Towards a trichotomy for quantified H-coloring.

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Abstract

Hell and Nešetřil proved that the H-coloring problem is NP-complete if, and only if, H is bipartite. In this paper, we investigate the complexity of the quantified H-coloring problem (a restriction of the quantified constraint satisfaction problem to undirected graphs). We introduce this problem using a new two player coloring game. We prove that the quantified H-coloring problem is:

1. tractable, if H is bipartite;
2. NP-complete, if H is not bipartite and not connected; and,
3. Pspace complete, if H is connected and has a unique cycle, which is of odd length.

We conjecture that the last case extends to all non-bipartite connected graphs.

1 Introduction

A very natural generalisation of graph coloring problems is defined in terms of graph homomorphism; the problem takes as input a graph G and accepts it if, and only if, there exists a homomorphism into a fixed graph H. This problem is known as the *H-coloring problem*. In [11], Hell and Nešetřil proved that the class of H-coloring problems has a dichotomy: the problem is tractable if H is bipartite and NP-complete otherwise. Constraint satisfaction problems are closely related to the H-coloring problem. In the case of constraint satisfaction problems with boolean domains, known as generalised satisfiability, a dichotomy has been proved [15] using an algebraic approach. This approach has been extended to arbitrary domains (see for example [13, 5]) and though some progress has been made, the *dichotomy conjecture* [9] that states that every constraint satisfaction problem is either tractable or NP-complete, is still open. Building on the result from [15], dichotomy results were proved independently in [7] and [8] for *quantified satisfiability problems without constants* (In [15], the result was proved only in the case where constants were allowed): the restrictions of the general problem are either tractable or Pspace-complete. A partial *trichotomy* result was proved for *quantified constraint satisfaction problems* in [4]: for a restricted, but large, class of quantified constraint satisfaction problems, the authors prove

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that every problem is either tractable, NP-complete or Pspace-complete. A full classification for this problem is even harder than proving the dichotomy conjecture. Hell and Nešetřil's dichotomy motivated us to concentrate for now on the restriction of quantified constraint satisfaction problems to graphs, which we call *quantified H-coloring*. This restriction ensures that we can deal directly with explicit graph properties rather than having to rely on the algebraic method used in the case of the constraint satisfaction problem [13, 5], which uses more indirect characterisations based on properties of closure functions of the structures, rather than "explicit" properties of the structures. The proof of Hell and Nešetřil uses more explicit properties of graphs and we believe that a similar approach can be adopted to study the quantified H-coloring problem. Generalisations of graph coloring problems which are (in their unrestricted form) Pspace-complete are investigated in [2, 3]. As usual with this complexity class, the problems are definable via two-player games. We adopt a similar stance and introduce the quantified H-coloring by means of a new coloring game. We show that the quantified H-coloring problem is tractable if H is bipartite; NP-complete if H is not bipartite and not connected; and, Pspace-complete if H is connected and contains a unique cycle which is of odd length. Thus, as a corollary, we obtain the main contribution of this paper: a trichotomy when H is a graph with at most one cycle. In the tractable and NP-complete cases, we use *rewriting rules* over the input which allow us to derive the existence of an equivalent input. We prove that if certain paths are present in the input then it must necessarily be rejected, otherwise it can be rewritten in a specific form which implies that the problem collapses to NP. Moreover, when H is a bipartite graph, we provide a reduction to 2-colorability, which proves the tractability of the problem. When H is not connected and not bipartite, we derive from Hell and Nešetřil's dichotomy theorem that the quantified H-coloring problem is NP-complete. In the Pspace-complete case, we first study *odd Catherine wheels*, which are graphs that consists of an odd cycle together with disjoint paths, where each path is attached to a different vertex of the cycle. Using some structural properties of these graphs, we prove that a variant of quantified satisfiability, known to be also Pspace-complete, reduces to the quantified H-coloring, whenever H is an odd Catherine wheel. As a corollary, we prove that the quantified H-coloring problem is Pspace-complete, whenever H has only one cycle, that is of odd length. The paper is organised as follows. In Section 2, we introduce a new coloring game and use this game to define the quantified H-coloring problem. In Section 3, we study cases where the quantified H-coloring problem is in NP. In Section 4, we prove that the quantified H-coloring problem is Pspace-complete, whenever H is a connected graph with a unique cycle which is of odd length. Finally, in Section 5, we prove the main result and conclude with a conjecture and a few open problems. The few omitted proofs are available in the appendix.

