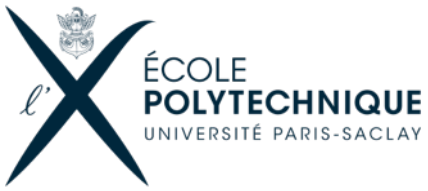
Computing and Processing Correspondences with Functional Maps

SIGGRAPH Asia 2016 course

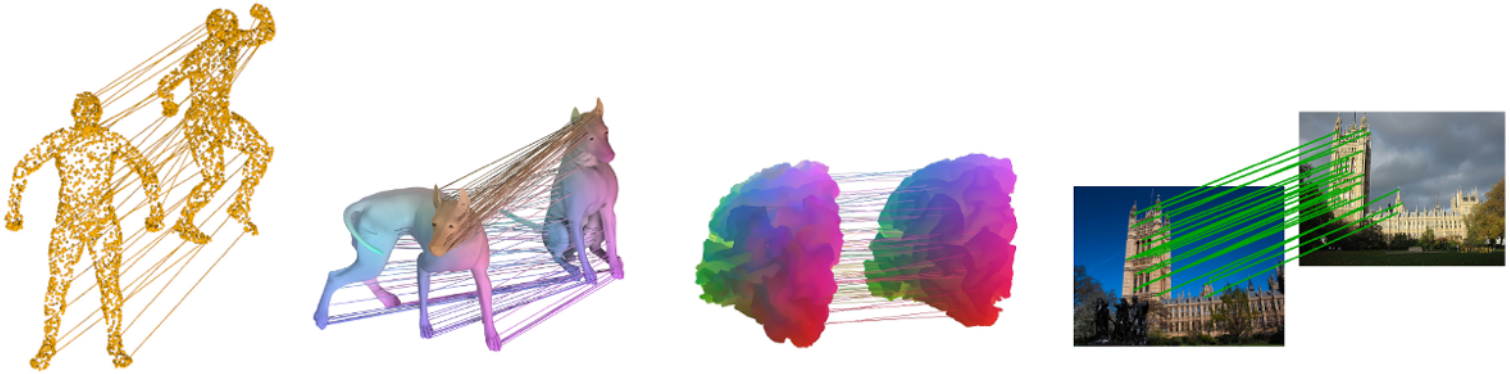
Maks Ovsjanikov, Etienne Corman, Michael Bronstein, Emanuele Rodolà,
Mirela Ben-Chen, Leonidas Guibas, Frederic Chazal, Alex Bronstein



General Overview

Overall Objective:

Create tools for computing and analyzing *mappings* between geometric objects.



General Overview

Overall Objective:

Create tools for computing and analyzing *mappings* between geometric objects.

Rather than comparing *points* on objects it is often easier to compare *real-valued functions* defined on them. Such maps can be represented as matrices.

Course Overview

- Course Website:

  http://www.lix.polytechnique.fr/~maks/fmaps_course/

  or http://bit.do/fmaps

- Course Notes:

  Linked from the website. Or use http://bit.do/fmaps_notes
  Attention: (significantly) more material than in the lectures

- Sample Code:

  See **Sample Code** link on the website.

Course Schedule

- 2:15pm – 3:10pm *Introduction* (Maks)
  - Introduction to the functional maps framework.

- 2:15pm – 3:10pm *Computing Functional Maps* (Michael)
  - Optimization methods for functional map estimation.

04:00pm - 04:15pm *Break*

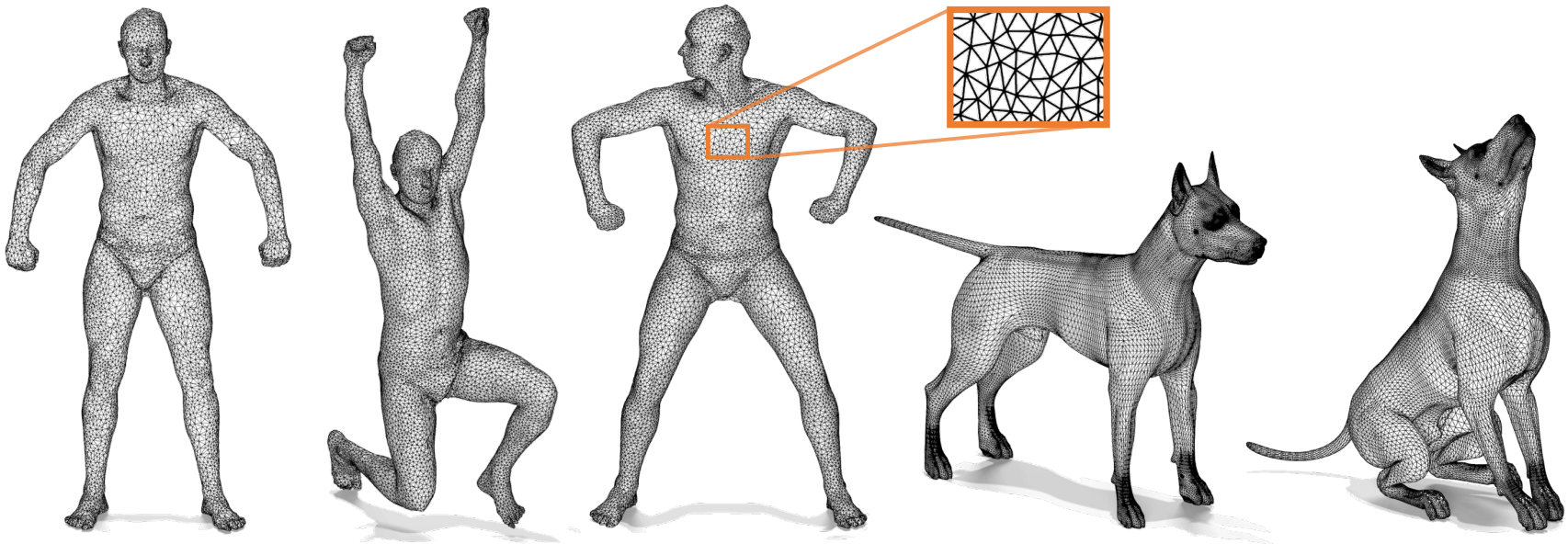
- 04:15pm - 05:05pm *Maps in Shape Collections* (Etienne)
  - Networks of Maps, Descriptor learning, Shape comparison.

- 05:10pm - 06:00pm *Conversion, Applications* (Emanuele)
  - Pointwise map recovery, Applications

06:00pm - *Wrapup, Q&A* (all)

What is a Shape?

- Continuous: a surface embedded in 3D.
- Discrete: a triangle mesh.

