# Computing and Processing Correspondences with Functional Maps


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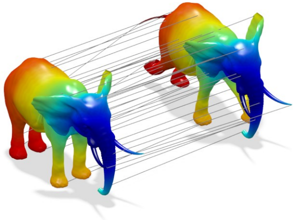
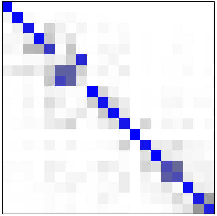
# Functional Maps by Simultaneous Diagonalization of Laplacians

# Choice of the basis





Functional correspondence matrix $\mathbf{C}$ expressed in the Laplacian eigenbases

Kovnatsky, Bronstein[2], Glashoff, Kimmel 2013

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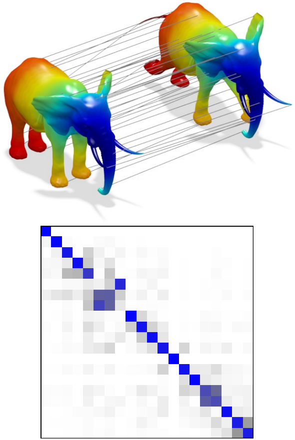

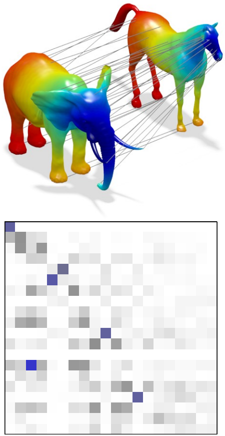

Functional correspondence matrix $\mathbf{C}$ expressed in the Laplacian eigenbases

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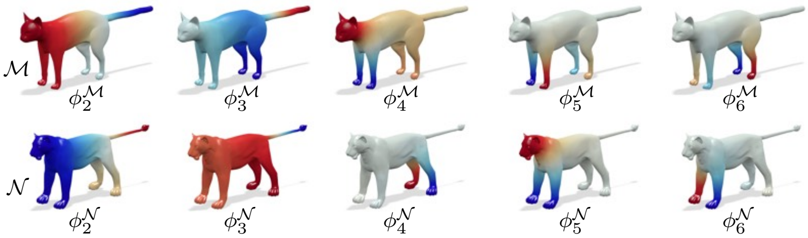
# Problem with Laplacian eigenbases

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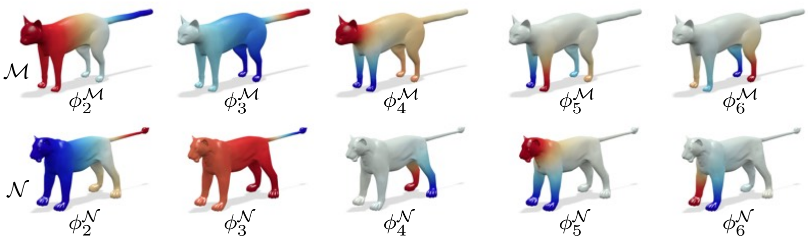


- Isometric manifolds with simple spectrum: sign ambiguity

$$T_F \phi_i^{\mathcal{M}} = \pm \phi_i^{\mathcal{N}}$$

Kovnatsky, Bronstein[2], Glashoff, Kimmel 2013

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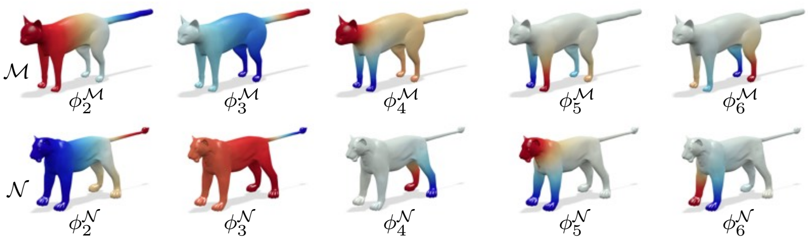


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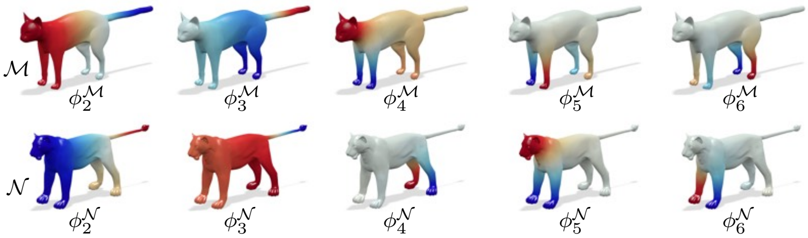


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- Non-isometric manifolds: eigenvectors can differ dramatically in order and form

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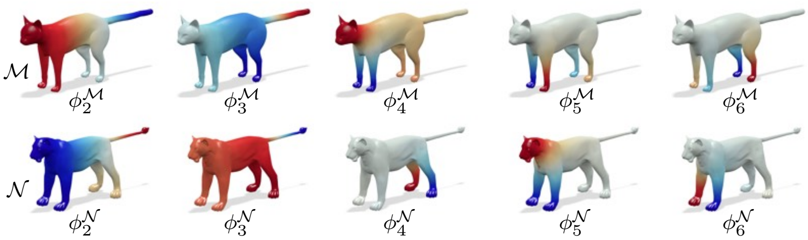


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- General spectrum: ambiguous rotation of eigenspace
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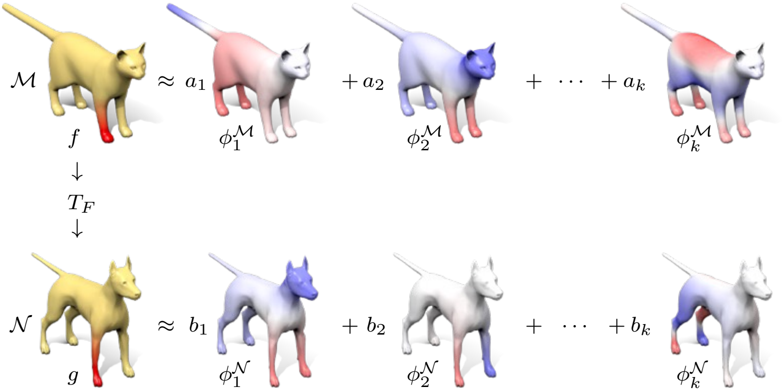
- Incompatibilities tend to increase with frequency

Kovnatsky, Bronstein$^2$, Glashoff, Kimmel 2013

# Coupled bases

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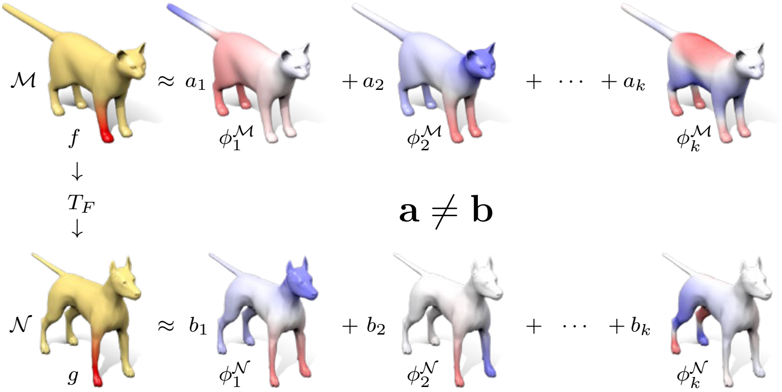


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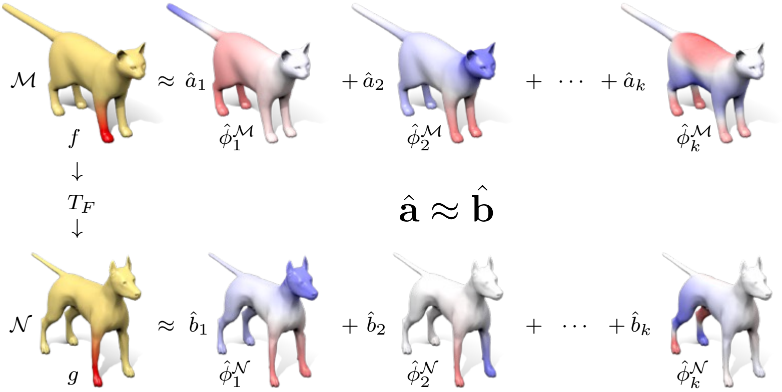


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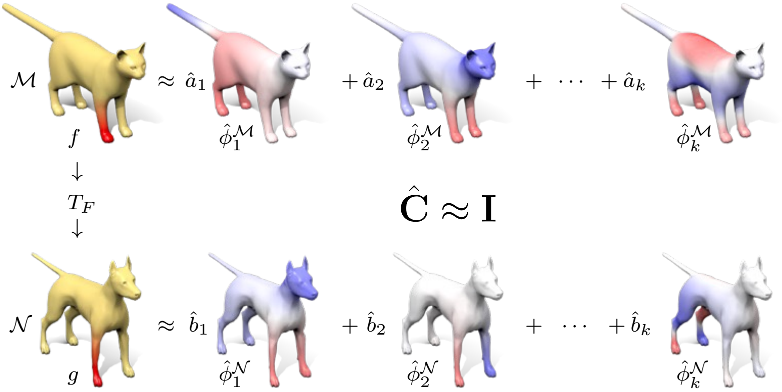


Kovnatsky, Bronstein$^2$, Glashoff, Kimmel 2013

# Coupled bases

Find a new pair of approximate orthonormal eigenbases

$$\hat{\phi}_i^{\mathcal{M}} = \sum_{j=1}^{k'} p_{ji}\phi_j^{\mathcal{M}} \qquad \hat{\phi}_i^{\mathcal{N}} = \sum_{j=1}^{k'} q_{ji}\phi_j^{\mathcal{N}} \qquad i = 1, \dots, k$$

parametrized by $k' \times k$ matrices $\mathbf{P} = (p_{ij})$ and $\mathbf{Q} = (q_{ij})$

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$$\delta_{ij} = \langle \hat{\phi}_i^{\mathcal{M}}, \hat{\phi}_j^{\mathcal{M}} \rangle_{L^2(\mathcal{M})} = \sum_{l,m=1}^{k'} p_{li}p_{mj}\langle \phi_l^{\mathcal{M}}, \phi_m^{\mathcal{M}} \rangle_{L^2(\mathcal{M})}$$

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- Approximate eigenbasis: approximately diagonalizes the Laplacian

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Kovnatsky, Bronstein[2], Glashoff, Kimmel 2013

# Joint diagonalization problem

$$\min_{\mathbf{P},\mathbf{Q}} \quad \mathrm{off}(\mathbf{P}^\top \mathbf{\Lambda}_{\mathcal{M},k'} \mathbf{P}) + \mathrm{off}(\mathbf{Q}^\top \mathbf{\Lambda}_{\mathcal{N},k'} \mathbf{Q}) + \mu \|\mathbf{P}^\top \mathbf{A} - \mathbf{Q}^\top \mathbf{B}\|$$
$$\text{s.t.} \quad \mathbf{P}^\top \mathbf{P} = \mathbf{I} \qquad \mathbf{Q}^\top \mathbf{Q} = \mathbf{I}$$

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Kovnatsky, Bronstein[2], Glashoff, Kimmel 2013

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Kovnatsky, Bronstein$^2$, Glashoff, Kimmel 2013

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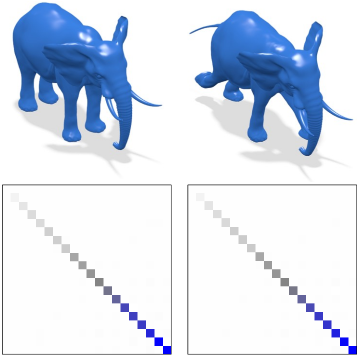
- If Frobenius norm is used and $k' = k$, due to rotation invariance $\mathbf{C} = \mathbf{Q}\mathbf{P}^\top$ is the functional correspondence matrix
- Robust norm $\|\mathbf{X}\|_{2,1} = \sum_j \|\mathbf{x}_j\|_2$ allows coping with outliers

Kovnatsky, Bronstein[2], Glashoff, Kimmel 2013

# Example of joint diagonalization



Isometric

Elements of $\mathbf{P}^\top \mathbf{\Lambda}_{\mathcal{M},k'} \mathbf{P}$ and $\mathbf{Q}^\top \mathbf{\Lambda}_{\mathcal{N},k'} \mathbf{Q}$

Kovnatsky, Bronstein[2], Glashoff, Kimmel 2013

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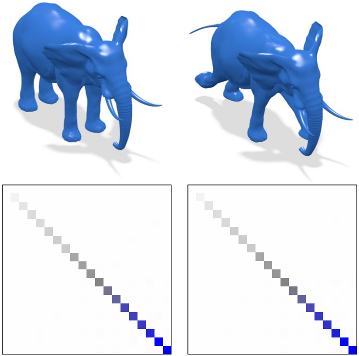

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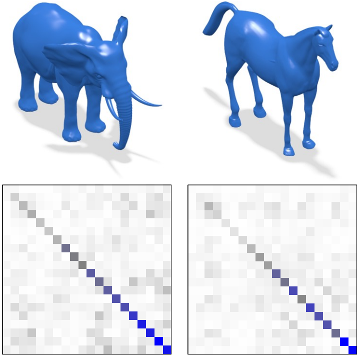

Non-isometric

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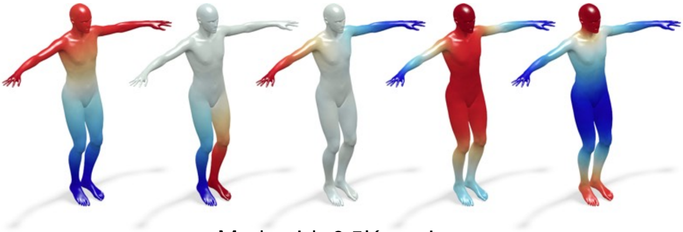

Mesh with 8.5K vertices

Mesh with 850 vertices

Kovnatsky, Bronstein$^2$, Glashoff, Kimmel 2013

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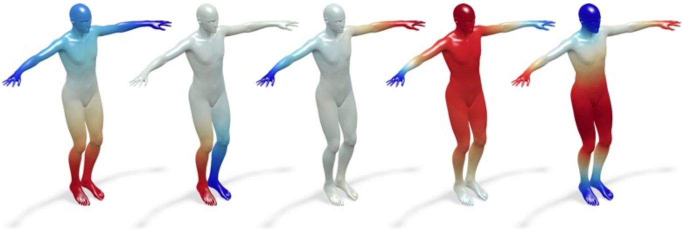

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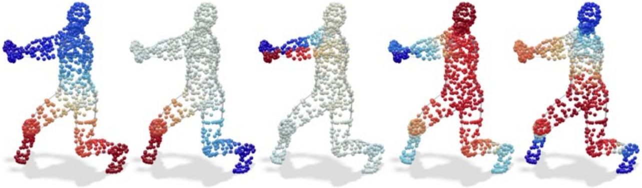
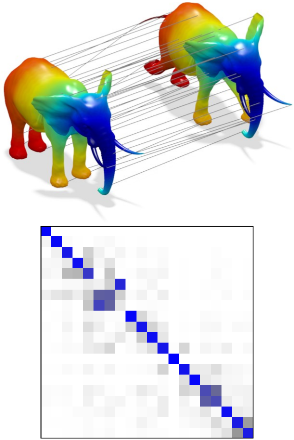
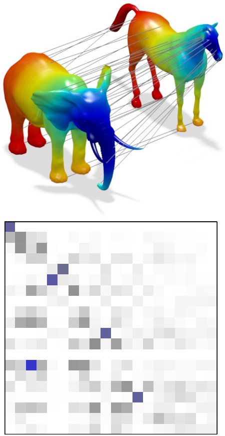


