Iterative Closest Point (ICP)

- Approach: iterate between finding correspondences and finding the transformation:

Given a pair of shapes, X and Y, iterate:

1. For each $x_i \in X$ find **nearest** neighbor $y_i \in Y$.
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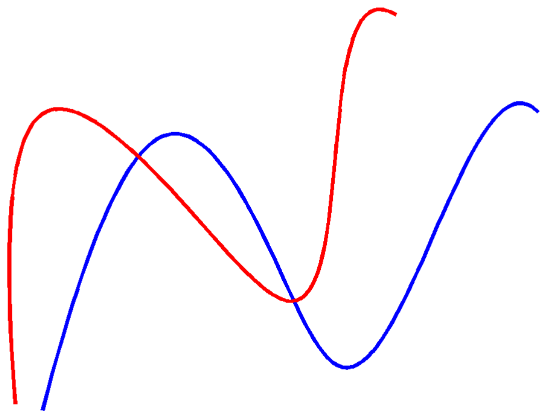

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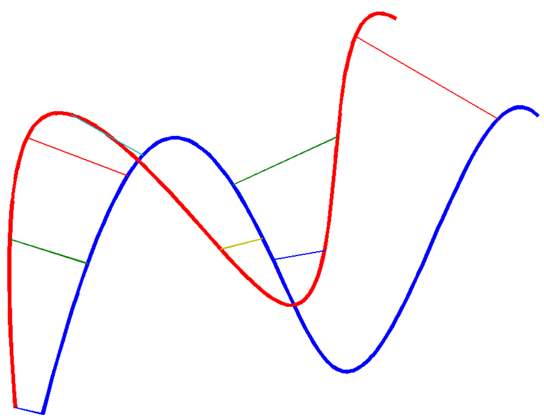

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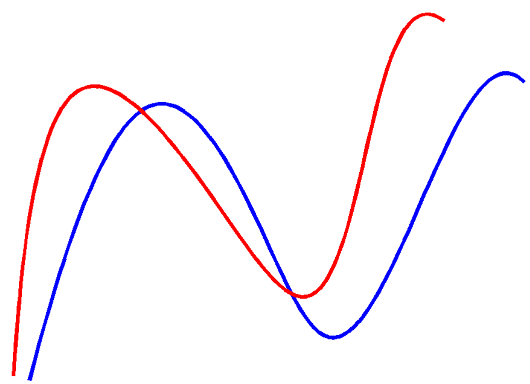

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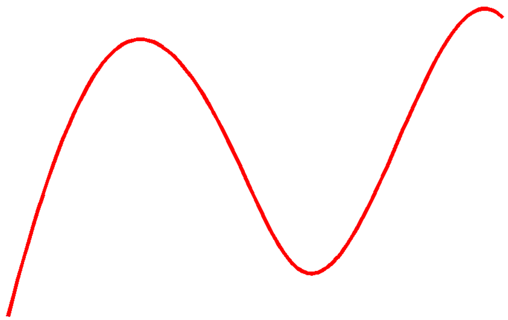

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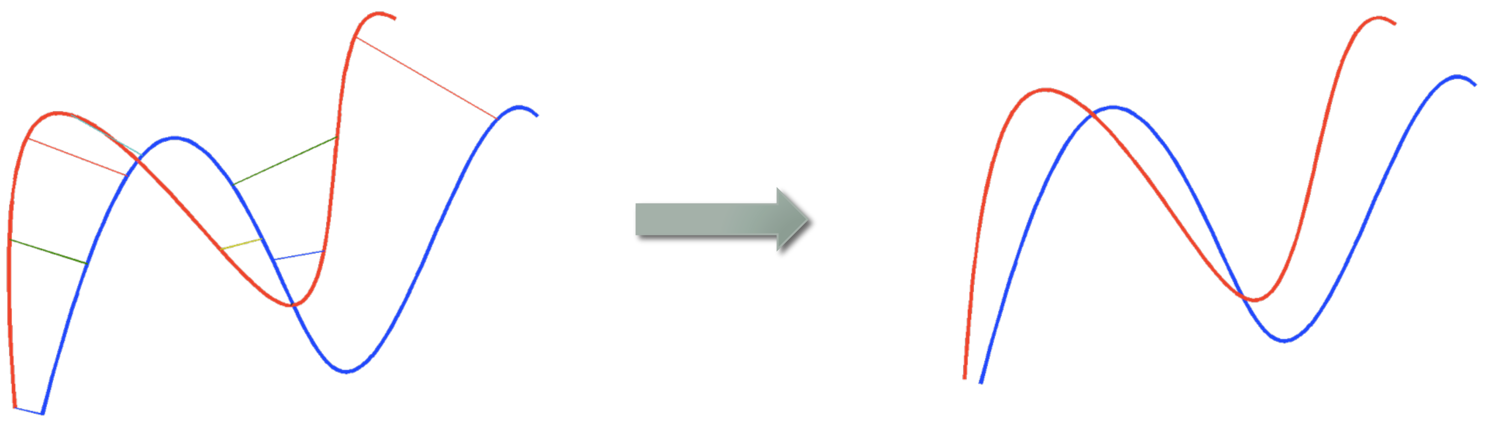
Iterative Closest Point

