Computational Geometry and Topology
Géométrie et topologie algorithmiques

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Acknowledgments of Involved Colleagues:

Jean Daniel Boissonnat, Frederic Chazal, Marc Glisse,
As well as Olivier Devillers and Luca Castelli

Computational Geometry & Topology

Today

- Geometric Representations (warm-up)
- Comparing geometric objects
- Comparing topological spaces
- Simplicial complexes
- Polytopes …

Toward Applications

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Toward Applications

Metrics & Distances

The terms **distance** and **metric** are closely related, but not identical.

A **metric** is a function $d : X \times X \to \mathbb{R}_{\geq 0}$ on a set X that satisfies four axioms:

1. **Non-negativity**: $d(x, y) \geq 0$.
2. **Identity of indiscernibles:** $d(x, y) = 0 \iff x = y$.
3. **Symmetry**: $d(x, y) = d(y, x)$.
4. **Triangle inequality**: $d(x, z) \leq d(x, y) + d(y, z)$.

When these hold, d is a **metric**, and (X, d) is called a **metric space**.

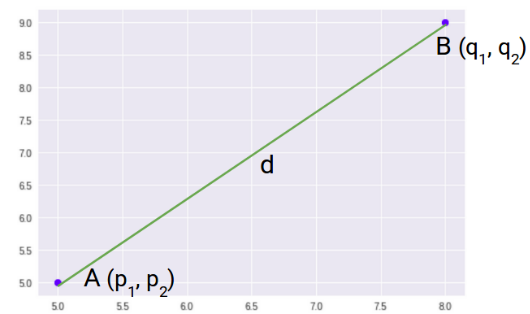
- In everyday or applied contexts, *distance* often just means some measure of dissimilarity between objects.
- A **distance** may fail to satisfy one or more of the strict axioms of a metric.

Distances

- For points
- For point sets
- For distributions
- For geometric elements
- For more complex shapes?

Only Comparison or Validation purposes

Distance between points



Euclidean distance

- In the plane:

$$d(p,q) = \sqrt{(p_1 - q_1)2 + (p_2 - q_2)2}$$

- Higher dimension:

$$d(\mathbf{p},\mathbf{q}) = d(\mathbf{q},\mathbf{p}) = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2 + \cdots + (q_n - p_n)^2}$$

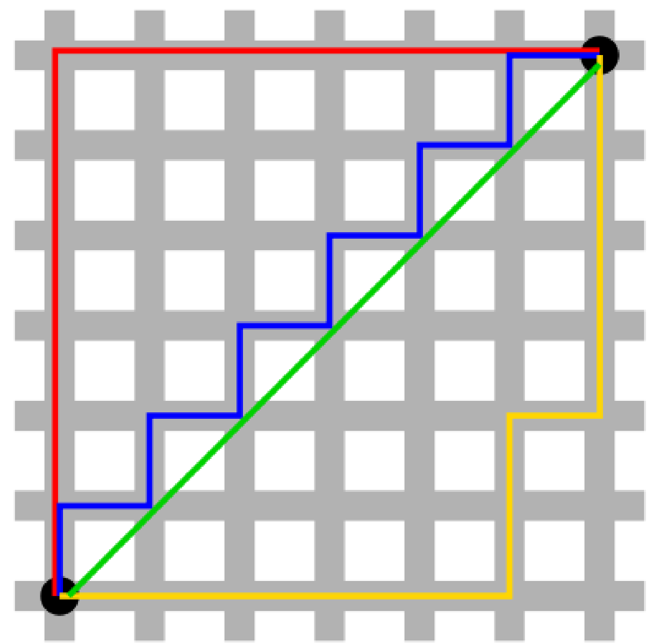
$$= \sqrt{\sum_{i=1}^{n} (q_i - p_i)^2}.$$

Distance between points

Manhattan distance

- In the plane: $|x_1 - x_2| + |y_1 - y_2|$

  Example in a grid ->



- Easily generalized to higher dimensions.

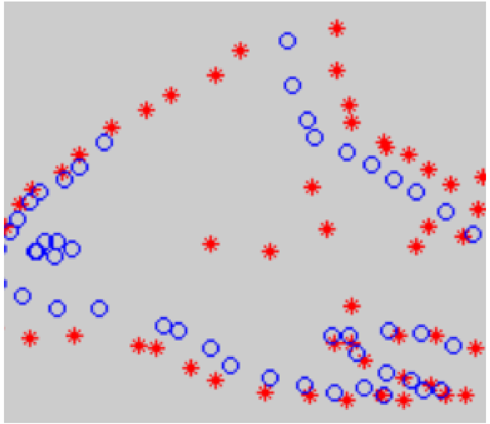
Distance between points

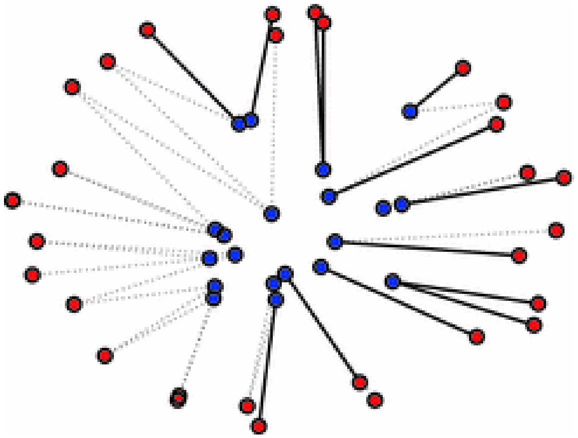
L_p distance

- In the plane: $(|x_1 - x_2|^p + |y_1 - y_2|^p)^{1/p}$

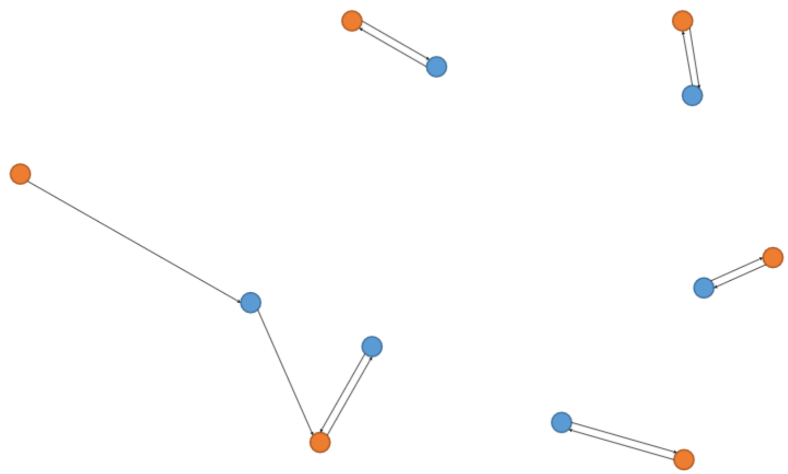
- *Easily generalized to higher dimensions.*
- *Euclidean distance is* ***L_2 distance****.*
- *Rectilinear or Manhattan distance is L_1 distance.*
- *L_∞ distance is $\max(|x_1 - x_2|, |y_1 - y_2|)$, also called* ***Chebychev*** distance.

Point Set Comparison / Registration



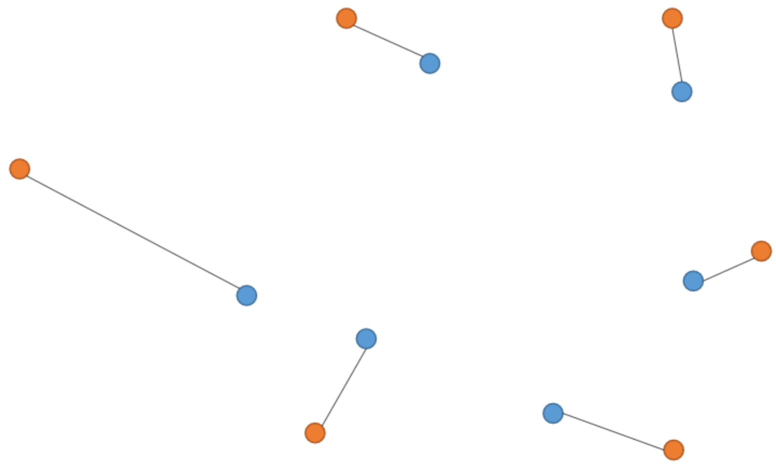


Chamfer Distance



$$d_{CD}(S_1, S_2) = \sum_{x \in S_1} \min_{y \in S_2} \|x - y\|_2^2 + \sum_{y \in S_2} \min_{x \in S_1} \|x - y\|_2^2$$

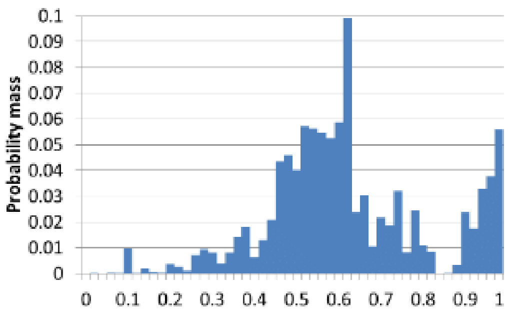
Earth Mover's Distance

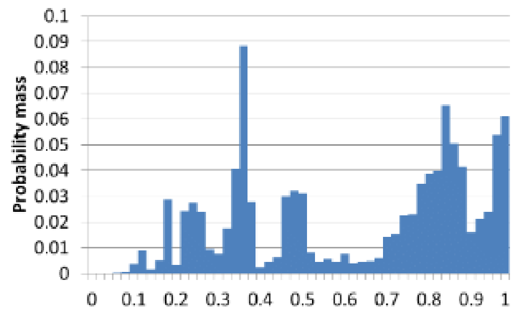


$$d_{EMD}(S_1, S_2) = \min_{\phi: S_1 \to S_2} \sum_{x \in S_1} \|x - \phi(x)\|_2$$

where $\phi : S_1 \to S_2$ is a bijection.