Point cloud with 850 vertices

Kovnatsky, Bronstein[2], Glashoff, Kimmel 2013

# Choice of the basis





Functional correspondence matrix $\mathbf{C}$ expressed in standard Laplacian eigenbases

Kovnatsky, Bronstein$^2$, Glashoff, Kimmel 2013

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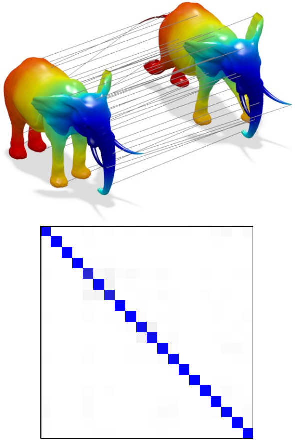
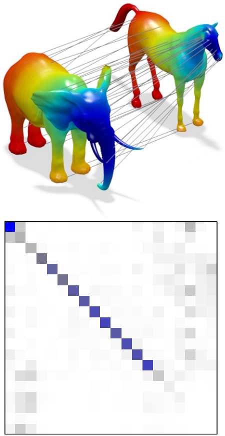
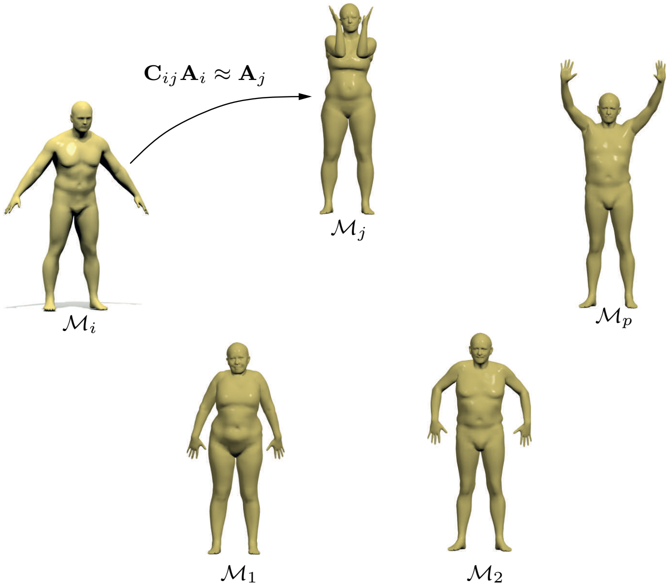




Functional correspondence matrix $\mathbf{C}$ expressed in
coupled approximate eigenbases

Kovnatsky, Bronstein[2], Glashoff, Kimmel 2013

# Multiple shapes




Kovnatsky, Bronstein[2], Glashoff, Kimmel 2013; Kovnatsky, Glashoff, Bronstein 2016

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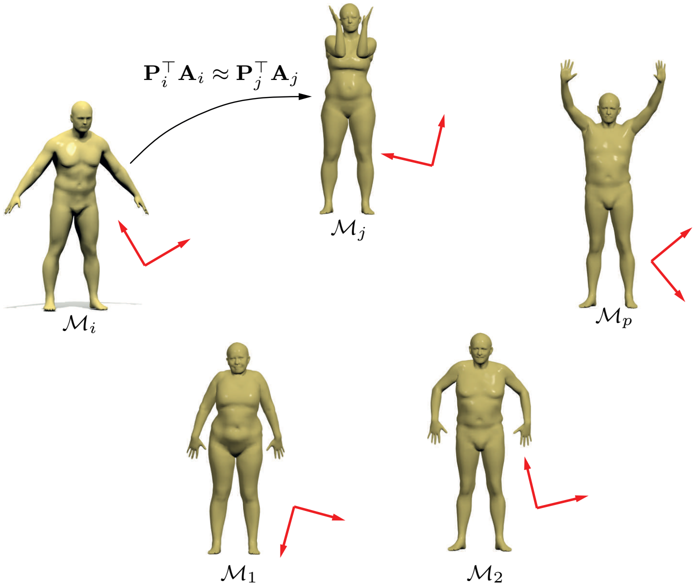


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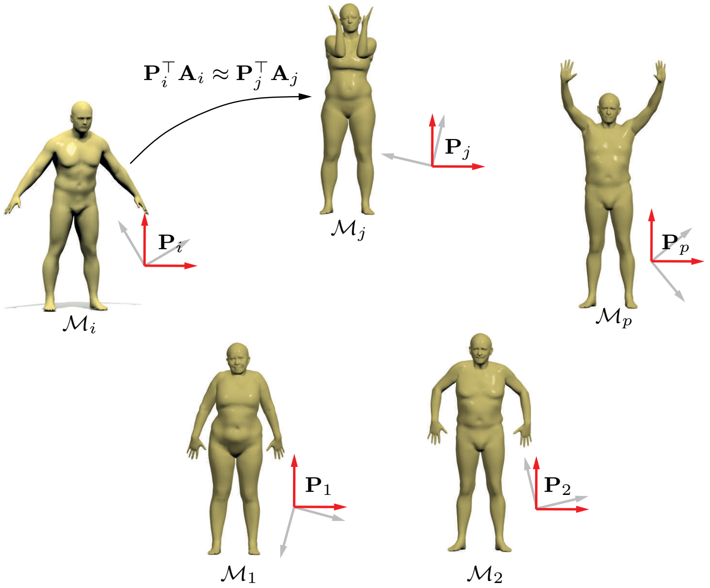



Kovnatsky, Bronstein$^2$, Glashoff, Kimmel 2013; Kovnatsky, Glashoff, Bronstein 2016

# Multiple shapes

$$\min_{\mathbf{P}_1,\ldots,\mathbf{P}_p} \quad \sum_{i=1}^{p} \mathrm{trace}(\mathbf{P}_i^\top \mathbf{\Lambda}_{\mathcal{M}_i} \mathbf{P}_i) + \mu \sum_{i \neq j} \|\mathbf{P}_i^\top \mathbf{A}_i - \mathbf{P}_j^\top \mathbf{A}_j\|$$
$$\text{s.t.} \quad \mathbf{P}_i^\top \mathbf{P}_i = \mathbf{I}$$

- 'Synchronization problem'
- Matrices $\mathbf{P}_1, \ldots, \mathbf{P}_p$ orthogonally align the $p$ eigenbases

Kovnatsky, Bronstein$^2$, Glashoff, Kimmel 2013; Kovnatsky, Glashoff, Bronstein 2016

# Computing Functional Maps with Manifold Optimization

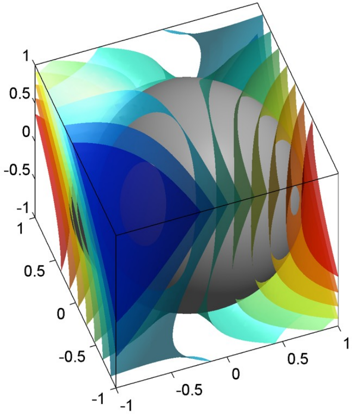
$$\min_{\mathbf{P}} \ \mathrm{trace}(\mathbf{P}^\top \mathbf{\Lambda} \mathbf{P}) + \mu \|\mathbf{P}\mathbf{A} - \mathbf{B}\| \quad \text{s.t.} \quad \mathbf{P}^\top \mathbf{P} = \mathbf{I}$$

$$\min_{\mathbf{P}} \ \mathrm{trace}(\mathbf{P}^\top \mathbf{\Lambda} \mathbf{P}) + \mu \|\mathbf{P}\mathbf{A} - \mathbf{B}\| \quad \text{s.t.} \quad \mathbf{P}^\top \mathbf{P} = \mathbf{I}$$

Optimization on the Stiefel manifold
of orthogonal matrices

# Manifold optimization toy example: eigenvalue problem

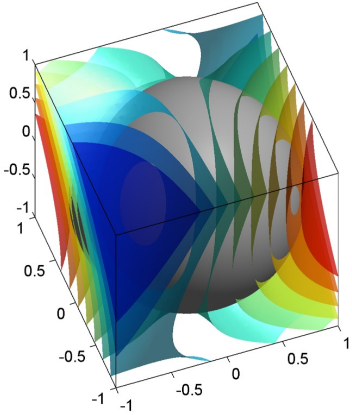
$$\min_{\mathbf{x}\in\mathbb{R}^3} \mathbf{x}^\top \mathbf{A}\mathbf{x} \quad \text{s.t.} \quad \mathbf{x}^\top\mathbf{x} = 1$$

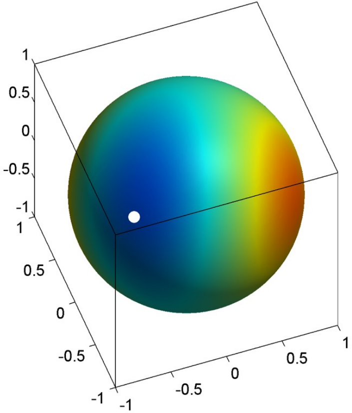



Minimization of a quadratic function on the sphere

# Manifold optimization toy example: eigenvalue problem

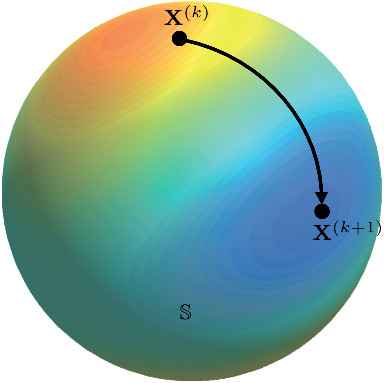
$$\min_{\mathbf{x}\in\mathbb{S}(3,1)} \mathbf{x}^\top \mathbf{A}\mathbf{x}$$



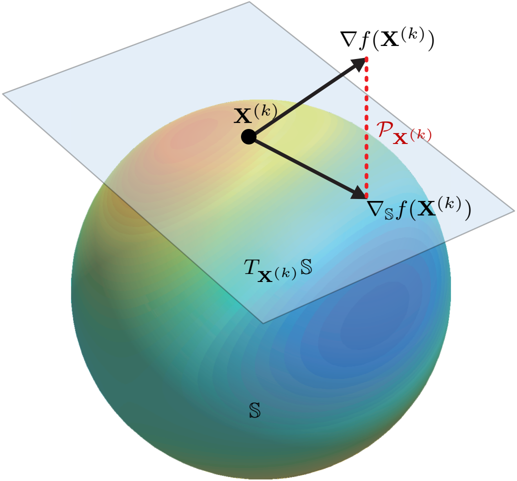

Minimization of a quadratic function on the sphere

# Optimization on the manifold: main idea




Absil et al. 2009

# Optimization on the manifold: main idea




Absil et al. 2009

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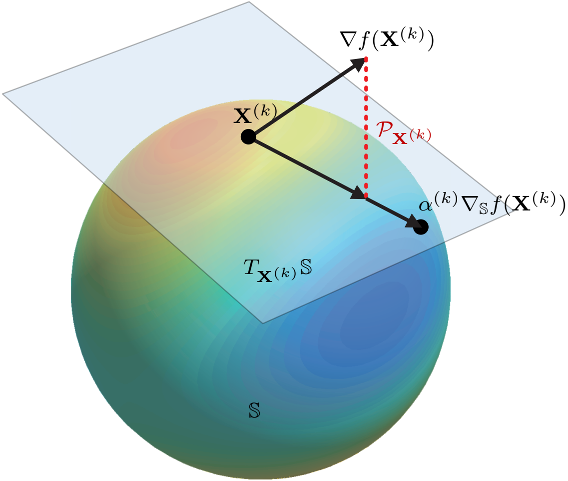
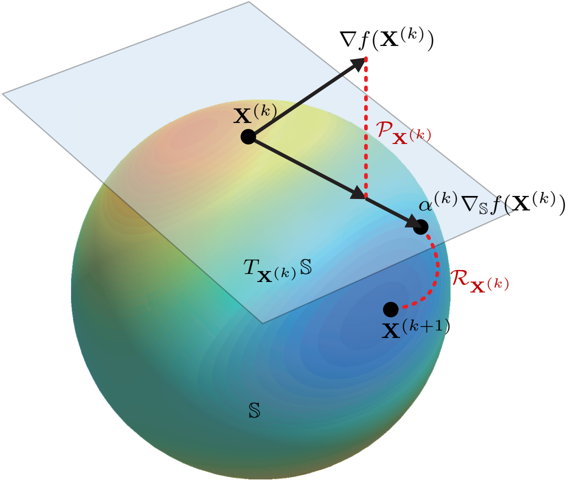



Absil et al. 2009

# Optimization on the manifold: main idea




Absil et al. 2009

# Optimization on the manifold

**repeat**

Compute extrinsic gradient $\nabla f(\mathbf{X}^{(k)})$

Projection: $\nabla_{\mathbb{S}} f(\mathbf{X}^{(k)}) = \mathcal{P}_{\mathbf{X}^{(k)}}(\nabla f(\mathbf{X}^{(k)}))$

Compute step size $\alpha^{(k)}$ along the descent direction $-\nabla_{\mathbb{S}} f(\mathbf{X}^{(k)})$

Retraction: $\mathbf{X}^{(k+1)} = \mathcal{R}_{\mathbf{X}^{(k)}}(-\alpha^{(k)} \nabla_{\mathbb{S}} f(\mathbf{X}^{(k)}))$

$k \leftarrow k+1$

**until** *convergence*;

Absil et al. 2009; Boumal et al. 2014

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- Projection $\mathcal{P}$ and retraction $\mathcal{R}$ operators are manifold-dependent

Absil et al. 2009; Boumal et al. 2014

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- Projection $\mathcal{P}$ and retraction $\mathcal{R}$ operators are manifold-dependent
- Typically expressed in closed form

Absil et al. 2009; Boumal et al. 2014

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$k \leftarrow k + 1$

**until** *convergence*;

- Projection $\mathcal{P}$ and retraction $\mathcal{R}$ operators are manifold-dependent
- Typically expressed in closed form
- "Black box": need to provide only $f(\mathbf{X})$ and gradient $\nabla f(\mathbf{X})$

Absil et al. 2009; Boumal et al. 2014

$$\min_{\mathbf{P}} \ \ \mathrm{trace}(\mathbf{P}^\top \mathbf{\Lambda} \mathbf{P}) + \mu \|\mathbf{PA} - \mathbf{B}\|_2^2 \quad \text{s.t.} \quad \mathbf{P}^\top \mathbf{P} = \mathbf{I}$$

Optimization on the Stiefel manifold

$$\min_{\mathbf{P}} \quad \underbrace{\mathrm{trace}(\mathbf{P}^\top \mathbf{\Lambda} \mathbf{P})}_{\text{smooth}} + \underbrace{\mu \|\mathbf{P}\mathbf{A} - \mathbf{B}\|_{2,1}}_{\text{non-smooth}} \ \text{s.t.} \quad \mathbf{P}^\top \mathbf{P} = \mathbf{I}$$