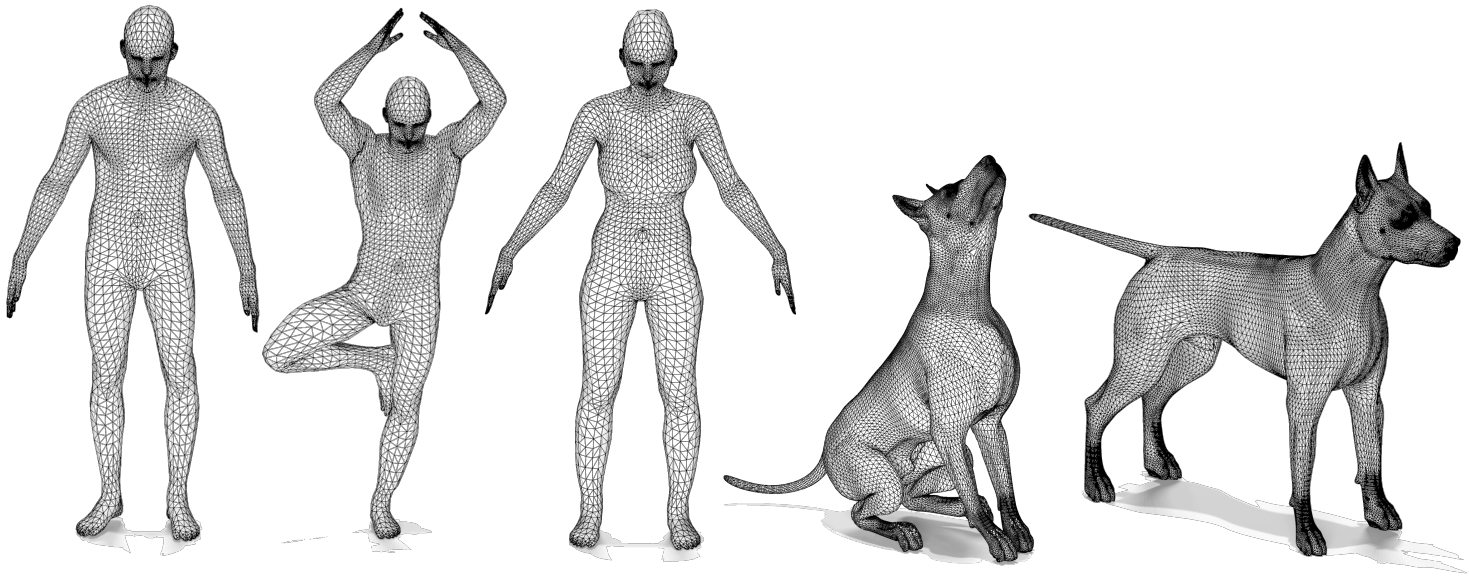


5k – 200k triangles

Shapes from the SCAPE, TOSCA and FAUST datasets

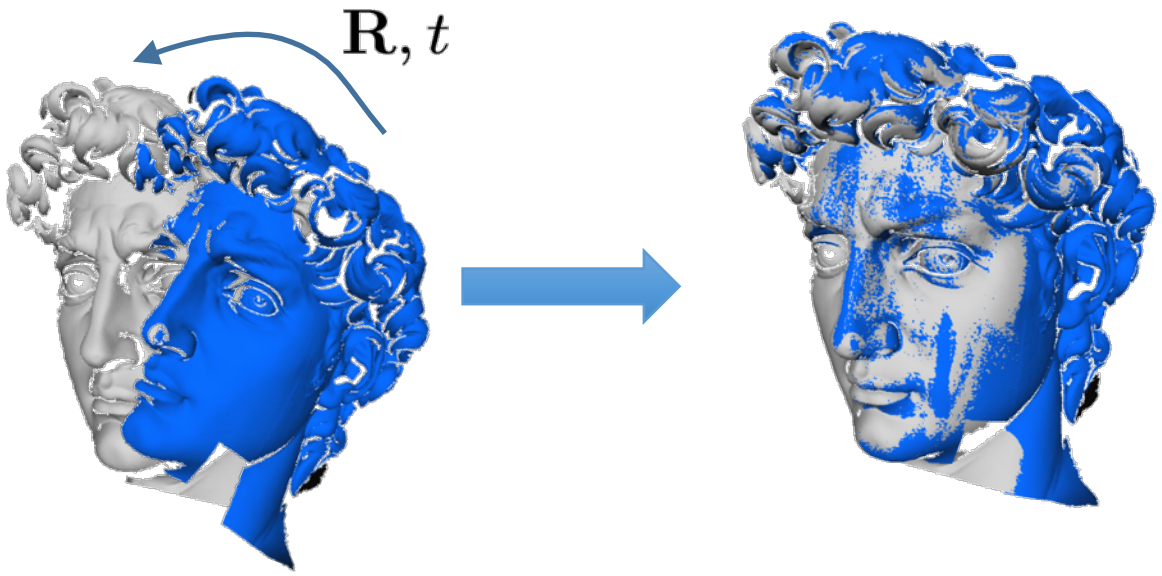
What is a Shape?

- A graph embedded in 3D: a triangle mesh.
  - Connected.
  - Manifold (each edge on at most 2 triangles).
  - Without boundary.



Shapes from the SCAPE, TOSCA and FAUST datasets

Rigid Shape Matching

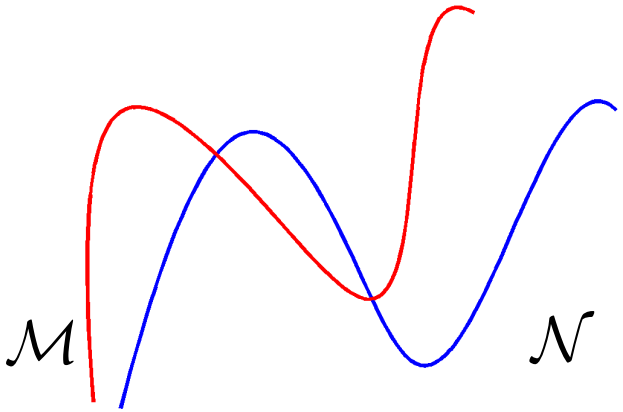


- Given a pair of shapes, find the optimal *Rigid Alignment* between them.

- The unknowns are the rotation/translation parameters of the source onto the target shape.

Iterative Closest Point (ICP)

- Classical approach: iterate between finding correspondences and finding the transformation:



example in 2D

Given a pair of shapes, $\mathcal{M}$ and $\mathcal{N}$, iterate:

1. For each $x_i \in \mathcal{M}$ find **nearest** neighbor $y_i \in \mathcal{N}$.
2. Find optimal transformation $\mathbf{R}, t$ minimizing:

$$\arg\min_{R,t} \sum_i \|Rx_i + t - y_i\|_2^2$$

Iterative Closest Point

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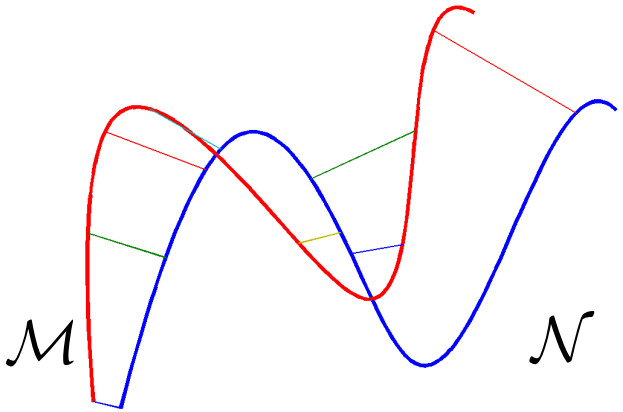

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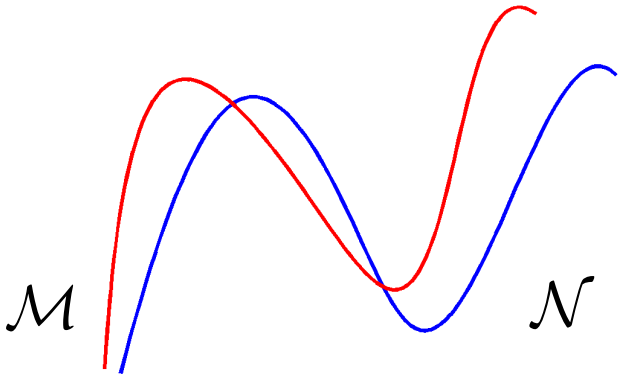

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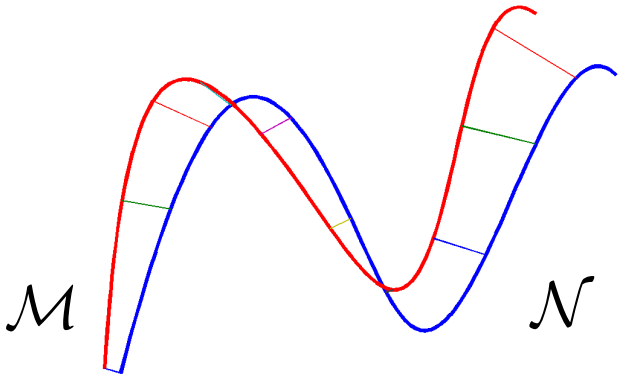

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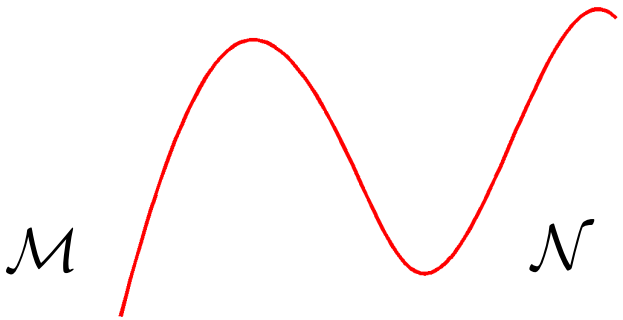

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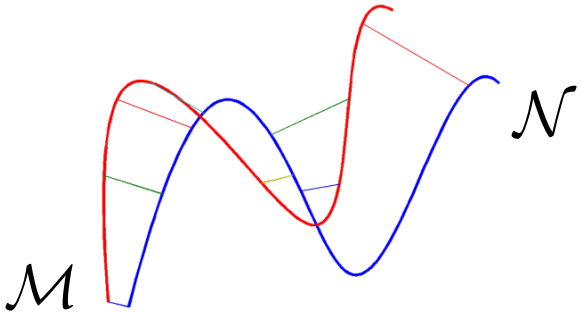


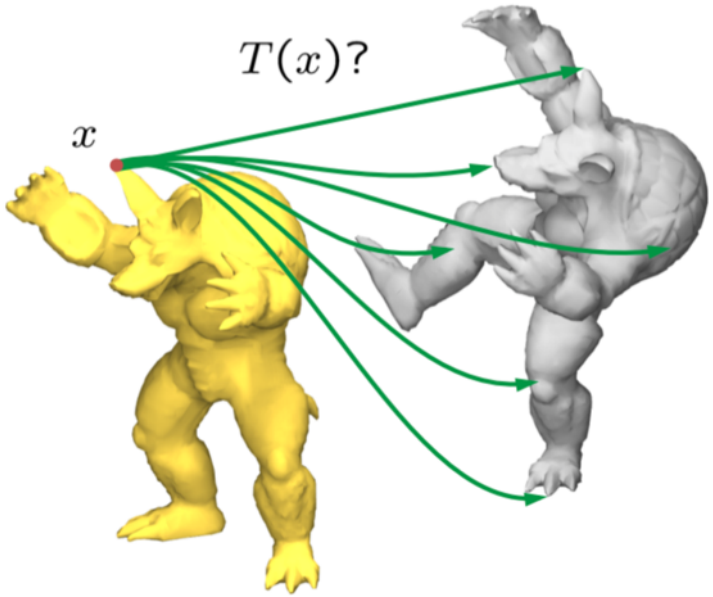
1. Finding nearest neighbors: can be done with space-partitioning data structures (e.g., KD-tree).
2. Finding the optimal transformation $\mathbf{R}, t$ minimizing:

$$\arg\min_{R,t} \sum_i \|Rx_i + t - y_i\|_2^2$$

Can be done efficiently via SVD decomposition.