Earth Mover's Distance





Comparison

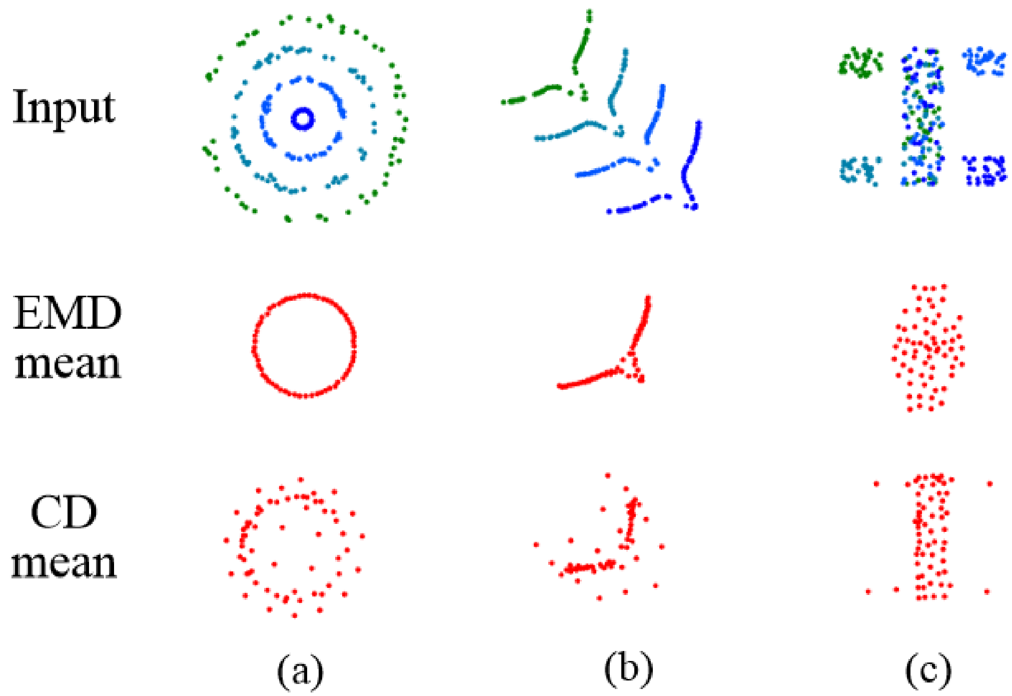
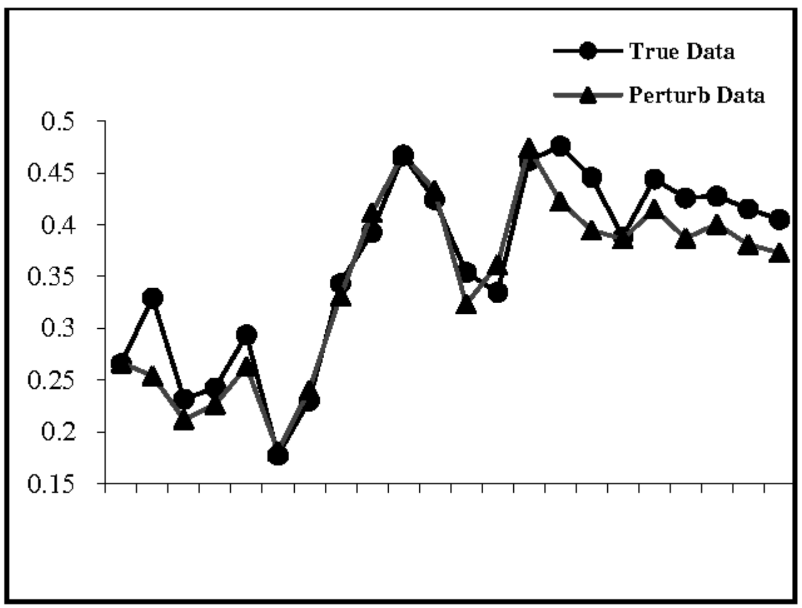
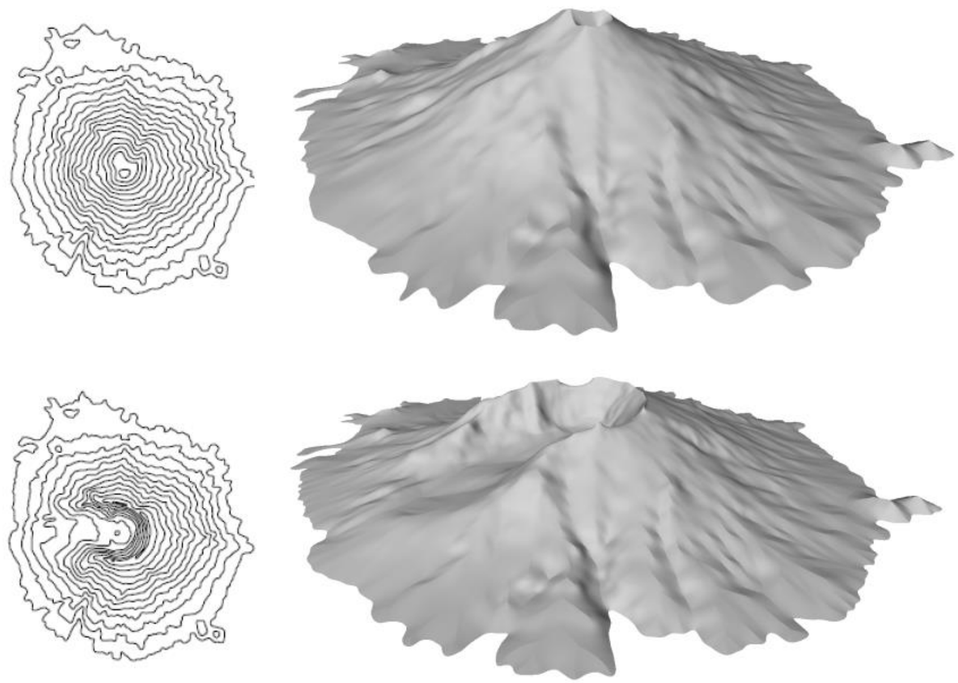


Fig. from Fan et al. 2016

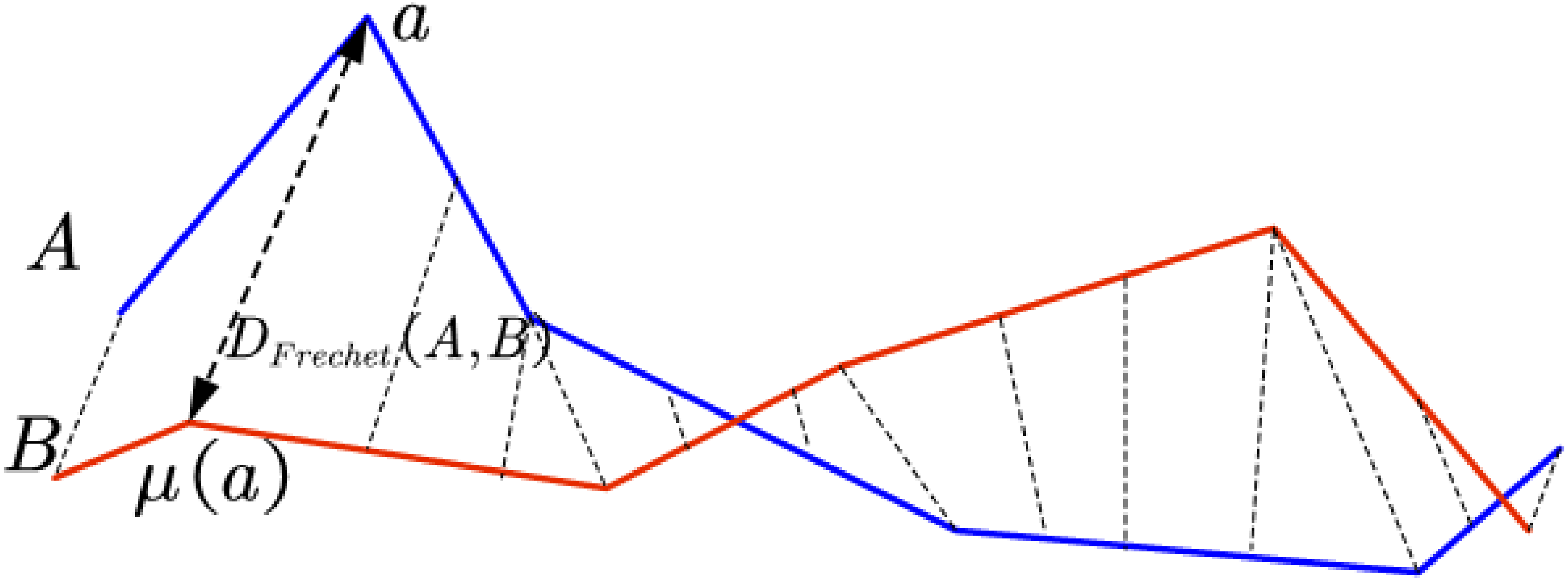
Mean-shape behavior of EMD and CD.

Curve Comparison / Validation Metrics





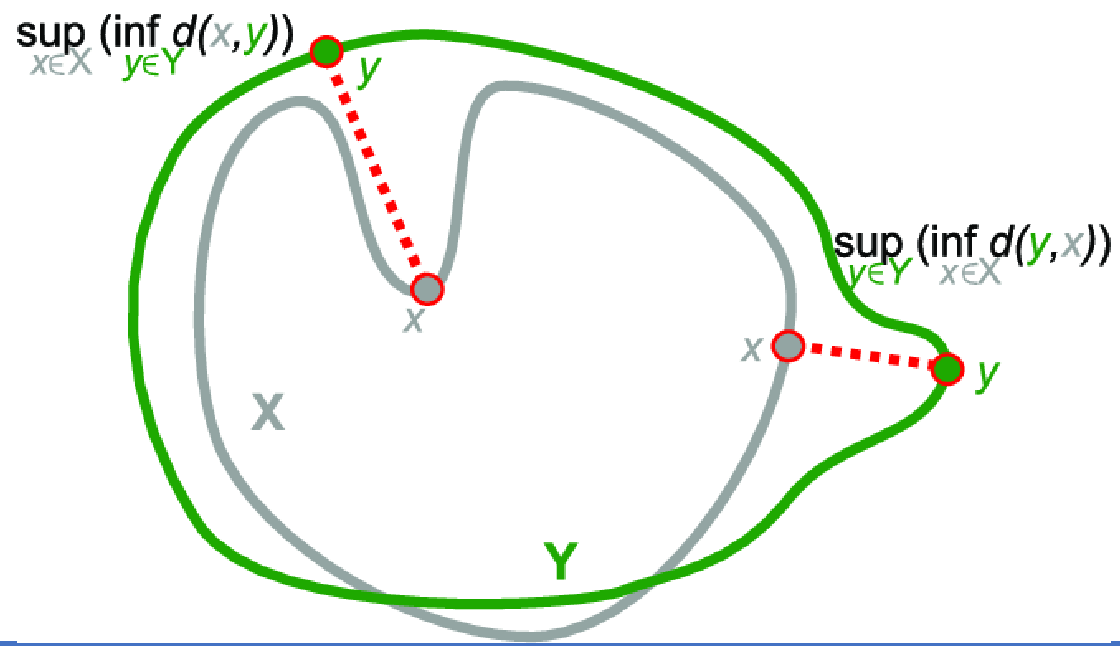
Fréchet distance between curves



Shape Comparison / Validation Metrics

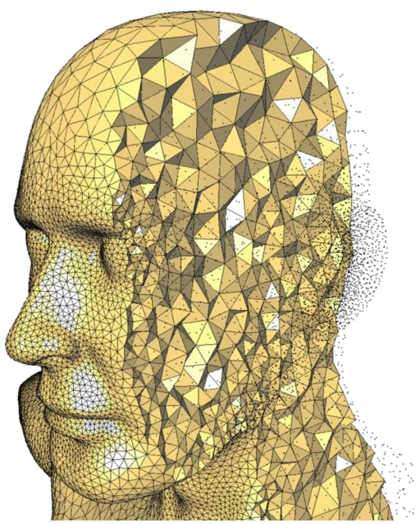
Hausdorff distance

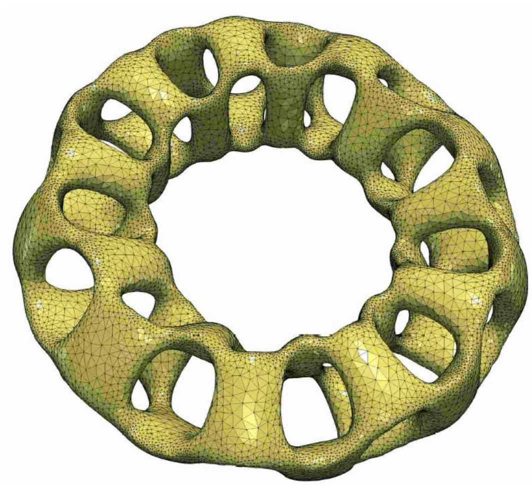
$$d_{\mathrm{H}}(X,Y) = \max\left\{ \sup_{x\in X} \inf_{y\in Y} d(x,y),\ \sup_{y\in Y} \inf_{x\in X} d(x,y) \right\}$$