Non-smooth optimization on the Stiefel manifold

# Manifold ADMM (MADMM)

$$\min_{\mathbf{X}\in\mathbb{S}(n,k)} \quad \underbrace{f(\mathbf{X})}_{\text{smooth}} + \underbrace{g(\mathbf{X})}_{\text{non-smooth}}$$

Hestenes 1969; Powell 1969; Kovnatsky, Glashoff, Bronstein 2016

# Manifold ADMM (MADMM)

$$\min_{\substack{\mathbf{X}\in\mathbb{S}(n,k)\\ \mathbf{Z}\in\mathbb{R}^{n\times k}}} \quad \underbrace{f(\mathbf{X})}_{\text{smooth}} + \underbrace{g(\mathbf{Z})}_{\text{non-smooth}} \qquad \text{s.t.} \quad \mathbf{Z} = \mathbf{X}$$

Hestenes 1969; Powell 1969; Kovnatsky, Glashoff, Bronstein 2016

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Apply the method of multipliers only to the constraint $\mathbf{Z} = \mathbf{X}$

$$\min_{\substack{\mathbf{X}\in\mathbb{S}(n,k)\\ \mathbf{Z}\in\mathbb{R}^{n\times k}}} f(\mathbf{X}) + g(\mathbf{Z}) + \tfrac{\rho}{2}\|\mathbf{X} - \mathbf{Z} + \mathbf{U}\|_{\mathrm{F}}^2$$

Solve alternating w.r.t. $\mathbf{X}$ and $\mathbf{Z}$ and updating $\mathbf{U} \leftarrow \mathbf{U} + \mathbf{X} - \mathbf{Z}$

Hestenes 1969; Powell 1969; Kovnatsky, Glashoff, Bronstein 2016

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Problem breaks into

- Smooth manifold optimization sub-problem w.r.t. $\mathbf{X}$, and
- Non-smooth unconstrained sub-problem w.r.t. $\mathbf{Z}$

Hestenes 1969; Powell 1969; Kovnatsky, Glashoff, Bronstein 2016

# Manifold ADMM (MADMM)

Initialize $k \leftarrow 1$, $\mathbf{Z}^{(1)} = \mathbf{X}^{(1)}$, $\mathbf{U}^{(1)} = 0$.

**repeat**

$\mathbf{X}$-step: $\mathbf{X}^{(k+1)} = \underset{\mathbf{X} \in \mathbb{S}}{\operatorname{argmin}} f(\mathbf{X}) + \frac{\rho}{2} \|\mathbf{X} - \mathbf{Z}^{(k)} + \mathbf{U}^{(k)}\|_{\mathrm{F}}^2$

$\mathbf{Z}$-step: $\mathbf{Z}^{(k+1)} = \underset{\mathbf{Z}}{\operatorname{argmin}} g(\mathbf{Z}) + \frac{\rho}{2} \|\mathbf{X}^{(k+1)} - \mathbf{Z} + \mathbf{U}^{(k)}\|_{\mathrm{F}}^2$

Update $\mathbf{U}^{(k+1)} = \mathbf{U}^{(k)} + \mathbf{X}^{(k+1)} - \mathbf{Z}^{(k+1)}$

$k \leftarrow k + 1$

**until** *convergence*;

Kovnatsky, Glashoff, Bronstein 2016

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- Solver/number of optimization iterations in $\mathbf{X}$- and $\mathbf{Z}$-steps

Kovnatsky, Glashoff, Bronstein 2016

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- Solver/number of optimization iterations in $\mathbf{X}$- and $\mathbf{Z}$-steps
- $\mathbf{X}$-step and $\mathbf{X}$-step in some problems have a closed form

Kovnatsky, Glashoff, Bronstein 2016

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Update $\mathbf{U}^{(k+1)} = \mathbf{U}^{(k)} + \mathbf{X}^{(k+1)} - \mathbf{Z}^{(k+1)}$

$k \leftarrow k + 1$

**until** *convergence*;

- Solver/number of optimization iterations in $\mathbf{X}$- and $\mathbf{Z}$-steps
- $\mathbf{X}$-step and $\mathbf{X}$-step in some problems have a closed form
- Parameter $\rho > 0$ can be chosen fixed or adapted

Kovnatsky, Glashoff, Bronstein 2016

# $L_2$ vs $L_{2,1}$ data term

Least squares

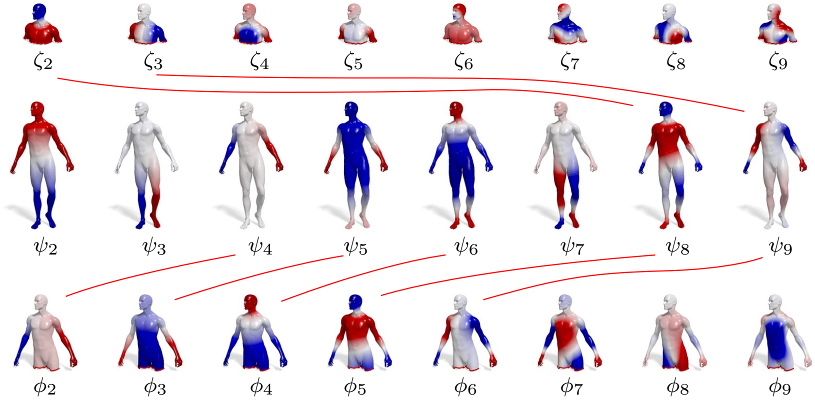
Robust (MADMM)

Correspondence computed with data containing 10% outliers

Kovnatsky, Glashoff, Bronstein 2016

# Partial Functional Maps

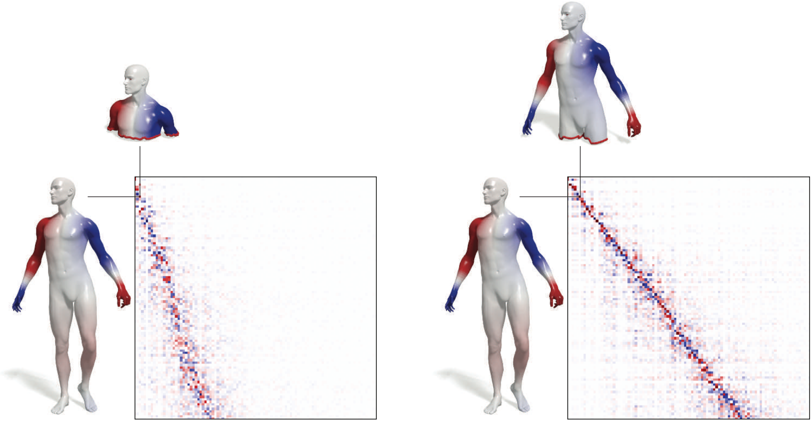
# Partial Laplacian eigenvectors




Laplacian eigenvectors of a shape with missing parts
(Neumann boundary conditions)

Rodolà, Cosmo, Bronstein, Torsello, Cremers 2016

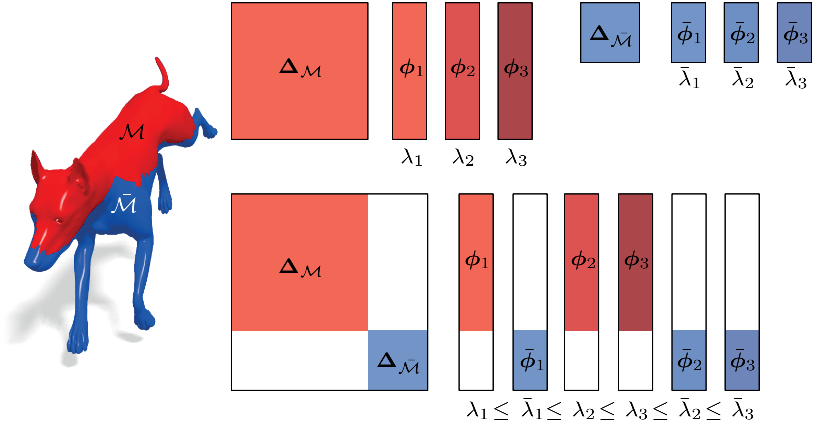
# Partial Laplacian eigenvectors



Functional correspondence matrix $\mathbf{C}$

Rodolà, Cosmo, Bronstein, Torsello, Cremers 2016

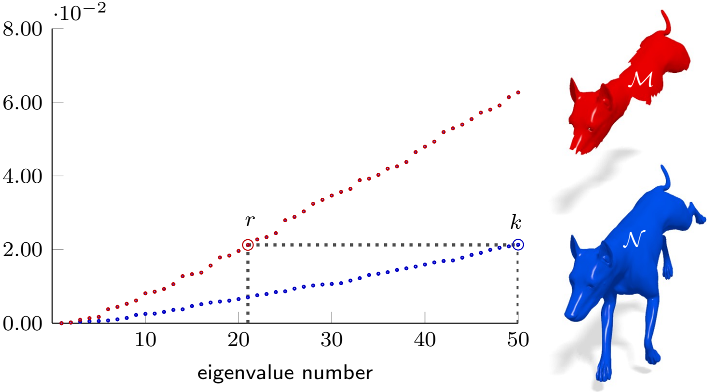
# Perturbation analysis: intuition




- Ignoring boundary interaction: disjoint parts (block-diagonal matrix)
- Eigenvectors = Mixture of eigenvectors of the parts

Rodolà, Cosmo, Bronstein, Torsello, Cremers 2016

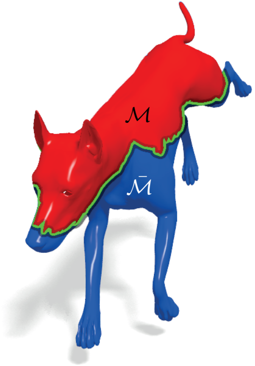
# Perturbation analysis: eigenvalues

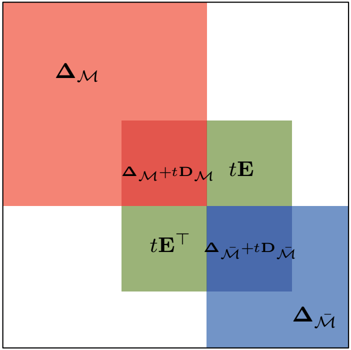



- Slope $\frac{r}{k} \approx \frac{|\mathcal{M}|}{|\mathcal{N}|}$ (depends on the area of the cut)
- Consistent with Weyl's law

Rodolà, Cosmo, Bronstein, Torsello, Cremers 2016

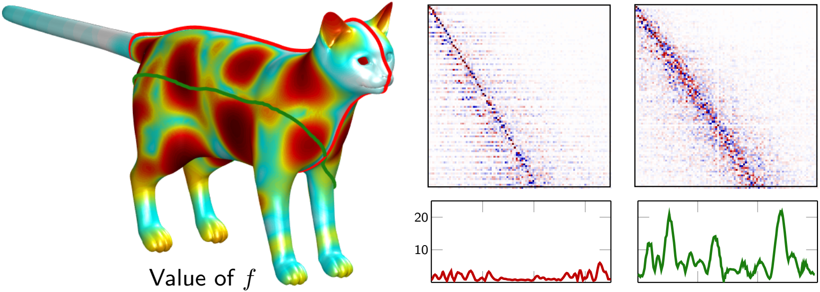
# Perturbation analysis: details




Rodolà, Cosmo, Bronstein, Torsello, Cremers 2016

# Perturbation analysis: boundary interaction strength




- Eigenvector perturbation depends on length and position of the boundary
- Perturbation strength $\leq c \int_{\partial\mathcal{M}} f(m)dm$, where

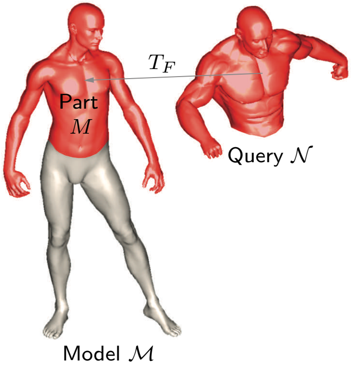
$$f(m) = \sum_{\substack{i,j=1 \\ j \neq i}}^{n} \left( \frac{\phi_i(m)\phi_j(m)}{\lambda_i - \lambda_j} \right)^2$$

Rodolà, Cosmo, Bronstein, Torsello, Cremers 2016

# Partial functional maps

- Model shape $\mathcal{M}$
- Query shape $\mathcal{N}$
- Part $M \subseteq \mathcal{M} \approx$ isometric to $\mathcal{N}$
- Data $f_1, \ldots, f_q \in L^2(\mathcal{N})$
  $g_1, \ldots, g_q \in L^2(\mathcal{M})$
- Partial functional map

$$(T_F f_i)(m) \quad \approx \quad g_i(m), \quad m \in M$$




Rodolà, Cosmo, Bronstein, Torsello, Cremers 2016

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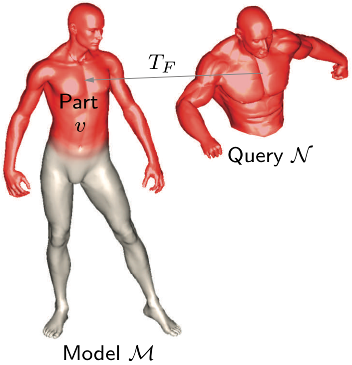
- Partial functional map

$$T_F f_i \quad \approx \quad g_i \cdot v, \quad v : \mathcal{M} \to [0, 1]$$




Rodolà, Cosmo, Bronstein, Torsello, Cremers 2016

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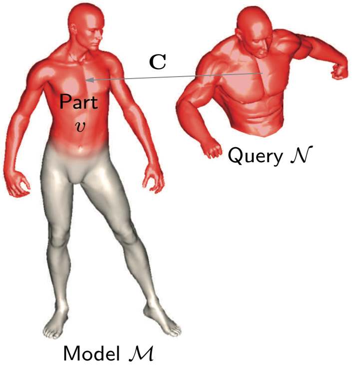
- Partial functional map

$$
\begin{aligned}
\mathbf{CA} &\approx \mathbf{B}(v), \quad v : \mathcal{M} \to [0,1] \\
\mathbf{A} &= \left( \langle \phi_i^{\mathcal{N}}, f_j \rangle_{L^2(\mathcal{N})} \right) \\
\mathbf{B}(v) &= \left( \langle \phi_i^{\mathcal{M}}, g_j \cdot v \rangle_{L^2(\mathcal{M})} \right)
\end{aligned}
$$




Rodolà, Cosmo, Bronstein, Torsello, Cremers 2016

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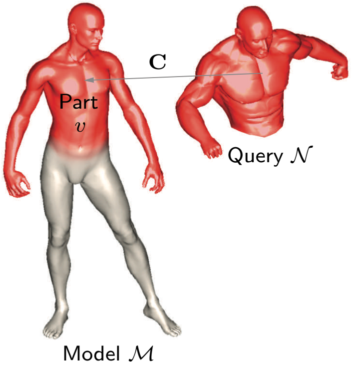