2 A new coloring game

In this paper, we only ever consider finite simple undirected graphs without self-loops. Given graphs G and H, a *homomorphism* f from G to H, denoted $G \xrightarrow{f} H$, is a vertex-mapping function $f : V(G) \to V(G)$ such that $\{x,y\} \in E(G)$ implies $\{f(x), f(y)\} \in E(H)$. We write $G \longrightarrow H$ if there exists a homomorphism from G to H. The *H-coloring problem* takes as input a graph G, which is a yes-instance if, and only if, there exists a homomorphism from G to H. Let $n \geq 0$. An *n-partitioned graph* $\mathscr{G}$ consists of a graph G together with a partition $\{U_1, X_2, U_3, X_4, \ldots, U_{2n+1}, X_{2n+2}\}$ of $V(G)$. In the following, G will always designate the underlying graph of $\mathscr{G}$. Let $\mathscr{G}$ be an n-partitioned graph and H be a (non-partitioned) graph. The *$(\mathscr{G}, H)$-game* is a two-player game, that pitches Opponent (male) against Proponent (female). Opponent plays on the universal partitions (the sets U_i) and Proponent plays on the existential partitions (the sets X_i). They play alternate partitions, in ascending order, until all the partitions have been played. The game goes as follows. For $0 \leq i \leq n$:

- for every vertex in partition U_{2i+1}, Opponent chooses a vertex in H: *i.e.* he gives a function

$f_{U_{2i+1}} : U_{2i+1} \to V(H)$; and,

- for every vertex in partition X_{2i+2}, Proponent chooses a vertex in H: *i.e.* she gives a function $f_{X_{2i+2}} : X_{2i+2} \to V(H)$.

Proponent wins if, and only if, the function $f := f_{U_1} \cup f_{X_2} \cup \ldots \cup f_{X_{2n+2}}$ is a homomorphism from G to H. We say that there exists an *alternating-homomorphism* from the n-partitioned graph $\mathscr{G}$ to the (non-partitioned) graph H, and we write $\mathscr{G} \xrightarrow{alt} H$ if, and only if, for all functions $f_{U_1} : U_1 \to |H|$, there exists a function $f_{X_2} : X_2 \to |H|$, such that, …, for all functions $f_{U_{2n+1}} : U_{2n+1} \to |H|$, there exists a function $f_{X_{2n+2}} : X_{2n+2} \to |H|$, such that, $f_{U_1} \cup f_{X_2} \cup \ldots f_{U_{2n+1}} \cup f_{X_{2n+2}}$ is a homomorphism from G to H. The following is a direct consequence of the above definitions.

Proposition 1. *Let $\mathscr{G}$ be a partitioned graph and H be a graph. $\mathscr{G} \xrightarrow{alt} H$ if, and only if, Proponent has a winning strategy in the $(\mathscr{G}, H)$-game.*

We define the *quantified H-coloring* problem as the decision problem which takes as input a partitioned graph $\mathscr{G}$ and whose yes-instances are those partitioned graph $\mathscr{G}$ such that $\mathscr{G} \xrightarrow{alt} H$.

2.1 Rewriting partitioned graphs

If H has no edges then the only yes-instances of the quantified H-coloring problem are edgeless partioned graphs. Thus, in the following we always assume that H has at least one edge. Note that if a partitioned graph has no edge then it is clearly a yes-instance (for any H). In the following, we ignore this trivial case and always assume that a partitioned graph has at least one edge.