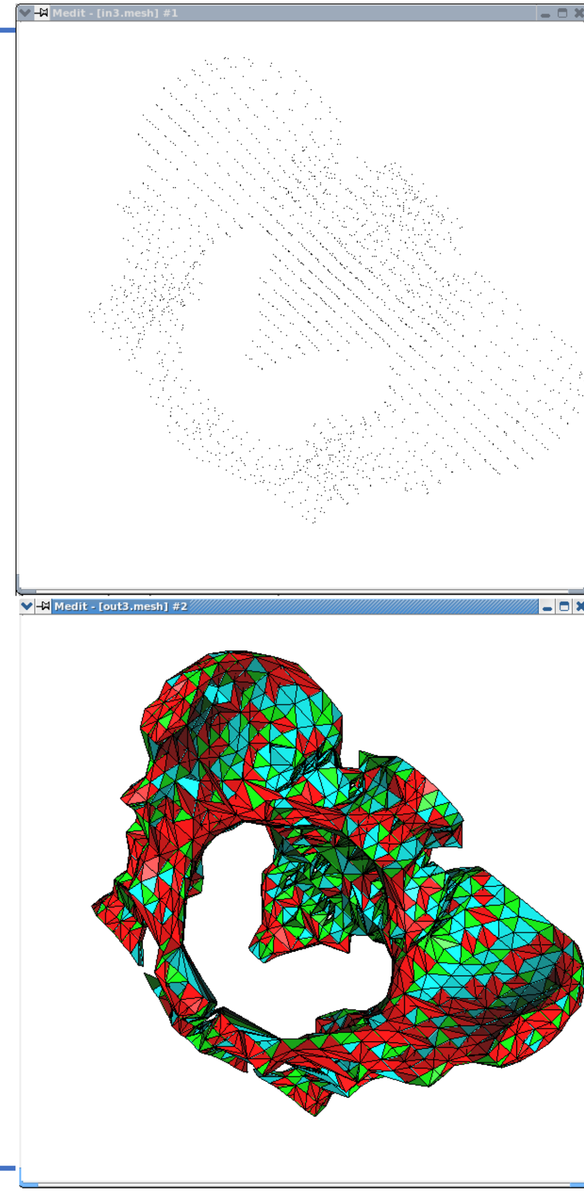


Geometric Algorithms

- **Discrete Shape Representation**

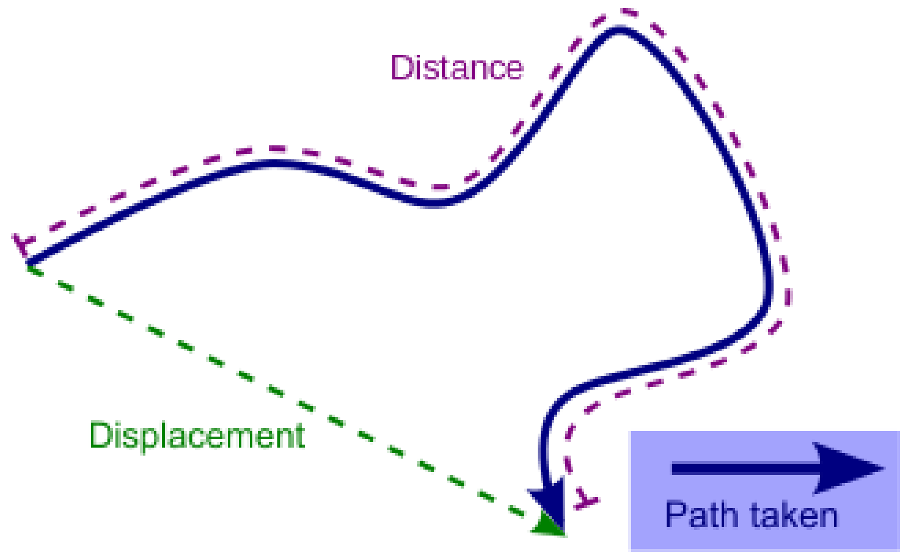




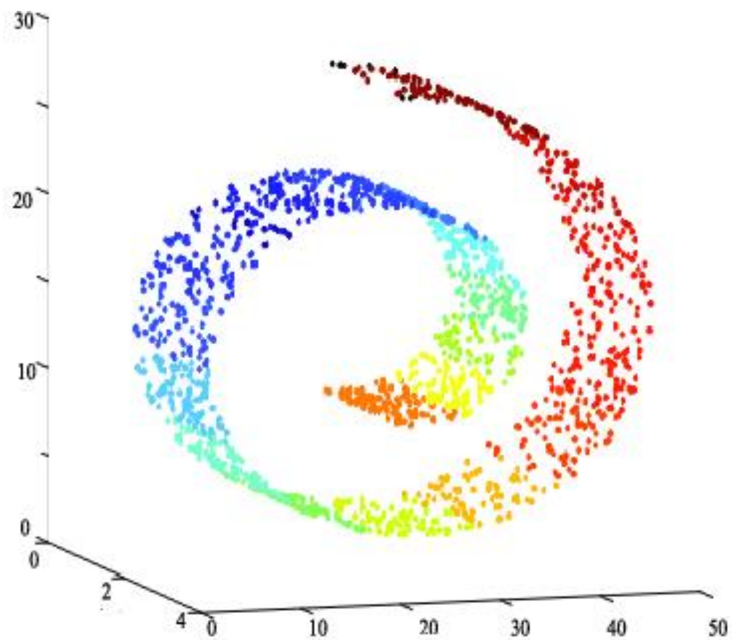


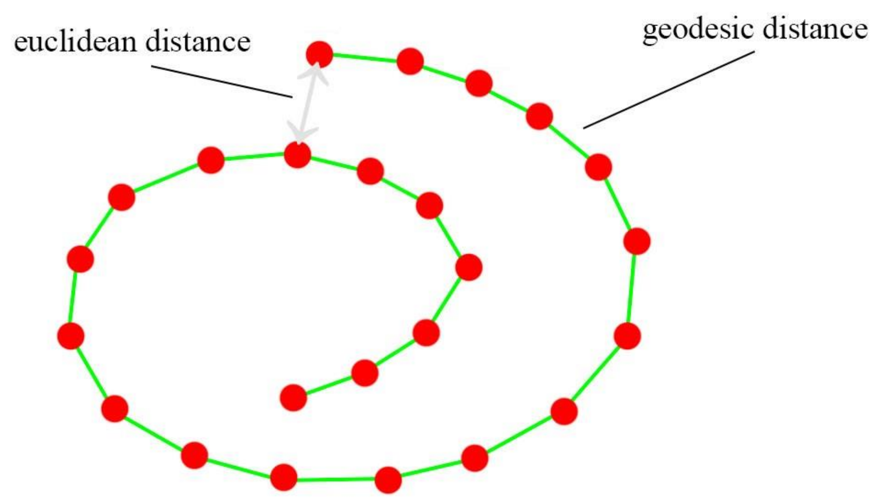
Distance along a path

Distance along a path



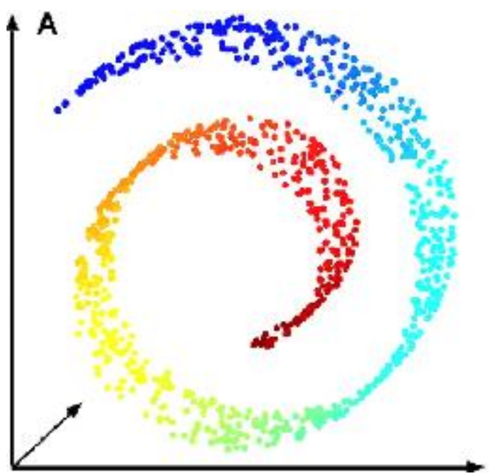
Geodesic Distance

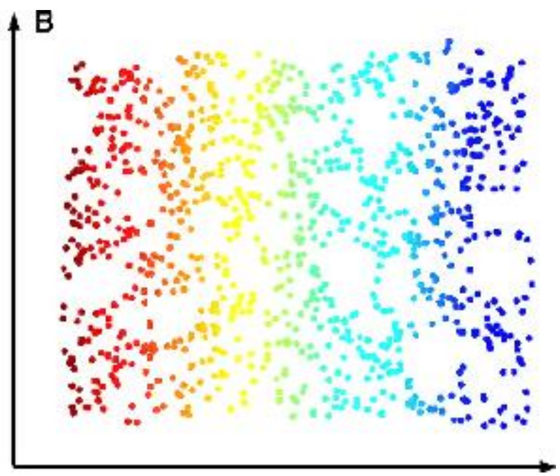


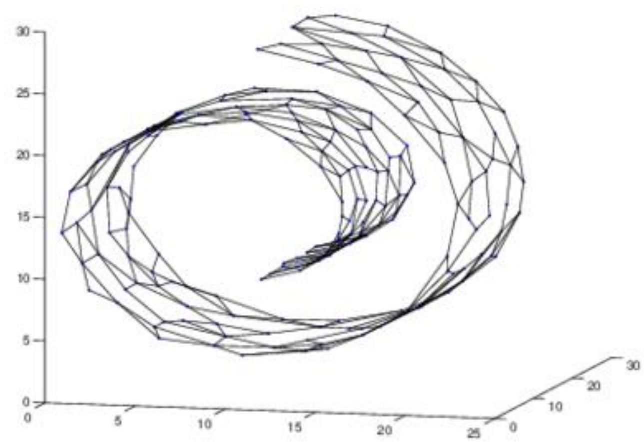


It’s important to find the underlying structure…

Geometric Data Analysis

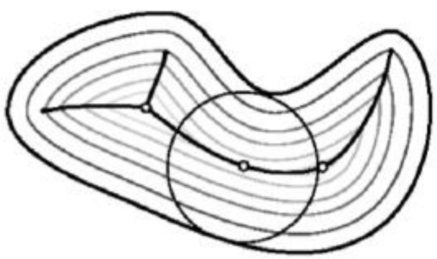


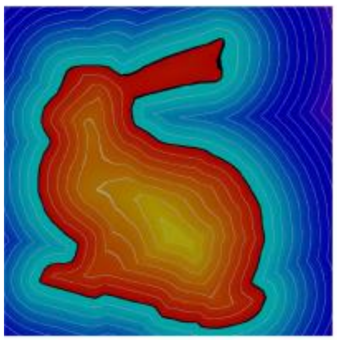


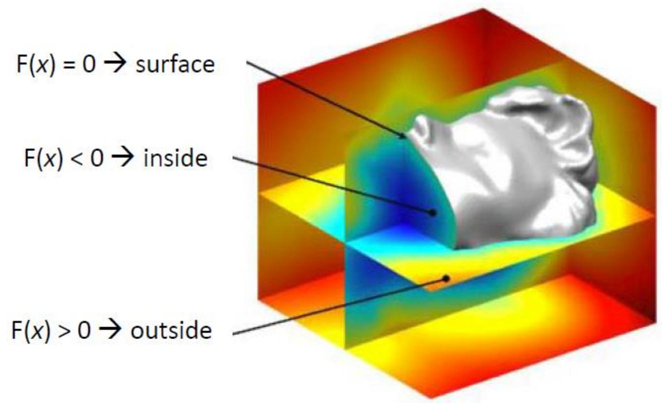


It’s important to find the underlying structure…
We will come back to it!

Distance Function

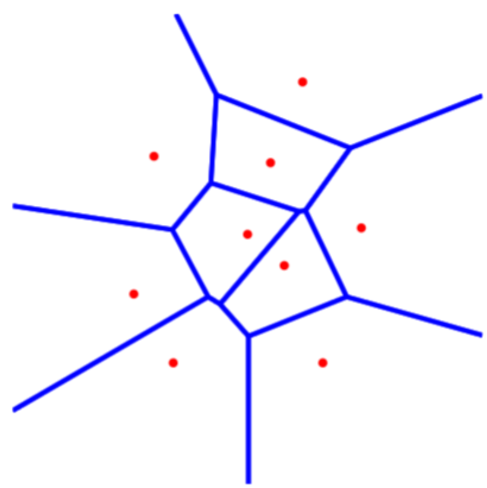






Bisectors and Voronoi daigram

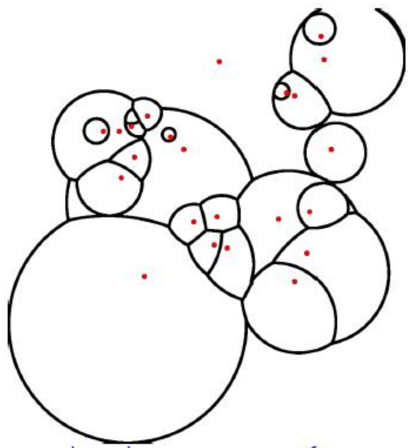
$\mathcal{P}$ a finite set of points in $\mathbb{R}^d$

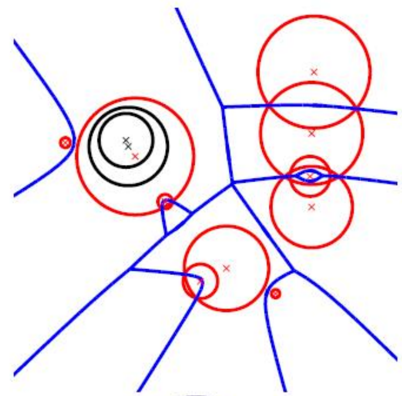


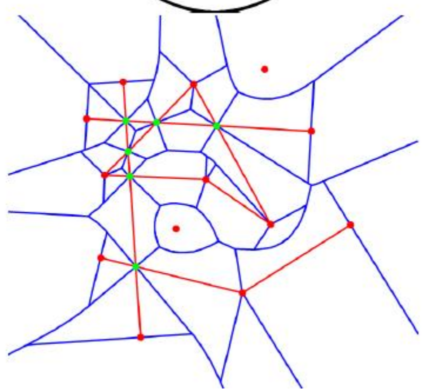
Voronoi cell $\quad V(p_i) = \{x : \|x - p_i\| \leq \|x - p_j\|, \ \forall j\}$

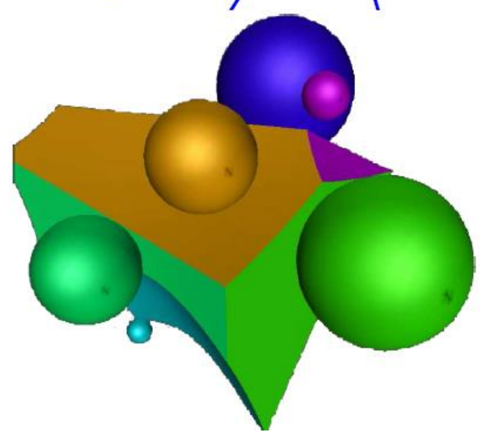
Voronoi diagram $\quad \mathrm{Vor}(\mathcal{P}) = \{$ cells $V(p_i)$ and their faces, $p_i \in \mathcal{P}$ $\}$

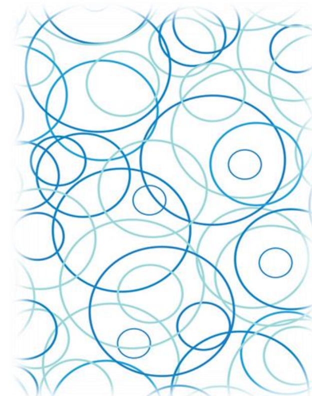
Voronoi diagram for different distances











Questions?