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\end{aligned}
$$

**Optimization problem w.r.t. correspondence $\mathbf{C}$ and part $v$**

$$\min_{\mathbf{C}, v} \|\mathbf{CA} - \mathbf{B}(v)\|_{2,1} + \rho_{\text{corr}}(\mathbf{C}) + \rho_{\text{part}}(v)$$

Rodolà, Cosmo, Bronstein, Torsello, Cremers 2016

# Partial functional maps

$$\min_{\mathbf{C},v} \|\mathbf{CA} - \mathbf{B}(v)\|_{2,1} + \rho_{\mathrm{corr}}(\mathbf{C}) + \rho_{\mathrm{part}}(v)$$

Rodolà, Cosmo, Bronstein, Torsello, Cremers 2016

# Partial functional maps

$$\min_{\mathbf{C},v} \|\mathbf{CA} - \mathbf{B}(v)\|_{2,1} + \rho_{\text{corr}}(\mathbf{C}) + \rho_{\text{part}}(v)$$

- **Part regularization**
  - Area preservation $\int_{\mathcal{M}} v(m)dx \approx |\mathcal{N}|$
  - Spatial regularity $=$ small boundary length (Mumford-Shah)

Rodolà, Cosmo, Bronstein, Torsello, Cremers 2016; Bronstein$^2$ 2008

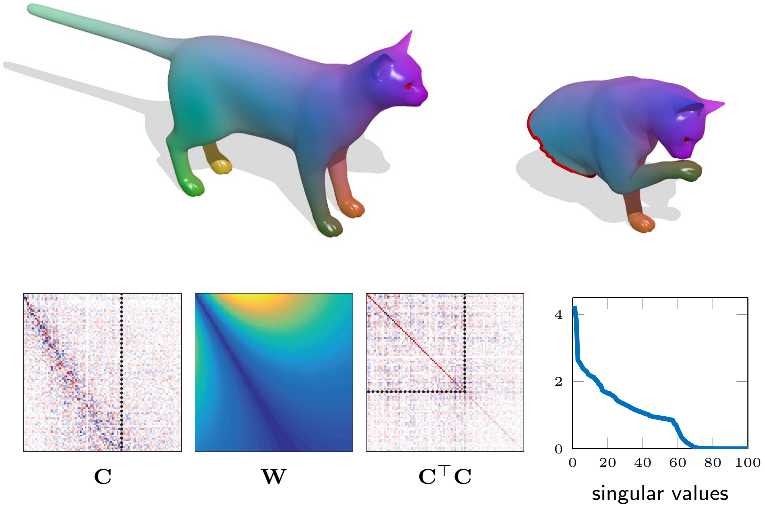
# Partial functional maps

$$\min_{\mathbf{C},v} \|\mathbf{CA} - \mathbf{B}(v)\|_{2,1} + \rho_{\mathrm{corr}}(\mathbf{C}) + \rho_{\mathrm{part}}(v)$$

- **Part regularization**
  - Area preservation $\int_{\mathcal{M}} v(m)dx \approx |\mathcal{N}|$
  - Spatial regularity $=$ small boundary length (Mumford-Shah)
- **Correspondence regularization**
  - Slanted diagonal structure
  - Approximate ortho-projection $(\mathbf{C}^\top \mathbf{C})_{i \neq j} \approx 0$
  - $\mathrm{rank}(\mathbf{C}) \approx r$

Rodolà, Cosmo, Bronstein, Torsello, Cremers 2016; Bronstein$^2$ 2008

# Structure of partial functional correspondence




Rodolà, Cosmo, Bronstein, Torsello, Cremers 2016

# Alternating minimization

- $\mathbf{C}$-**step:** fix $v^*$, solve for correspondence $\mathbf{C}$

$$\min_{\mathbf{C}} \|\mathbf{CA} - \mathbf{B}(v^*)\|_{2,1} + \rho_{\text{corr}}(\mathbf{C})$$

- $v$-**step:** fix $\mathbf{C}^*$, solve for part $v$

$$\min_{v} \|\mathbf{C}^*\mathbf{A} - \mathbf{B}(v)\|_{2,1} + \rho_{\text{part}}(v)$$

Rodolà, Cosmo, Bronstein, Torsello, Cremers 2016

# Alternating minimization

- **$\mathbf{C}$-step:** fix $v^*$, solve for correspondence $\mathbf{C}$

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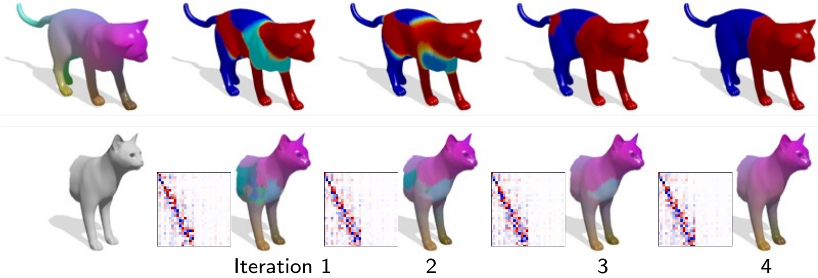
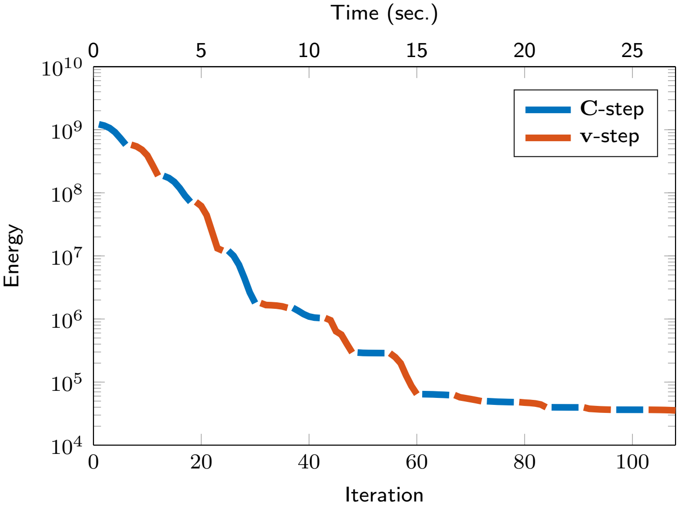



Rodolà, Cosmo, Bronstein, Torsello, Cremers 2016

# Example of convergence

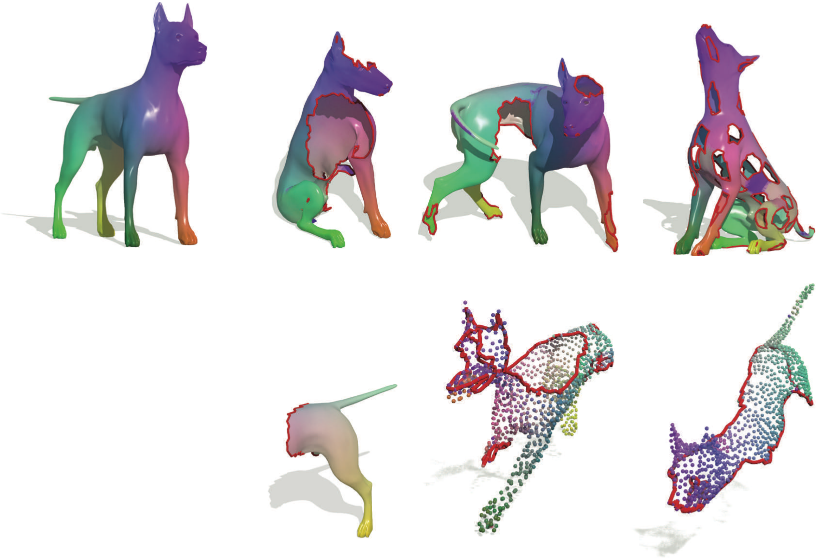



Rodolà, Cosmo, Bronstein, Torsello, Cremers 2016

# Examples of partial functional maps

Rodolà, Cosmo, Bronstein, Torsello, Cremers 2016

# Examples of partial functional maps

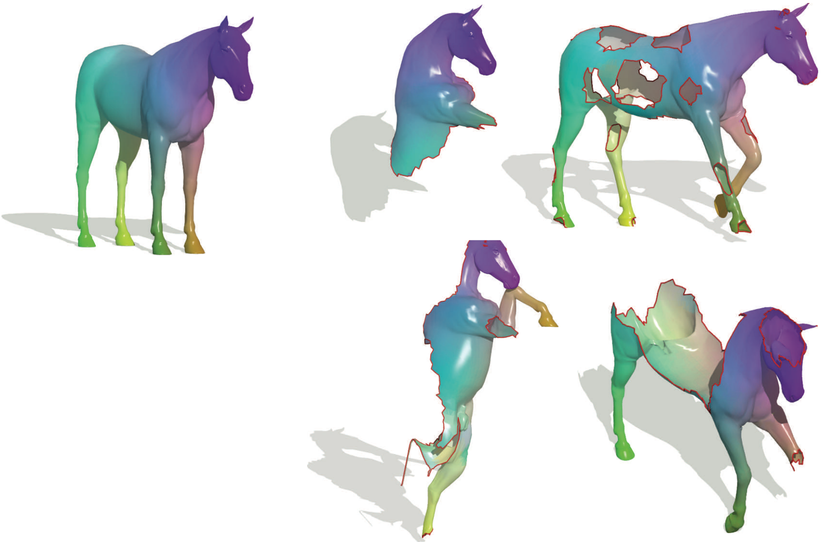


Rodolà, Cosmo, Bronstein, Torsello, Cremers 2016

# Examples of partial functional maps

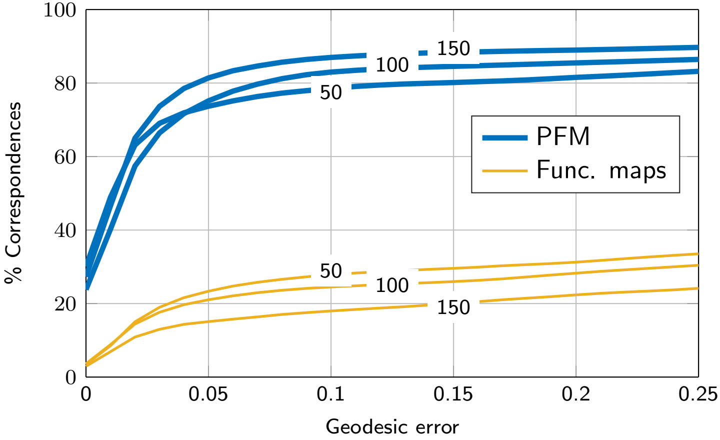
Rodolà, Cosmo, Bronstein, Torsello, Cremers 2016

# Examples of partial functional maps



Rodolà, Cosmo, Bronstein, Torsello, Cremers 2016

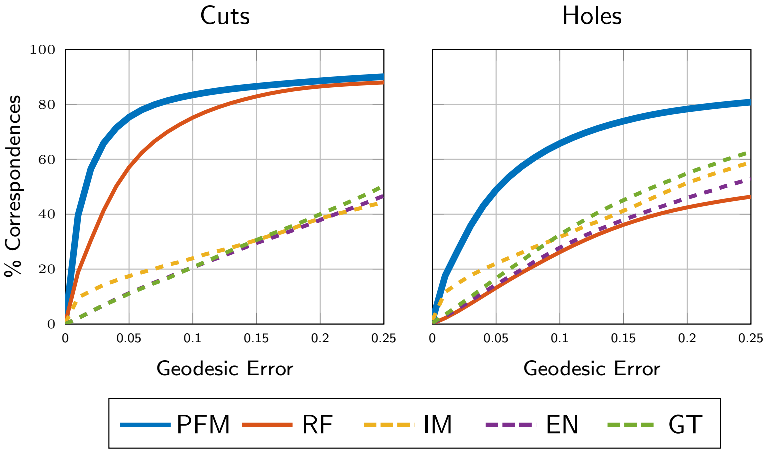
# Partial functional maps vs Functional maps




Correspondence performance for different basis size $k$

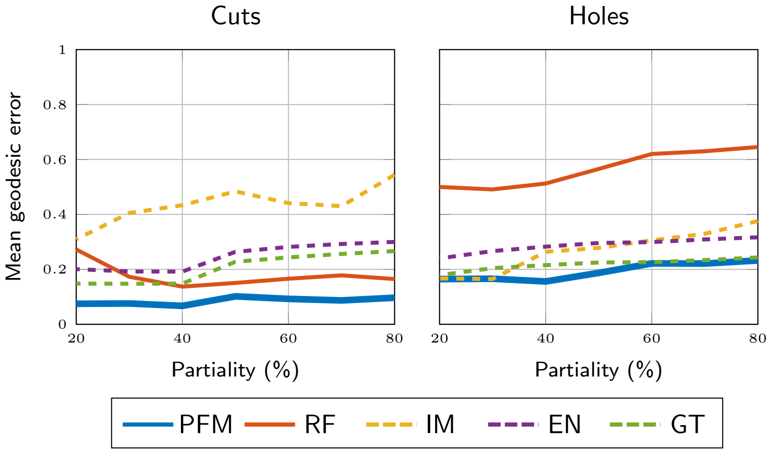
Rodolà, Cosmo, Bronstein, Torsello, Cremers 2016

# Partial correspondence performance




SHREC'16 Partial Matching benchmark Rodolà et al. 2016; Methods: Rodolà, Cosmo, Bronstein, Torsello, Cremers 2016 (**PFM**); Sahillioğlu, Yemez 2012 (IM); Rodolà, Bronstein, Albarelli, Bergamasco, Torsello 2012 (GT); Rodolà et al. 2013 (EN); Rodolà et al. 2014 (RF)

# Partial correspondence performance




SHREC'16 Partial Matching benchmark Rodolà et al. 2016; Methods: Rodolà, Cosmo, Bronstein, Torsello, Cremers 2016 (**PFM**); Sahillioğlu, Yemez 2012 (IM); Rodolà, Bronstein, Albarelli, Bergamasco, Torsello 2012 (GT); Rodolà et al. 2013 (EN); Rodolà et al. 2014 (RF)

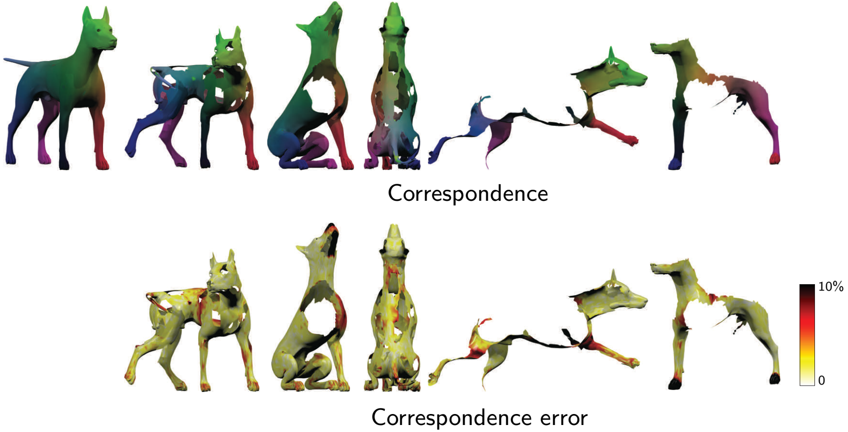
# Geometric deep learning + Partial functional maps