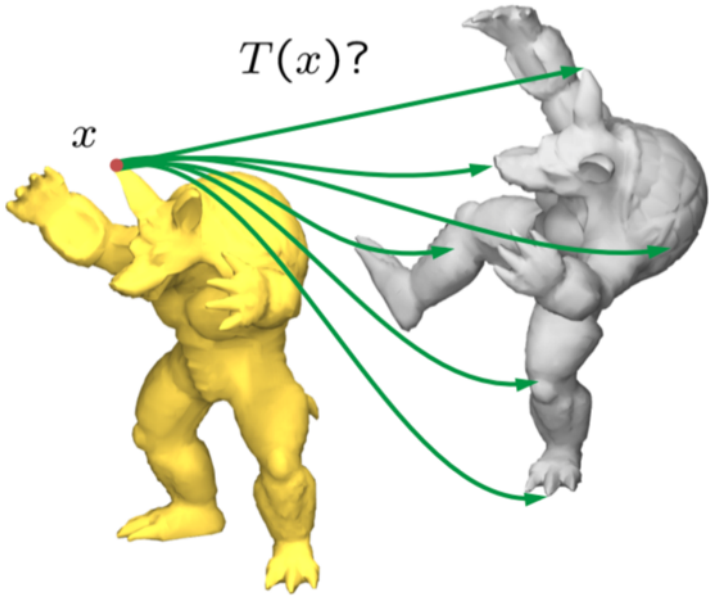
Arun et al., Least-Squares Fitting of Two 3-D Point Sets

Non-Rigid Shape Matching



Unlike rigid matching with rotation/translation, there is no compact representation to optimize for in non-rigid matching.

Non-Rigid Shape Matching



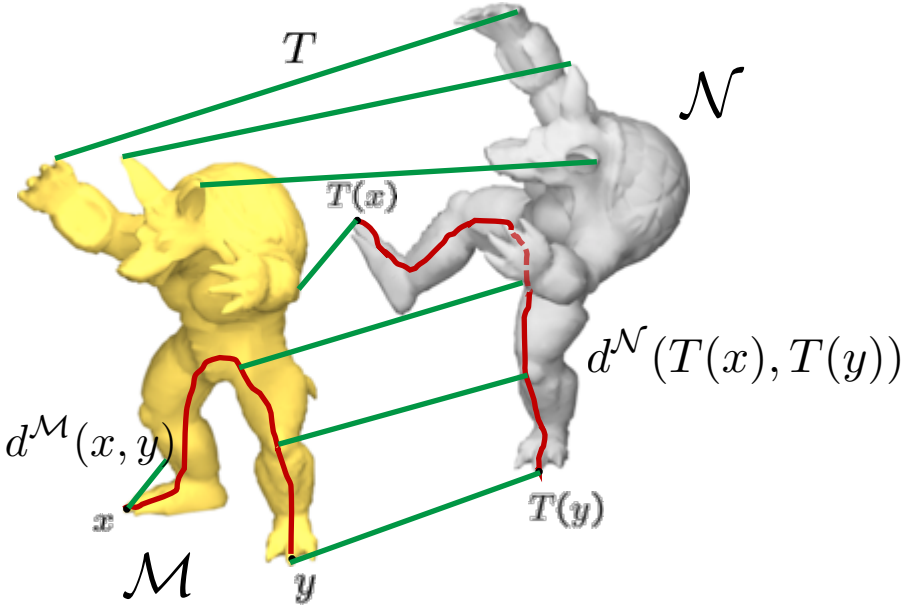
Main Problems:

- What does it mean for a correspondence to be “good”?
- How to compute it efficiently in practice?

Isometric Shape Matching

Possible Model:

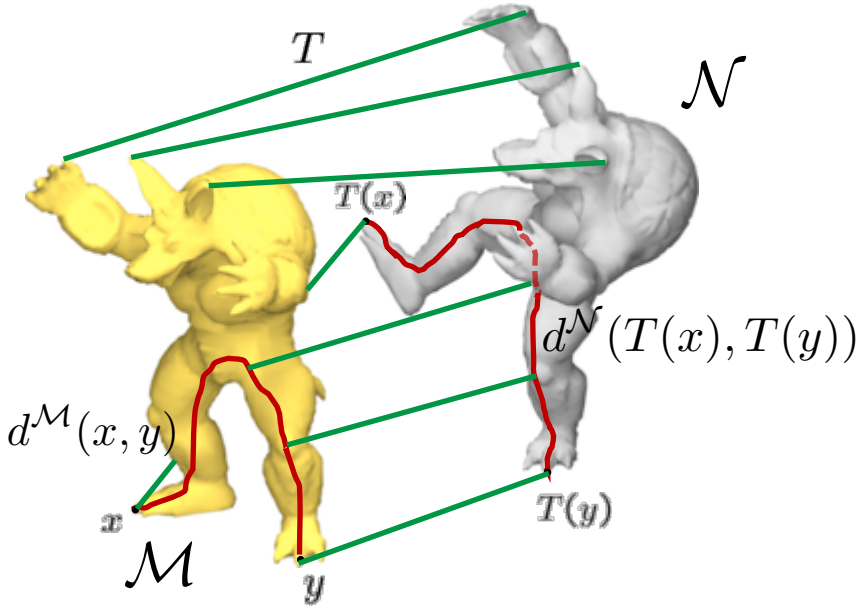
Good maps must preserve geodesic distances.



Geodesic: length of shortest path lying entirely on the surface.

Isometric Shape Matching

Approach:



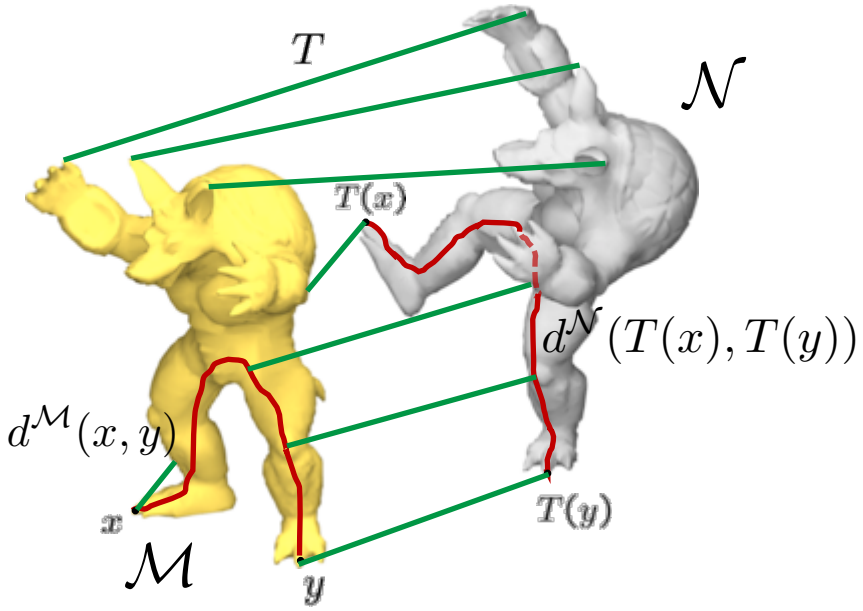
Find the point mapping by minimizing the distance distortion:

$$T_{\text{opt}} = \arg\min_{T} \sum_{x,y} \| d^{\mathcal{M}}(x, y) - d^{\mathcal{N}}(T(x), T(y)) \|$$

The unknowns are point correspondences.

Isometric Shape Matching

Approach:



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$$T_{\text{opt}} = \arg\min_{T} \sum_{x,y} \|d^{\mathcal{M}}(x, y) - d^{\mathcal{N}}(T(x), T(y))\|$$

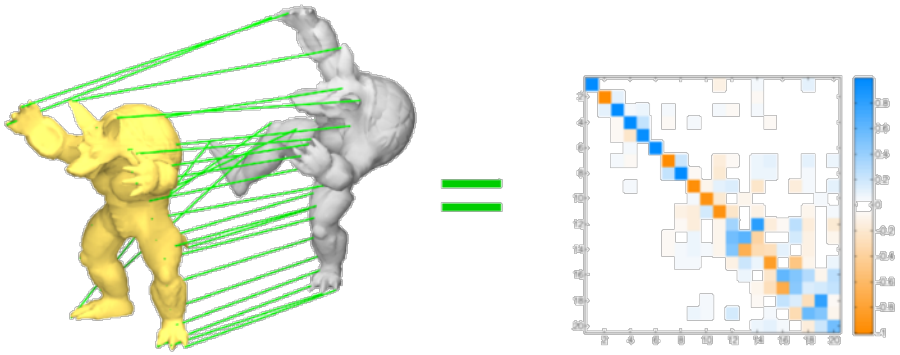
Problem:

The space of possible solutions is highly non-linear, non-convex.

