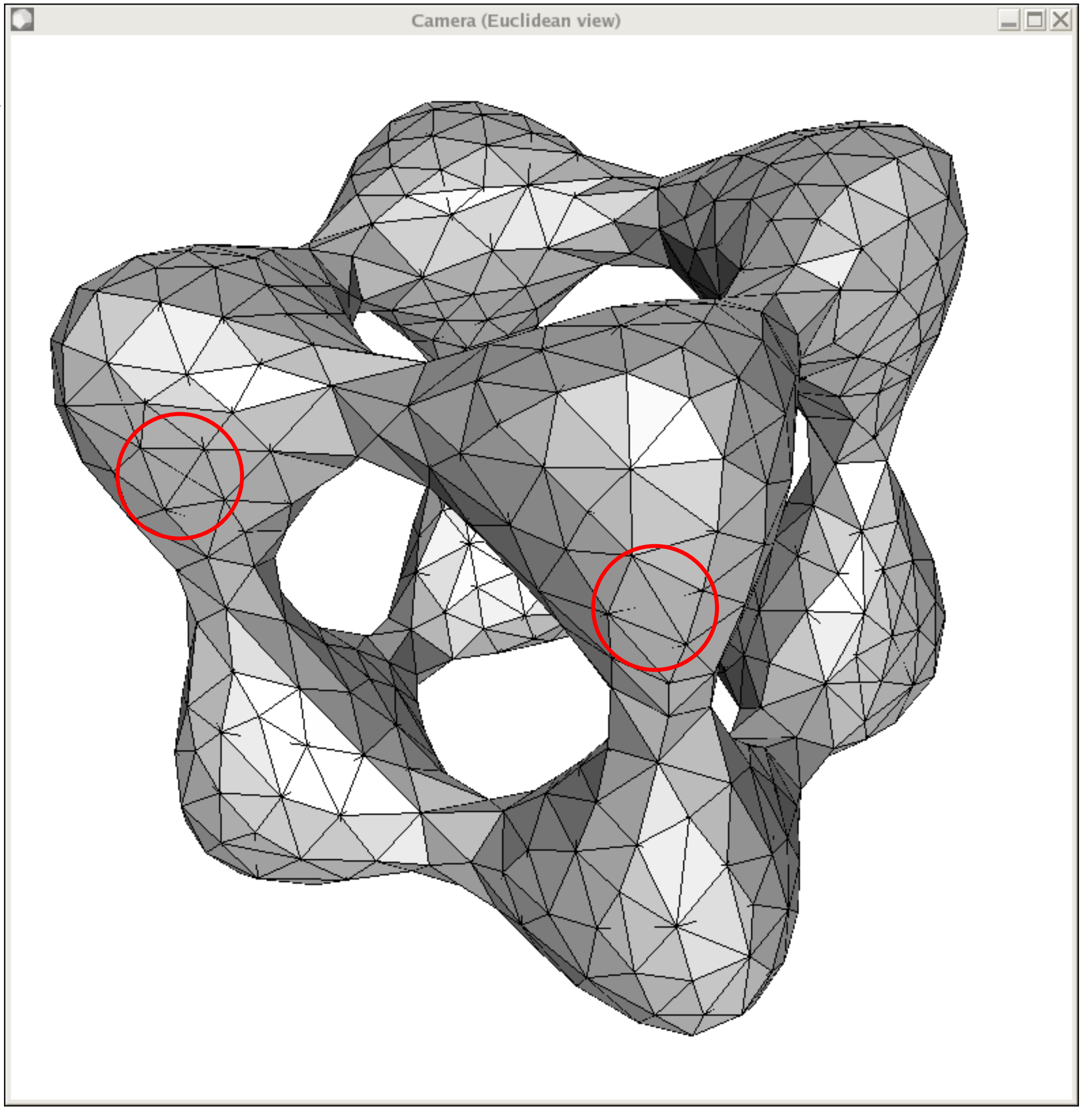

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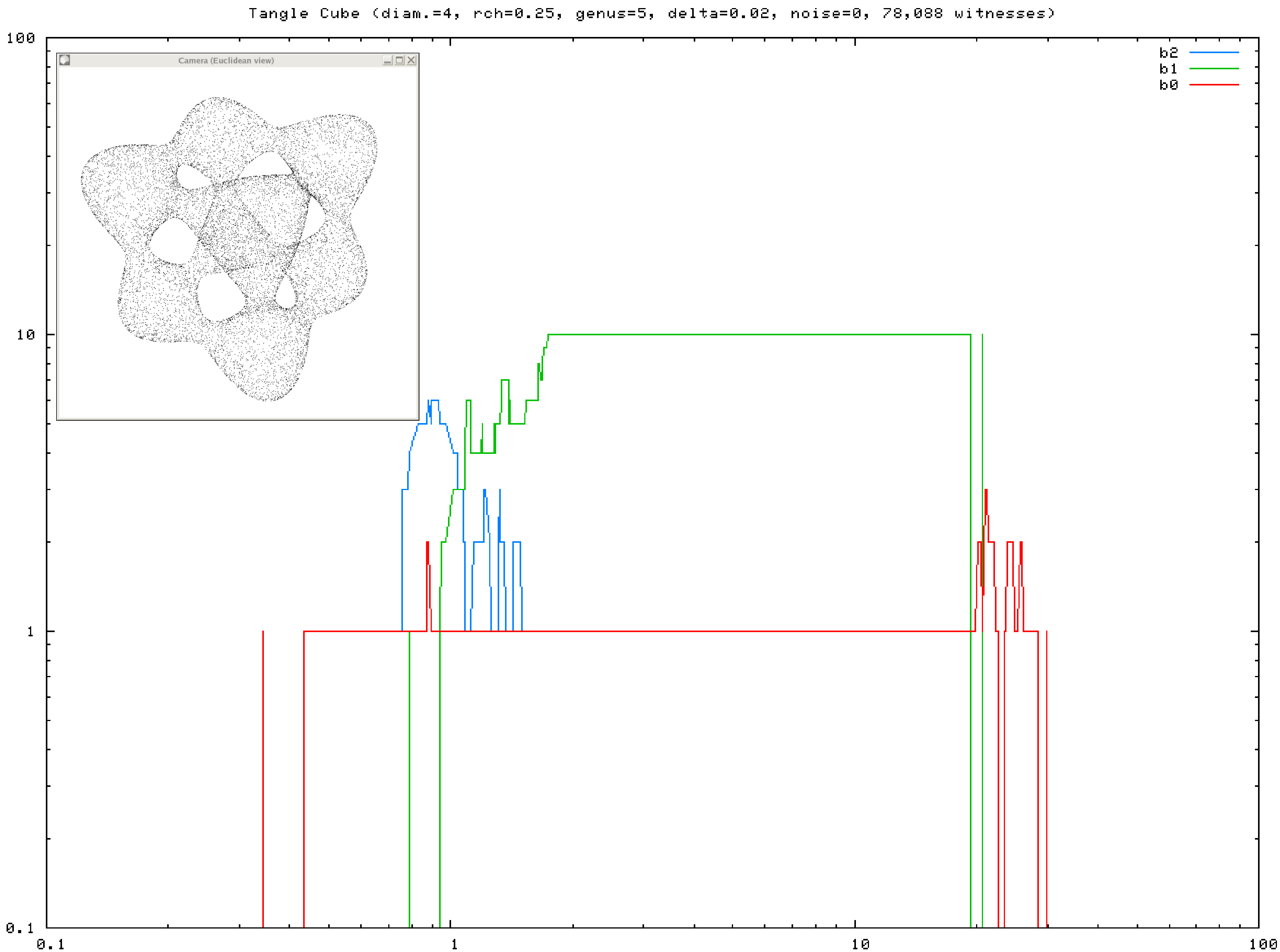
Solution relax witness test
[Guibas, Oudot]