Correspondence


Correspondence error

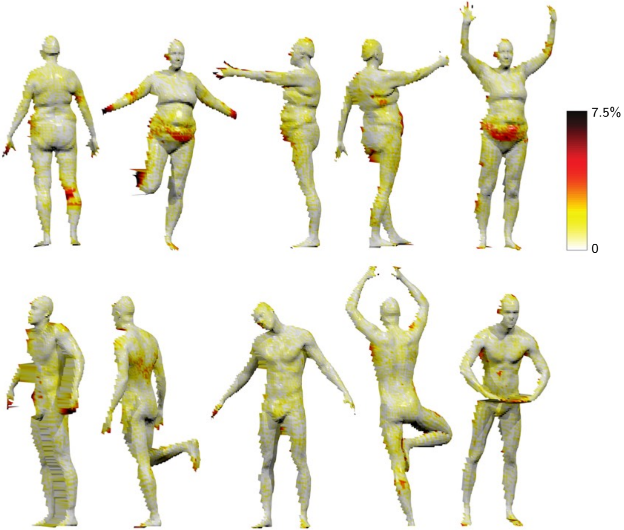
Boscaini, Masci, Rodolà, Bronstein 2016

# Geometric deep learning + Partial functional maps




Boscaini, Masci, Rodolà, Bronstein 2016

# Geometric deep learning + Partial functional maps




Pointwise geodesic error (in % of geodesic diameter)

Monti, Boscaini, Masci, Rodolà, Svoboda, Bronstein 2016

# Geometric deep learning + Partial functional maps


Correspondence visualization (similar colors encode corresponding points)
Training: FAUST / Testing: FAUST

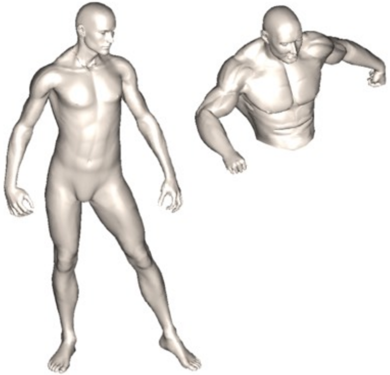
Monti, Boscaini, Masci, Rodolà, Svoboda, Bronstein 2016

# Geometric deep learning + Partial functional maps


Correspondence visualization (similar colors encode corresponding points)
Training: FAUST / Testing: SCAPE+TOSCA

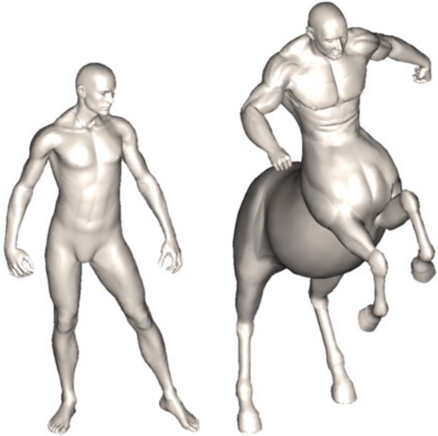
Monti, Boscaini, Masci, Rodolà, Svoboda, Bronstein 2016

# Partial correspondence (part-to-full)



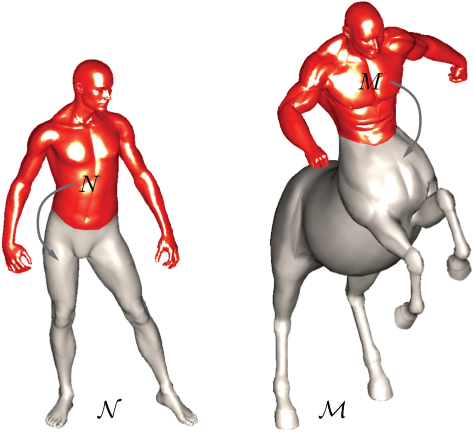
Rodolà, Cosmo, Bronstein, Torsello, Cremers 2016

# Partial correspondence (part-to-part)



Litany, Rodolà, Bronstein$^2$, Cremers 2016

# Key observation

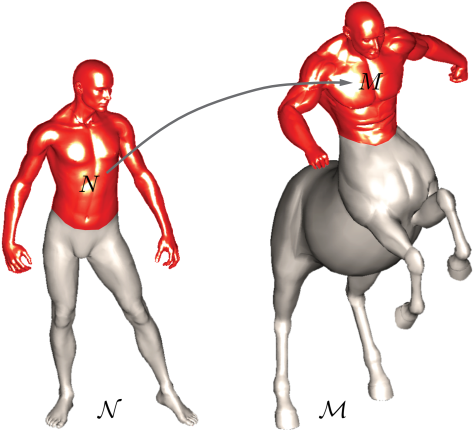



$$\mathbf{C}_{N\mathcal{N}} \qquad \mathbf{C}_{M\mathcal{M}}$$

$$\text{slant} \propto \frac{|N|}{|\mathcal{N}|} \qquad \text{slant} \propto \frac{|M|}{|\mathcal{M}|}$$

Litany, Rodolà, Bronstein[2], Cremers 2016

# Key observation

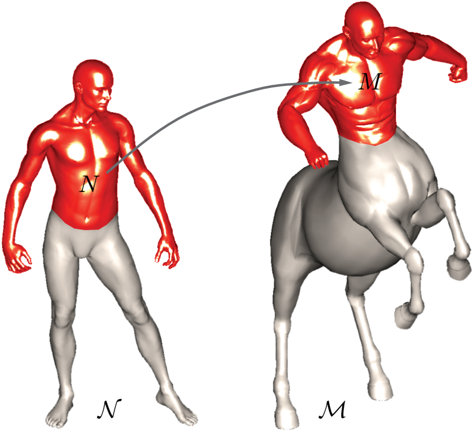



$$\mathbf{C}_{NM} = \mathbf{C}_{\mathcal{M}M}\mathbf{C}_{\mathcal{N}\mathcal{M}}\mathbf{C}_{N\mathcal{N}}$$

$$\text{slant} \propto \frac{|N|}{|\mathcal{N}|}\frac{|\mathcal{M}|}{|M|}$$

Litany, Rodolà, Bronstein$^2$, Cremers 2016

# Key observation

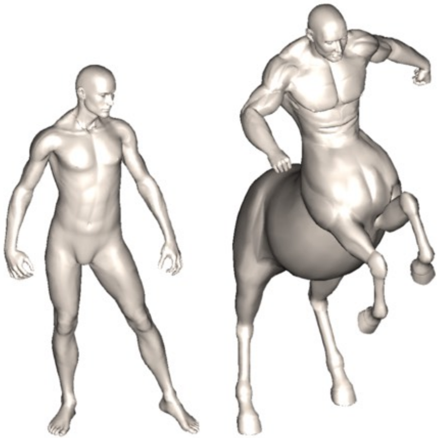



$$\mathbf{C}_{NM} = \mathbf{C}_{\mathcal{M}M}\mathbf{C}_{\mathcal{N}\mathcal{M}}\mathbf{C}_{N\mathcal{N}}$$

$$\text{slant} \propto \frac{|N|}{|\mathcal{N}|}\frac{|\mathcal{M}|}{|M|} = \frac{|\mathcal{M}|}{|\mathcal{N}|}$$

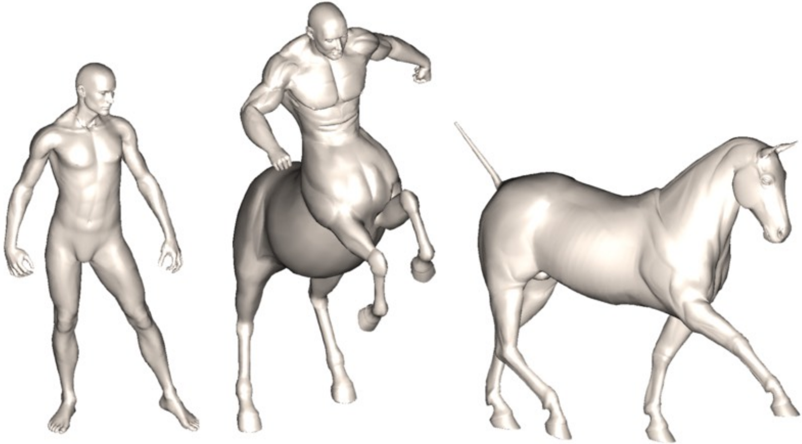
Litany, Rodolà, Bronstein$^2$, Cremers 2016

# Partial correspondence (part-to-part)



Litany, Rodolà, Bronstein$^2$, Cremers 2016

# Non-rigid puzzle (multi-part)



Litany, Rodolà, Bronstein$^2$, Cremers 2016

Litany, Bronstein$^2$ 2012

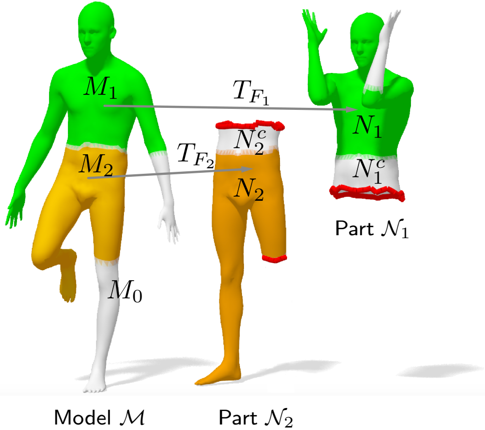
# Non-rigid puzzles problem formulation

**Input**

- Model $\mathcal{M}$
- Parts $\mathcal{N}_1, \ldots, \mathcal{N}_p$

**Output**

- Segmentation $M_i \subseteq \mathcal{M}$
- Located parts $N_i \subseteq \mathcal{N}_i$
- Clutter $N_i^c$
- Missing parts $M_0$
- Correspondences $T_{F_i}$




Litany, Rodolà, Bronstein[2], Cremers 2016

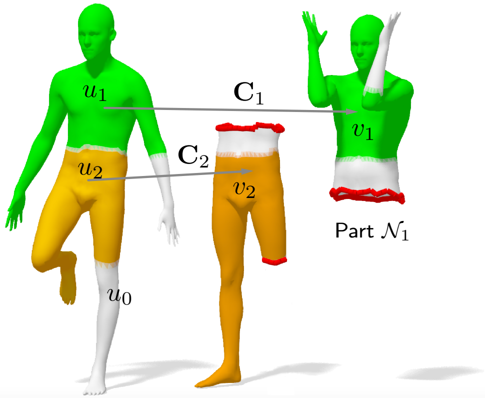
# Non-rigid puzzles problem formulation

**Input**

- Model $\mathcal{M}$
- Parts $\mathcal{N}_1, \ldots, \mathcal{N}_p$

**Output**

- Segmentation $u_i : \mathcal{M} \rightarrow [0, 1]$
- Located parts $v_i : \mathcal{N}_i \rightarrow [0, 1]$
- Clutter $1 - v_i$
- Missing parts $u_0$
- Correspondences $\mathbf{C}_i$



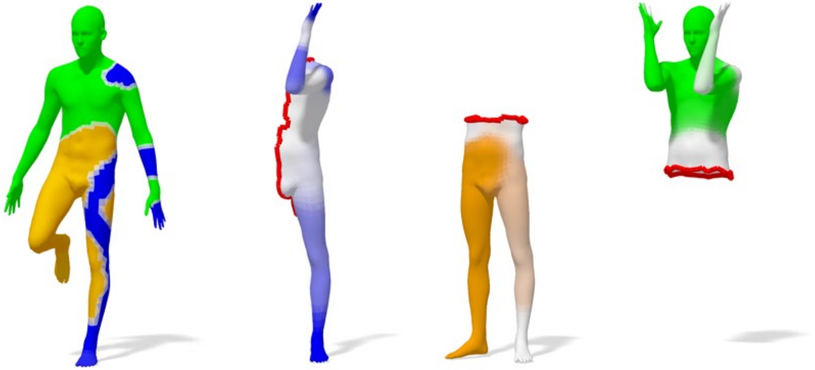

Litany, Rodolà, Bronstein[2], Cremers 2016

# Non-rigid puzzles problem formulation

$$\begin{aligned} \min_{\mathbf{C}_i, u_i, v_i} \quad & \sum_{i=1}^{p} \|\mathbf{C}_i \mathbf{A}_i(v_i) - \mathbf{B}(u_i)\|_{2,1} + \sum_{i=0}^{p} \rho_{\text{part}}(u_i, v_i) + \sum_{i=1}^{p} \rho_{\text{corr}}(\mathbf{C}_i) \\ \text{s.t.} \quad & \sum_{i=0}^{p} u_i = 1 \end{aligned}$$

Litany, Rodolà, Bronstein[2], Cremers 2016

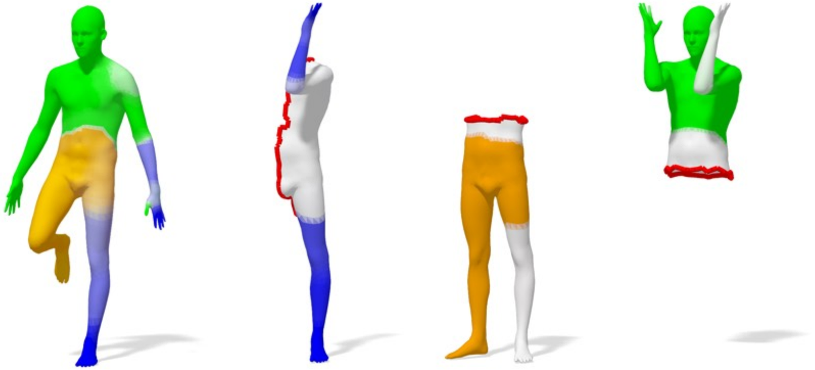
# Convergence example



Outer iteration $1$

Litany, Rodolà, Bronstein$^2$, Cremers 2016

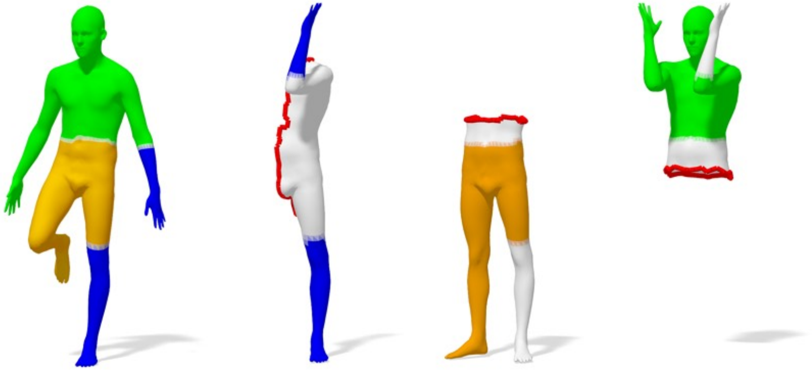
# Convergence example



Outer iteration $2$

Litany, Rodolà, Bronstein$^2$, Cremers 2016

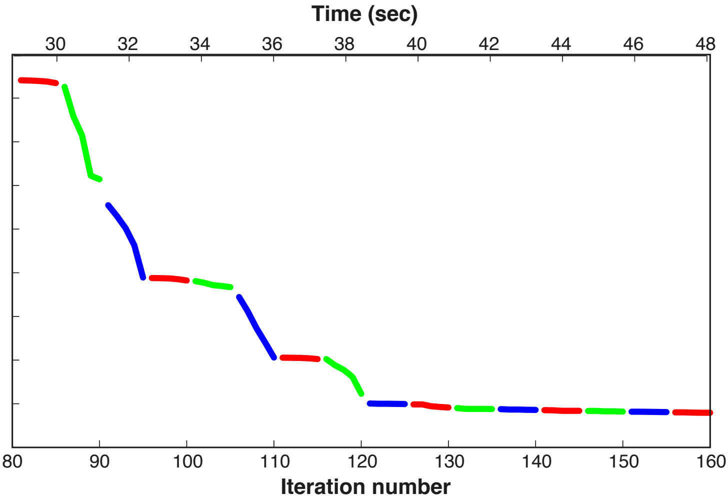
# Convergence example



Outer iteration $3$

Litany, Rodolà, Bronstein$^2$, Cremers 2016

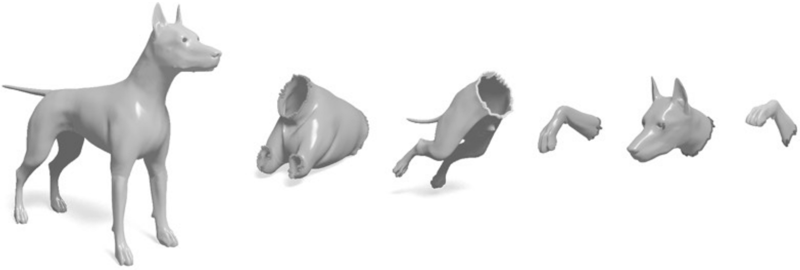
# Convergence example




Litany, Rodolà, Bronstein$^2$, Cremers 2016

# Example: "Perfect puzzle"

| | |
|---|---|
| Model/Part | Synthetic (TOSCA) |
| Transformation | Isometric |
| Clutter | No |
| Missing part | No |
| Data term | Dense (SHOT) |