Functional Map Representation

We would like to define a representation of shape maps that is more amenable to direct optimization.

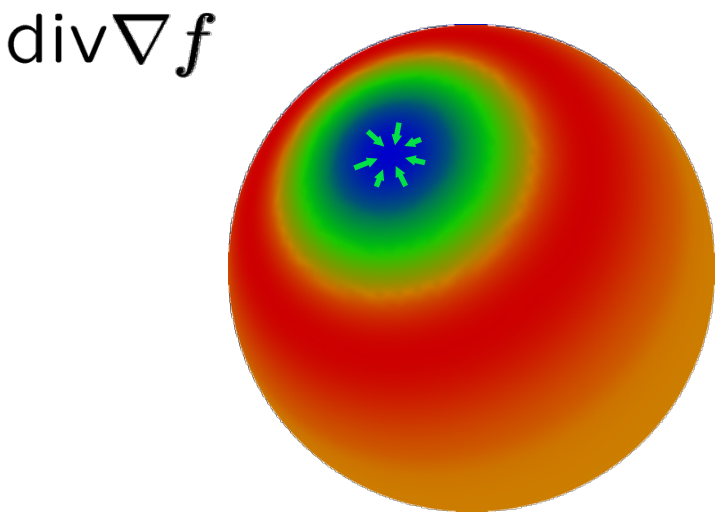
1. A compact representation for “natural” maps.
2. Inherently global and multi-scale.
3. Handles uncertainty and ambiguity gracefully.
4. Allows efficient manipulations (averaging, composition).
5. Leads to simple (linear) optimization problems.



Background: Laplace-Beltrami Operator

Given a compact Riemannian manifold $\mathcal{M}$ without boundary, the Laplace-Beltrami operator Δ :

$$\Delta : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}), \quad \Delta f = \text{ div } \nabla f$$



Laplace-Beltrami Operator

Given a compact surface $\mathcal{M}$ without boundary, the Laplace-Beltrami operator Δ :

1. Is invariant under isometric deformations.

2. Has a countable eigendecomposition:

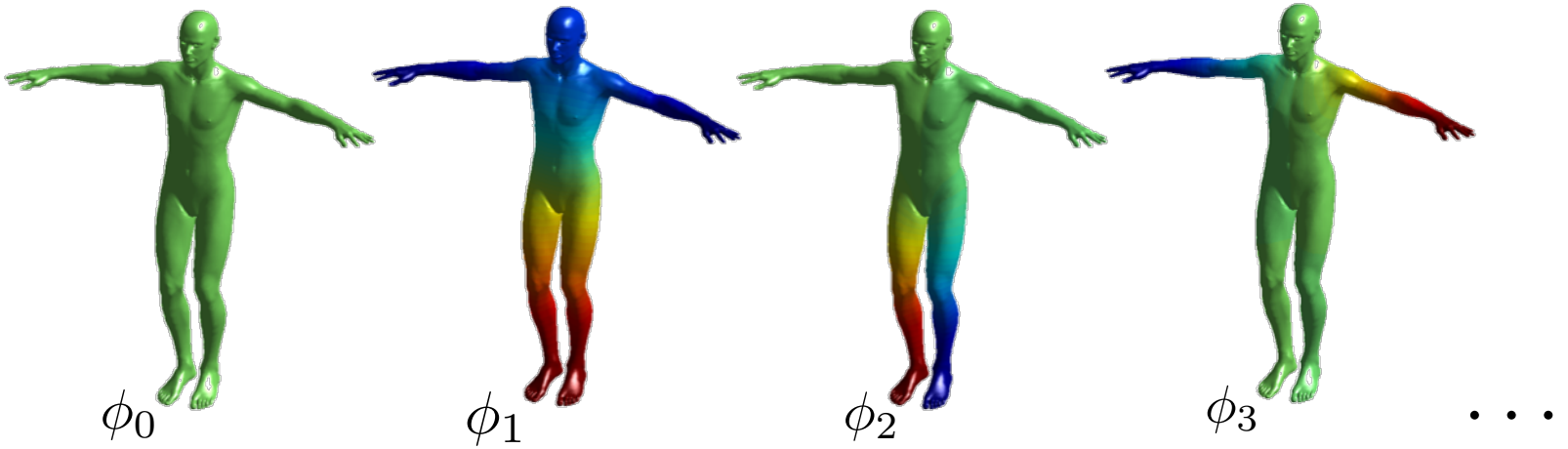
$$\Delta\phi_i = \lambda_i\phi_i$$

   that forms an orthonormal basis for $L^2(\mathcal{M})$.

3. Characterizes the geodesic distances fully.

Laplace-Beltrami Eigenfunctions

The Laplace-Beltrami operator Δ has an eigendecomposition: $\Delta\phi_i = \lambda_i\phi_i$

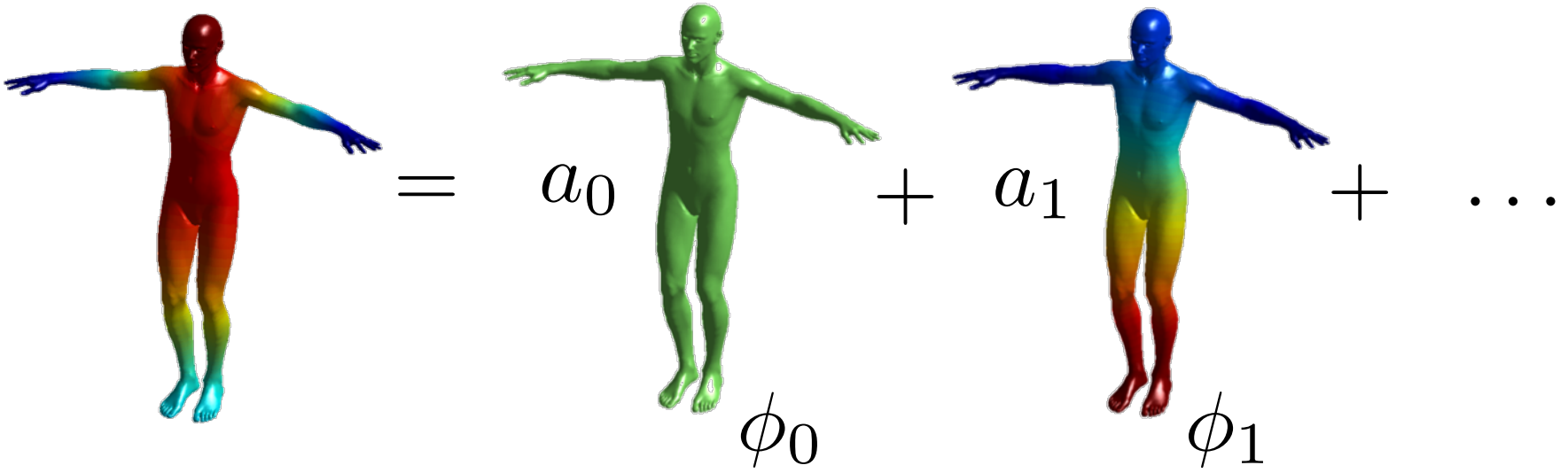


Ordered from low frequency (smoothest) to higher frequency (oscillating).

Laplace-Beltrami Eigenfunctions

…. that forms an orthonormal basis for $L^2(\mathcal{M})$:

Any (square-integrable) $f : \mathcal{M} \to \mathbb{R}$ can be represented as a linear combination of the LB eigenfunctions.

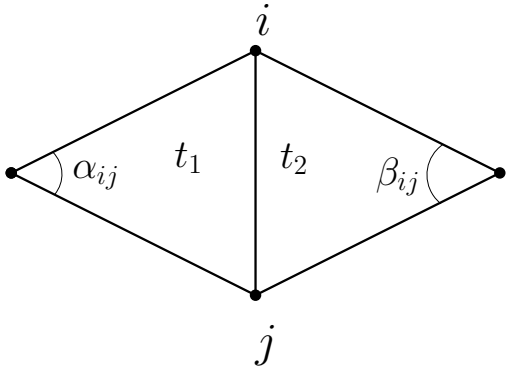


$$f = \sum_{i=0}^{\infty} a_i \phi_i \qquad a_i = \int_{\mathcal{M}} f(x)\phi_i(x)d\mu(x)$$

In the Discrete World

- Functions are defined at vertices of the mesh.
- Integration is defined with respect to a discrete volume measure: $\|f\|_2^2 = f^T A f$

  A - diagonal matrix of area weights.
- Laplacian is discretized as a matrix $L = A^{-1}W$



$$L_{ij} = \frac{1}{2A(j)} \left(\cot(\alpha_{ij}) + \cot(\beta_{ij})\right)$$

Can be derived from 1st order FEM.