Computational Geometry & Topology

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- Geometric Representations (warm-up)
- Comparing geometric objects
- Comparing topological spaces
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- Polytopes …

Toward Applications

Basic definitions

Let X be a set. A **topology** on X is a collection $\tau \subseteq \mathcal{P}(X)$ of subsets of X (called **open sets**) such that the following **axioms** hold:

1. **Inclusion of trivial opens:**
   $\varnothing \in \tau$ and $X \in \tau$.
2. **Stability under arbitrary unions:**
   If $\{U_i\}_{i \in I} \subseteq \tau$ is any family of sets indexed by an arbitrary set I, then
   $$\bigcup_{i \in I} U_i \in \tau.$$
3. **Stability under finite intersections:**
   If $U_1, \ldots, U_n \in \tau$ with $n \in \mathbb{N}$, then
   $$\bigcap_{j=1}^{n} U_j \in \tau.$$

Basic definitions

- (X, τ) is then called a **topological space**.
- The axioms ensure that open sets behave in a way compatible with our geometric/analytic intuition of "regions without boundary points included":
  - You can always take unions of open sets (like merging regions).
  - You can always take finite intersections (like overlapping regions).
  - You must always have the "empty region" and the whole space available.

Trivial examples on any set X

- **Indiscrete (trivial) topology**:
  $\tau = \{\varnothing, X\}$.
  → Only the whole space and the empty set are open.
- **Discrete topology**:
  $\tau = \mathcal{P}(X)$ (all subsets of X).
  → Every subset is open. This is the "finest" possible topology.

Euclidean topology in higher dimensions

On $\mathbb{R}^n$, open sets are unions of open balls

$$B_r(x) = \{y \in \mathbb{R}^n : \|y - x\| < r\}.$$

→ Fundamental for geometry, calculus, and physics.

Topological spaces in geometric modeling

1. **Euclidean space** $\mathbb{R}^n$ – ambient space for curves, surfaces, and solids; open sets are unions of open balls. **Applications:** general modeling, CAD, physics simulations.
2. **Subspaces of** $\mathbb{R}^n$ – subsets like curves or surfaces (e.g., $S^2 \subset \mathbb{R}^3$) with the induced topology; open sets are intersections with Euclidean opens. **Applications:** surface modeling, mesh design.
3. **Manifolds** – spaces locally homeomorphic to $\mathbb{R}^n$ (e.g., circle, sphere, torus); open sets are those that look Euclidean under local charts. **Applications:** smooth shape representation, differential geometry, NURBS.
4. **Product spaces** – Cartesian products like $[0, 1]^2$; open sets are products of opens in each factor. **Applications:** parametric surfaces, UV mapping, texture domains.
5. **Quotient spaces** – formed by identifying boundaries (e.g., circle as $[0, 1]$ with $0 \sim 1$, torus as a square with opposite edges glued); open sets are preimages of opens under the quotient map. **Applications:** periodic surfaces, closed shapes, topology of meshes.

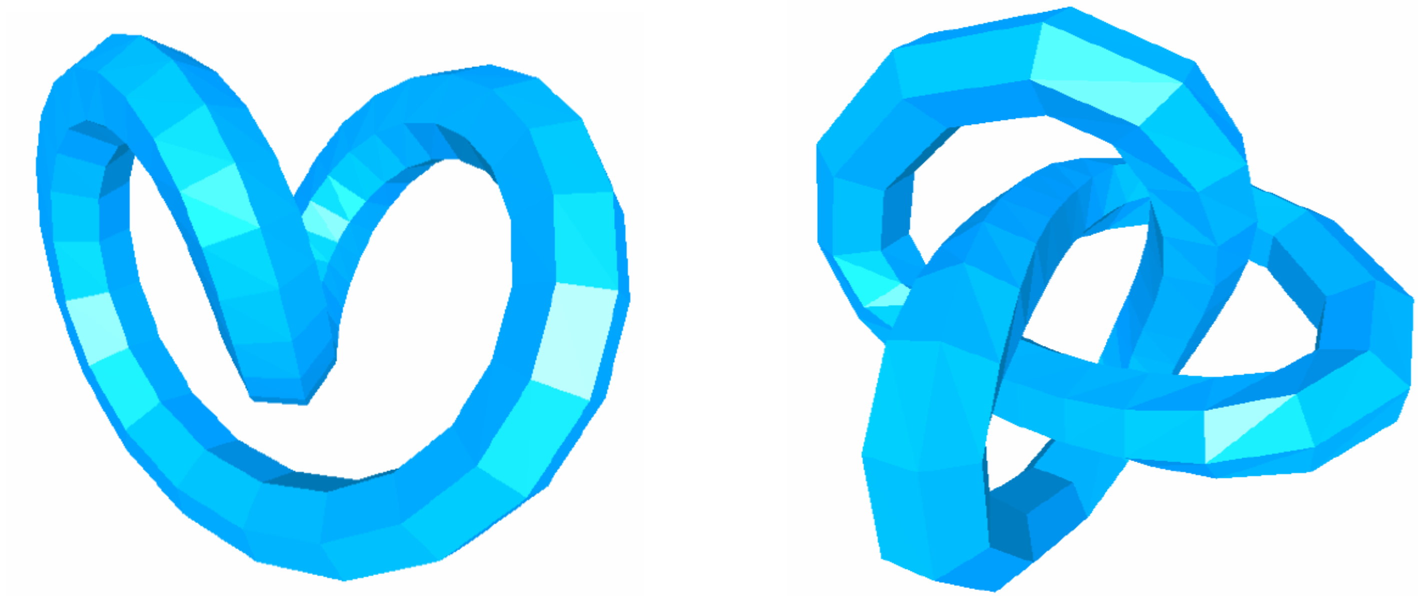
Basic definitions

Let (X, τ_X) and (Y, τ_Y) be **topological spaces**, i.e. X and Y are sets and τ_X, τ_Y are topologies on them (collections of subsets called *open sets* satisfying the usual axioms).

Notation. A map $f : X \to Y$ is called a **bijection** if it is one-to-one and onto. For a map f that is bijective we denote its inverse by $f^{-1} : Y \to X$.

Continuity (topological). A function $f : X \to Y$ is **continuous** iff for every open set $V \in \tau_Y$ the preimage $f^{-1}(V) \in \tau_X$. Equivalently, f is continuous iff for every $x \in X$ and every neighbourhood V of $f(x)$ there exists a neighbourhood U of x with $f(U) \subseteq V$.

Topology: homeomorphy and isotopy

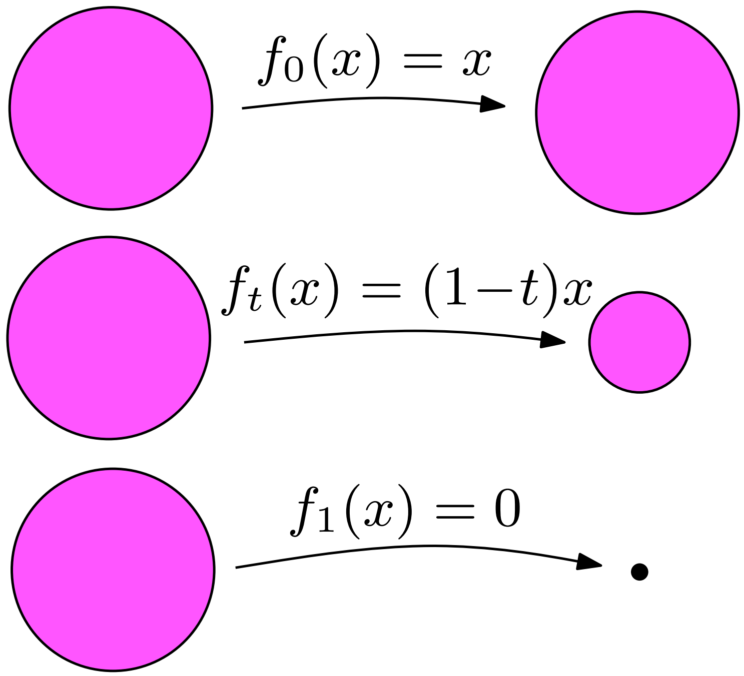


- X and Y are homeomorphic if there exists a bijection $h : X \to Y$ s. t. h and h^{-1} are continuous.

- $X, Y \subset \mathbb{R}^d$ are isotopic if there exists a continuous map $F : X \times [0, 1] \to \mathbb{R}^d$ s. t. $F(., 0) = Id_X$, $F(X, 1) = Y$ and $\forall t \in [0, 1]$, $F(., t)$ is an homeomorphism on its image.

- $X, Y \subset \mathbb{R}^d$ are ambient isotopic if there exists a continuous map $F : \mathbb{R}^d \times [0, 1] \to \mathbb{R}^d$ s. t. $F(., 0) = Id_{\mathbb{R}^d}$, $F(X, 1) = Y$ and $\forall t \in [0, 1]$, $F(., t)$ is an homeomorphim of $\mathbb{R}^d$.

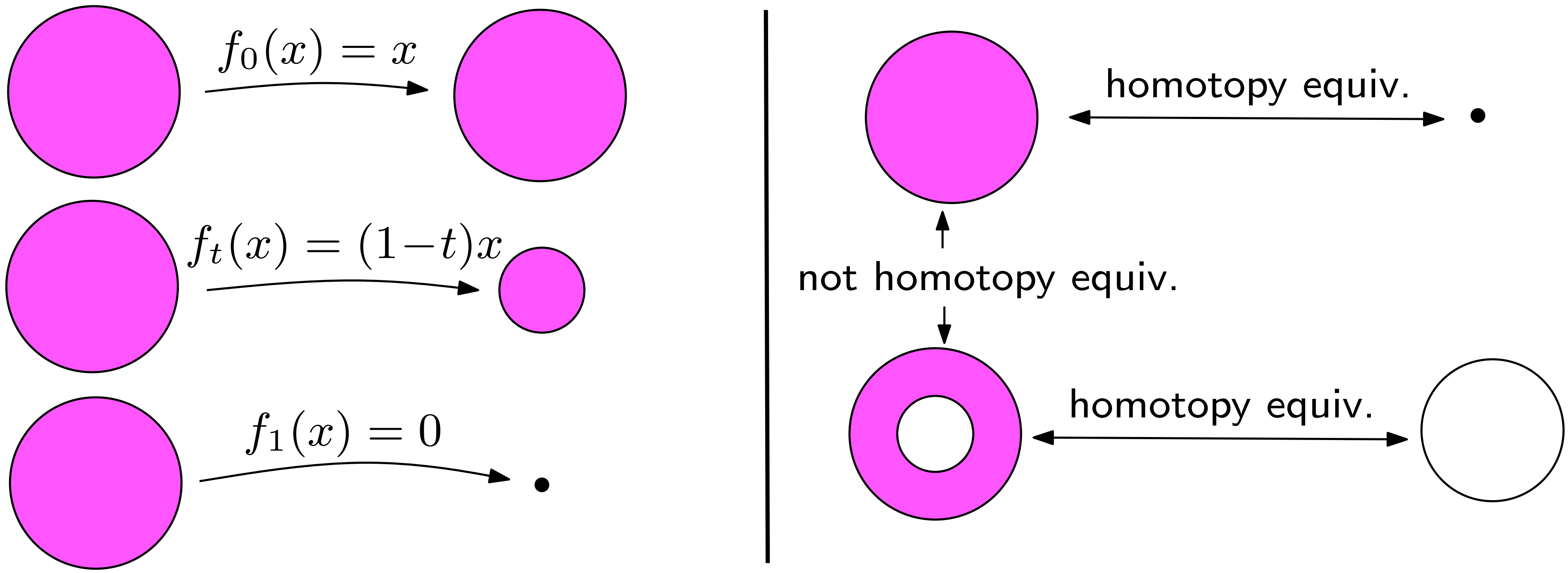
Topology: homotopy type



- Two maps $f_0 : X \to Y$ and $f_1 : X \to Y$ are homotopic if there exists a continuous map $H : [0,1] \times X \to Y$ s. t. $\forall x \in X$, $H(0,x) = f_0(x)$ and $H_1(1,x) = f_1(x)$.