Litany, Rodolà, Bronstein$^2$, Cremers 2016

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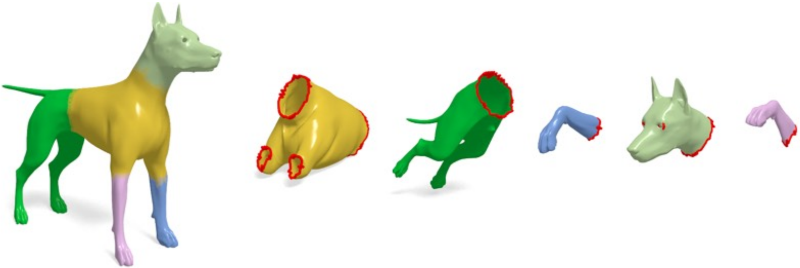

Segmentation

Litany, Rodolà, Bronstein$^2$, Cremers 2016

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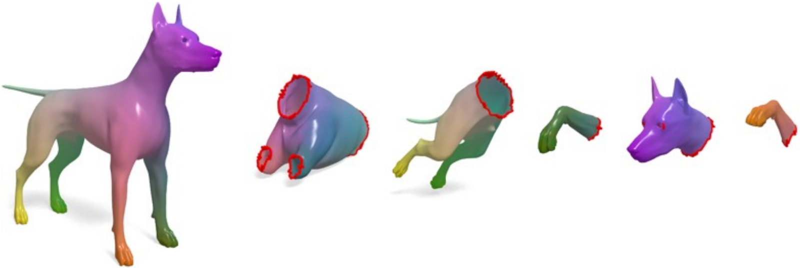
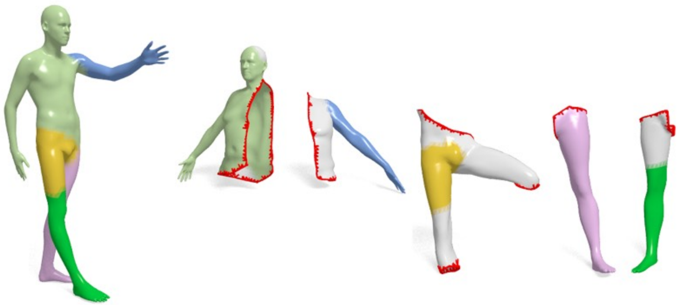


Correspondence

Litany, Rodolà, Bronstein$^2$, Cremers 2016

# Example: Overlapping parts

| | |
|---|---|
| Model/Part | Synthetic (FAUST) |
| Transformation | Near-isometric |
| Clutter | **Yes (overlap)** |
| Missing part | No |
| Data term | Dense (SHOT) |

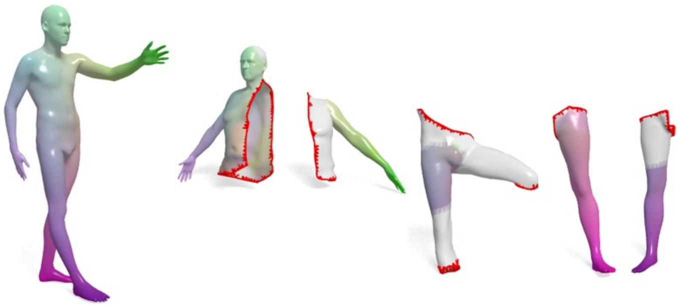


Segmentation

Litany, Rodolà, Bronstein[2], Cremers 2016

# Example: Overlapping parts

| | |
|---|---|
| Model/Part | Synthetic (FAUST) |
| Transformation | Near-isometric |
| Clutter | **Yes (overlap)** |
| Missing part | No |
| Data term | Dense (SHOT) |

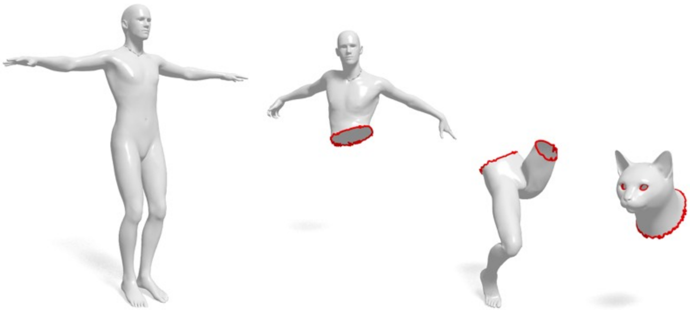


Correspondence

Litany, Rodolà, Bronstein[2], Cremers 2016

# Example: Missing parts

| | |
|---|---|
| Model/Part | Synthetic (TOSCA) |
| Transformation | Isometric |
| Clutter | **Yes (extra part)** |
| Missing part | **Yes** |
| Data term | Dense (SHOT) |



Litany, Rodolà, Bronstein[2], Cremers 2016

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|---|---|
| Model/Part | Synthetic (TOSCA) |
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| Clutter | **Yes (extra part)** |
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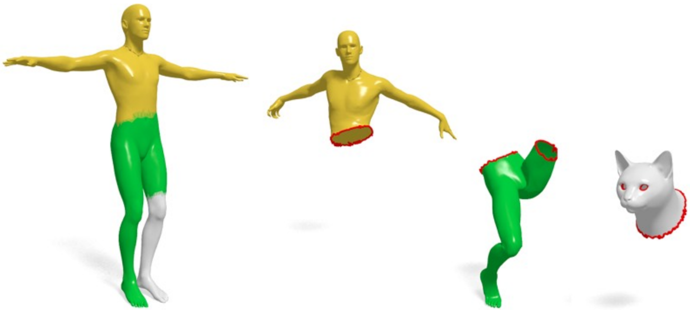


Segmentation

Litany, Rodolà, Bronstein$^2$, Cremers 2016

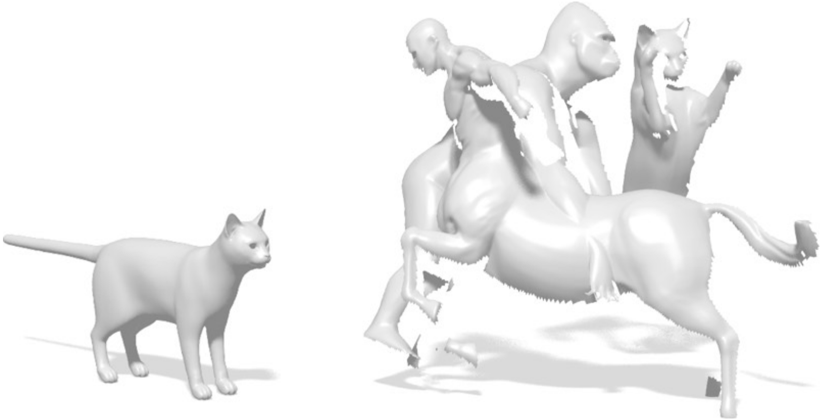
# Example: Missing parts

| | |
|---|---|
| Model/Part | Synthetic (TOSCA) |
| Transformation | Isometric |
| Clutter | **Yes (extra part)** |
| Missing part | **Yes** |
| Data term | Dense (SHOT) |

Correspondence

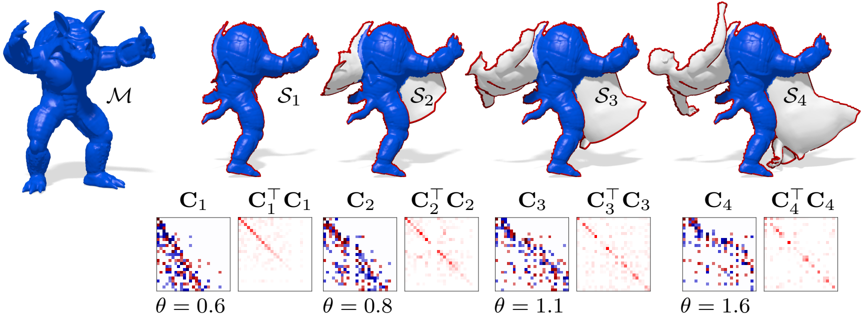
Litany, Rodolà, Bronstein$^2$, Cremers 2016

# Non-rigid clutter



Cosmo, Rodolà, Torsello, Bronstein 2016

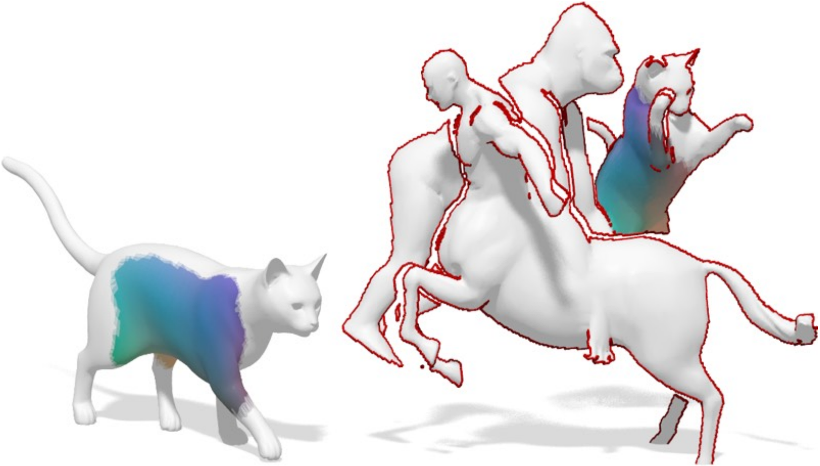
# Problem structure




- Slanted diagonal structure (angle $\theta$ has to be estimated)
- $\mathbf{C}^\top \mathbf{C}$ has sparse diagonal
- Good descriptor + initialization is crucial! (learned descriptor)

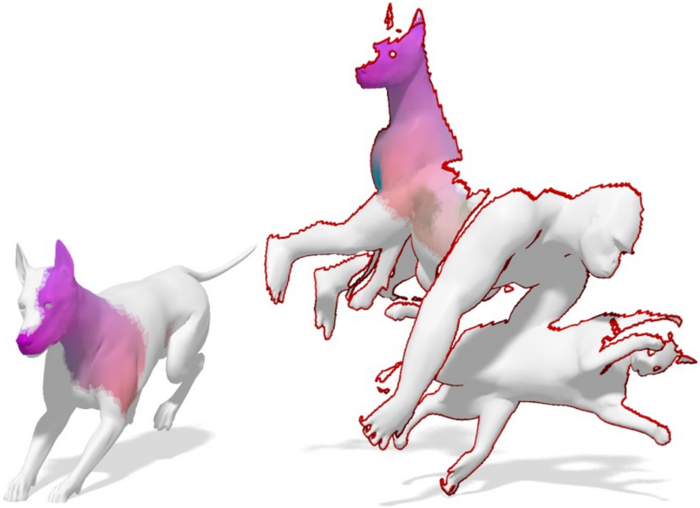
Cosmo, Rodolà, Torsello, Bronstein 2016

# Examples of matching in cluttered scenes



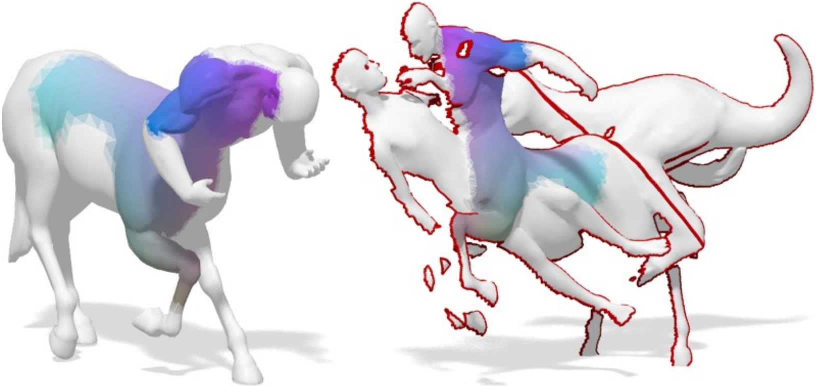
Cosmo, Rodolà, Torsello, Bronstein 2016

# Examples of matching in cluttered scenes



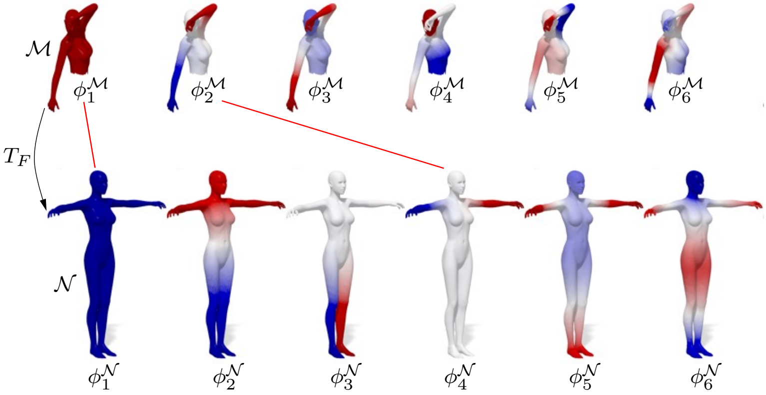
Cosmo, Rodolà, Torsello, Bronstein 2016

# Examples of matching in cluttered scenes



Cosmo, Rodolà, Torsello, Bronstein 2016

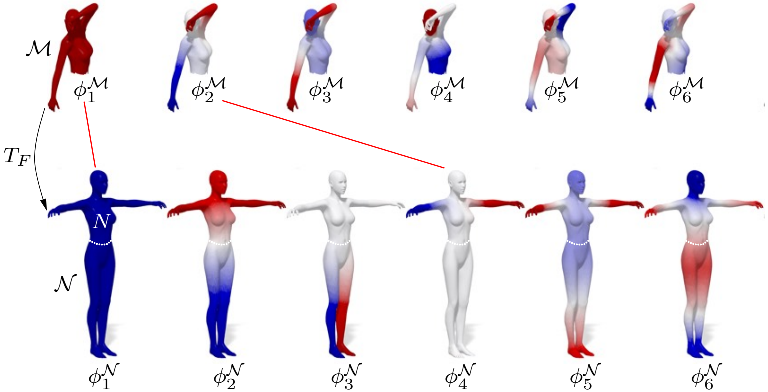
# Partial functional correspondence with spatial part model