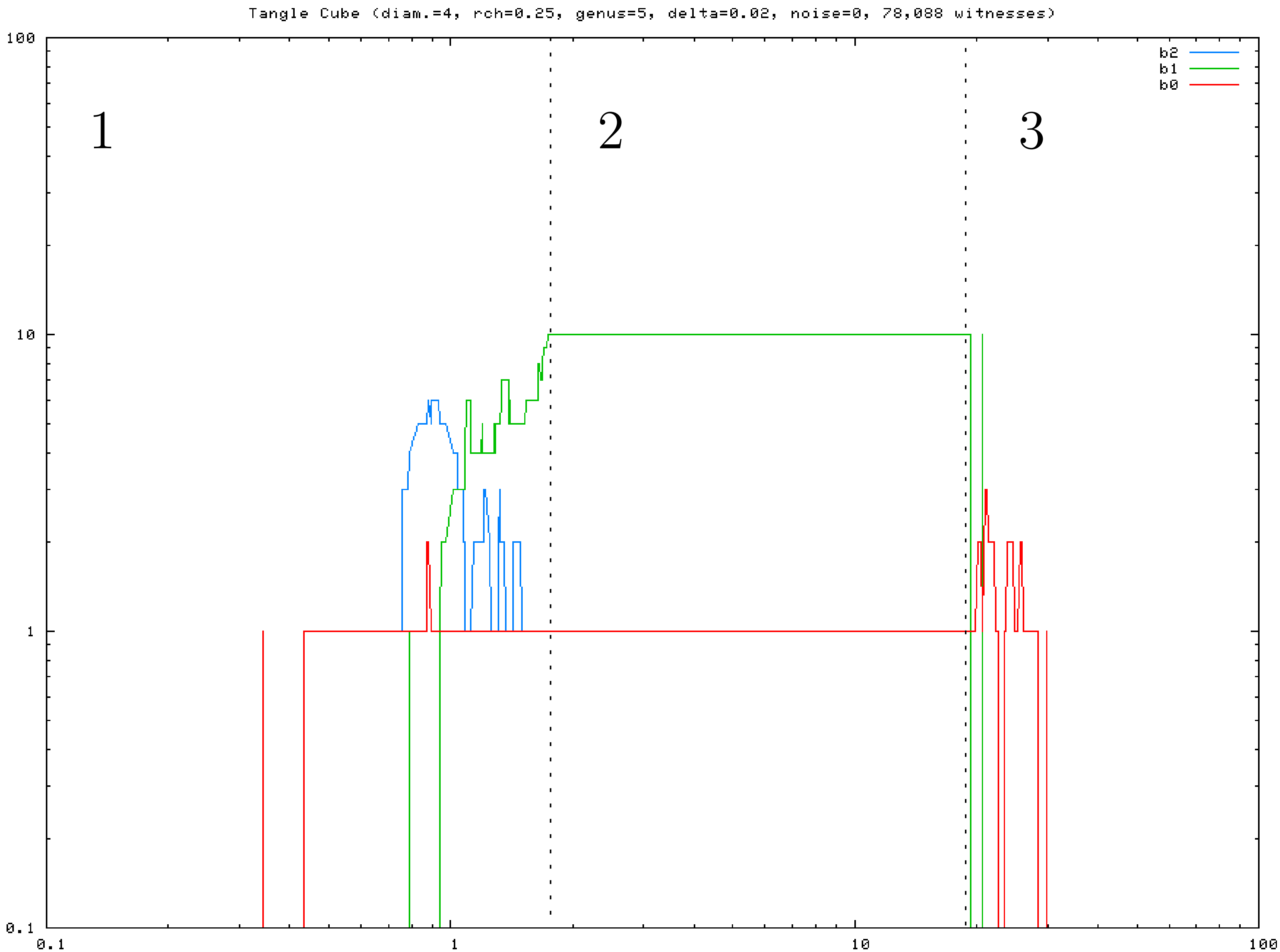
$\Rightarrow \mathrm{C}^W_\nu(L) = \mathrm{Del}_{\mathcal{S}}(L)+$ slivers
$\Rightarrow \mathrm{C}^W_\nu(L) \nsubseteq \mathrm{Del}(L)$
$\Rightarrow \mathrm{C}^W_\nu(L)$ not embedded.

Post-process extract manifold M
from $\mathrm{C}^W_\nu(L) \cap \mathrm{Del}(L)$
[Amenta, Choi, Dey, Leekha]

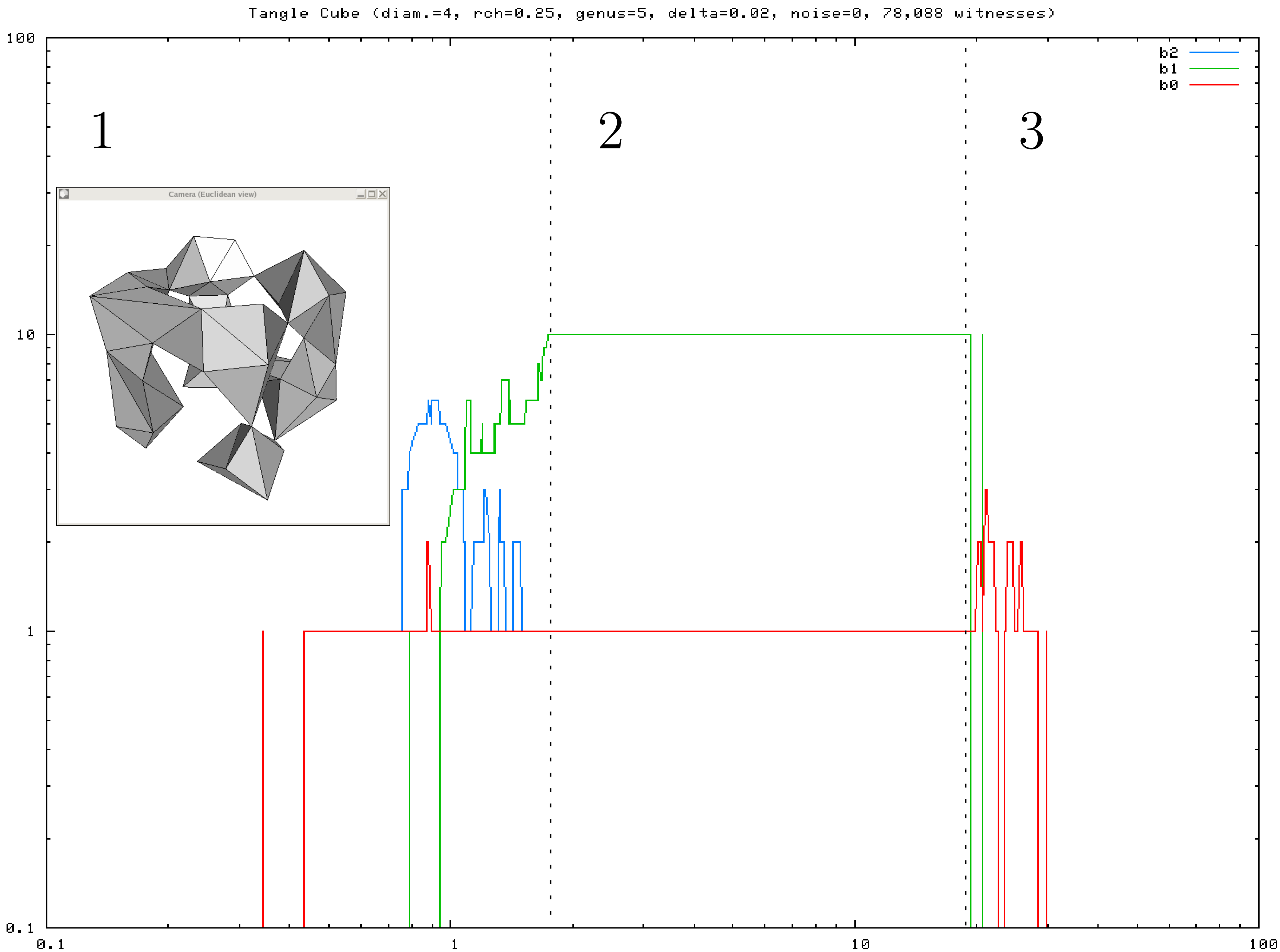
Some results



Some results



Some results



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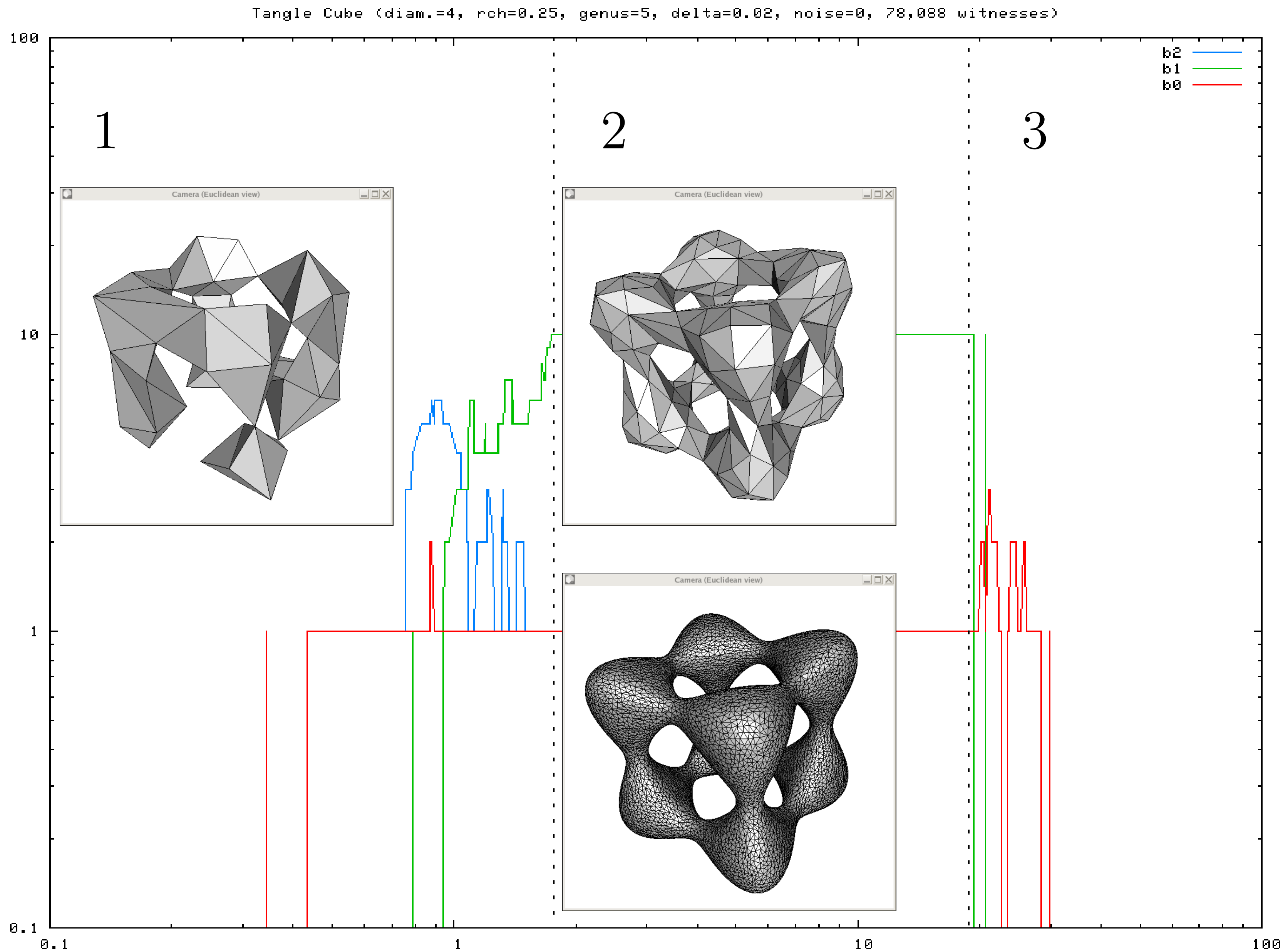