- X and Y have the same homotopy type (or are homotopy equivalent) if there exists continuous maps $f : X \to Y$ and $g : Y \to X$ s. t. $g \circ f$ is homotopic to Id_X and $f \circ g$ is homotopic to Id_Y.

Topology: homotopy type



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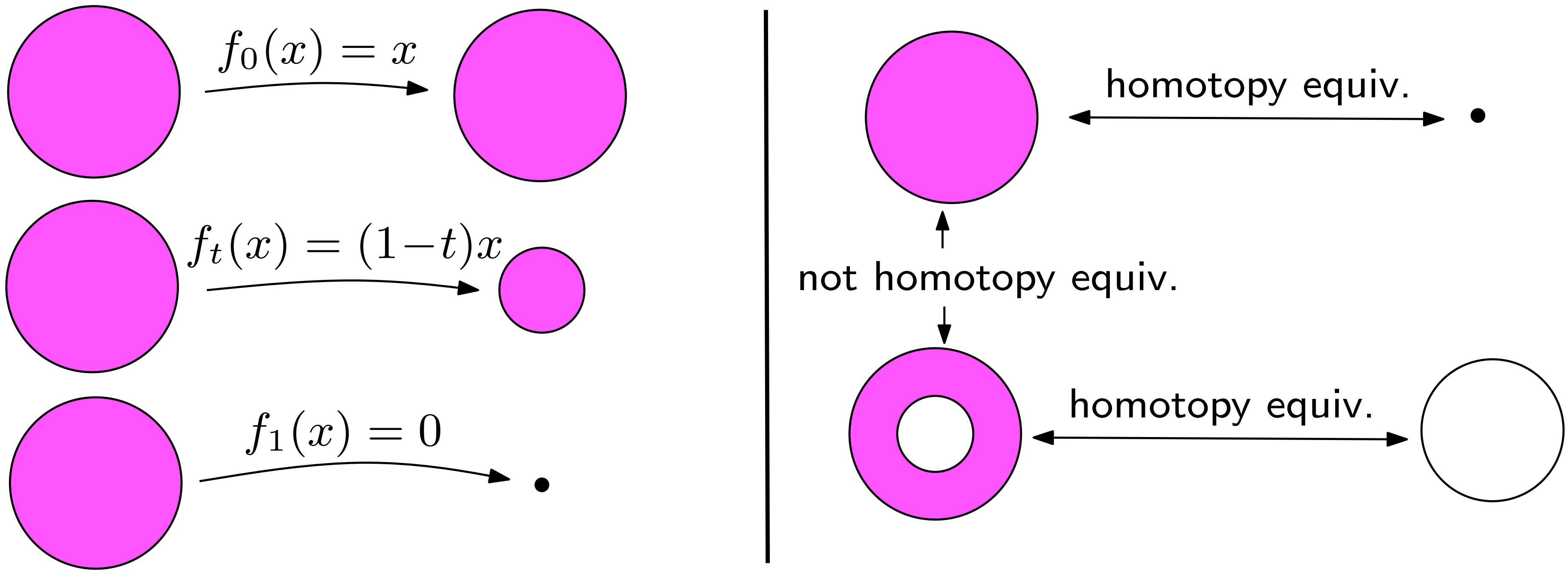

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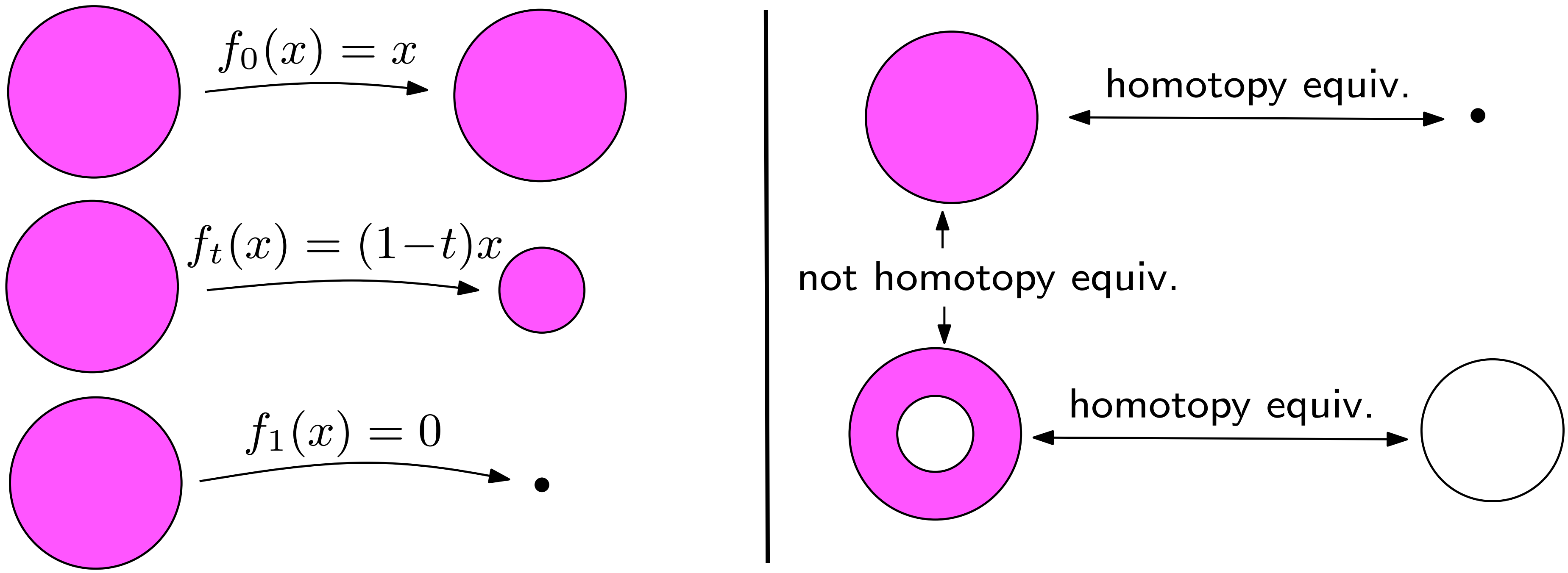
X and Y homotopy equivalent $\Rightarrow$ X and Y have isomorphic homotopy and homology groups.

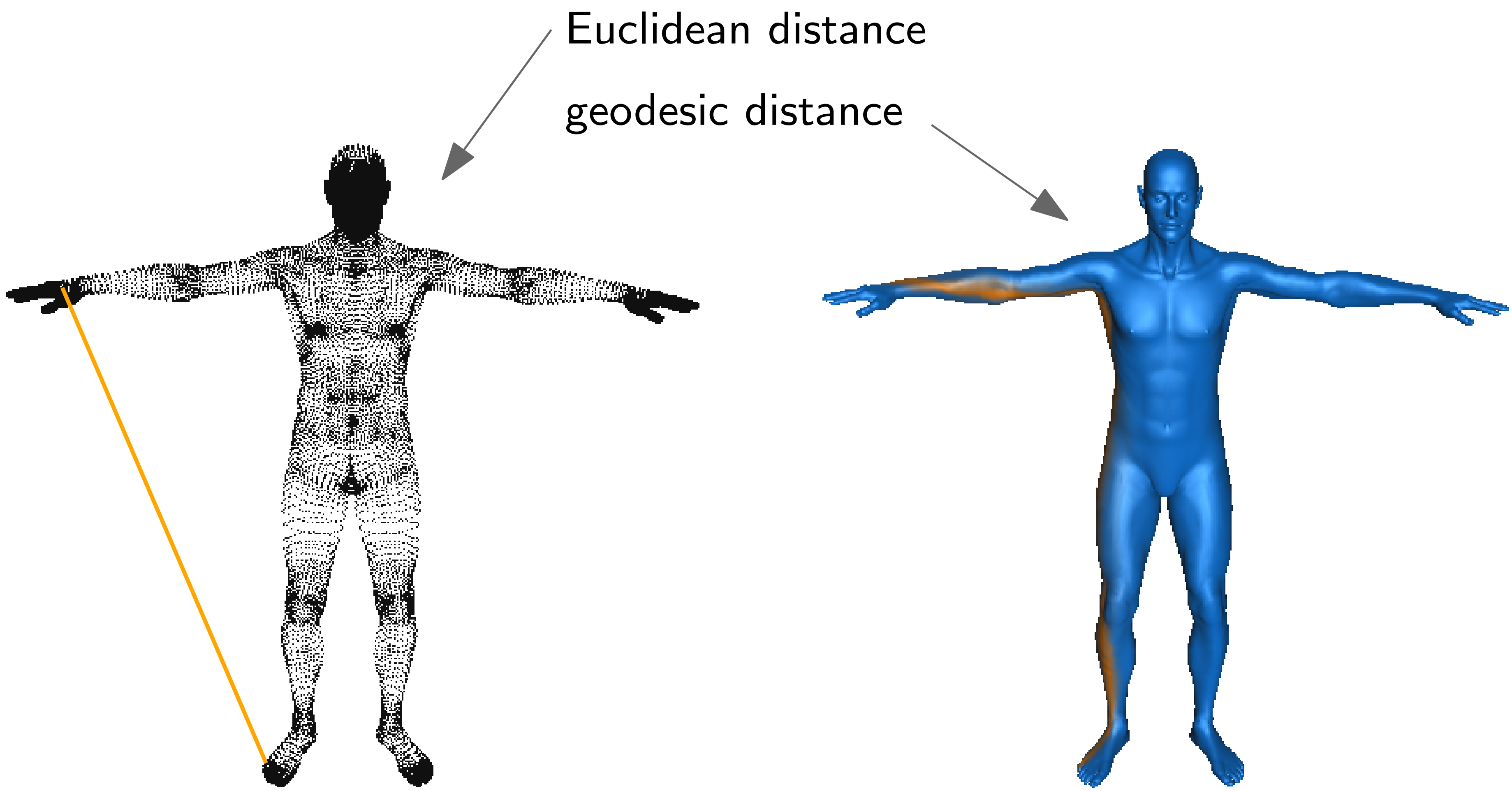
Topology: homotopy type



If $Y \subset X$ and if there exists a continuous map $H : [0,1] \times X \to X$ s.t.:
i) $\forall x \in X$, $H(0,x) = x$,
ii) $\forall x \in X$, $H(1,x) \in Y$
iii) $\forall y \in Y$, $\forall t \in [0,1]$, $H(t,y) \in Y$,
then X and Y are homotopy equivalent. If one replaces condition *iii)* by $\forall y \in Y$, $\forall t \in [0,1]$, $H(t,y) = y$ then H is a deformation retract of X onto Y.

Mathematical Framework

- geometric data set $\equiv$ compact metric space



Mathematical Framework

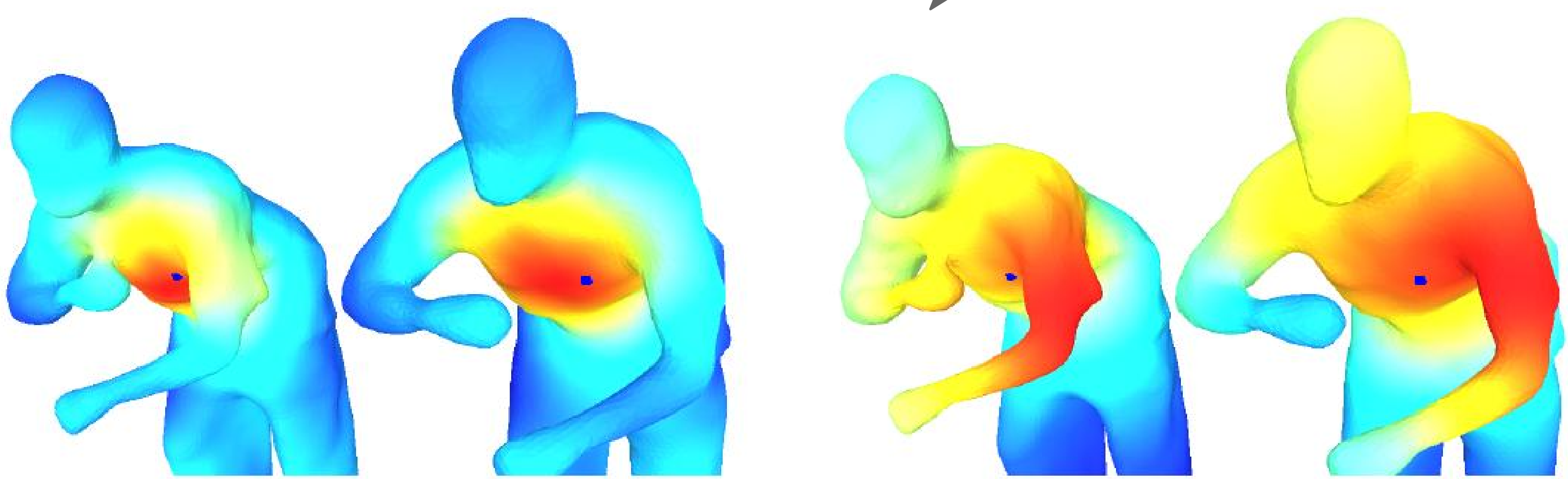
- geometric data set $\equiv$ compact metric space