**Slanted diagonal:** $\langle T_F \phi_i^{\mathcal{M}}, \phi_j^{\mathcal{N}} \rangle_{L^2(N)} \approx \pm\delta_{i,\pi_j}$ $\qquad \pi_j \approx j \frac{|\mathcal{N}|}{|\mathcal{M}|}$

Litany, Rodolà, Bronstein[2] 2016

# Partial functional correspondence with spatial part model




**Slanted diagonal:** $\langle T_F \phi_i^{\mathcal{M}}, \phi_j^{\mathcal{N}} \rangle_{L^2(N)} \approx \pm \delta_{i,\pi_j}$ $\qquad \pi_j \approx j \frac{|\mathcal{N}|}{|\mathcal{M}|}$

Litany, Rodolà, Bronstein$^2$ 2016

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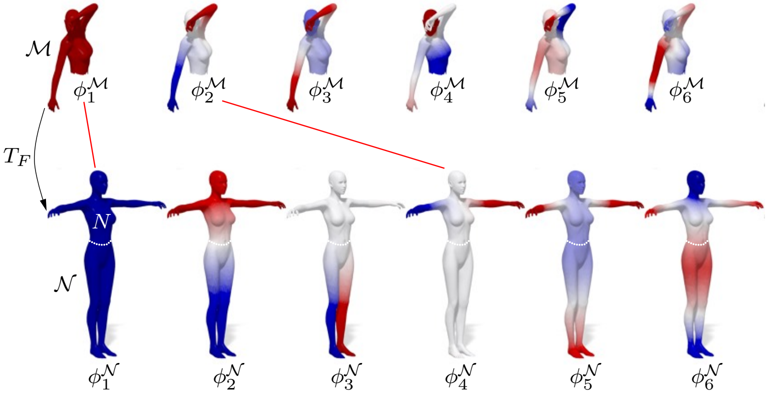


**Slanted diagonal:** $\langle T_F \phi_i^{\mathcal{M}}, v \cdot \phi_j^{\mathcal{N}} \rangle_{L^2(\mathcal{N})} \approx \pm \delta_{i,\pi_j}$ $\qquad \pi_j \approx j \frac{|\mathcal{N}|}{|\mathcal{M}|}$

Litany, Rodolà, Bronstein[2] 2016

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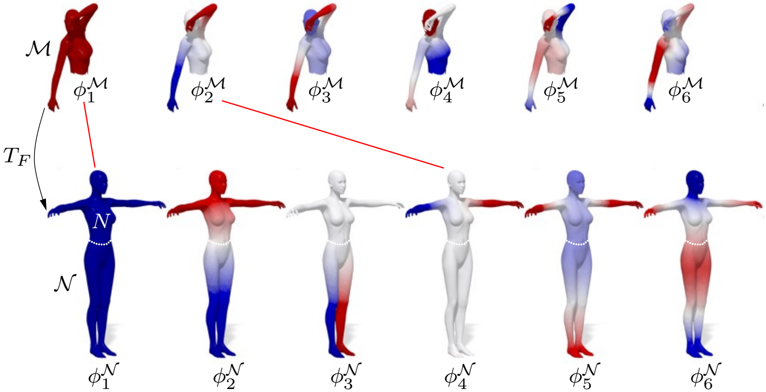
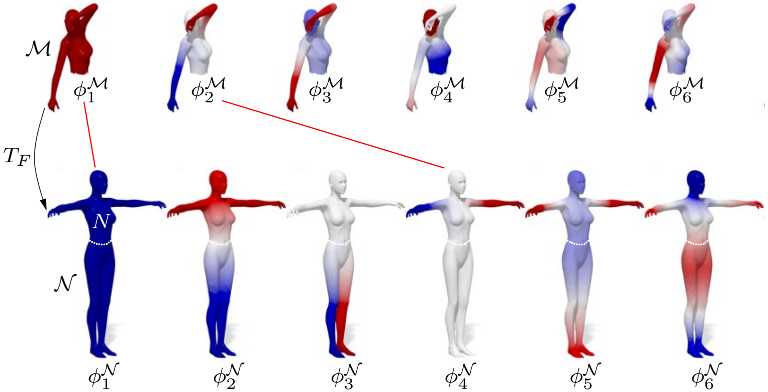



**Slanted diagonal:** $\langle T_F \phi_i^{\mathcal{M}}, v \cdot \phi_j^{\mathcal{N}} \rangle_{L^2(\mathcal{N})} \approx \pm \delta_{i,\pi_j}$ $\qquad \pi_j \approx j \frac{|\mathcal{N}|}{|\mathcal{M}|}$

- Complicated alternating optimization w.r.t. $v$ and $\mathbf{C}$

Litany, Rodolà, Bronstein[2] 2016

# Partial functional correspondence with spatial part model

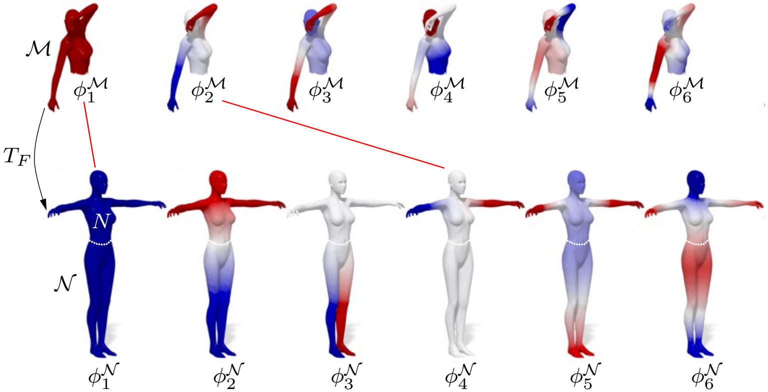



**Slanted diagonal:** $\langle T_F \phi_i^{\mathcal{M}}, v \cdot \phi_j^{\mathcal{N}} \rangle_{L^2(\mathcal{N})} \approx \pm \delta_{i,\pi_j}$ $\qquad \pi_j \approx j \frac{|\mathcal{N}|}{|\mathcal{M}|}$

- Complicated alternating optimization w.r.t. $v$ and $\mathbf{C}$
- Explicit spatial model $v$ of the part

Litany, Rodolà, Bronstein[2] 2016

# Partial functional correspondence with spatial part model

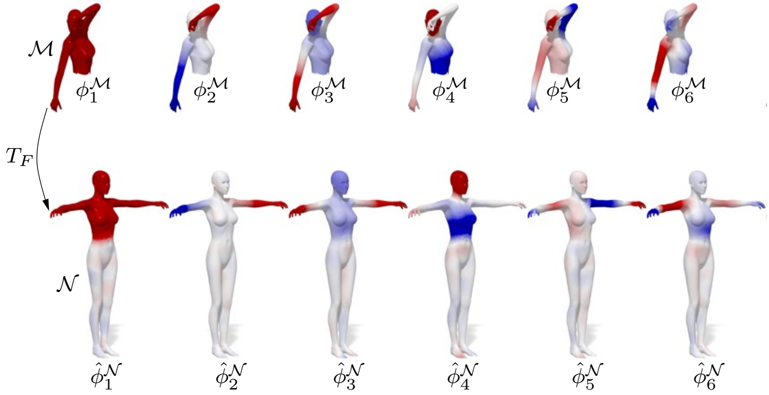



**Slanted diagonal:** $\langle T_F \phi_i^{\mathcal{M}}, v \cdot \phi_j^{\mathcal{N}} \rangle_{L^2(\mathcal{N})} \approx \pm \delta_{i,\pi_j}$ $\qquad \pi_j \approx j \frac{|\mathcal{N}|}{|\mathcal{M}|}$

- Complicated alternating optimization w.r.t. $v$ and $\mathbf{C}$
- Explicit spatial model $v$ of the part $\Rightarrow$ $\mathcal{O}(n)$ **complexity!**

Litany, Rodolà, Bronstein[2] 2016

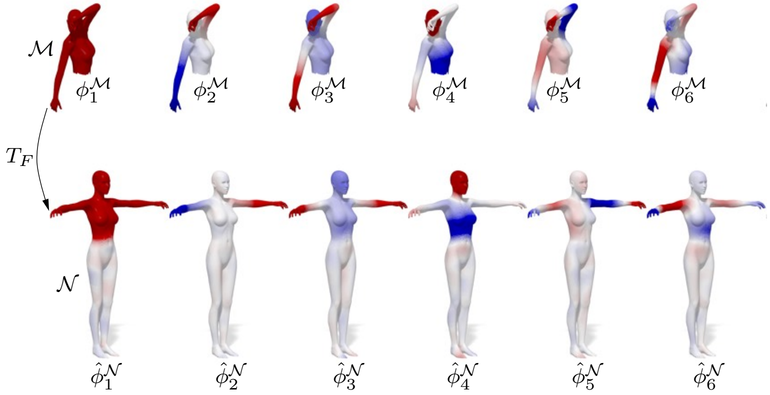
# Spectral partial functional correspondence




**Find a new basis $\{\hat{\phi}_i^{\mathcal{N}}\}_{i=1}^{k}$ such that** $\langle T_F \phi_i^{\mathcal{M}}, \hat{\phi}_j^{\mathcal{N}} \rangle_{L^2(\mathcal{N})} \approx \delta_{ij}$

Litany, Rodolà, Bronstein[2] 2016

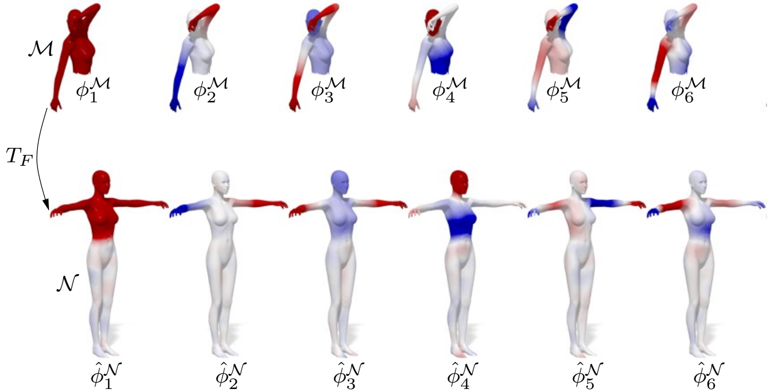
# Spectral partial functional correspondence




**Find a new basis** $\{\hat{\phi}_i^{\mathcal{N}}\}_{i=1}^{k}$ **such that** $\langle T_F \phi_i^{\mathcal{M}}, \sum_{l=1}^{k} q_{lj} \phi_l^{\mathcal{N}} \rangle_{L^2(\mathcal{N})} \approx \delta_{ij}$

Litany, Rodolà, Bronstein$^2$ 2016

# Spectral partial functional correspondence

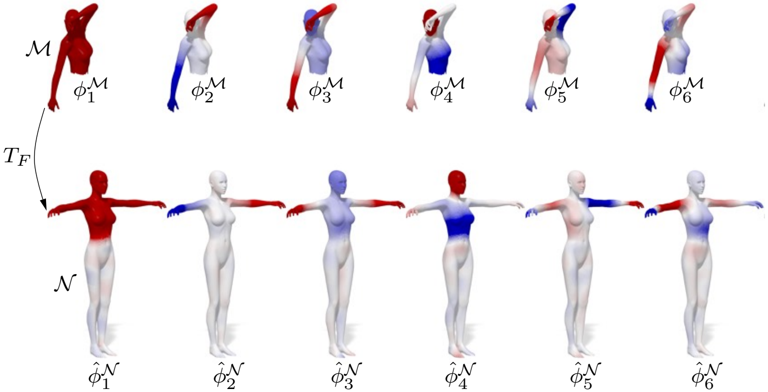



**Find a new basis** $\{\hat{\phi}_i^{\mathcal{N}}\}_{i=1}^{k}$ **such that** $\langle T_F \phi_i^{\mathcal{M}}, \sum_{l=1}^{k} q_{lj} \phi_l^{\mathcal{N}} \rangle_{L^2(\mathcal{N})} \approx \delta_{ij}$

- New basis functions $\{\hat{\phi}_i^{\mathcal{N}}\}_{i=1}^{k}$ are localized on $N$

Litany, Rodolà, Bronstein[2] 2016

# Spectral partial functional correspondence

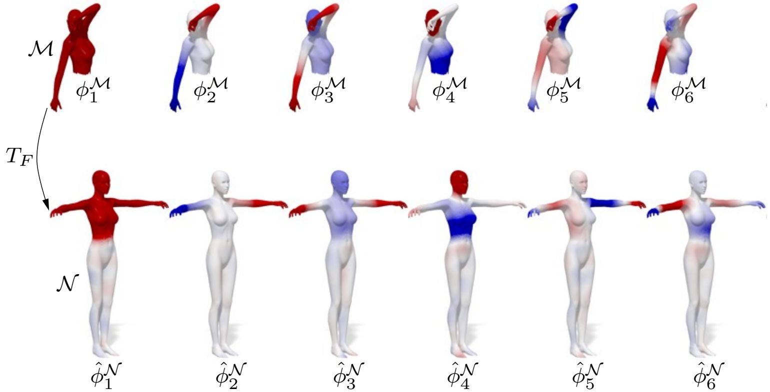



**Find a new basis $\{\hat{\phi}_i^{\mathcal{N}}\}_{i=1}^k$ such that** $\langle T_F \phi_i^{\mathcal{M}}, \sum_{l=1}^k q_{lj} \phi_l^{\mathcal{N}} \rangle_{L^2(\mathcal{N})} \approx \delta_{ij}$

- New basis functions $\{\hat{\phi}_i^{\mathcal{N}}\}_{i=1}^k$ are localized on $N$
- Optimization over coefficients $\mathbf{Q} = (q_{ij})$

Litany, Rodolà, Bronstein[2] 2016

# Spectral partial functional correspondence

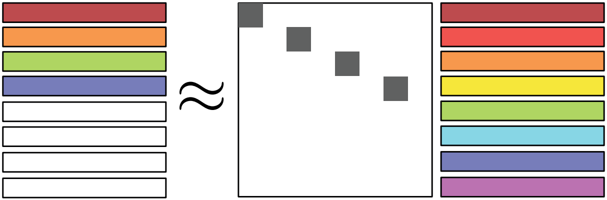



**Find a new basis** $\{\hat{\phi}_i^{\mathcal{N}}\}_{i=1}^k$ **such that** $\langle T_F \phi_i^{\mathcal{M}}, \sum_{l=1}^k q_{lj} \phi_l^{\mathcal{N}} \rangle_{L^2(\mathcal{N})} \approx \delta_{ij}$