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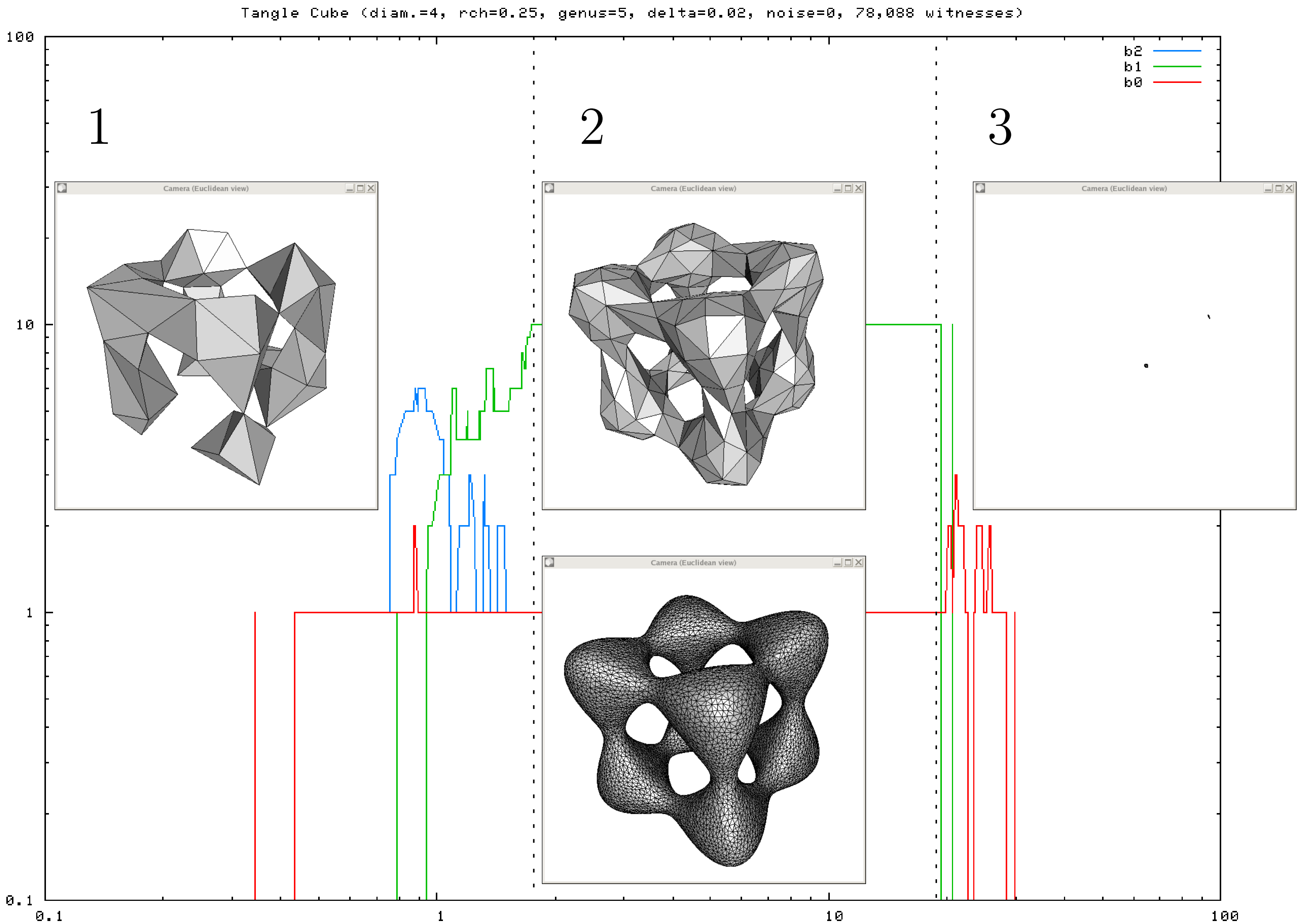
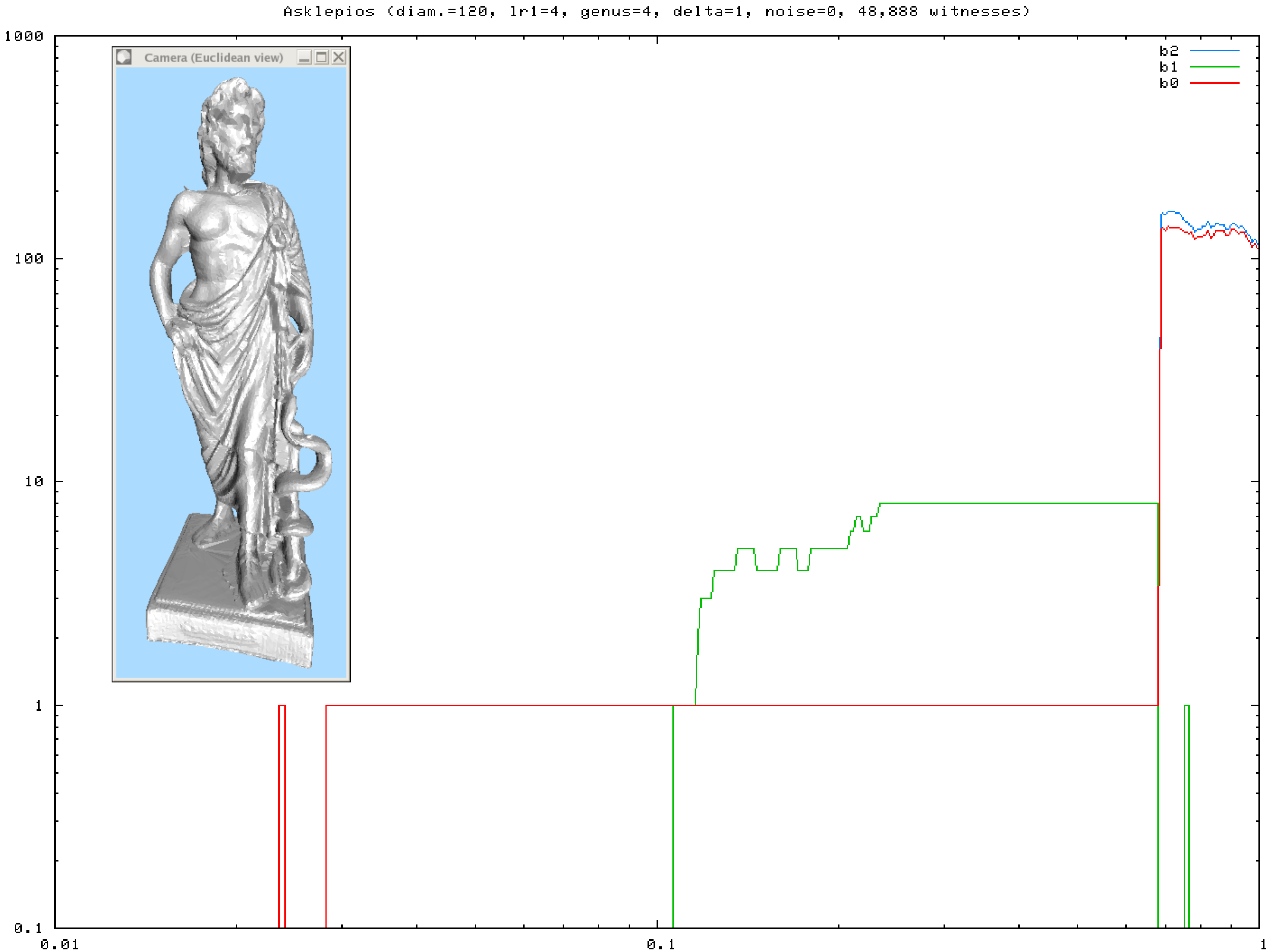
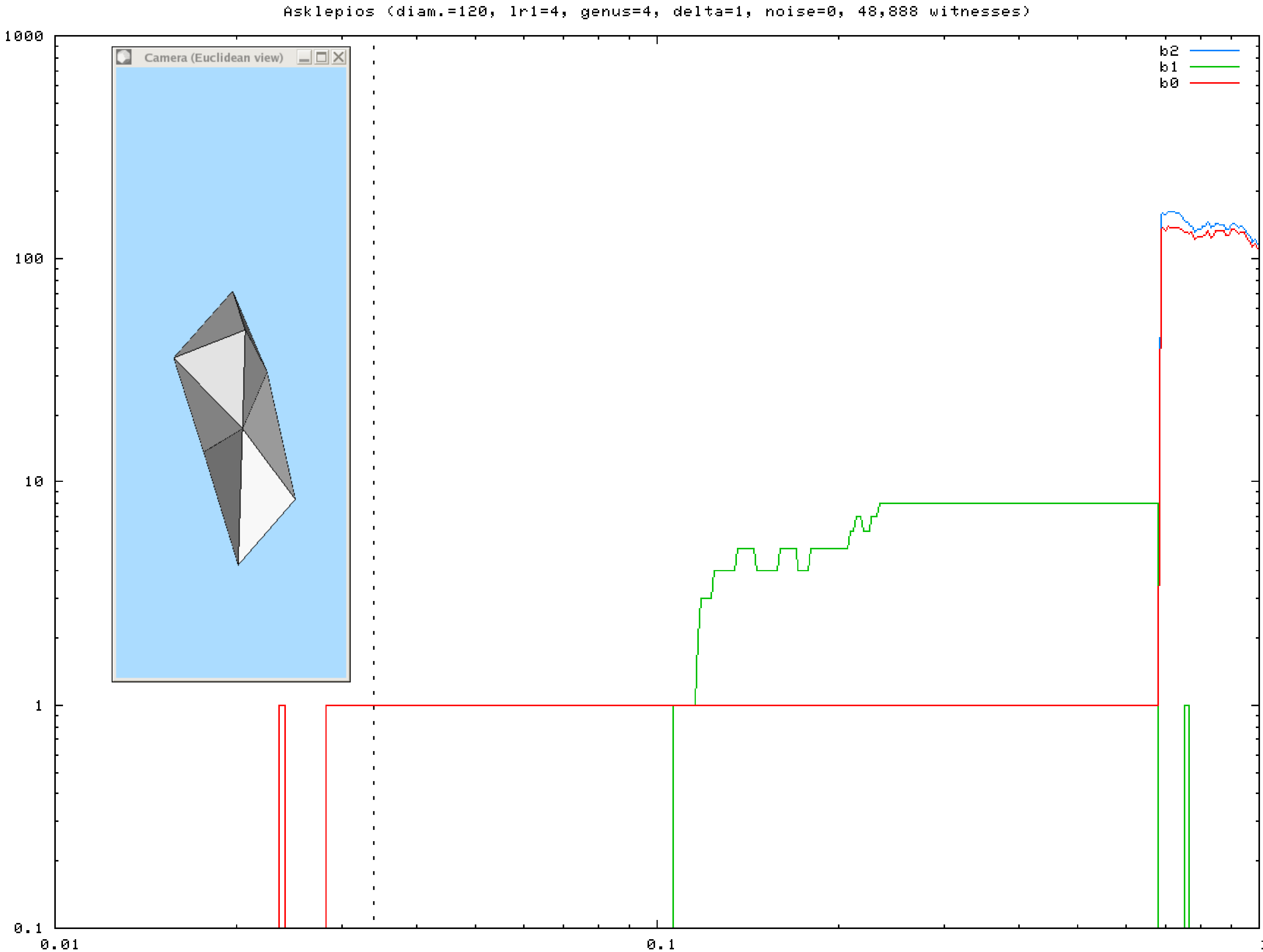