Euclidean distance

geodesic distance

diffusion distance

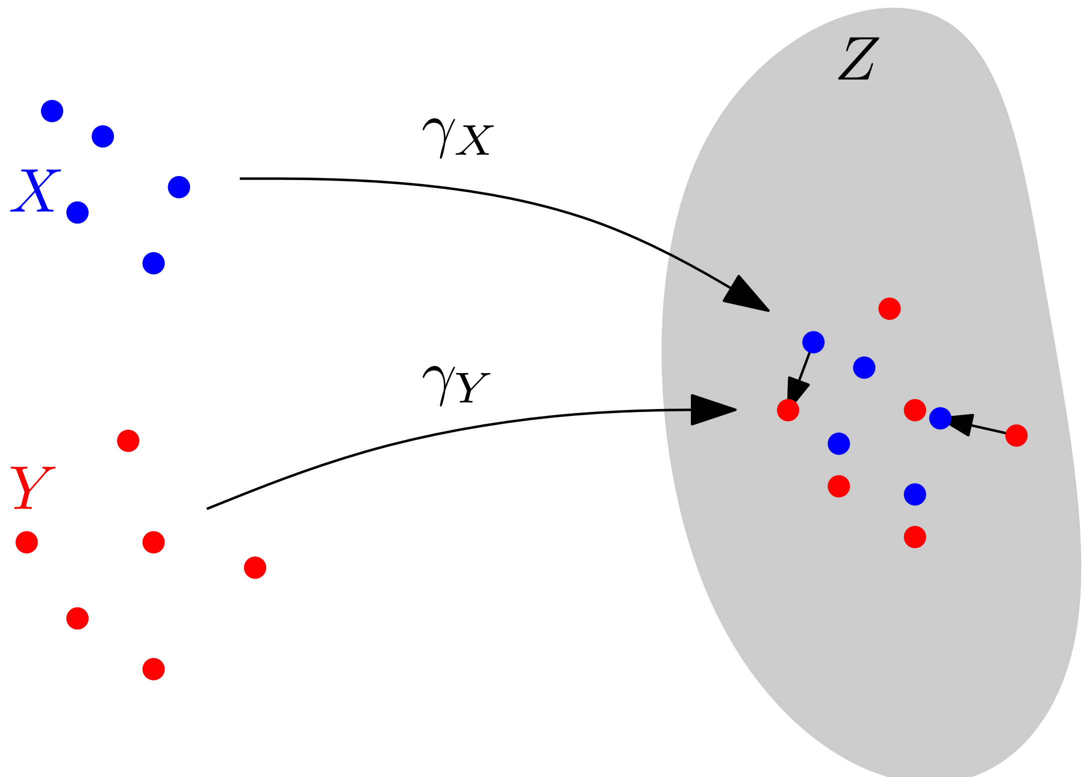
...



Mathematical Framework

- geometric data set $\equiv$ compact metric space

- distance between data sets $\equiv$ Gromov-Hausdorff (GH) distance



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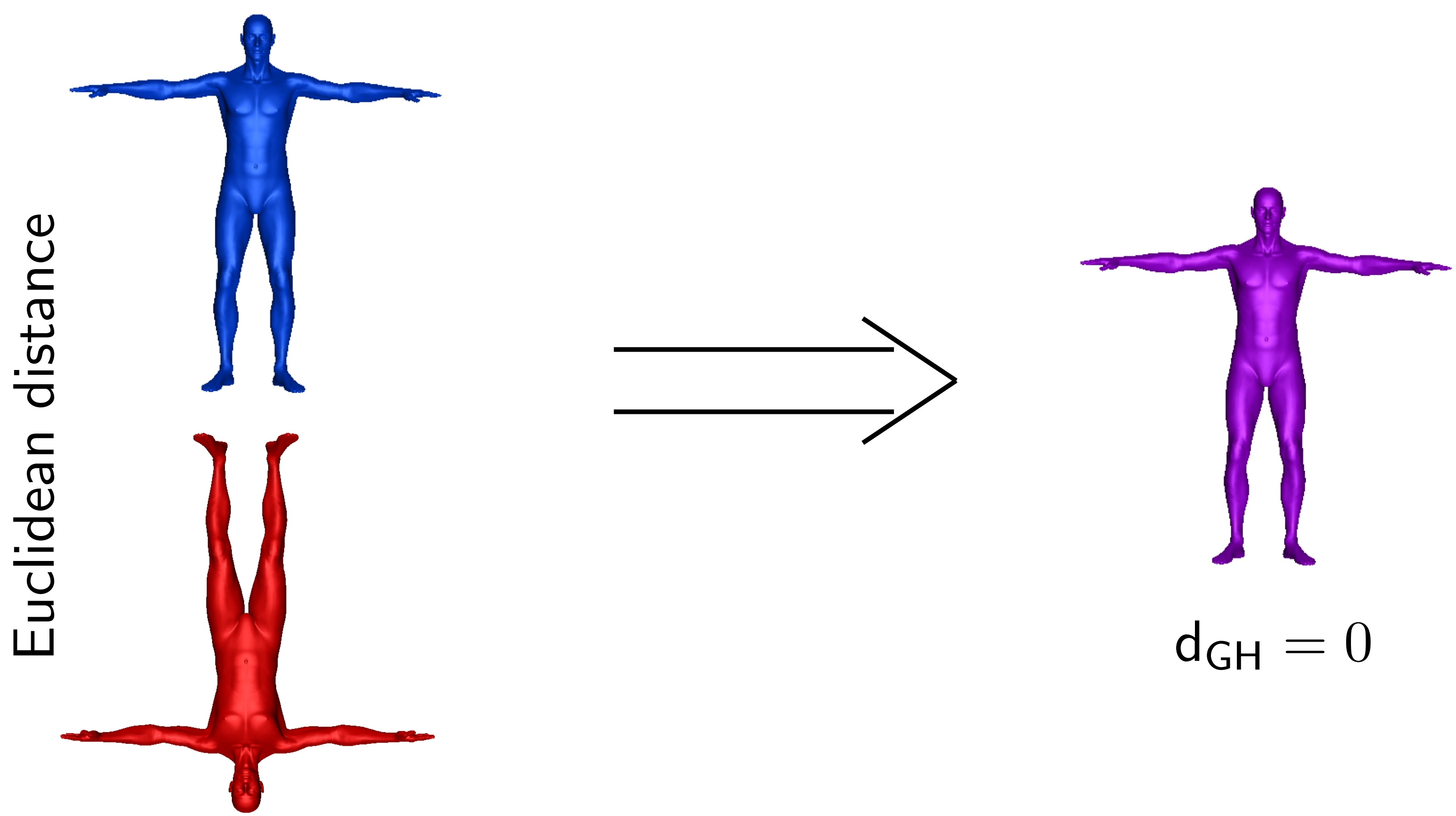

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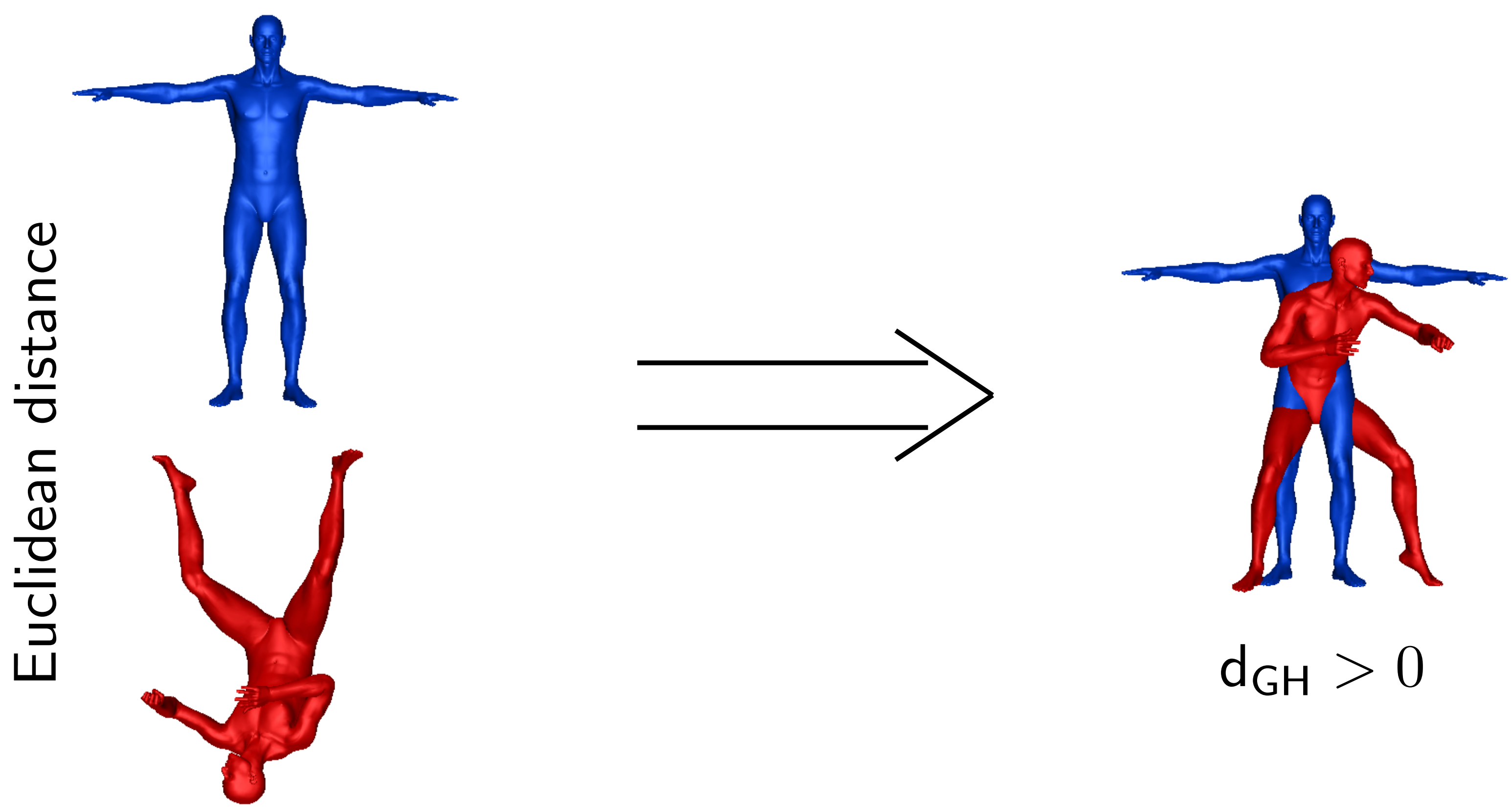