In the Discrete World

- Computing the eigenfunctions of the Laplacian reduces to solving the generalized eigenvalue problem:

$$L\phi = \lambda\phi \quad \Leftrightarrow \quad W\phi = \lambda A\phi$$

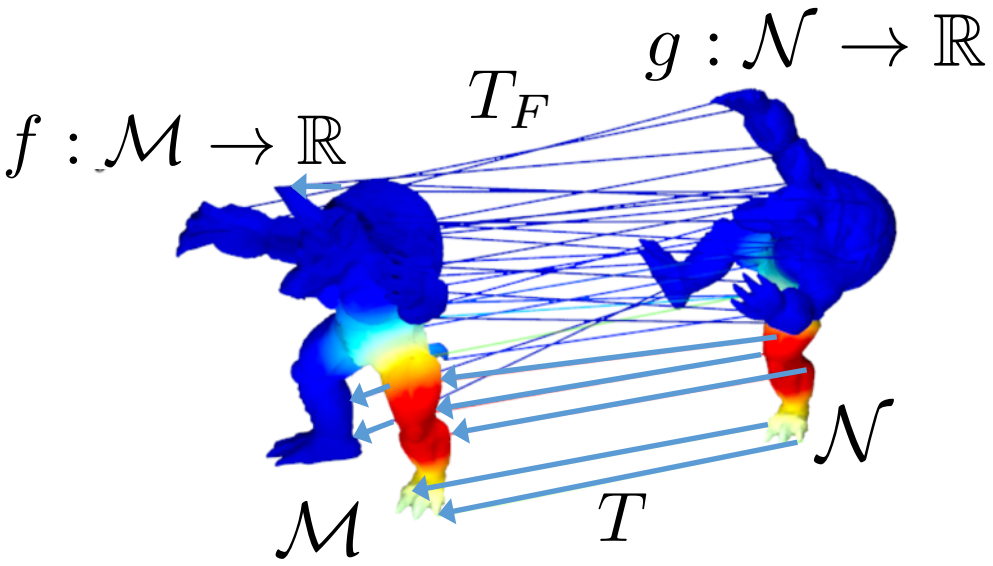
- **`eigs`** function in Matlab
- Both A and W are sparse positive semidefinite.

Number of triangles	**Computation time** (in s)
5000	0.65
25000	2.32
50868	3.6
105032	10

Time to compute 100 basis functions.

Functional Approach to Mappings

Given a pair of shapes and a point-to-point map $T : \mathcal{N} \to \mathcal{M}$



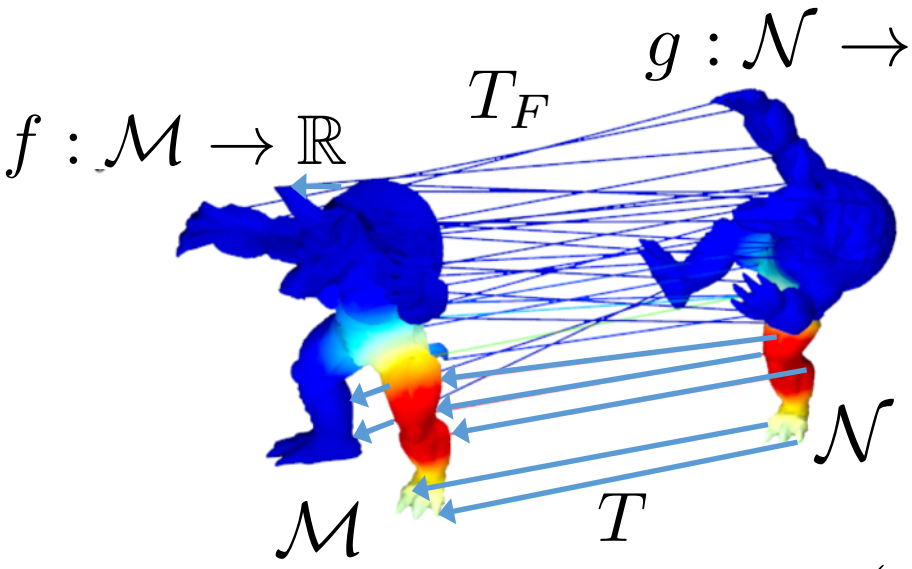
The induced functional correspondence:

$$T_F(f) = g, \quad g = f \circ T$$

Attention: the functional map is in the opposite direction.

Approach

Given a pair of shapes and a point-to-point map $T : \mathcal{N} \to \mathcal{M}$



$$T_F(f) = g, \quad g = f \circ T$$

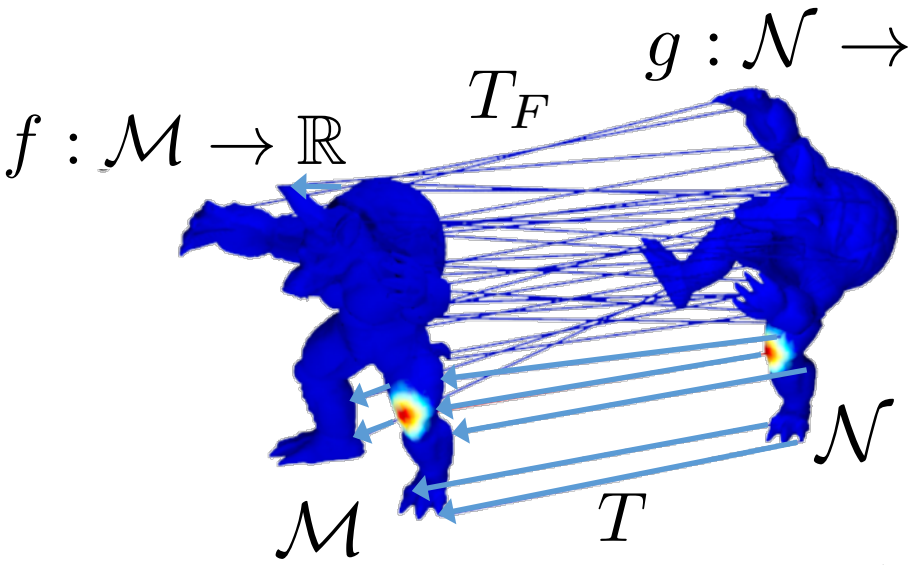
Note that $T_F(f)$ is:

1) Linear $T_F(\alpha_1 f_1 + \alpha_1 f_1) = \alpha_1 T_F(f_1) + \alpha_2 T_F(f_2)$

2) Complete (recover T from indicator functions)

Approach

Given a pair of shapes and a point-to-point map $T : \mathcal{N} \to \mathcal{M}$



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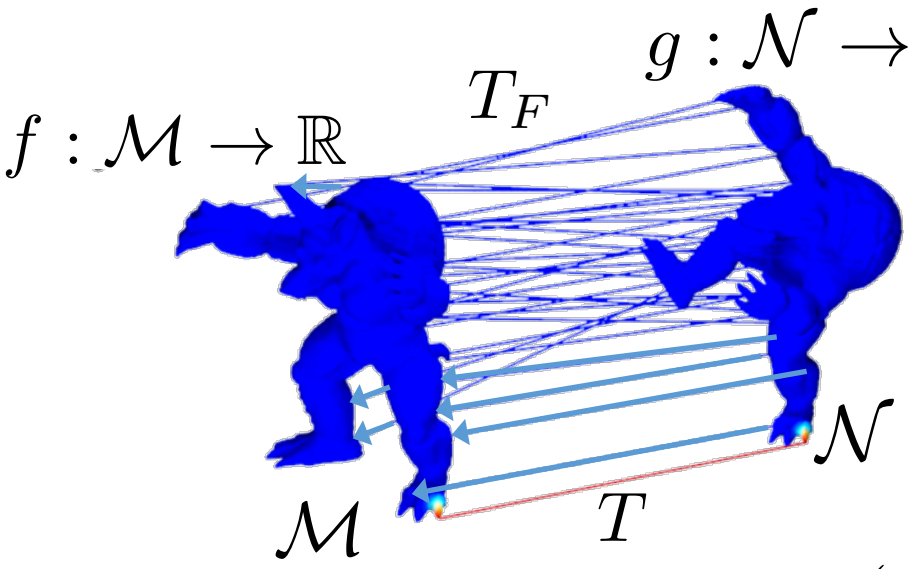