Some results (cont'd)



input model provided courtesy of IMATI by the Aim@Shape repository

Some results (cont'd)



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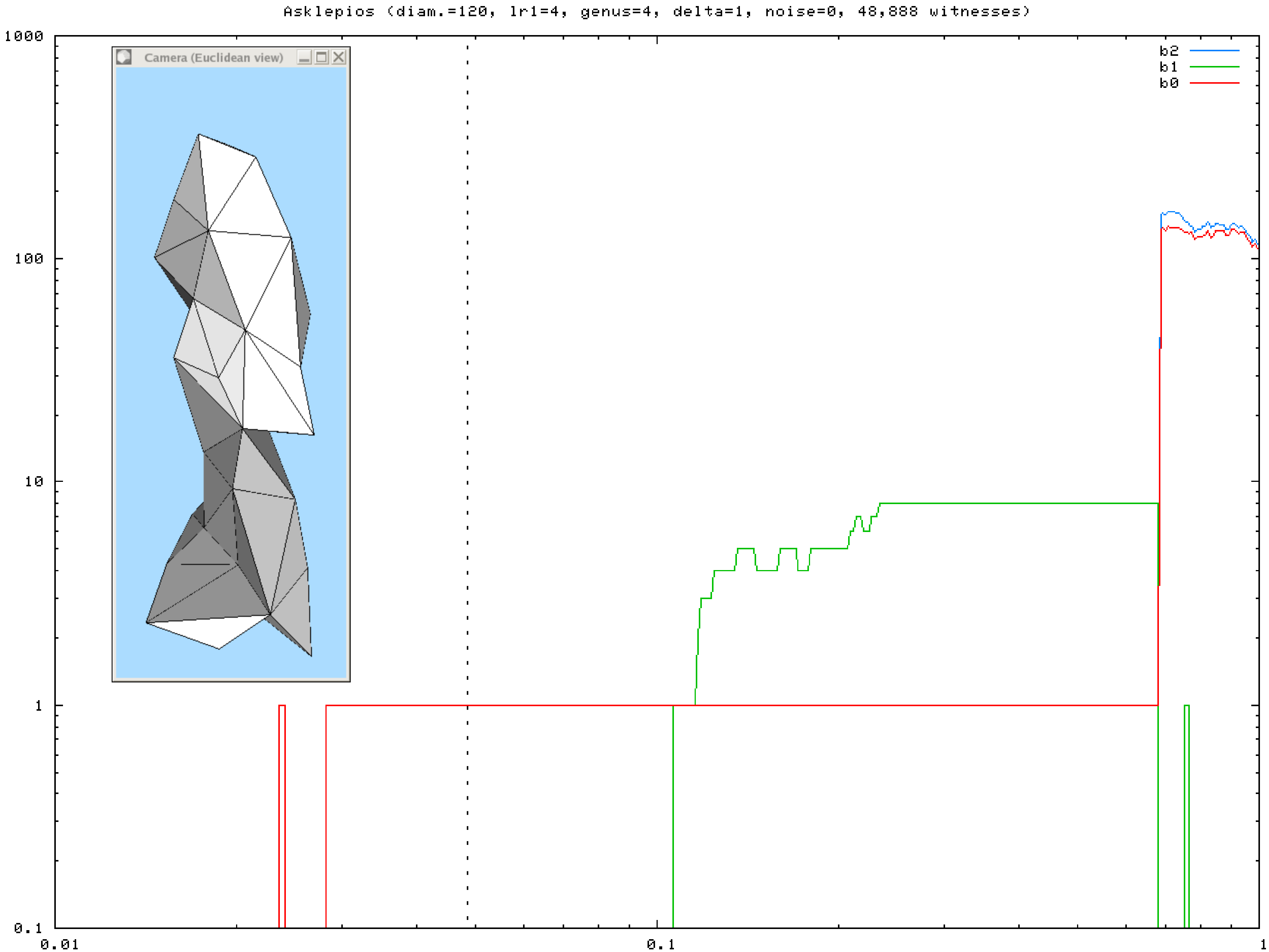