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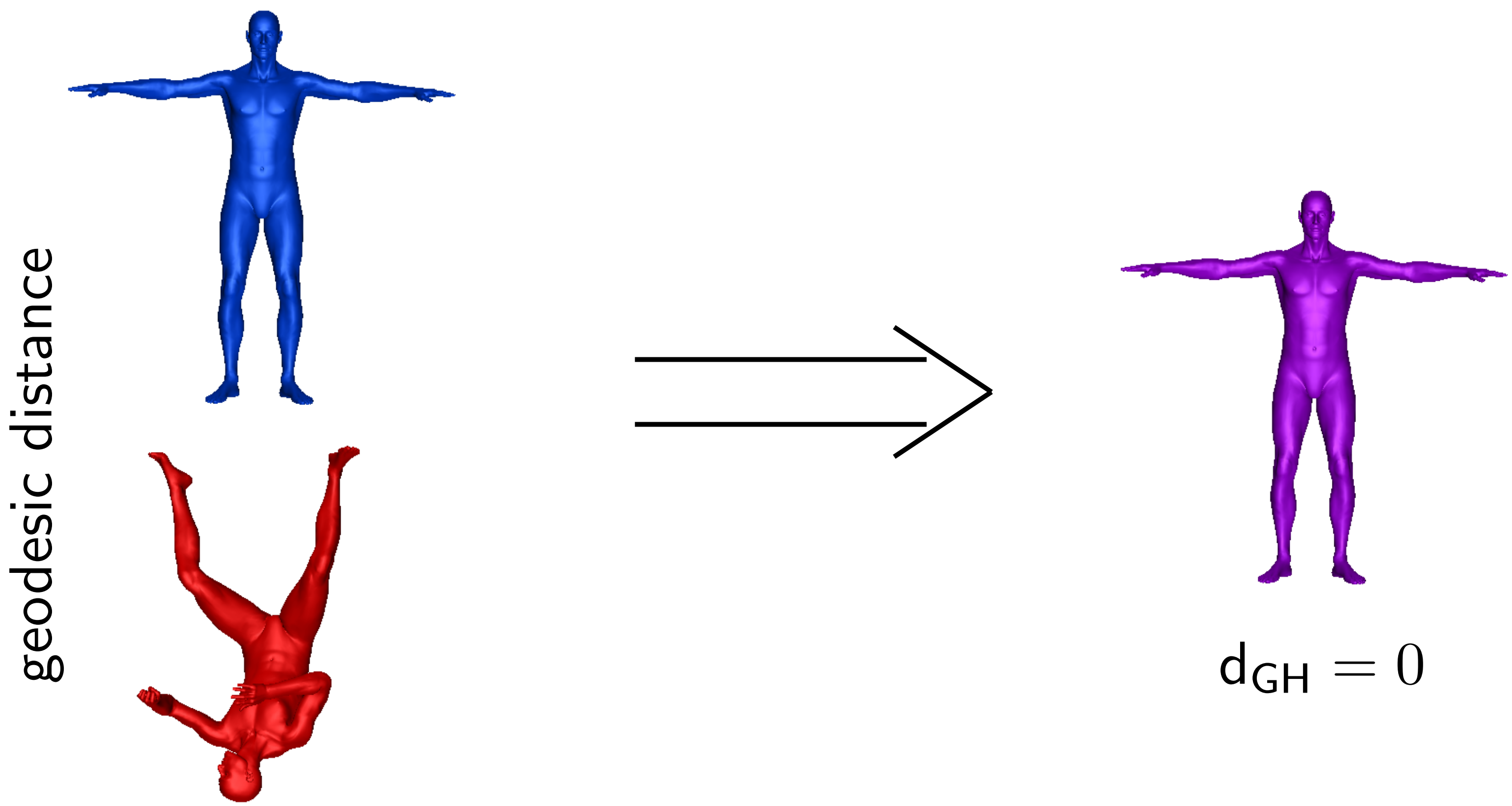

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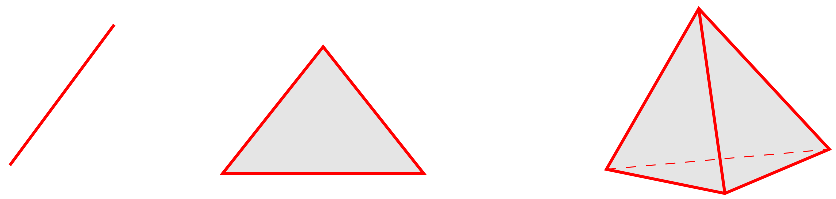
Geometric simplices

A k-simplex σ is the convex hull of $k+1$ points of $\mathbb{R}^d$ that are affinely independent

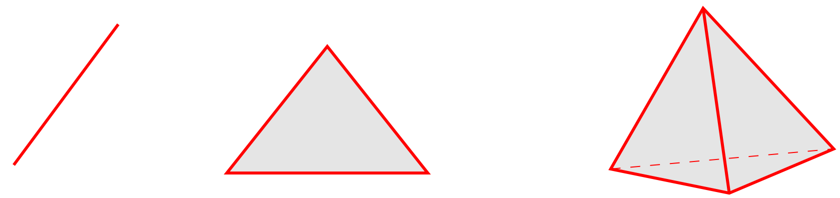
$$\sigma = \text{conv}(p_0, ..., p_k) = \{x \in \mathbb{R}^d,\ x = \sum_{i=0}^{k} \lambda_i\, p_i, \quad \lambda_i \in [0,1],\ \sum_{i=0}^{k} \lambda_i = 1\}$$

$k = \dim(\text{aff}(\sigma))$ is called the dimension of σ

1-simplex = line segment
2-simplex = triangle
3-simplex = tetrahedron



Faces of a simplex



$V(\sigma)$ = set of vertices of a k-simplex σ

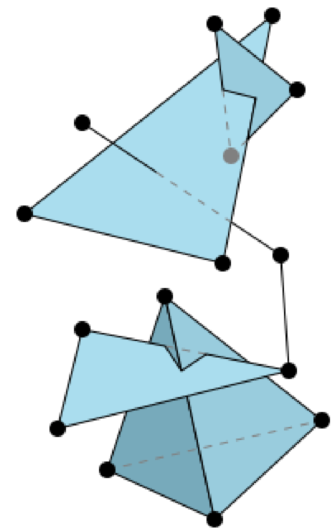
$\forall V' \subseteq V(\sigma)$, $\mathrm{conv}(V')$ is a face of σ

a k-simplex has $\begin{pmatrix} k+1 \\ i+1 \end{pmatrix}$ faces of dimension i

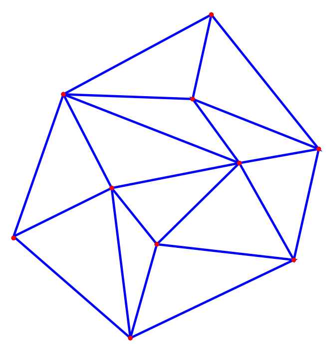
$2^{k+1} - 1$ faces in total

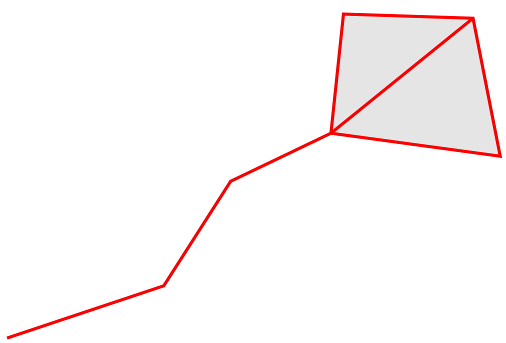
Geometric simplicial complexes

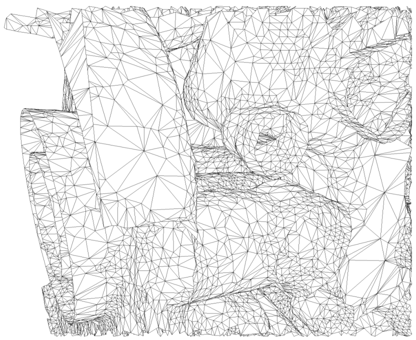
A finite collection of simplices K called the faces of K such that



- $\forall \sigma \in K$, σ is a simplex
- $\sigma \in K$, $\tau \subset \sigma \Rightarrow \tau \in K$
- $\forall \sigma, \tau \in K$, either $\sigma \cap \tau = \emptyset$ or $\sigma \cap \tau$ is a common face of both







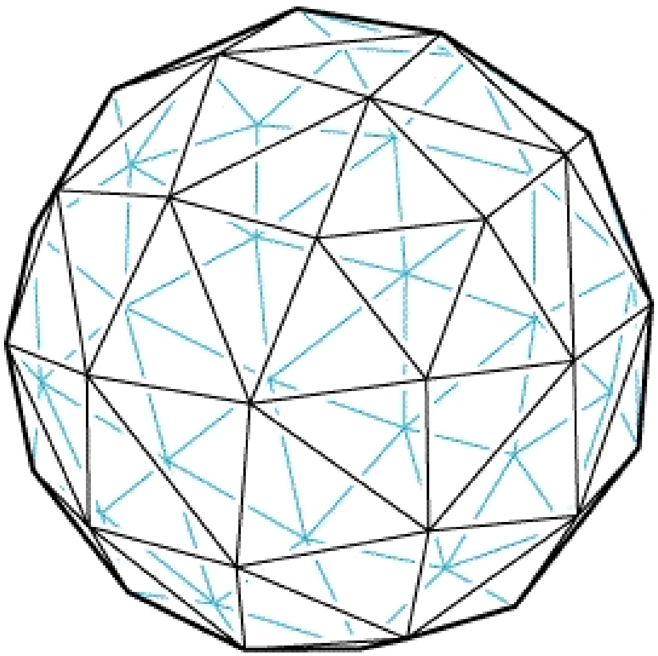
Geometric simplicial complexes

The dimension of a simplicial complex K is the max dimension of its simplices

A subset of K which is a complex is called a subcomplex of K

The underlying space $|K| \subset \mathbb{R}^d$ of K is the union of the simplices of K

An important example : the boundary complex of the convex hull of a finite set of points in general position



Polytope

$$\mathrm{conv}(P) = \{x \in \mathbb{R}^d,\ x = \sum_{i=0}^{k} \lambda_i\, p_i,\ \lambda_i \in [0,1],\ \sum_{i=0}^{k} \lambda_i = 1\}$$

Supporting hyperplane H :
$H \cap P \neq \emptyset$, P on one side of H

Faces : $\mathrm{conv}(P) \cap H$, H supp. hyp.

- P is in general position iff no subset of $k+2$ points lie in a k-flat
- If P is in general position, all faces of $\mathrm{conv}(P)$ are simplices

Abstract simplicial complexes

Given a finite set of elements P, an abstract simplicial complex K with vertex set P is a set of subsets of P s.t.

1. $\forall p \in P, \quad p \in K$
2. if $\sigma \in K$ and $\tau \subseteq \sigma$, then $\tau \in K$

The elements of K are called the (abstract) simplices or faces of K

The dimension of a simplex σ is $\dim(\sigma) = \sharp\mathrm{vert}(\sigma) - 1$

Realization of an abstract simplicial complex

- A realization of an abstract simplicial complex K is a geometric simplicial complex K_g whose corresponding abstract simplicial complex is isomorphic to K, i.e.

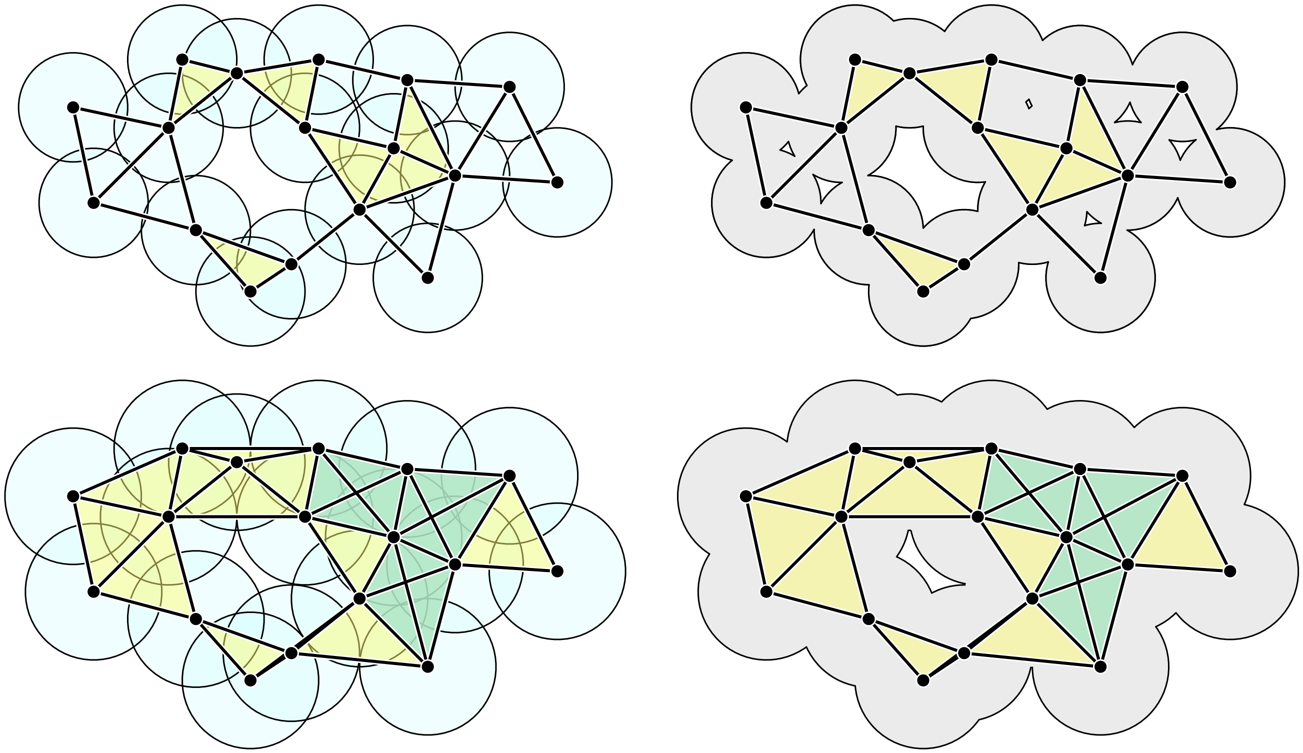
$$\exists \text{ bijective } f : \text{vert}(K) \to \text{vert}(K_g) \quad \text{s.t.} \quad \sigma \in K \quad \Rightarrow \quad f(\sigma) \in K_g$$