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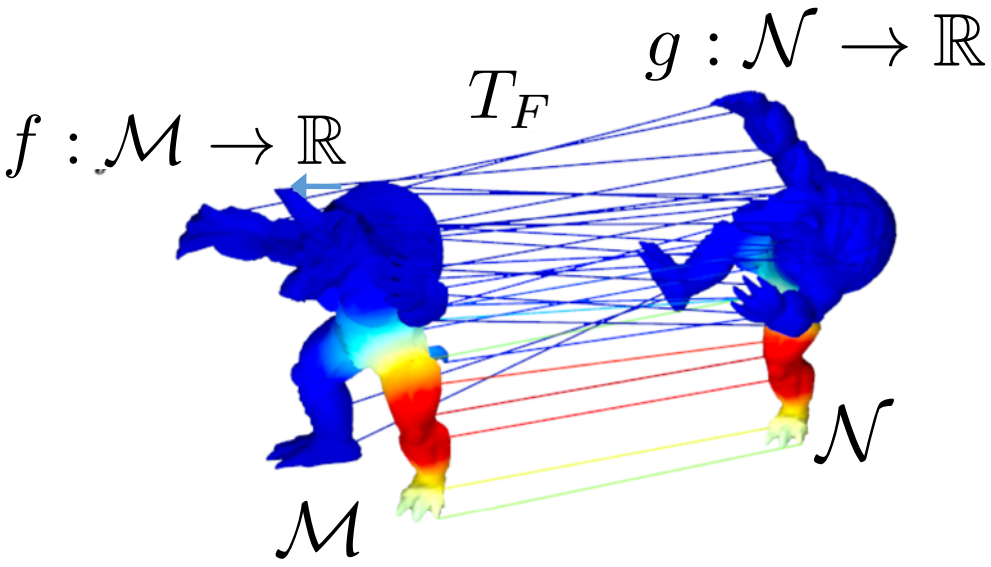
Observation

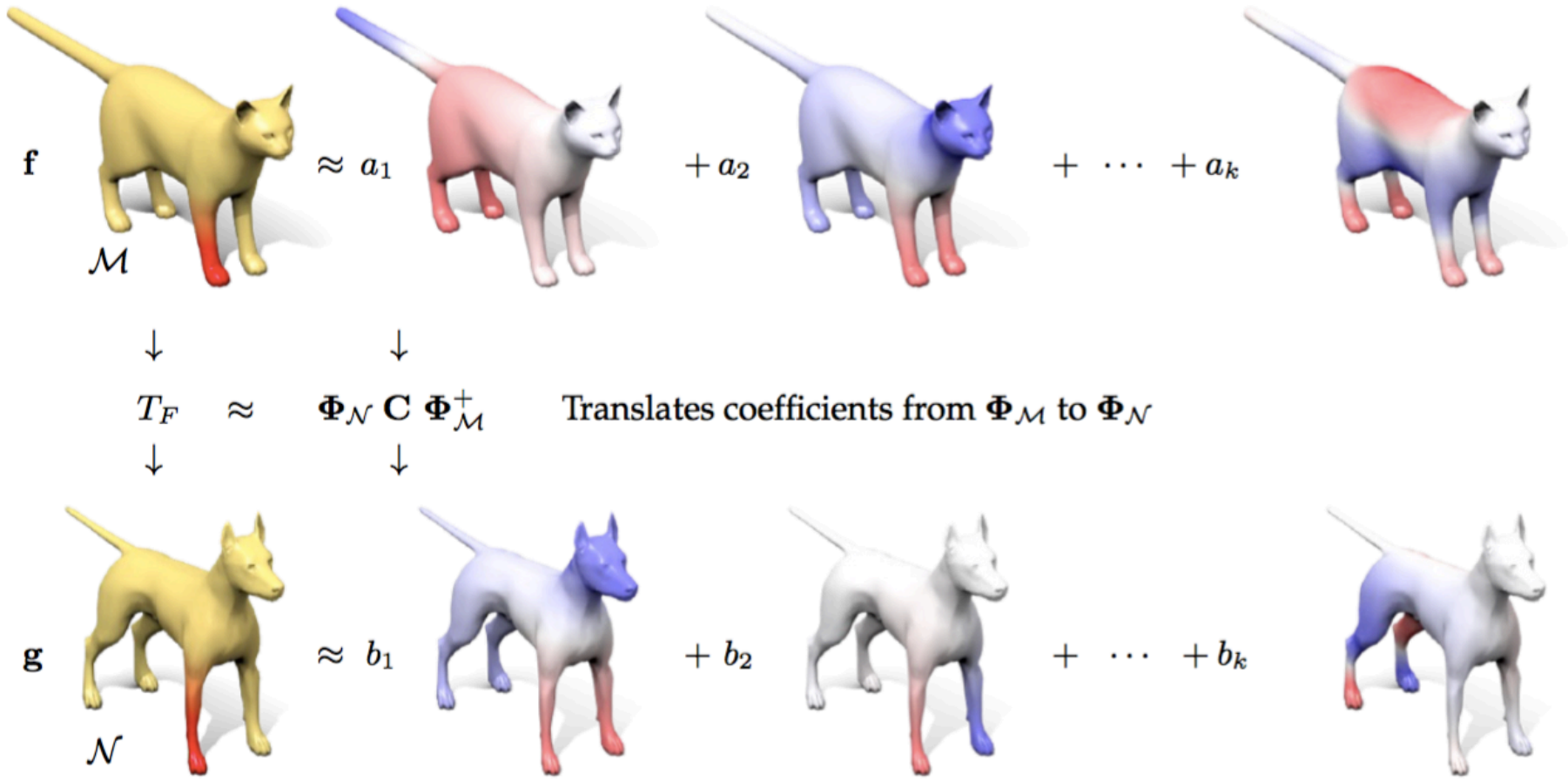


Express both f and $T_F(f)$ in terms of *basis functions:*

$$f = \sum_i a_i \phi_i^{\mathcal{M}} \qquad g = T_F(f) = \sum_i b_i \phi_i^{\mathcal{N}}$$

Since T_F is linear, there is a linear transformation from $\{a_i\}$ to $\{b_j\}$.

Functional Map Definition



Functional map:

matrix C that translates coefficients from $\Phi_{\mathcal{M}}$ to $\Phi_{\mathcal{N}}$.

Functional Maps

Definition:

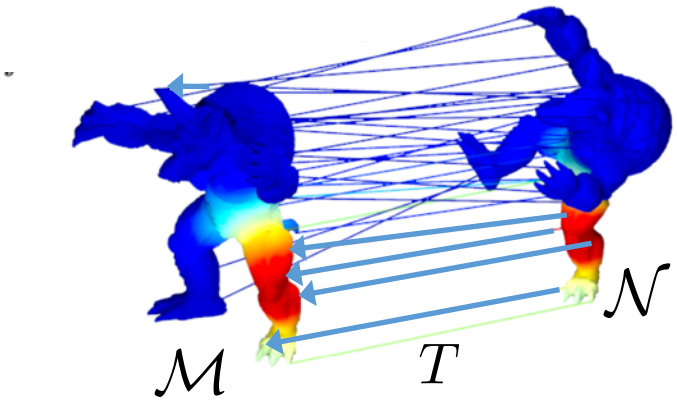
For a fixed choice of basis functions $\{\phi^{\mathcal{M}}\}, \{\phi^{\mathcal{N}}\}$, and a linear transformation T_F between functions, a functional map is a matrix C, s.t. for any $f = \sum_i a_i \phi_i^{\mathcal{M}}$ if $T(f) = \sum_i b_i \phi_i^{\mathcal{N}}$, then:

$$\mathbf{b} = C\mathbf{a}$$

C_{ij} : coefficient of $T_F(\phi_j^{\mathcal{M}})$ in the basis of $\phi_i^{\mathcal{N}}$.

In an orthonormal basis: $C_{ij} = \int_{\mathcal{N}} T_F(\phi_j^{\mathcal{M}}) \phi_i^{\mathcal{N}} \, d\mu$

Example 1



Given two shapes with $n_{\mathcal{M}}, n_{\mathcal{N}}$ points and a map: $T : \mathcal{N} \to \mathcal{M}$

$\mathbf{T} : n_{\mathcal{N}} \times n_{\mathcal{M}}$ matrix encoding the map T,
one 1 per column with zeros everywhere else.

If functions are represented as vectors (in the hat basis), the functional map is given by matrix-vector product:

$$g = \mathbf{T}^T f \qquad C = \mathbf{T}^T$$

Example 2

Given two shapes with $n_{\mathcal{M}}, n_{\mathcal{N}}$ points and a map: $T : \mathcal{N} \to \mathcal{M}$

$\mathbf{T} : n_{\mathcal{N}} \times n_{\mathcal{M}}$ matrix encoding the map T,
one 1 per column with zeros everywhere else.

If functions are represented in the reduced basis:

$\Phi_{\mathcal{M}} : n_{\mathcal{M}} \times k_{\mathcal{M}}$ matrix of the first $k_{\mathcal{M}}$ eigenfunctions of $\Delta_{\mathcal{M}}$ as columns.

$\Phi_{\mathcal{N}} : n_{\mathcal{N}} \times k_{\mathcal{N}}$ matrix of the first $k_{\mathcal{N}}$ eigenfunctions of $\Delta_{\mathcal{N}}$ as columns.