Two partitioned graphs $\mathscr{G}$ and $\mathscr{G}'$ are *problem-equivalent* if, for every graph H, $\mathscr{G} \xrightarrow{alt} H$ if, and only if $\mathscr{G}' \xrightarrow{alt} H$.

We define a *rewrite* scheme that transforms an n-partitioned graph $\mathscr{G}$ to a partitioned graph $\overline{\mathscr{G}}$ as follows.

1. Remove all isolated vertices of $\mathscr{G}$.
2. Suppose $\mathscr{G}$ is the disjoint union of m connected partitioned subgraphs $\mathscr{G}_1, \ldots, \mathscr{G}_m$. For each $1 \leq l \leq m$, construct $\overline{\mathscr{G}}_l$ from $\mathscr{G}_l$ thus:
   (a) while there is a minimal $i \leq n$ such that U_{2i+1} is empty, move every vertex of X_{2j+2} into X_{2j}, for all $i \leq j \leq n$ and every vertex of U_{2j+3} into U_{2j+1}, for all $i \leq j \leq n-1$.
   (b) while there is a minimal $i \leq n$ such that X_{2i} is empty, move every vertex of X_{2j+2} into X_{2j}, for all $i \leq j \leq n$ and every vertex of U_{2j+1} into U_{2j-1}, for all $i \leq j \leq n$.
3. Set $\overline{\mathscr{G}}$ to be the disjoint union of the graphs $\overline{\mathscr{G}}_1, \overline{\mathscr{G}}_2, \ldots \overline{\mathscr{G}}_m$.

We say that $\mathscr{G}$ is *rewrite-reduced* if $\mathscr{G} = \overline{\mathscr{G}}$. We say that two partitioned graphs $\mathscr{G}$ and $\mathscr{G}'$ are *rewrite-equivalent* if $\overline{\mathscr{G}} = \overline{\mathscr{G}}'$.

Proposition 2. *Let $\mathscr{G}$ and $\mathscr{G}'$ be two partitioned graphs. If $\mathscr{G}$ and $\mathscr{G}'$ are rewrite-equivalent then they are problem-equivalent.*

2.2 Restricting partitions

Let $\mathscr{G}$ be a partitioned graph. We say that $\mathscr{G}$ is in Σ_1-*form* (respectively, Π_2-*form*), if the only non-empty partition is X_2 (respectively, if the only non-empty partitions are among $\{U_1, X_2\}$). If $\mathscr{G}$ is in in Π_2-form and there is at most one vertex in U_1, then we say that $\mathscr{G}$ is in Π_2-*fan form*. If there is exactly one vertex

x in $U1$ and it is adjacent to at least one vertex y in X_2 then we say that $\mathscr{G}$ is in *strict* Π_2*-fan form*. Finally, we say that $\mathscr{G}$ is in Π_2*-multifan form*, if $\mathscr{G}$ is the finite disjoint union of graphs in Π_2-fan form.

Theorem 3. *Let H be a graph that is not bipartite. The restriction of the quantified H-coloring problem to partitioned graphs in Π_2-multifan form is NP-complete.*

Proof. Let $\mathscr{G}$ be the disjoint union of $\mathscr{G}_1, \mathscr{G}_2, \dots, \mathscr{G}_m$, all in Π_2-fan form. Note that $\mathscr{G} \xrightarrow{alt} H$ if, and only if, $\mathscr{G}_i \xrightarrow{alt} H$, for every $1 \le i \le m$. Let $1 \le i \le m$. If $\mathscr{G}_i$ is not in strict Π_2-fan form then it is rewrite-equivalent to $\overline{\mathscr{G}_i}$ in Σ_1-form, and we may simply guess a homomorphism and verify in polynomial time. If $\mathscr{G}_i$ is in strict Π_2-fan form then we test all possible maps for the single element in U_1, guessing the rest of the homomorphism and verifying in polynomial time. This proves that this restriction of the problem belongs to NP.