- Requires two main computations:

1. Computing nearest neighbors.
2. Computing the optimal transformation



ICP: Nearest Neighbor Computation

Closest points

$$y_i = \arg\min_{y \in Y} \|y - x_i\|$$

- How to find closest points **efficiently**?
- Straightforward complexity: $\mathcal{O}(MN)$

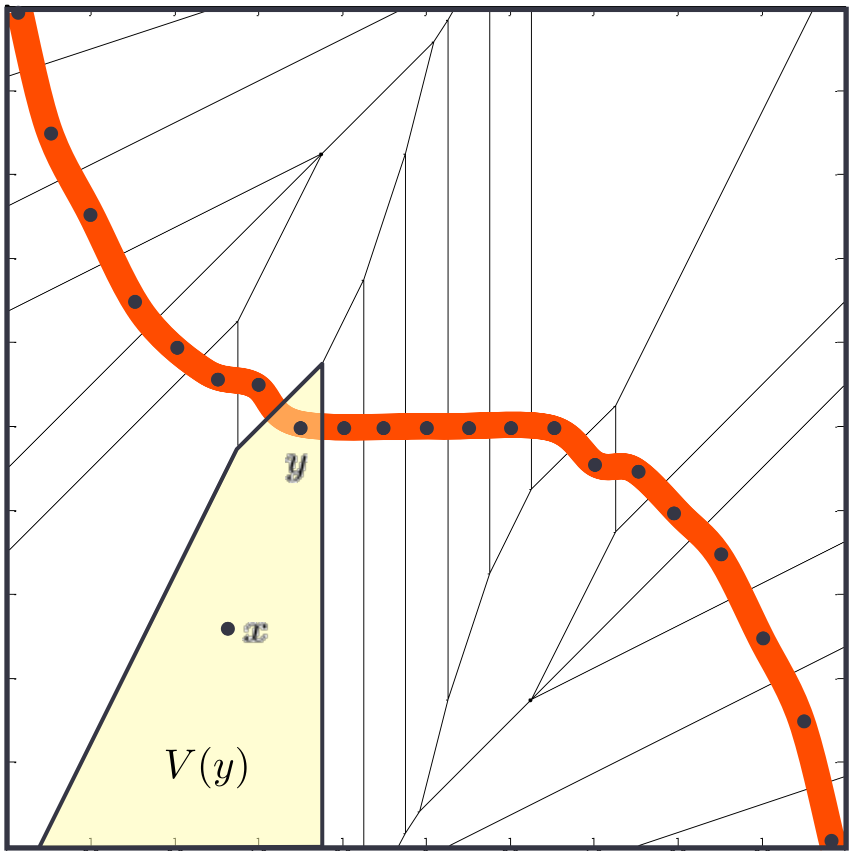
  M number of points on X, N number of points on Y.

- Y divides the space into **Voronoi cells**

$$V(y \in Y) = \{z \in \mathbb{R}^3 : \|y - z\| < \|y' - z\| \; \forall \; y' \in Y \neq y\}$$

- Given a query point y, **determine to which cell it belongs**.