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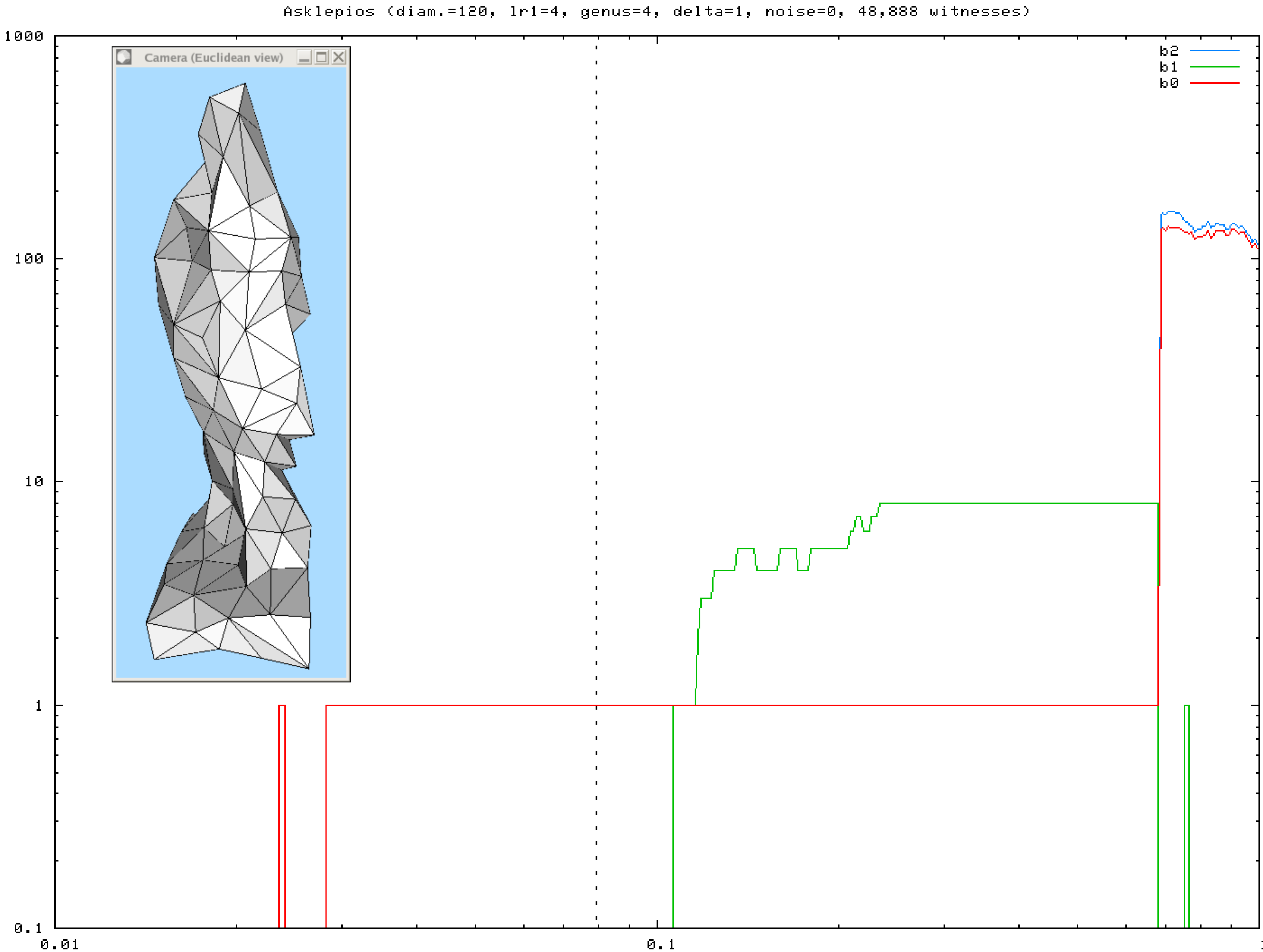

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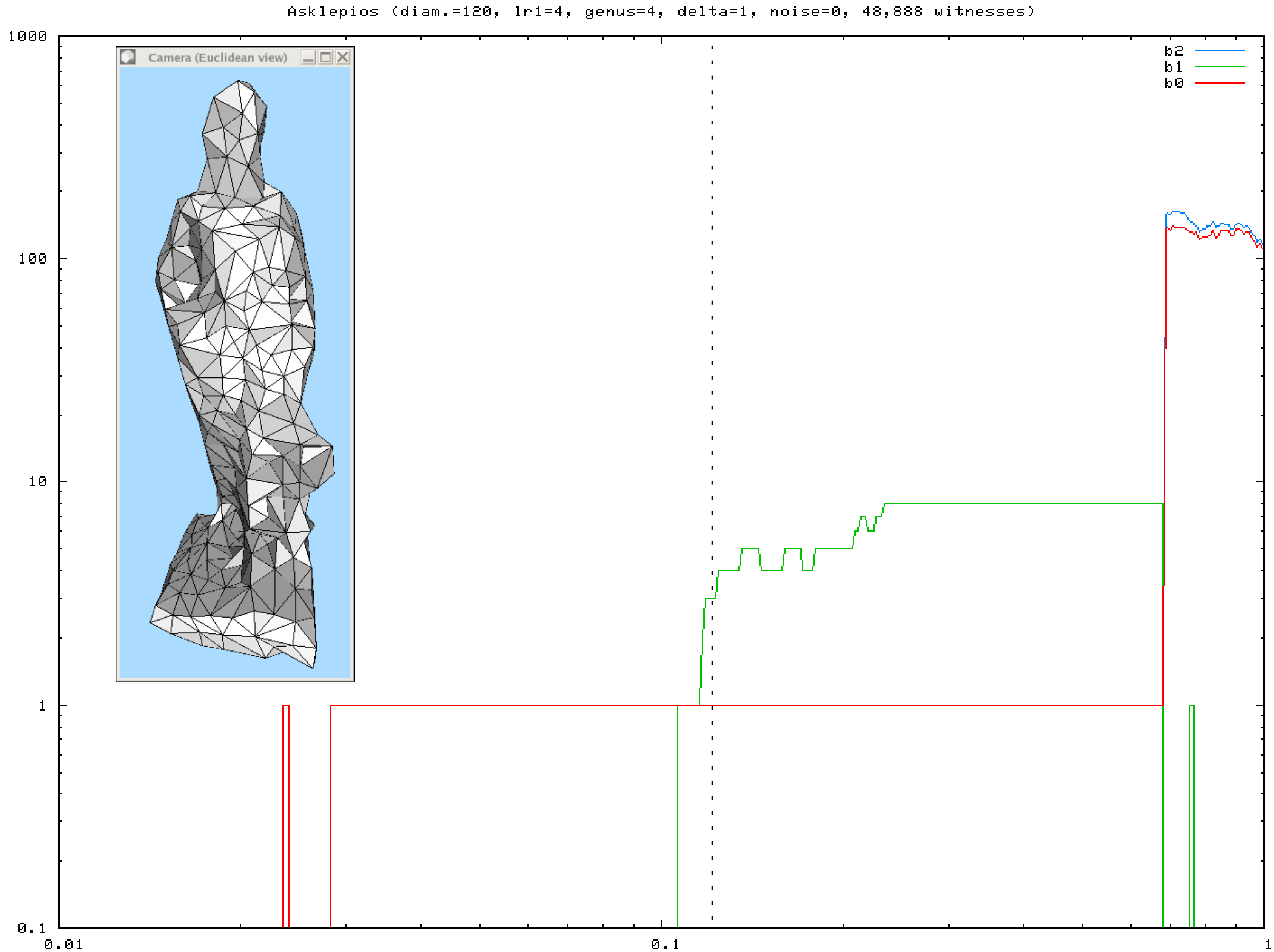

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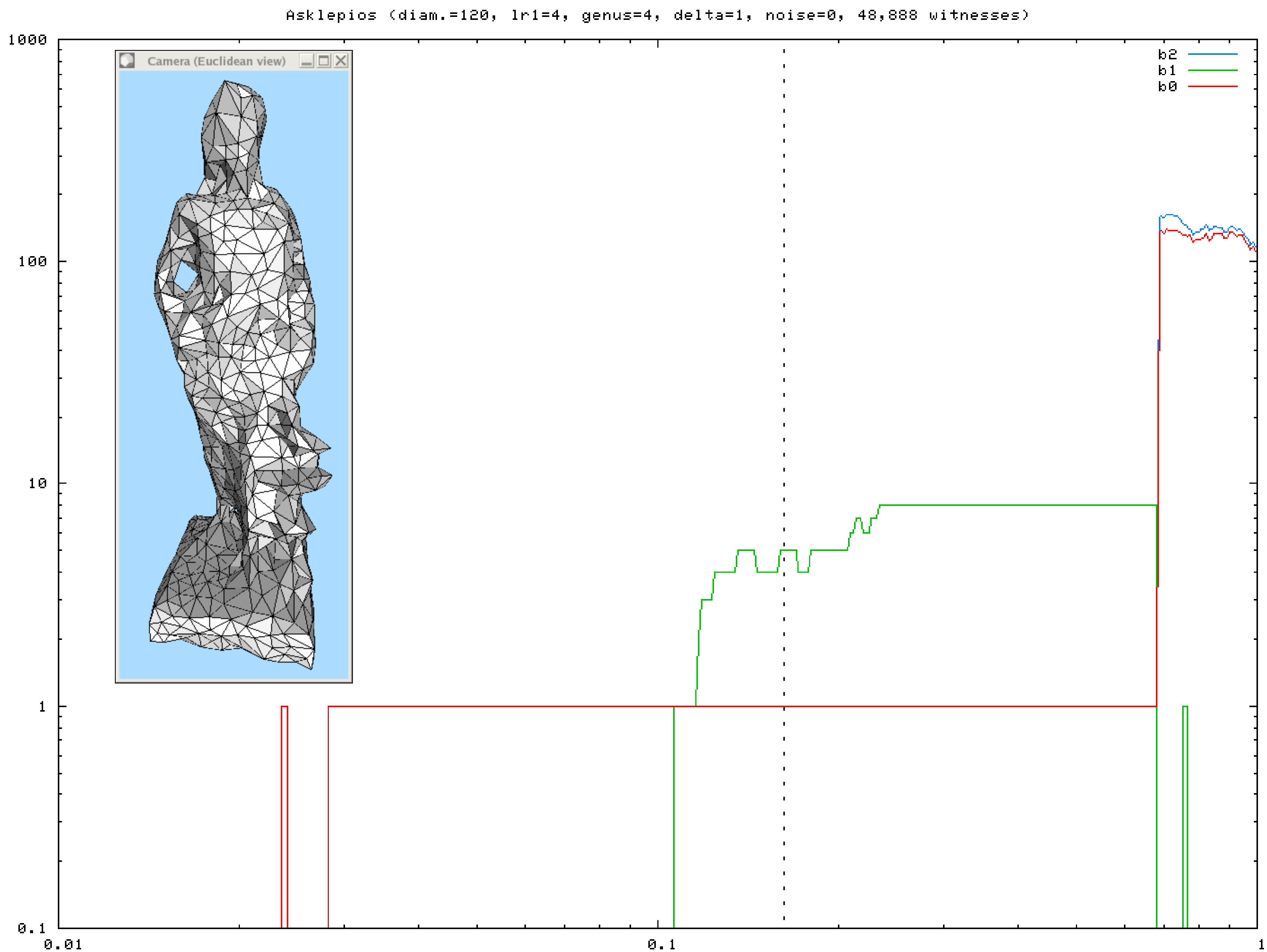
Some results (cont'd)