If $\mathscr{G}$ is in Σ_1-form then $\mathscr{G} \xrightarrow{alt} H$ if, and only if, $G \longrightarrow H$. Since H is not bipartite, by Hell and Nešetřil's dichotomy result [11], the H-coloring problem is NP-hard and the NP-hardness of the restriction of quantified H-coloring to Π_2-multifan form follows. □

3 Cases in NP

In this section, we will find that for certain H, every input which is not in Π_2-multifan form can be discarded. The proof techniques are similar in all cases. Firstly, we study the case when H has an isolated vertex. Secondly, we look at the case when H is bipartite. Finally, we turn to the cases where H is not connected. Note that the first case is subsumed by the last one. The proof of the first case presents the advantage of being simpler and we present it in order to familiarise the reader with the proof technique.

3.1 H has an isolated vertex

Let H be a graph with an isolated vertex s and let $\mathscr{G}$ be a partitioned graph. We first prove that certain edges are forbidden in $\mathscr{G}$, due to the isolated vertex in H.

Lemma 4. *If there is an edge in $\mathscr{G}$ between x in U_i and y in X_j (for any i, j), or between x in U_i and y in U_j (for any i, j) then $\mathscr{G} \stackrel{alt}{\not\longrightarrow} H$.*

Proof. Regardless of what is played before, when Opponent plays x on s, there is no way that partial function can extend to homomorphism, regardless of where Proponent will map, or has mapped, y. Similarly, in the second case, Opponent plays both vertices on s and wins. □

Theorem 5. *If H has an isolated vertex then the Quantified H-colouring problem is equivalent to the H-coloring problem under logspace reduction.*

Proof. The H-coloring problem reduces trivially to the quantified H coloring problem.

We define the converse reduction as follows. Let N be a fixed no-instance of the H-coloring problem (say, H augmented with one vertex adjacent to every vertex of H). If $\mathscr{G}$ has an edge as in the previous lemma then we know that it is a no-instance and we reduce $\mathscr{G}$ to N (with all vertices in X_2). If $\mathscr{G}$ has no such edge then every element in a universal partition is isolated. Thus, Opponent's moves have no bearing on Proponent's moves, and we may disregard every element occurring in a universal partition. Indeed, the rewritten-reduced graph $\overline{\mathscr{G}}$ will be in Σ_1-form. We reduce $\mathscr{G}$ to its underlying graph G. □

Example. Let H_0 be the graph that consists of a triangle and an isolated vertex, *i.e.* $H_0 := K_3 \cup K_1$. By Theorem 5, the quantified H_0-coloring problem is logspace equivalent to K_3-coloring, which is known to be NP-complete. Consequently, the quantified H_0-coloring problem is NP-complete.

3.2 Bipartite Templates

Let H be a bipartite graph and let $\mathscr{G}$ be a partitioned graph. We first prove that certain paths are forbidden in $\mathscr{G}$ because H is bipartite. Next, we show that if $\mathscr{G}$ does not have such paths then it must be rewrite equivalent to a partioned graph in Π_2-multifan form. Finally, we use this important structural property to prove that, for such instances, the quantified H-coloring problem essentially collapses to the H-coloring problem.

Lemma 6. *If there is a path in $\mathscr{G}$ between any x in X_i and y in U_j (for $i < j$) then $\mathscr{G} \overset{alt}{\not\longrightarrow} H$. If there is a path in $\mathscr{G}$ between any x in U_i and y in U_j (for any i, j), then $\mathscr{G} \overset{alt}{\not\longrightarrow} H$.*

Proof. We only prove the first case as the second may be done similarly. Let a be any vertex in H on which Proponent plays x. If a is an isolated vertex then Opponent wins (*cf.* proof of Lemma 4). Assume that a is not isolated. If the path in $\mathscr{G}$ between x and y is of even length then Opponent plays y on b, where b is adjacent to a. A winning strategy for Proponent would imply the existence of an odd cycle in H, which would contradict the fact that H is bipartite. Thus, it follows that Opponent wins. If the path in $\mathscr{G}$ is of odd length then Opponent plays y on a and wins by the same argument. $\square$

The following results shows that every partitioned graph that does not have the paths mentioned in the above lemma can be rewritten into a partitioned graph that is in Π_2-multifan form.