Closest points: Voronoi Cells



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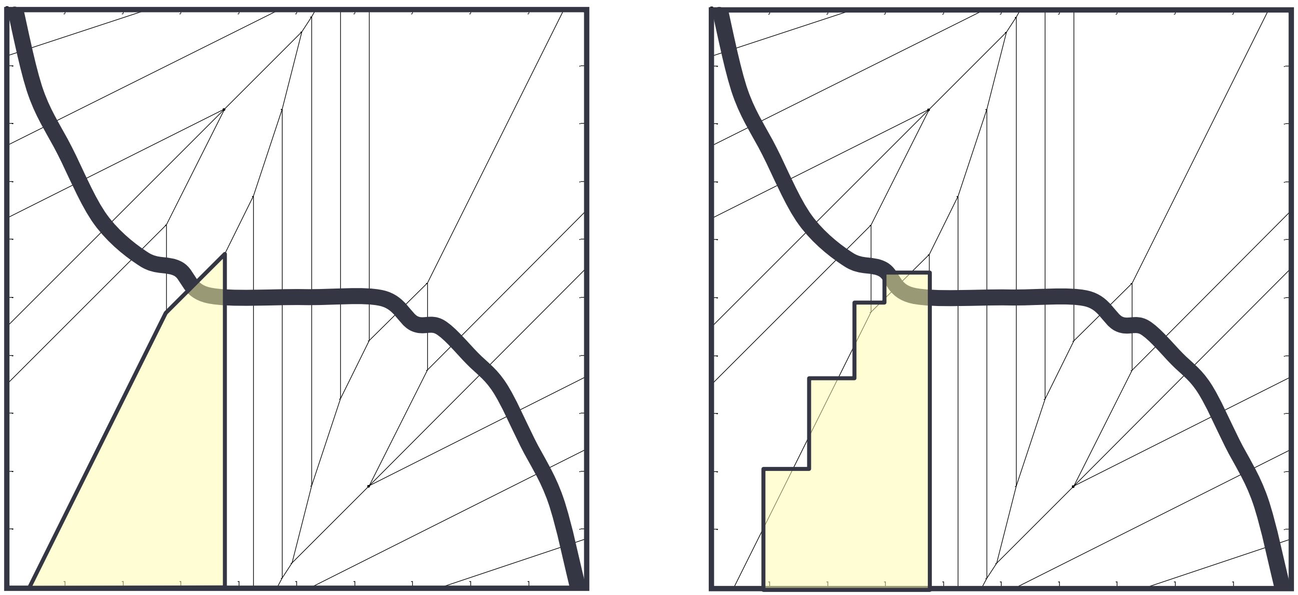
Source:
M. Bronstein

Closest points: Voronoi Cells

Approximate nearest neighbors



M. Bronstein

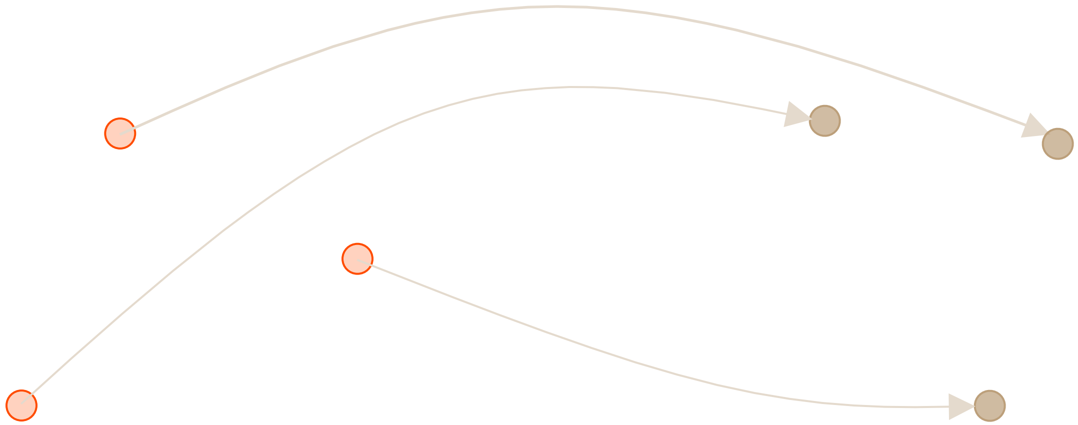
- To reduce search complexity, **approximate** Voronoi cells.
- Use **binary space partition trees** (e.g. **kd-trees** or **octrees**).
- Approximate nearest neighbor search complexity: $\mathcal{O}(N \log M)$.