The functional map matrix:

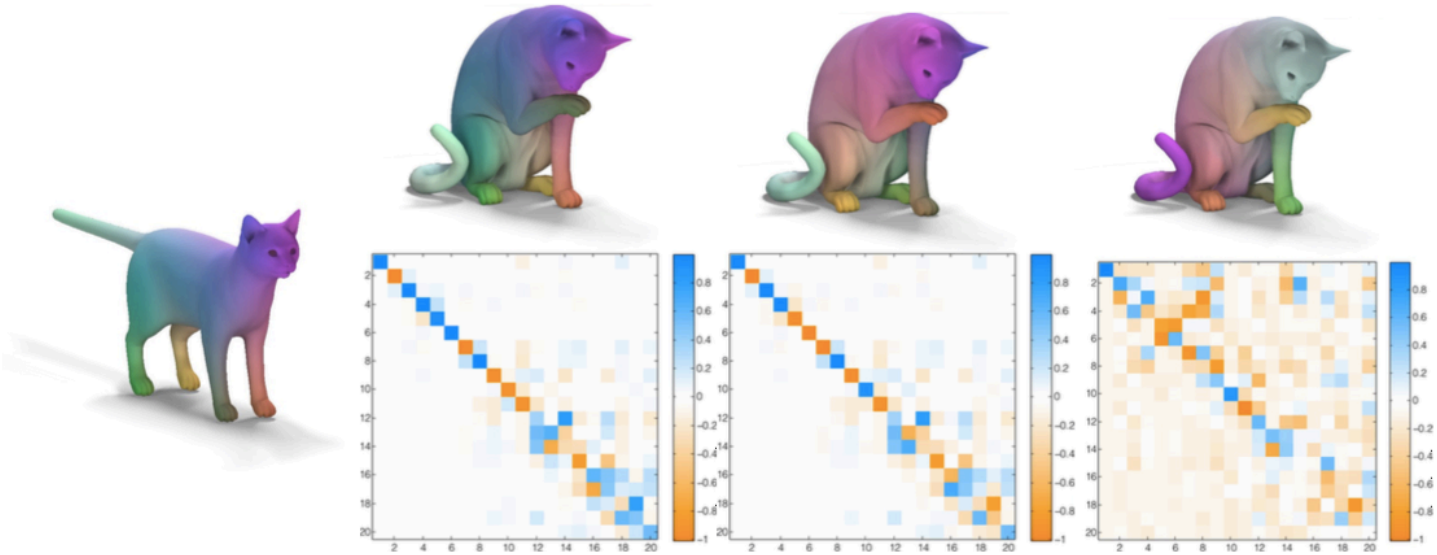
$$C = \Phi_{\mathcal{N}}^{+} \mathbf{T}^T \Phi_{\mathcal{M}} \qquad {}^{+} \text{: left pseudo-inverse.}$$

$$C = \Phi_{\mathcal{N}}^{T} \mathbf{T}^T \Phi_{\mathcal{M}} \qquad \text{if} \quad \Phi_{\mathcal{N}}^{T} \Phi_{\mathcal{N}} = Id$$

$$C = \Phi_{\mathcal{N}}^{T} A_{\mathcal{N}} \mathbf{T}^T \Phi_{\mathcal{M}} \qquad \text{if} \quad \Phi_{\mathcal{N}}^{T} A_{\mathcal{N}} \Phi_{\mathcal{N}} = Id$$

Example Maps in a Reduced Basis

Triangle meshes with pre-computed ponitwise maps



(a) source (b) ground-truth map (c) left to right map (d) head to tail map

"Good" maps are close to being diagonal

Functional Map algebra

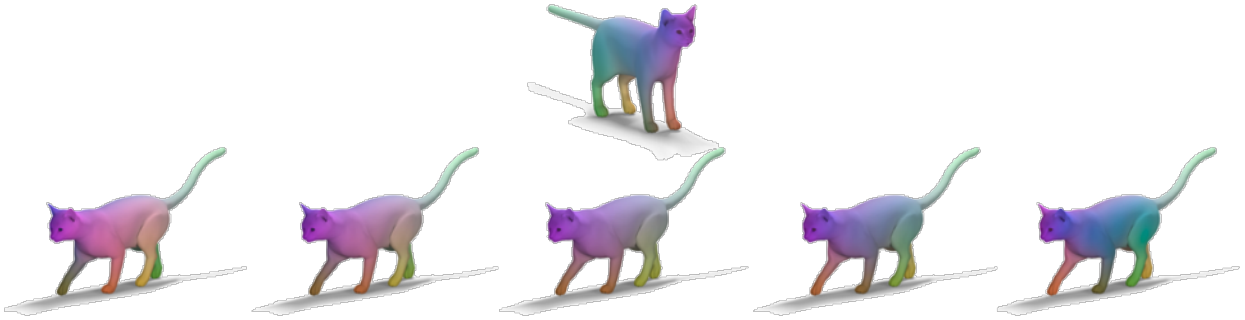
1. Map composition becomes matrix multiplication.

2. Map inversion is matrix inversion (in fact, transpose).

3. Algebraic operations on functional maps are possible.

E.g. interpolating between two maps with

$$C = \alpha C_1 + (1-\alpha) C_2.$$



(a) $\alpha = 0$ (b) $\alpha = 0.25$ (c) $\alpha = 0.5$ (d) $\alpha = 0.75$ (e) $\alpha = 1$

Shape Matching

In practice we do not know C. Given two objects our goal is to find the correspondence.

How can the functional representation help to compute the map in practice?

Matching via Function Preservation

Suppose we don't know C. However, we expect a pair of functions $f : \mathcal{M} \to \mathbb{R}$ and $g : \mathcal{N} \to \mathbb{R}$ to correspond. Then, C must be s.t.

$$C\mathbf{a} \approx \mathbf{b}$$

where $f = \sum_i a_i \phi_i^{\mathcal{M}}, \quad g = \sum_i b_i \phi_i^{\mathcal{N}}.$

Given enough $\{\mathbf{a}, \mathbf{b}\}$ pairs, we can recover C through *a linear least squares system*.

Map Constraints

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$$C\mathbf{a} \approx \mathbf{b}$$

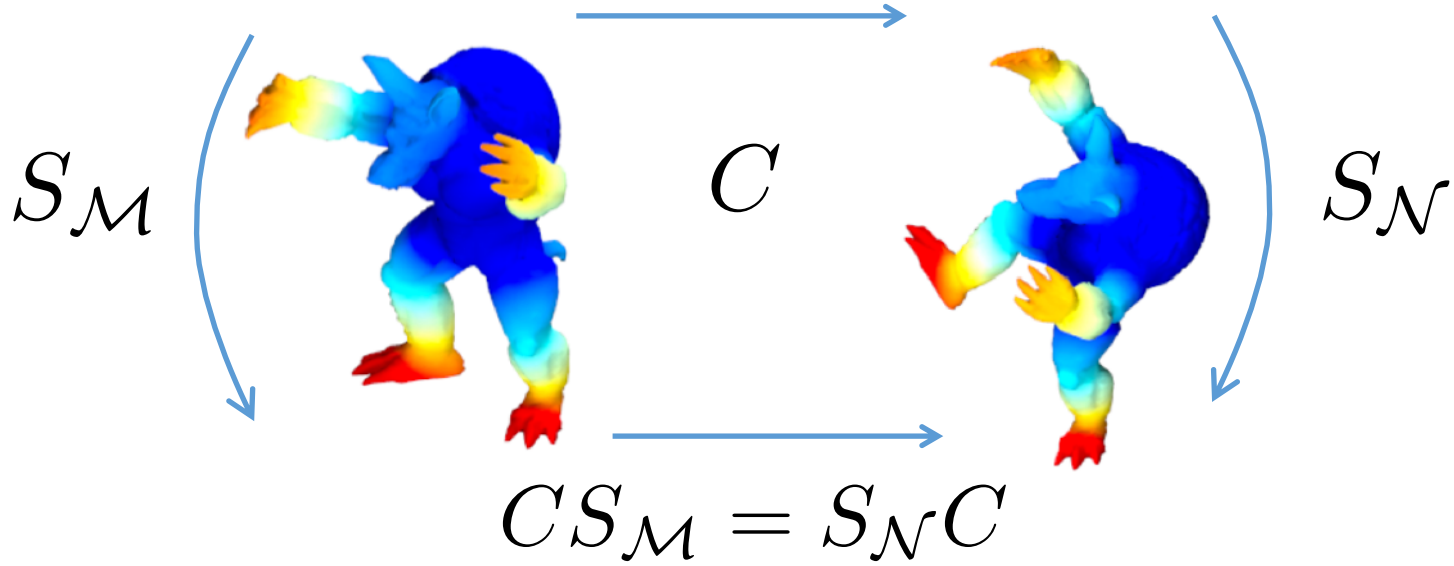
Function preservation constraint is general and includes:

- Attribute (e.g., color) preservation.
- Descriptor preservation (e.g. Gauss curvature).
- Landmark correspondences (e.g. distance to the point).
- Part correspondences (e.g. indicator function).

Commutativity Constraints

Regularizations:

Commutativity with other operators:



Note that the energy: $\|CS_{\mathcal{M}} - S_{\mathcal{N}}C\|_F^2$ is *quadratic* in C.