Lemma 7. *If there is no path in a partitioned graph $\mathscr{G}$ between any x in X_i and y in U_j (for $i < j$), or between any x in U_i and y in U_j (for any i, j) then the rewrite-reduced $\overline{\mathscr{G}}$ is in Π_2-multifan form.*

Proof. By definition of the rewrite rules, it suffices to prove that for every connected component $\mathscr{G}'$ of $\mathscr{G}$ that is not an isolated vertex, the rewrite-reduced $\overline{\mathscr{G}'}$ is in Π_2-fan form. Let $\mathscr{G}'$ be such a component and let $0 < i \leq n$ be the largest integer such that U_{2i+1} is non-empty. Take x in U_{2i+1}, and let y be any element of $\mathscr{G}'$ connected to x via a path. It follows from the second assumption that y can not be in a universal partition (U_{2i+1} included). Thus, it follows from the first assumption that y belongs to an existential partition of index at least $2i+2$. The rewrite rules may be applied to move x to U_1 and all other vertices of $\mathscr{G}'$ to X_2. Consequently, $\overline{\mathscr{G}'}$ is in Π_2-fan form. $\square$

The following result is essential. We show that, provided that the input is in Π_2-fan form and H has no isolated vertex, the quantified H-coloring problem essentially coincides with the K_2-coloring problem.

Lemma 8. *Let H be a bipartite graph that has no isolated vertex and let $\mathscr{G}$ in Π_2-multifan form. The following are equivalent:*

(i) $\mathscr{G} \xrightarrow{alt} H$

(ii) $\mathscr{G} \xrightarrow{alt} K_2$

(iii) $G \longrightarrow K_2$

Proof. Recall that H is not edgeless, thus H is homomorphically equivalent to K_2. Thus (iii) is equivalent to $G \longrightarrow H$. It suffices to prove the result for each connected component of $\mathscr{G}$. Thus, assume w.l.o.g. that $\mathscr{G}$ is connected. If $\mathscr{G}$ is an isolated vertex or has no vertex in U_1 then the result holds trivially. Otherwise, $\mathscr{G}$ is in strict Π_2-fan form, where x in U_1 is adjacent to y in X_2. Denote by 0 and 1 the two vertices of K_2.

- $(i) \implies (ii)$: By assumption H is non-trivial and let a and b be two adjacent vertices in $V(H)$. Assume that Opponent plays x on a in $V(H)$ (respectively, on b). Since $\mathscr{G} \xrightarrow{alt} H$, there exists a homomorphism $G \xrightarrow{s_a} H$ such that $s_a(x) = a$ (respectively, $G \xrightarrow{s_b} H$ such that $s_b(x) = b$). Let $H \xrightarrow{h} K_2$ such that $h(a) = 0$ and $h(b) = 1$. We construct a winning strategy for Proponent in the $(\mathscr{G}, K_2)$-game as follows. If Opponent plays x on 0 (respectively, on 1) in K_2 then all remaining moves are Proponent's: she plays every vertex y in X_2 on $h \circ s_a(y)$ (respectively, $h \circ s_b(y)$) and wins.
- $(ii) \implies (i)$: We proceed similarly to before. If Opponent plays x on 0 in K_2 (respectively, 1) there exists a homomorphism $G \xrightarrow{s_0} H$ (respectively, $G \xrightarrow{s_1} H$) satisfying $s_0(x) = 0$ (respectively, $s_1(x) = 1$).

  We know that a is not isolated, and there exists a vertex b in $V(H)$ adjacent to a. Let $K_2 \xrightarrow{h'} H$ be the homomorphism defined by $h'(0) = a$ and $h'(b) = 1$.
  Suppose that Opponent plays x on a (respectively, on b) then Proponent plays every vertex y in X_2 on $h' \circ s_0(y)$ (respectively, $h' \circ s_1(y)$) and wins.
- $(ii) \implies (i)$: clear.
- $(i) \implies (ii)$: follows from the symmetry of K_2.