- Any abstract simplicial complex K can be realized in $\mathbb{R}^n$

  Hint : $v_i \to p_i = (0, ..., 0, 1, 0, ...0) \in \mathbb{R}^n$ $\qquad (n = \sharp\text{vert}(K))$
  $\sigma = \text{conv}(p_1, ..., p_n)$ $\qquad$ (canonical simplex)
  $K_g \subseteq \sigma$

- Realizations are not unique but are all topologically equivalent (homeomorphic)

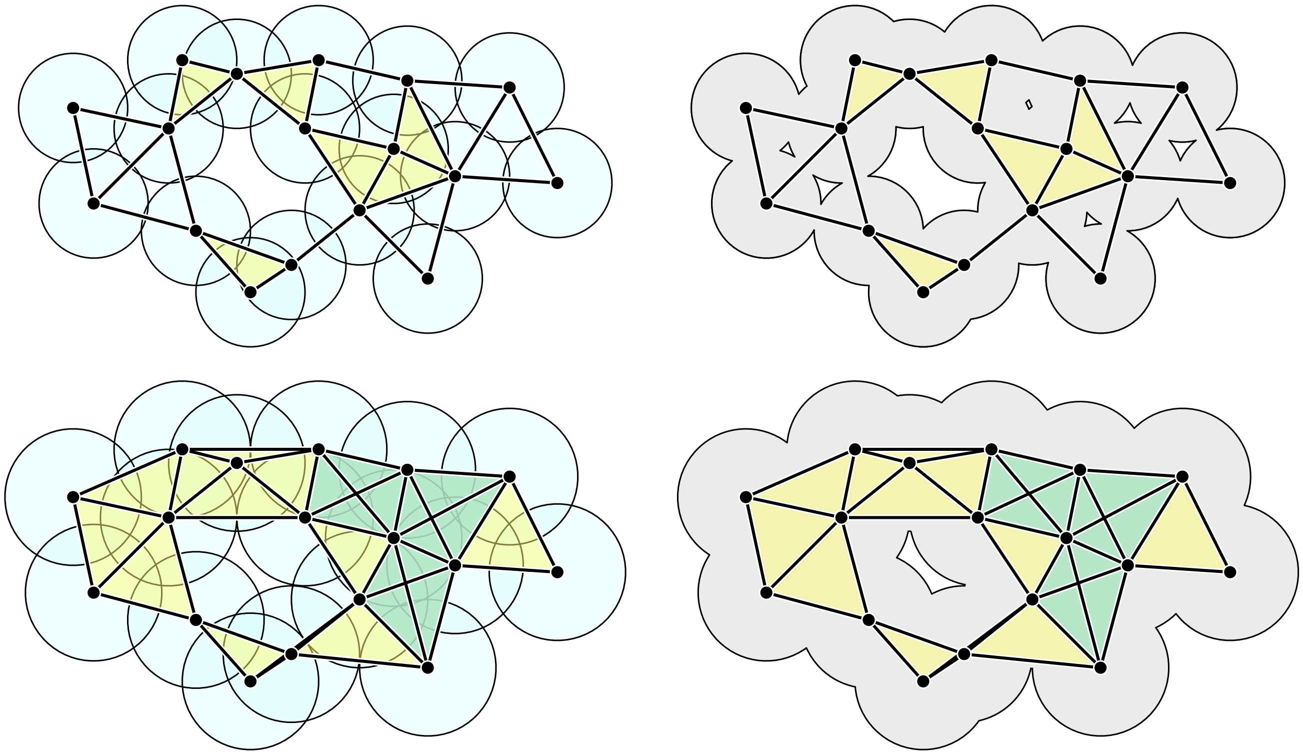
Nerve of a finite cover $\mathcal{U} = \{U_1, \ldots, U_n\}$ of X



The nerve of $\mathcal{U}$ is the simplicial complex $K(U)$ defined by

$$\sigma = [U_{i_0}, \ldots, U_{i_k}] \in K(U) \quad \Leftrightarrow \quad \cap_{i=1}^{k} U_{i_j} \neq \emptyset$$

Nerve of a cover

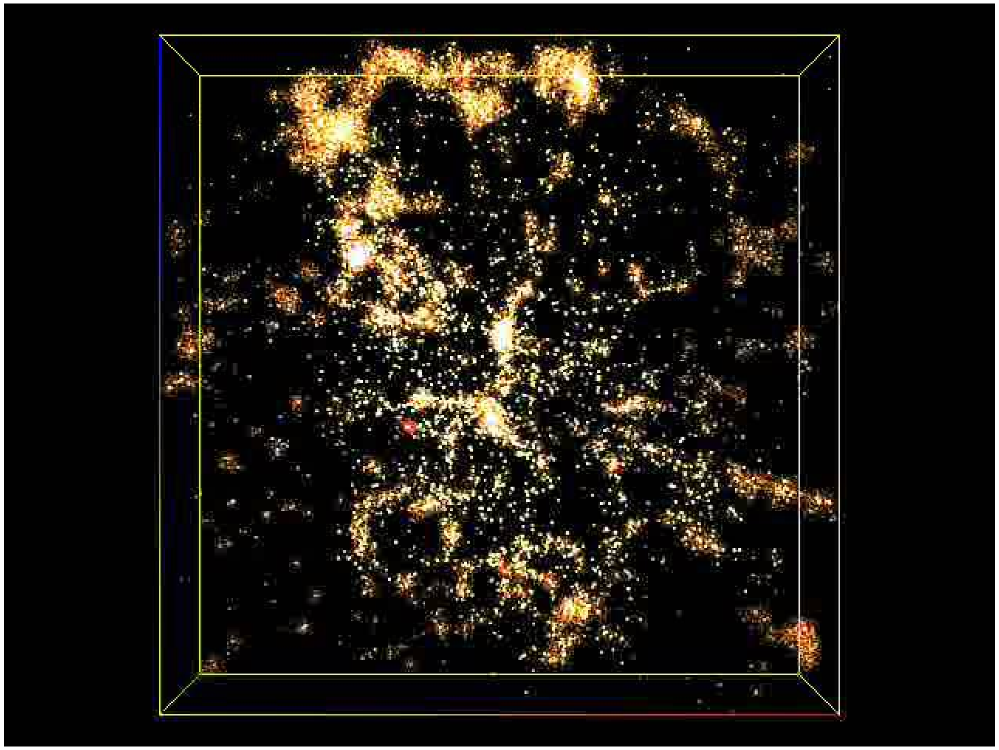


Nerve Theorem (Leray)

If any intersection of the U_i is either empty or contractible, then X and $K(U)$ have the same homotopy type

Geometric data analysis

Images, text, speech, neural signals, GPS traces,...



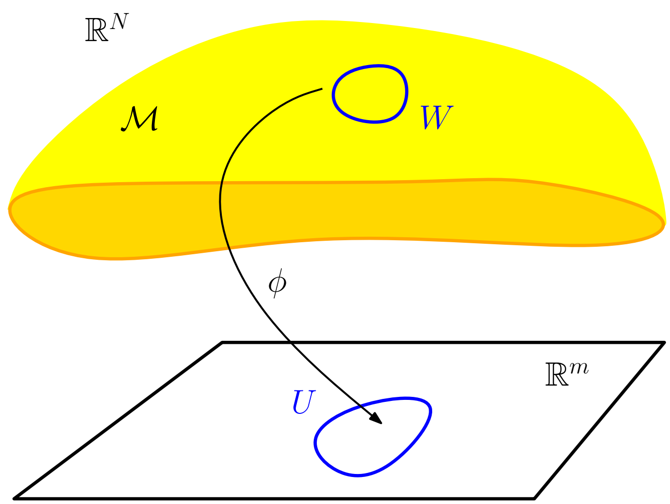
Geometrisation : Data = points + distances between points

Hypothesis : Data lie close to a structure of “small” intrinsic dimension

Problem : Infer the structure from the data

Submanifolds of $\mathbb{R}^d$

A compact subset $\mathbb{M} \subset \mathbb{R}^d$ is a submanifold without boundary of (intrinsic) dimension $k < d$, if any $p \in \mathbb{M}$ has an open (topological) k-ball as a neighborhood in $\mathbb{M}$



Intuitively, a submanifold of dimension k is a subset of $\mathbb{R}^d$ that looks locally like an open set of an affine space of dimension k

A curve a 1-dimensional submanifold
A surface is a 2-dimensional submanifold

Triangulation of a submanifold

We call triangulation of a submanifold $\mathbb{M} \subset \mathbb{R}^d$ a simplicial complex $\hat{\mathbb{M}}$ such that

- $\hat{\mathbb{M}}$ is embedded in $\mathbb{R}^d$
- its vertices are on $\mathbb{M}$
- it is homeomorphic to $\mathbb{M}$

Submanifold reconstruction

The problem is to construct a triangulation $\hat{\mathbb{M}}$ of some unknown submanifold $\mathbb{M}$ given a finite set of points $P \subset \mathbb{M}$

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Issues in high-dimensional geometry

- Dimensionality severely restricts our intuition and ability to visualize data
  $\Rightarrow$ need for automated and provably correct methods methods
- Complexity of data structures and algorithms rapidly grow as the dimensionality increases

  $\Rightarrow$ no subdivision of the ambient space is affordable

  $\Rightarrow$ data structures and algorithms should be sensitive to the intrinsic dimension (usually unknown) of the data
- Inherent defects : sparsity, noise, outliers

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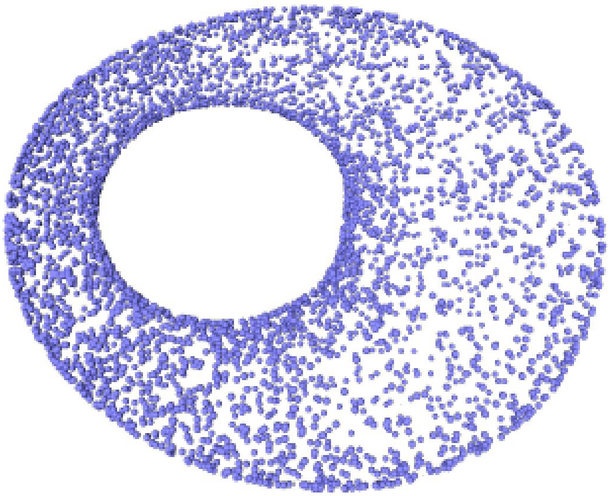
Looking for small and faithful simplicial complexes

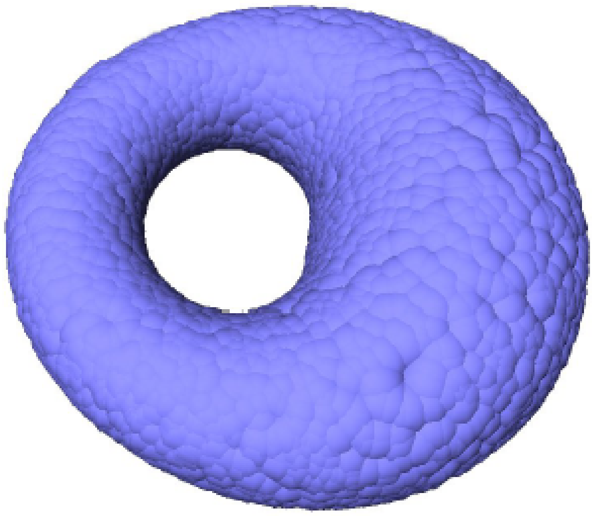
Need to compromise

- Size of the complex
  - can we have $\dim \hat{\mathbb{M}} = \dim \mathbb{M}$?
- Efficiency of the construction algorithms and of the representations
  - can we avoid the exponential dependence on d ?
  - can we minimize the number of simplices ?
- Quality of the approximation
  - Homotopy type & homology (Cech and α complexes, persistence)
  - Homeomorphism (Delaunay-type complexes)

Sampling and distance functions [Niyogi et al.], [Chazal et al.]

Distance to a compact K : $d_K : x \to \inf_{p \in K} \|x - p\|$





Stability

If the data points C are close (Hausdorff) to the geometric structure K, the topology and the geometry of the offsets $K_r = d^{-1}([0, r])$ and $C_r = d^{-1}([0, r])$ are close

Questions?

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