Regularization

Lemma 1:

The mapping is *isometric*, if and only if the functional map matrix commutes with the Laplacian:

$$C\Delta_{\mathcal{M}} = \Delta_{\mathcal{N}}C$$

Implies that isometries result in diagonal functional maps.

Regularization

Lemma 2:

The mapping is *locally volume preserving*, if and only if the functional map matrix is *orthonormal*.

Basic Pipeline

Given a pair of shapes $\mathcal{M}, \mathcal{N}$:

1. Compute the first *k* (~80-100) eigenfunctions of the Laplace-Beltrami operator. Store them in matrices: $\Phi_{\mathcal{M}}, \Phi_{\mathcal{N}}$

2. Compute descriptor functions (e.g., Wave Kernel Signature) on $\mathcal{M}, \mathcal{N}$. Express them in $\Phi_{\mathcal{M}}, \Phi_{\mathcal{N}}$, as columns of : $\mathbf{A}, \mathbf{B}$

3. Solve $C_{\text{opt}} = \arg\min_{C} \|C\mathbf{A} - \mathbf{B}\|^2 + \|C\Delta_{\mathcal{M}} - \Delta_{\mathcal{N}}C\|^2$

   $\Delta_{\mathcal{M}}, \Delta_{\mathcal{N}}$: diagonal matrices of eigenvalues of LB operator

4. Convert the functional map C_{opt} to a point to point map T.

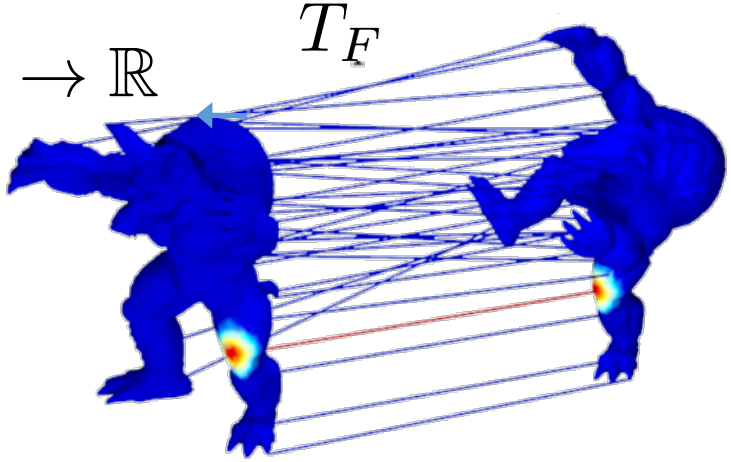
Conversion to point-to-point

Given a functional map C, we would like to convert to to a point-to-point map.

$$g : \mathcal{N} \to \mathbb{R}$$

$$f : \mathcal{M} \to \mathbb{R}$$

$$T_F$$



Option 1: declare $T(x) = \arg\max_y \Phi_{\mathcal{N}} C \delta_x$

Problems: high computational complexity $O(n_{\mathcal{M}} n_{\mathcal{N}})$, low accuracy.

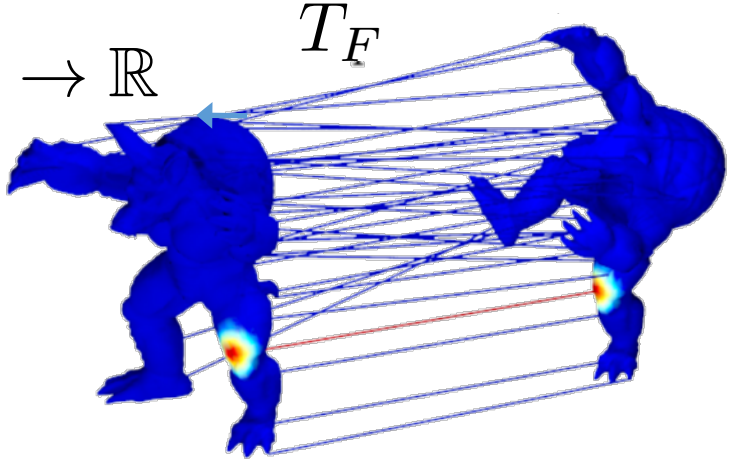
Conversion to point-to-point

Given a functional map C, we would like to convert to to a point-to-point map.

$g : \mathcal{N} \to \mathbb{R}$

$f : \mathcal{M} \to \mathbb{R}$

T_F



Option 2: declare $T(x) = \arg\min_y \|\delta_y - C\delta_x\|_2$

Advantages: computational complexity $O(n_{\mathcal{M}} \log n_{\mathcal{N}})$, higher accuracy (e.g., works with the identity map).

Incorporating Orthonormality

In many practical situations we would expect a volume-preserving map, which implies:

$$C^T C = Id$$

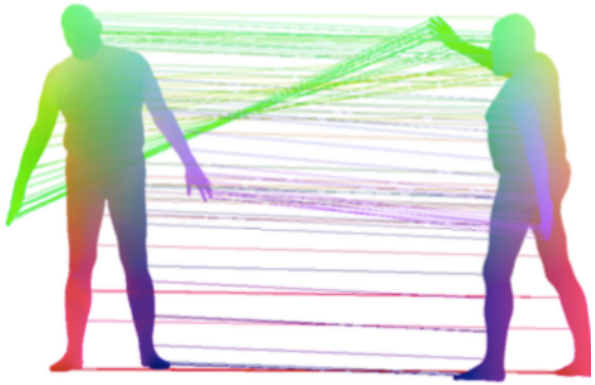
Option: use post-processing to enforce this constraint.

Iterate:

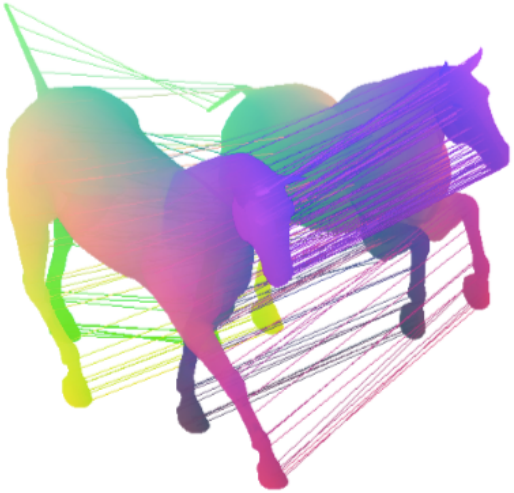
1. Compute the point-to-point map T.
2. Solve for the functional map: $\underset{C,\ \text{s.t.}\ C^T C = Id}{\arg\min} \sum_{x \in \mathcal{M}} \|C\delta_x - \delta_{T(x)}\|_2^2$

Exactly the same objective as ICP, but in higher dimension. Can use the same method!

Results



FAUST [Bogo et al. ’14]

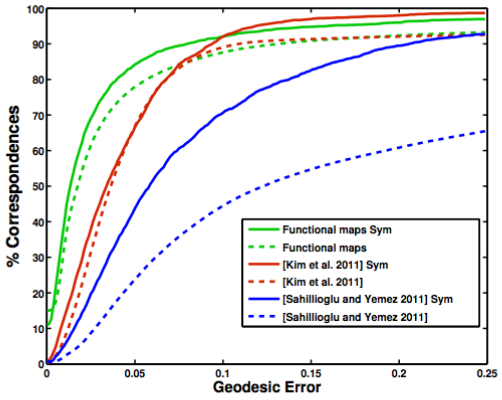


TOSCA [Bronstein et al. ‘08]

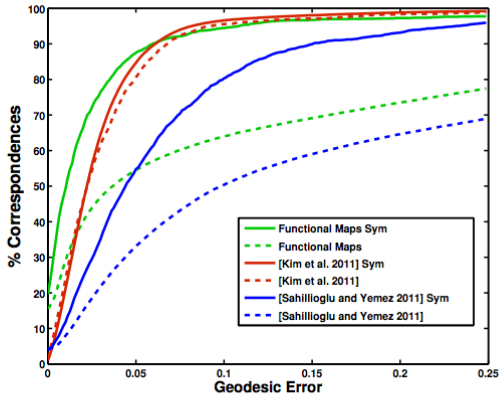
SCAPE [Anguelov et al. ‘05]

Results

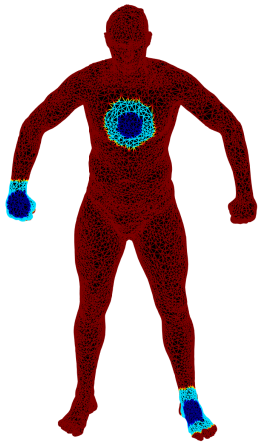
A very simple method that puts together many constraints and uses 100 basis functions gives reasonable results:



SCAPE



TOSCA



radius 0.025

radius 0.05

O., Ben-Chen, Solomon, Butscher, Guibas, Functional maps: a flexible representation of maps between shapes, 2012

Segmentation Transfer without P2P

To transfer functions we do not need to convert functional to pointwise maps.

E.g. we can also transfer segmentations: for each segment, transfer its indicator function, and for each point pick the segment that gave the highest value.