□

Theorem 9. *Let H be a bipartite graph. The Quantified H-coloring problem is tractable.*

Proof. We propose the following algorithm to solve the Quantified H-coloring problem. If H has an isolated vertex then by Theorem 5 the problem reduces to the H-coloring problem, which is tractable by Hell and Nešetřil's theorem. This provides us with a polynomial-time algorithm for this case. If H has no isolated vertex then we use the following algorithm. The input $\mathscr{G}$ is first scanned to check whether it has any of the forbidden paths of Lemma 6. If there are any then the input is rejected. Otherwise, the algorithm accepts an input if, and only if, it is bipartite.

This algorithm is clearly polynomial. We now prove its correctness. We know that if $\mathscr{G}$ has none of the forbidden paths, then it is rewrite-equivalent to the reduced $\overline{\mathscr{G}}$ in Π_2-multifan form, by Lemma 7. We write $\overline{G}$ to denote the underlying graph of $\overline{\mathscr{G}}$. Thus, by Proposition 2, it follows that $\mathscr{G} \xrightarrow{alt} H$ if, and only if, $\overline{\mathscr{G}} \xrightarrow{alt} H$. By Lemma 8, $\overline{\mathscr{G}} \xrightarrow{alt} H$ holds if, and only if $\overline{G} \longrightarrow K_2$. Since the underlying graphs of $\overline{\mathscr{G}}$ and $\mathscr{G}$ may differ only up to some isolated vertices, they are homomorphically-equivalent: this means that $\mathscr{G}$ is a yes-instance if, and only if, G is bipartite. □

3.3 Non-connected Templates

As in the previous case, we prove that certain paths prevent an instance from being accepted, for every template H that is not connected. Since any instance without such paths can be rewritten into an equivalent instance that is in Π_2-multifan form, we deduce that the quantified H-coloring problem is in NP. Finally, using the dichotomy of H-coloring we deduce that the quantified H-coloring problem is NP-complete for non bipartite non connected template.

Lemma 10. *Let H be a non-connected graph. If there is a path in $\mathscr{G}$ between any x in X_i and y in Uj (for $i < j$) then $\mathscr{G} \overset{alt}{\not\longrightarrow} H$. If there is a path in $\mathscr{G}$ between any x in Ui and y in Uj (for any i, j) then $\mathscr{G} \overset{alt}{\not\longrightarrow} H$.*

Proof. We prove the first case; the second may be done similarly. Wherever Proponent plays x in H, Opponent answers y in a different connected component of H. Clearly this can never extend to homomorphism, so Opponent wins. □

Theorem 11. *Let H be a graph that is not connected. The quantified H-coloring problem is in NP. Moreover, if H is not bipartite then the quantified H-coloring problem is NP-complete.*

Proof. Let $\mathscr{G}$ be the input partitioned graph. If $\mathscr{G}$ has any of the paths of Lemma 10 then we know it is a no-instance. Otherwise, by Lemma 7, we can compute $\overline{\mathscr{G}}$, which is in Π_2-multifan form. Finally, we can use the algorithm of Theorem 3.

If H is not bipartite, by Hell and Nešetřil's theorem [11] then the H-coloring problem is NP-complete. Since it reduces trivially to the quantified H-coloring problem, the completeness of the latter follows. □

4 Pspace-complete cases

In this section, we prove that the quantified H-coloring problem is Pspace-complete, whenever H is a connected graph with a unique cycle, which is of odd length. First, we define and study the special case of *odd Catherine wheels*. Secondly, we reduce a variant of quantified sat that is also Pspace-complete, *quantified not-all equal 3-sat* to the quantified H-coloring problem, where H is an odd Catherine wheel. Finally, we extend this reduction to any connected graph with a unique cycle, that is of odd length.