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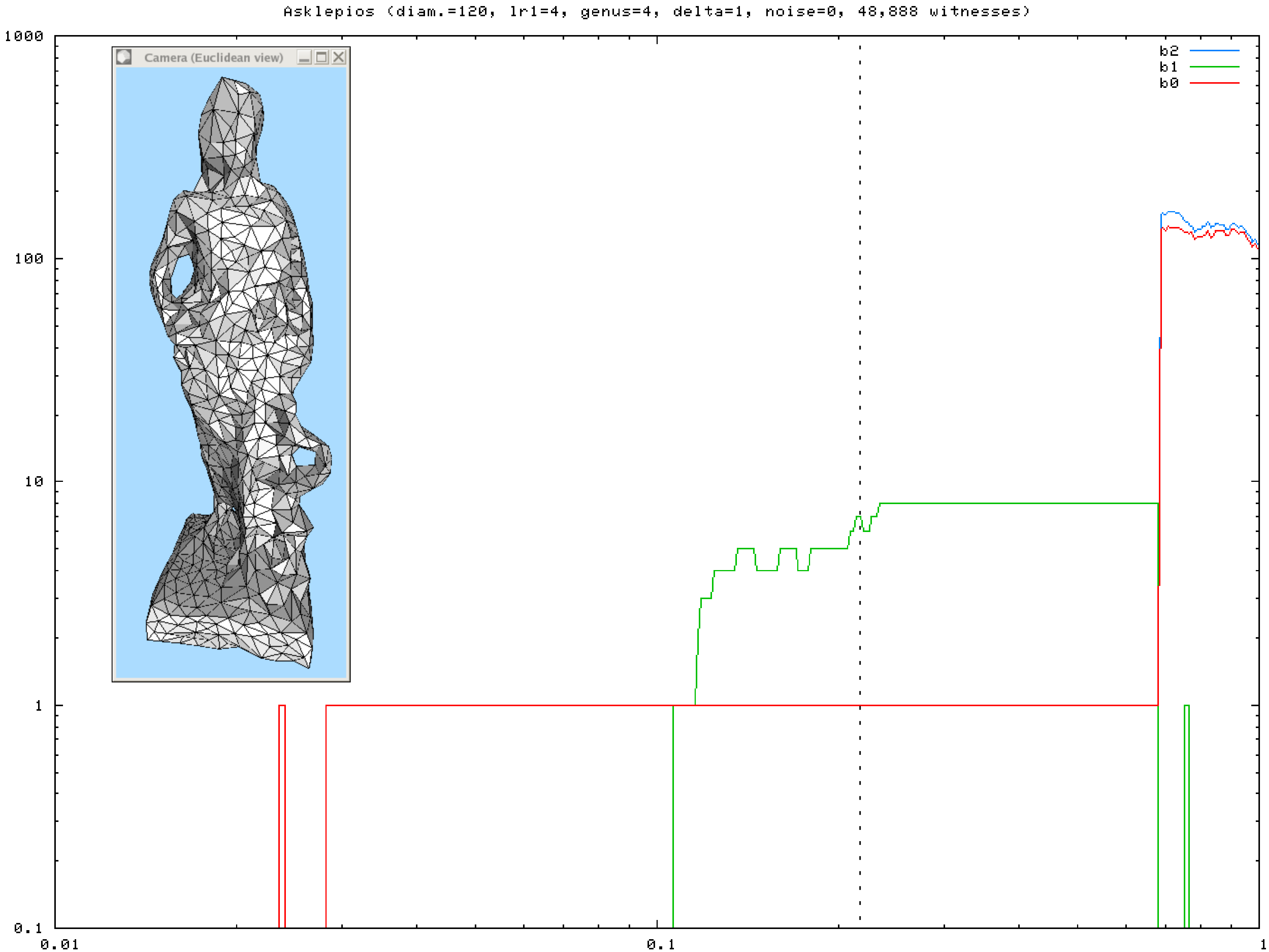

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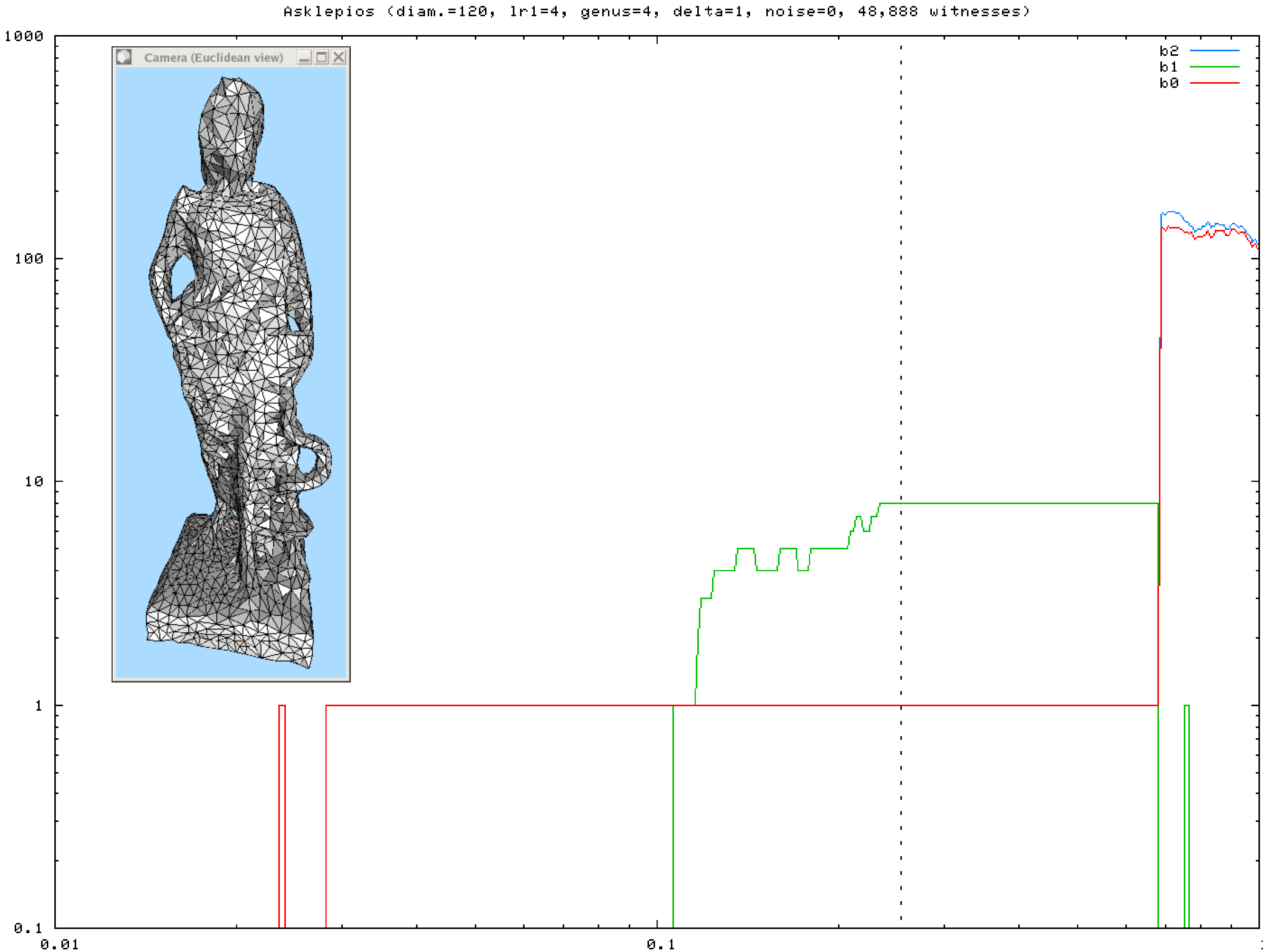