ICP: Optimal Transformation

Problem Formulation:

1. Given two sets points: $\{x_i\}, \{y_i\}, i = 1..n$ in $\mathbb{R}^3$. Find the rigid transform: $\mathbf{R}, t$ that minimizes:

$$\sum_{i=1}^{N} \|\mathbf{R}x_i + t - y_i\|_2^2$$



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$$\sum_{i=1}^{N} \|\mathbf{R}x_i + t - y_i\|_2^2$$
2. Closed form solution with rotation matrices:
    1. Construct: $C = \sum_{i=1}^{N}(y_i - \mu^Y)(x_i - \mu^X)^T$ where $\mu^X = \frac{1}{N}\sum_i x_i,$ $\mu^Y = \frac{1}{N}\sum_i y_i$
    2. Compute the SVD of C: $C = U\Sigma V^T$
        1. If $\det(UV^T) = 1, R_{\text{opt}} = UV^T$
        2. Else $R_{\text{opt}} = U\tilde{\Sigma}V^T, \tilde{\Sigma} = \text{diag}(1, 1, \ldots, -1)$
    3. Set $t_{\text{opt}} = \mu^Y - R_{\text{opt}}\mu^X$

Note that C is a 3x3 matrix. SVD is very fast.

Arun et al., Least-Squares Fitting of Two 3-D Point Sets

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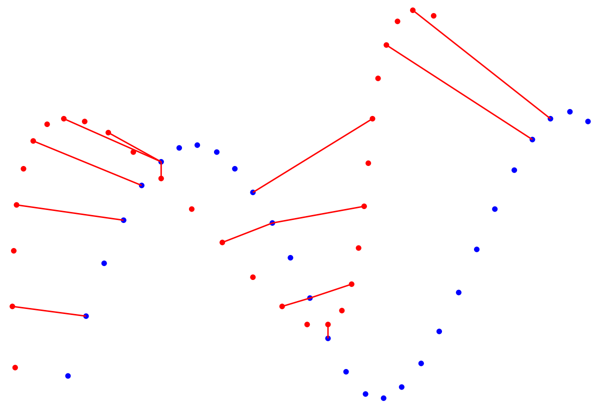
Convergence:

- at each iteration $\sum_{i=1}^{N} d^2(x_i, Y)$ decreases.
- Converges to local minimum
- Good initial guess: global minimum.

[Besl&McKay92]

Variations of ICP

1. Selecting source points (from one or both scans): sampling
2. Matching to points in the other mesh
3. Weighting the correspondences
4. Rejecting certain (outlier) point pairs
5. Assigning an error metric to the current transform
6. Minimizing the error metric w.r.t. transformation



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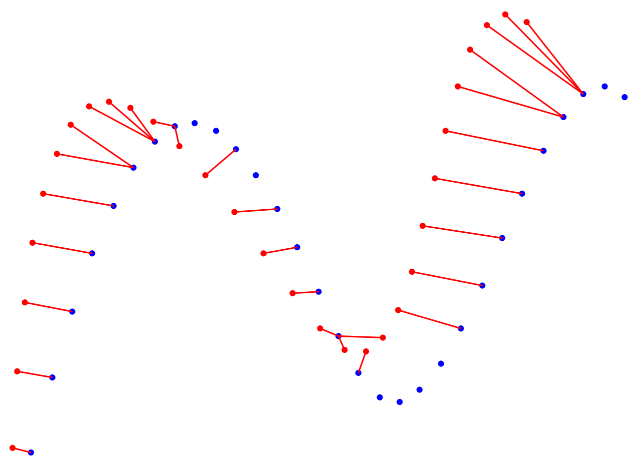

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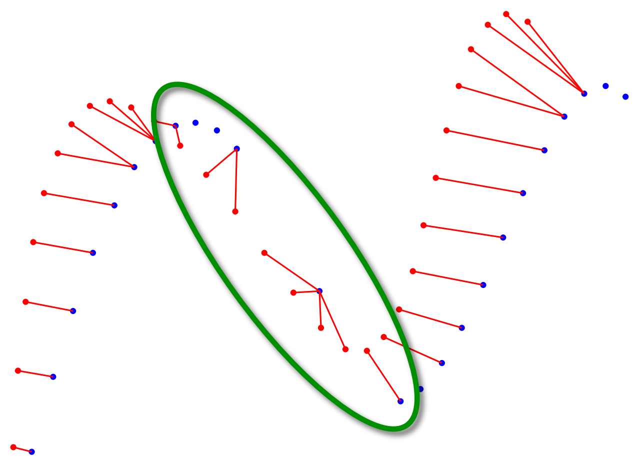
Problem:
uneven sampling