4.1 Odd Catherine Wheels

Let W be a graph that consists of an odd cycle C_{2k+1} (where $k > 0$) together with $2k+1$ disjoint paths $P_0, P_1 \dots, P_{2k}$, each of any finite length (but potentially trivial, *i.e.* the path of length 0 consists of a single vertex) such that the end of each path is identified with a different vertex of the odd cycle. We say that W is an *odd Catherine wheel*. Ordering the cycle and choosing an initial vertex, we represent W by the corresponding sequence of paths lengths, which we call a *listing*. For any distinct vertices u and v, we denote by $d(u,v)$ the length of the shortest path from u to v. Set

$$\omega := \min\{D(\{p,q\}) : \{p,q\} \text{ is an edge on the cycle of } W\}$$

where $D\{u,v\} := \max\{\min\{d(u,w), d(v,w)\} | w \in V(G)\}$, for any $\{u,v\}$ in $E(W)$. We say that a listing of W is *minimal* if $D\{k,k+1\} = \omega$. We need the following two lemmas to prove the completeness of the quantified W-coloring problem. The proofs, though technical, present no difficulties and due to space restrictions are only available in the appendix.

Lemma 12. *Let W be an odd Catherine wheel with a cycle of size $2k+1$. Let $P^0, P^1, \dots, P^{2k}$ be a minimal listing of W. There exists t in $V(W)$ such that there is an m-walk from t to k, but no m-walk from t to $k+1$, and there exists s in $V(W)$ such that there is an m-walk from s to $k+1$, but no m-walk from s to k.*

Lemma 13. *Let W be an odd Catherine wheel with a cycle of size $2k+1$. Let x and y be any vertex on this cycle. There is a walk from x to y of length $2k-1$ if, and only if x is distinct from y.*

4.2 Reduction from not-all-equal 3-sat.

Let W be an odd Catherine wheel with a cycle of size $2k+1$. In this section we prove that the quantified W-coloring problem is Pspace-complete. The proof involve a reduction from the problem *quantified not-all-equal* $2k+1$*-sat*, inspired by that given in [4]. This problem takes as input a formula φ of the form

$$Q_1 v_1, Q_2 v_2, \ldots, Q_n v_n \bigwedge_{i=1}^{c} N(v_{i_1}, v_{i_2}, \ldots, v_{i_{2k+1}}),$$

where $n, c \geq 1$ and for every $1 \leq j \leq n$, Q_j is either the quantifier $\exists$ or $\forall$ and v_i is a variable symbol. Such a formula is accepted if, and only if, it is true, where the semantics of N is "not all equal" (that is, under a Boolean valuation a clause is satisfied if, and only if, at least two variables are given different values).

Let ω be defined as in the previous section. Throughout this section, let φ be an input of quantified not-all-equal $2k+1$-sat. We build a partitioned graph $\mathscr{G}$ as follows.

The underlying graph.

- For each variable v_i of φ, we add a copy of C_{2k+1}; we fix a vertex w_i and two adjacent vertices x_i and z_i, both at distance exactly k from w_i; and, we add a path of length ω which we attach at the vertex y_i, and call z_i the other extremity of this path. We call this graph, together with the specified vertices x_i, y_i and w_i, *the variable gadget* associated with v_i. Next, we identify every vertex w_i, of every variable gadget into a single vertex w, common to every cycle of every variable gadget.
- For each clause $N(v_{i_1}, v_{i_2}, \ldots, v_{i_{2k+1}})$ of φ, we add a copy of C_{2k+1}, whose vertices are labelled by $1, 2 \ldots, 2k+1$; each vertex of the cycle represents a position in the clause; and, for $1 \leq j \leq 2k+1$, we add a path of length $2k+1$ from the vertex j of the cycle to the vertex x_{i_j}. We call this graph the *clause gadget* associated with $N(v_{i_1}, v_{i_2}, \ldots, v_{i_{2k+1}})$.