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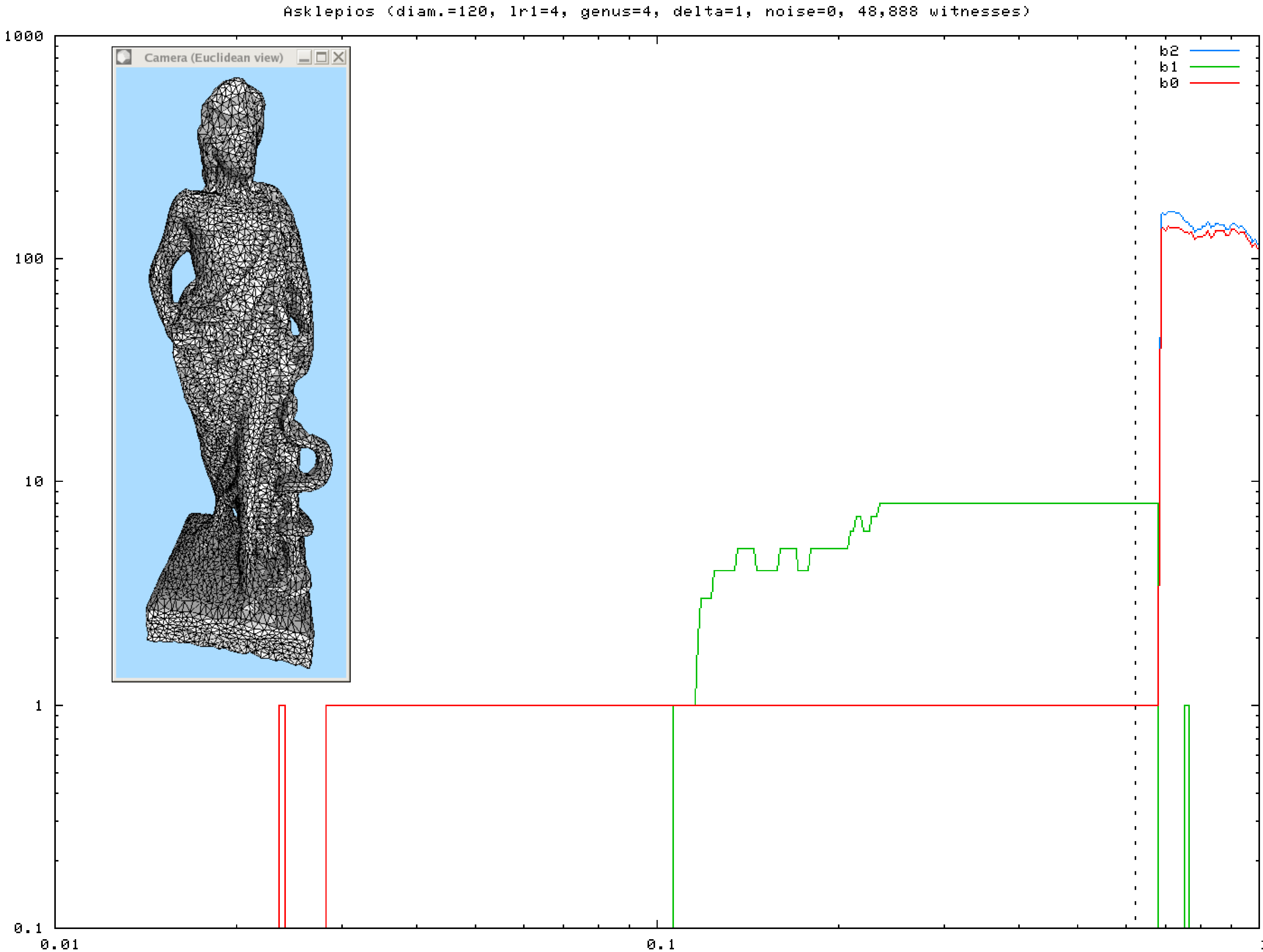

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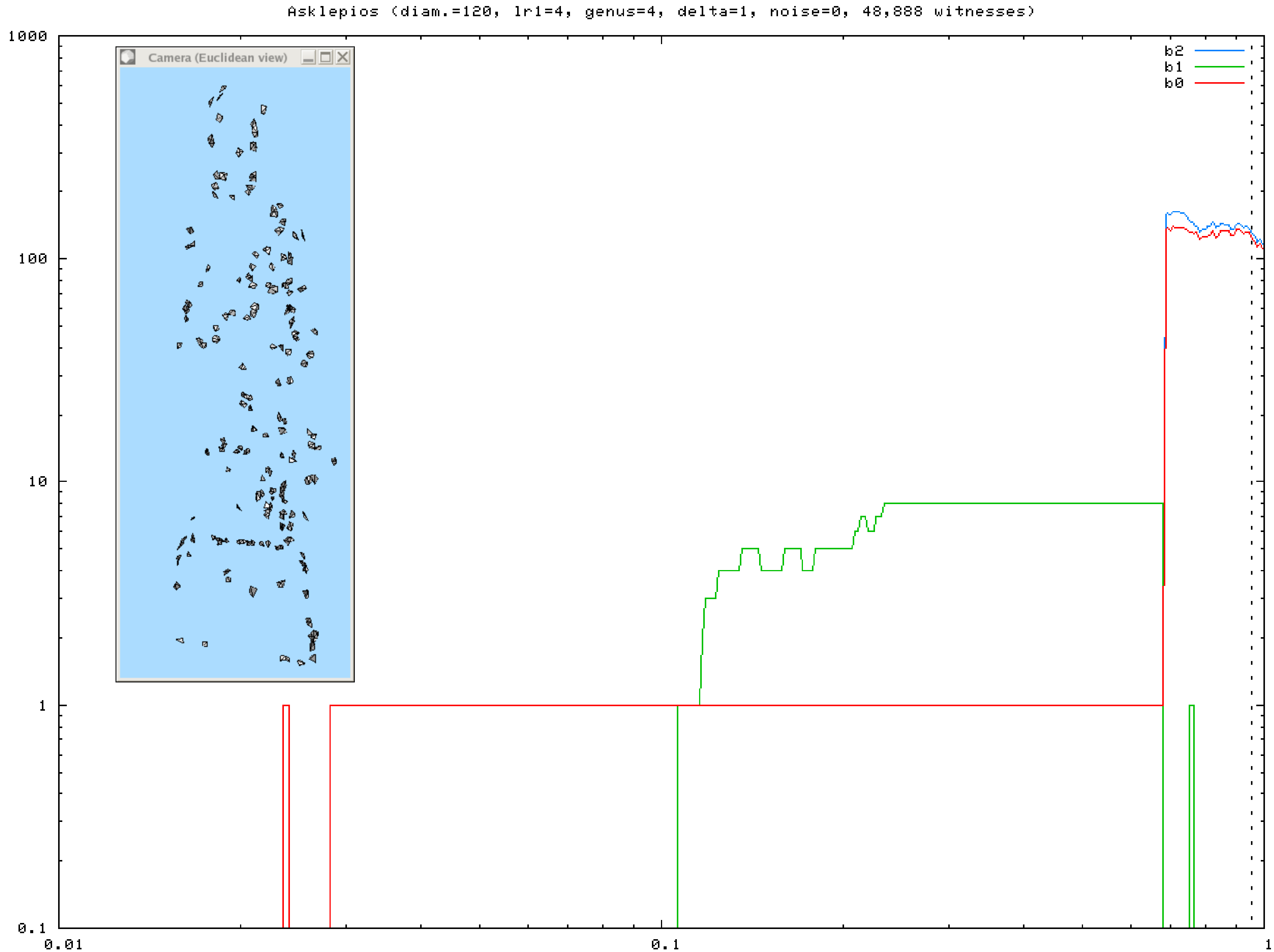

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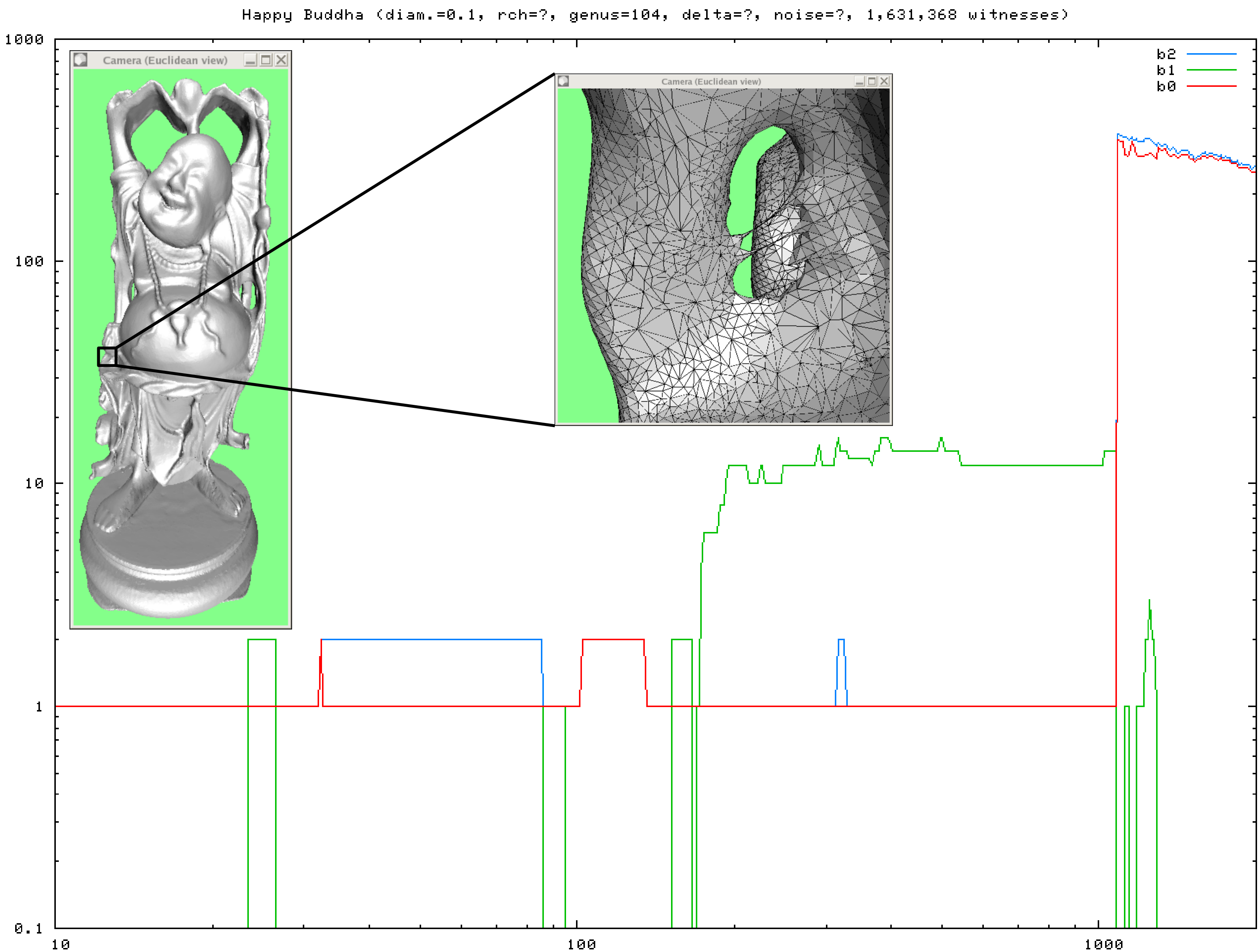