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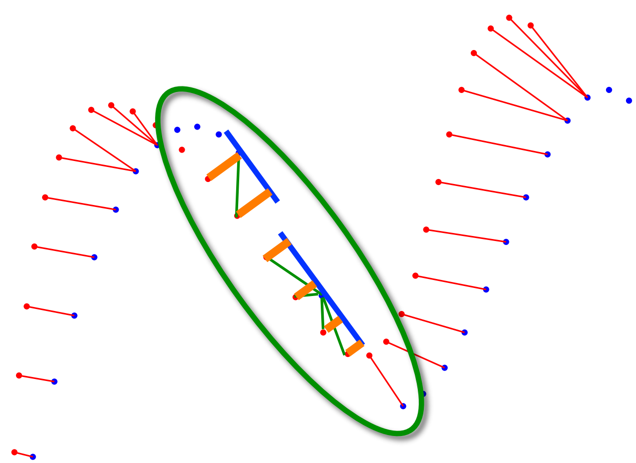
Solution:
Minimize distance to the tangent plane



Chen, Medioni, '91

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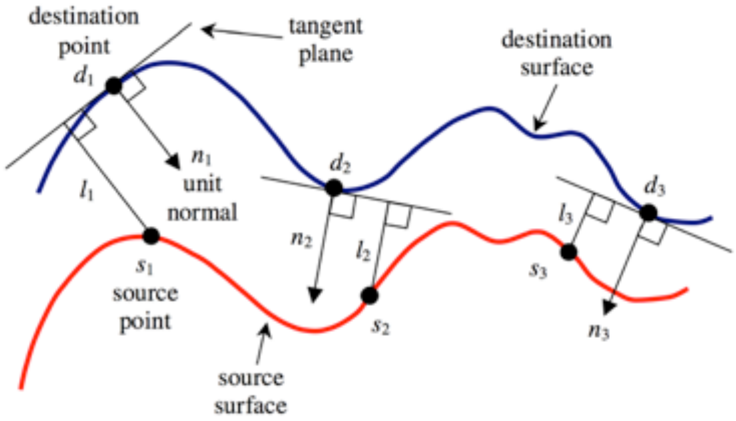

Kok-Lim Low, '04

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Question:

How to minimize the error?

Challenge:

Although the error is **quadratic** (linear derivative), the space of rotation matrices is **not linear.**

Problem:

No closed form solution!

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Common Solution:

Linearize rotation. Assume rotation angle is small.

$$\mathbf{R}x_i \approx x_i + r \times x_i$$

r : axis,

$\|r\|_2$: angle of rotation.

Note: follows from Rodrigues's formula

$$R(r,\alpha)x_i = x_i \cos(\alpha) + (r \times x_i)\sin(\alpha) + r(r^T x_i)(1 - \cos(\alpha))$$

And first order approximations: $\sin(\alpha) \approx \alpha, \quad \cos(\alpha) \approx 1$

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$$E(r,t) = \sum_{i=1}^{N} \left((x_i + r \times x_i + t - y_i)^T \mathbf{n}_{y_i}\right)^2$$
$$= \sum_{i=1}^{N} \left((x_i - y_i)^T \mathbf{n}_{y_i} + r^T (x_i \times \mathbf{n}_{y_i}) + t^T \mathbf{n}_{y_i}\right)^2$$

Setting: $\frac{\partial}{\partial r} E(r,t) = 0$ and $\frac{\partial}{\partial t} E(r,t) = 0$ leads to a 6x6 linear system

$$Ax = b$$

$$x = \begin{pmatrix} r \\ t \end{pmatrix} \quad A = \sum \begin{pmatrix} x_i \times \mathbf{n}_{y_i} \\ \mathbf{n}_{y_i} \end{pmatrix} \begin{pmatrix} x_i \times \mathbf{n}_{y_i} \\ \mathbf{n}_{y_i} \end{pmatrix}^T \quad b = \sum (y_i - x_i)^T \mathbf{n}_{y_i} \begin{pmatrix} x_i \times \mathbf{n}_{y_i} \\ \mathbf{n}_{y_i} \end{pmatrix}$$