- New basis functions $\{\hat{\phi}_i^{\mathcal{N}}\}_{i=1}^k$ are localized on $N$
- Optimization over coefficients $\mathbf{Q} = (q_{ij}) \Rightarrow \mathcal{O}(k^2)$ **complexity!**

Litany, Rodolà, Bronstein[2] 2016

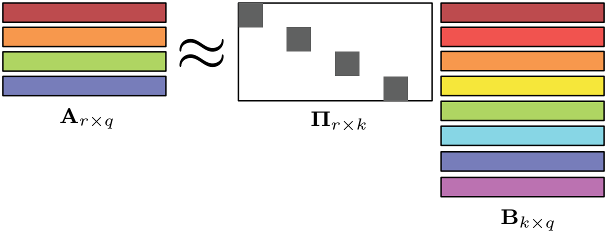
# Spectral partial functional correspondence



$\mathbf{A}_{k\times q} \approx \mathbf{\Pi}_{k\times k}\, \mathbf{B}_{k\times q}$

Litany, Rodolà, Bronstein[2] 2016

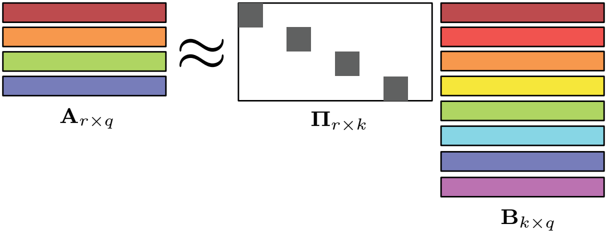
# Spectral partial functional correspondence




$\mathbf{\Pi}$ is $k \times r$ partial permutation with elements $(\pi_i, i) = \pm 1$ and $r \approx k\frac{|\mathcal{M}|}{|\mathcal{N}|}$

Litany, Rodolà, Bronstein$^2$ 2016

# Spectral partial functional correspondence



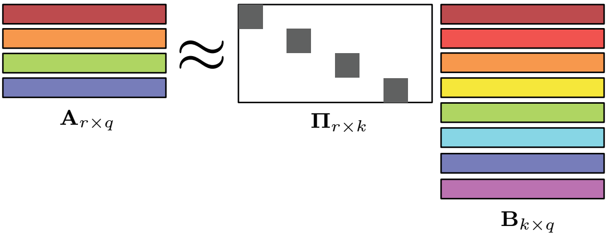

$\mathbf{\Pi}$ is $k \times r$ partial permutation with elements $(\pi_i, i) = \pm 1$ and $r \approx k\frac{|\mathcal{M}|}{|\mathcal{N}|}$

Relax $\mathbf{\Pi} \approx \mathbf{Q}^\top$ s.t. $\mathbf{Q}^\top\mathbf{Q} = \mathbf{I}$ ($k \times r$ ortho-projection)

$$\min_{\mathbf{Q}} \ \mathrm{trace}(\mathbf{Q}^\top \mathbf{\Lambda}_{\mathcal{N},k}\mathbf{Q}) + \mu\|\mathbf{A}_r - \mathbf{Q}^\top\mathbf{B}_k\|_{2,1} \quad \text{s.t.} \quad \mathbf{Q}^\top\mathbf{Q} = \mathbf{I}$$

Litany, Rodolà, Bronstein$^2$ 2016; Kovnatsky, Glashoff, Bronstein$^2$, Kimmel 2013 (Joint diag)

# Spectral partial functional correspondence

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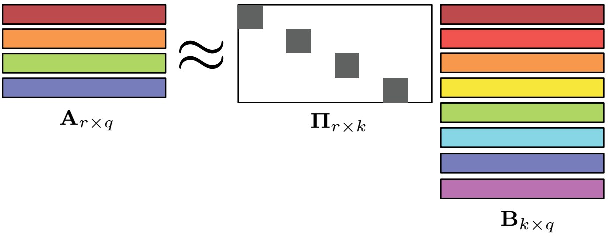
- Optimization on the Stiefel manifold with $k^2$ variables

Litany, Rodolà, Bronstein$^2$ 2016; Kovnatsky, Glashoff, Bronstein$^2$, Kimmel 2013 (Joint diag)

# Spectral partial functional correspondence

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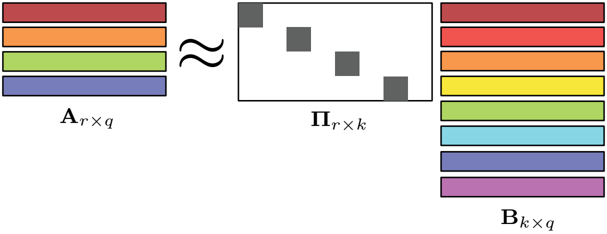
- Non-smooth optimization on the Stiefel manifold with $k^2$ variables

Litany, Rodolà, Bronstein[2] 2016; Kovnatsky, Glashoff, Bronstein[2], Kimmel 2013 (Joint diag); Kovnatsky, Glashoff, Bronstein 2016 (MADMM)

# Spectral partial functional correspondence

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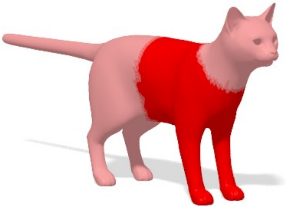
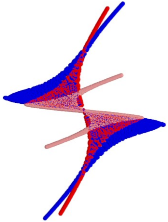
- Non-smooth optimization on the Stiefel manifold with $k^2$ variables
- Non-rigid alignment of eigenfunctions

Litany, Rodolà, Bronstein$^2$ 2016; Kovnatsky, Glashoff, Bronstein$^2$, Kimmel 2013 (Joint diag); Kovnatsky, Glashoff, Bronstein 2016 (MADMM)

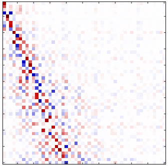
# Geometric interpretation



Full shape $\mathcal{N}$

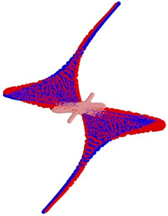


$\phi_2^{\mathcal{M}}, \phi_3^{\mathcal{M}}$ and $\phi_2^{\mathcal{N}}, \phi_3^{\mathcal{N}}$

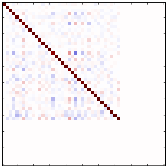


Laplacian eigenbasis

Part $\mathcal{M}$



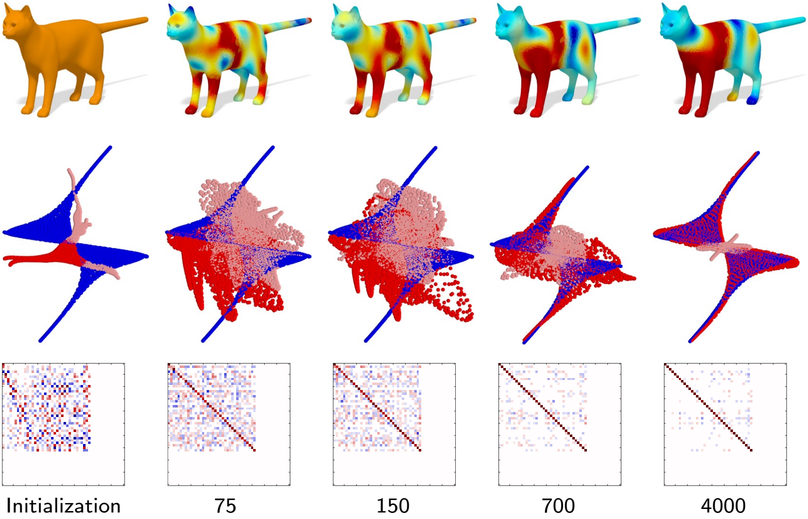
$\phi_2^{\mathcal{M}}, \phi_3^{\mathcal{M}}$ and $\hat{\phi}_2^{\mathcal{N}}, \hat{\phi}_3^{\mathcal{N}}$



**New basis**

Litany, Rodolà, Bronstein$^2$ 2016

# Convergence example

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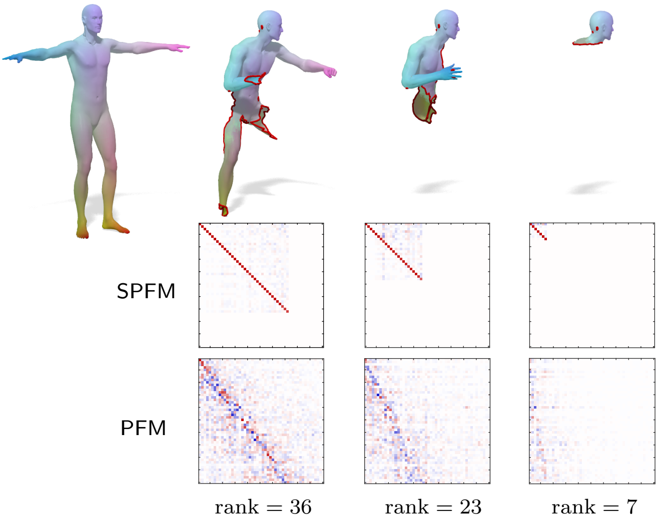
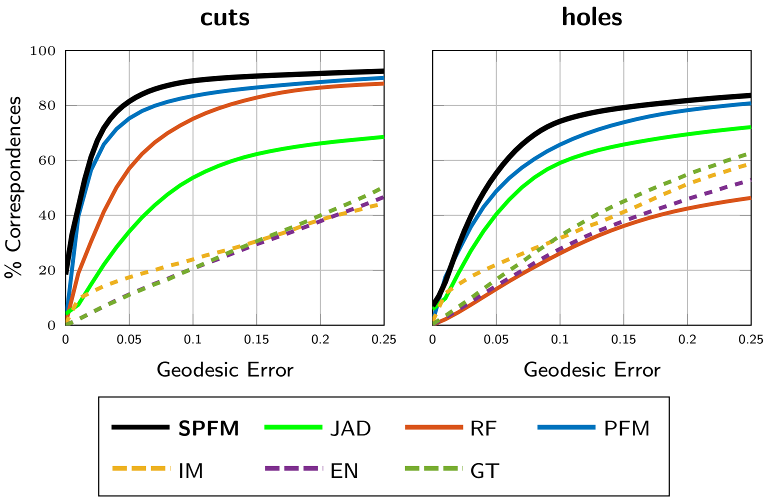
# Increasing partiality



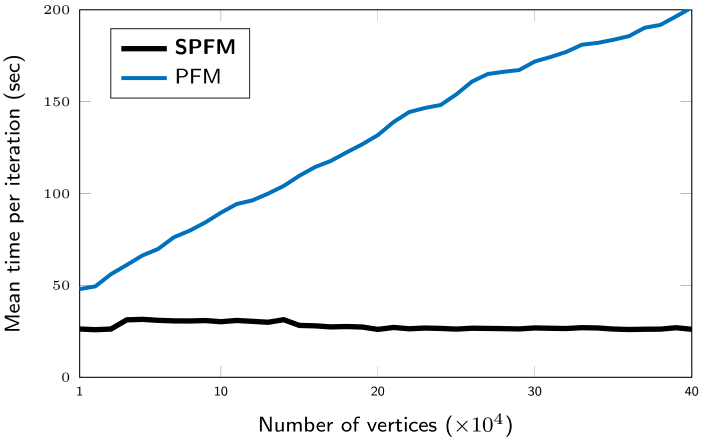

Litany, Rodolà, Bronstein[2] 2016

# SHREC'16 Partiality




SHREC'16 Partial Matching benchmark: Rodolà et al. 2016; Methods: Unpublished work (**SPFM**); Rodolà, Cosmo, Bronstein, Torsello, Cremers 2016 (PFM); Sahillioğlu, Yemez 2012 (IM); Rodolà, Bronstein, Albarelli, Bergamasco, Torsello 2012 (GT); Rodolà et al. 2013 (EN); Rodolà et al. 2014 (RF)

# Runtime

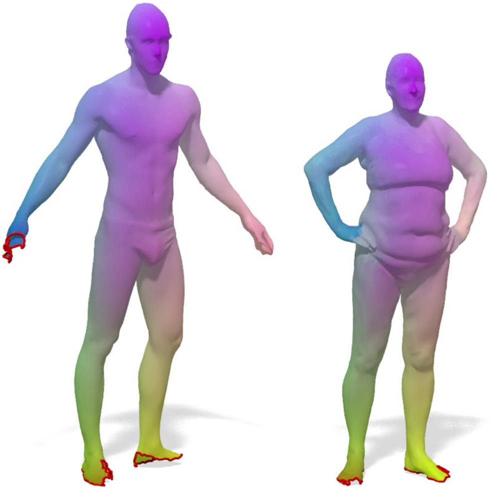



Litany, Rodolà, Bronstein$^2$ 2016

# Correspondence examples: topological noise

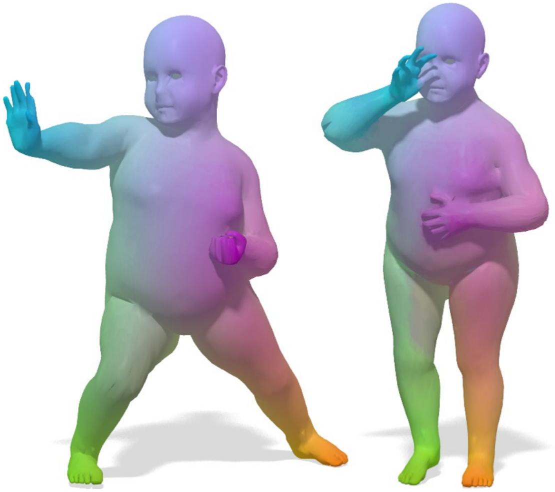
Litany, Rodolà, Bronstein$^2$ 2016; data: Bogo et al. 2014 (FAUST)

# Correspondence examples: topological noise



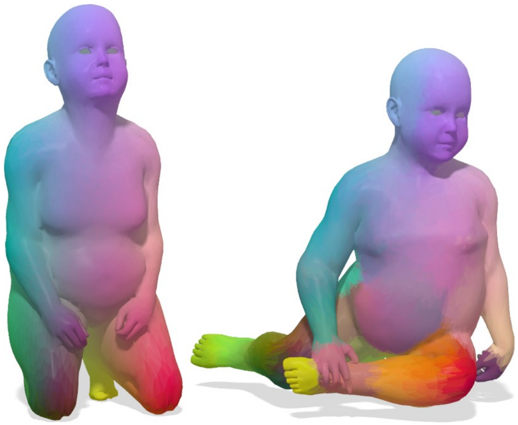
Litany, Rodolà, Bronstein$^2$ 2016; data: Bogo et al. 2014 (FAUST)

# Correspondence examples: topological noise



Litany, Rodolà, Bronstein[2] 2016; data: Rodola et al. 2016 (SHREC)

# Correspondence examples: topological noise



Litany, Rodolà, Bronstein[2] 2016; data: Rodola et al. 2016 (SHREC)