input data set courtesy of the Graphics Lab@Stanford

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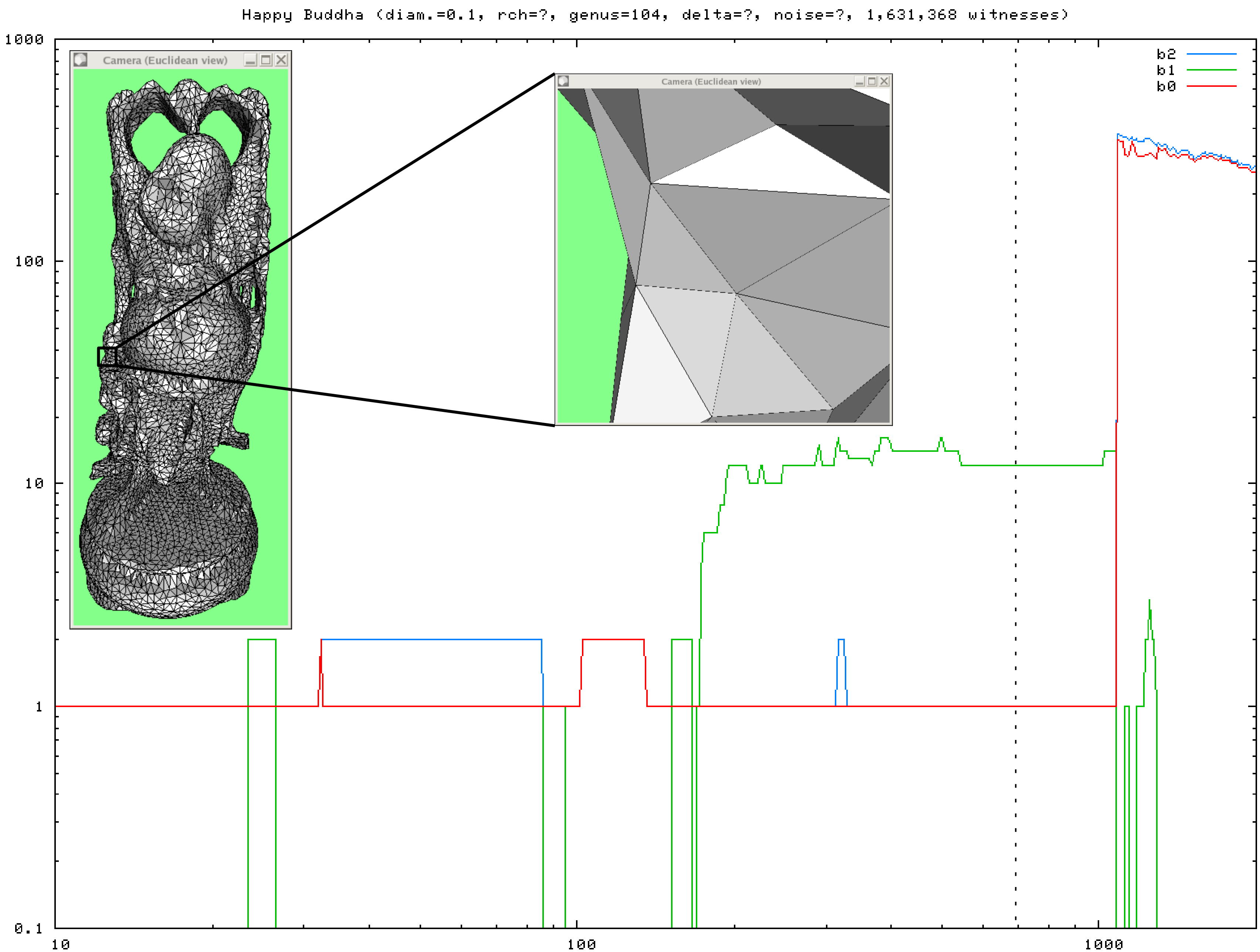

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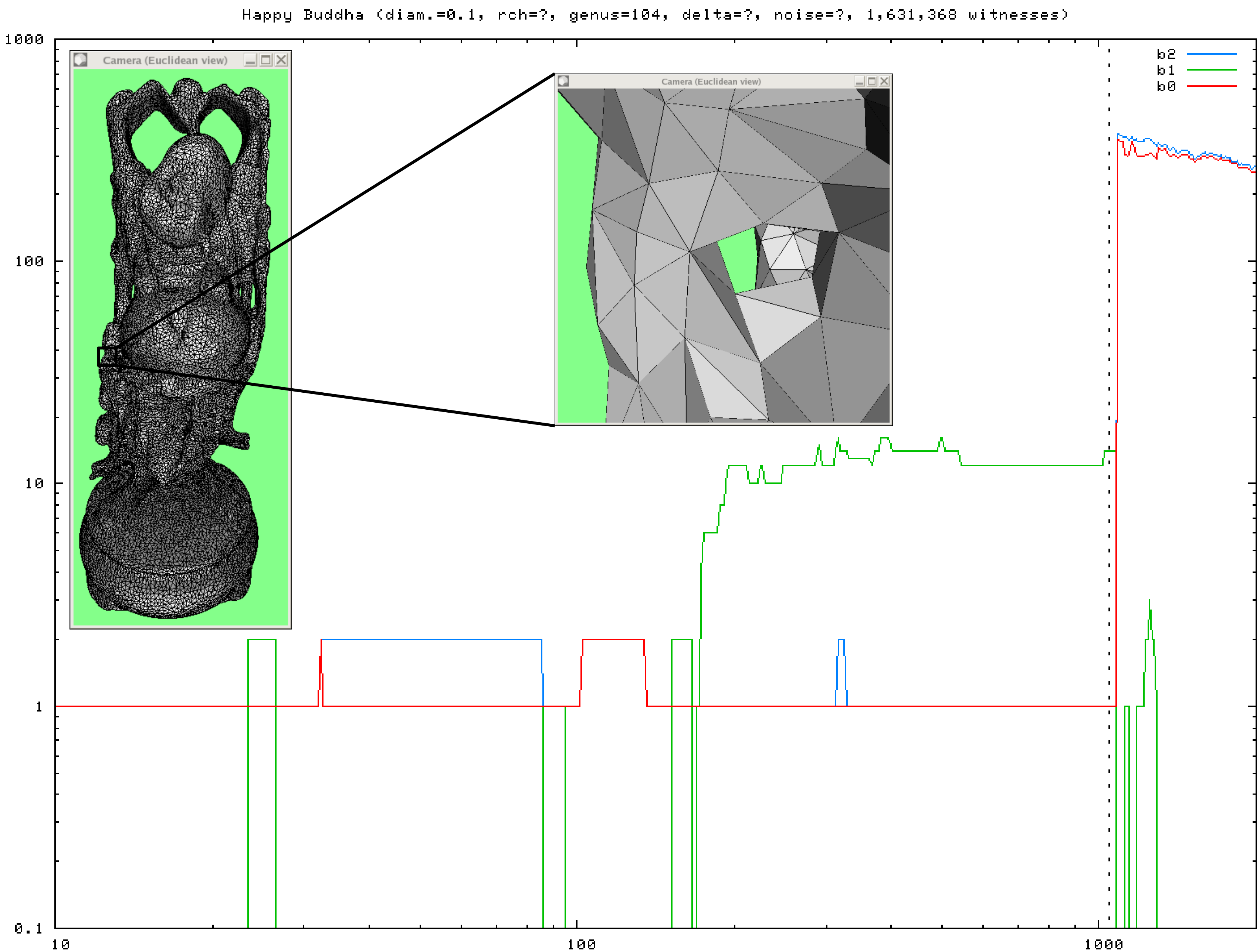

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