The partition. We set U_1 to be empty. We add w to X_2. Next, we read the quantifiers in φ from left to right, and proceed inductively. Let X_{l_i} be the current last (existential) partition used. Let v_i be the next quantified variable. There are two cases:

1. v_i is universally quantified. We add a universal partition U_{l_i+1} and set $U_{l_i+1} := \{z_i\}$; and, we add an existential partition X_{l_i+2} and add to X_{l_i+2} the rest of the variable gadget associated with v_i: that is, the rest of z_i's path, and every vertex of the cycle C_{2k+1} (apart from $w_i = w$, already in X_2).
2. v_i is existentially quantified. We add the rest of the variable gadget associated to v_i to X_{l_i}: that is, z_i, its path, and the rest of the corresponding C_{2k+1} (just like above $w_i = w$ is already in X_2).

Once we have read all the quantifiers of φ, we add all of the remaining vertices, *i.e.* those in the cycles and paths that encode clauses, to the last existential partition X_l.

Preliminary Observations. The construction given above is well-defined and the partitioned structure can be built in time polynomial in the size of the formula φ. It remains to prove that it is indeed a reduction. We may assume that φ has at least one universal variable, since Not all equal $2k+1$-sat reduces trivially to this restriction, under the reduction which adds a dummy universal variable.
Note that almost all vertices of $\mathscr{G}$ are in existential partitions: the only vertices in universal partitions are the z_i that belong to the variable gadgets associated with universal variables v_i of φ. Recall that Proponent plays these existential elements and wants to build a homomorphism to W, for each choice made by Opponent. In particular, every cycle must be played to the cycle of W, whether they are cycles from a clause gadget or a variable gadget, otherwise we would not have a homomorphism. Moreover, since C_{2k+1} is a core (see [12]) every endomorphism of C_{2k+1} is an automorphism. Thus, distinct elements of

a cycle in $\mathscr{G}$ must be played on distinct elements of the cycle in W.
We will need only to consider the case where w is played on the vertex labelled by 0 in a minimal listing of W. Indeed, if Proponent has a winning strategy then w must be played on such a vertex. Otherwise, by minimality of ω, there would exists a vertex u in W that does not have an ω-walk to either k or $k+1$ and if Opponent plays a vertex z_i (from a variable gadget associated with a universal variable v_i) on this vertex u then Proponent would not be able to map y_i to k or $k+1$ (which she has to, see previous remark on the play of cycles), and she would lose. Conversely, as we shall prove shortly, if φ is a yes-instance then we can build a winning strategy for Proponent, where w is played on the vertex labelled by 0 in a minimal listing of W.

Assignment vs. homomorphism In the following we assume that w is played on a vertex labelled 0, according to some minimal listing of W. Consider the variable gadget associated with a variable v_i of φ and ignore, for now, the variable z_i and its path of length ω. A homomorphism from this partial gadget to W corresponds to a truth assignment of v_i according to the following encoding: if the vertex x_i is played on k then the variable is assigned to true and if the vertex x_i is played on $k+1$ then the variable is assigned to false. Note that this encoding establishes a one-to-one correspondence between assignment to the variables of φ and the homomorphism h from the partial variable gadgets that satisfy $h(w)=0$. Thus, under the assumption that w is played on the vertex labelled 0, for notational simplicity, given such a homomorphism h, we feel free to also write h for the corresponding boolean assignment. The proof of the following result uses Lemma 13 and is available in the appendix.

Lemma 14. *Let $N(v_{i_1}, v_{i_2}, \dots, v_{i_{2k+1}})$ be a clause of φ. $N(v_{i_1}, v_{i_2}, \dots, v_{i_{2k+1}})$ is satisfied by h if, and only if, there exists a homomorphism from the clause gadget associated with $N(v_{i_1}, v_{i_2}, \dots, v_{i_{2k+1}})$, together with the partial gadgets associated with the variables $v_{i_1}, v_{i_2}, \dots, v_{i_{2k+1}}$ such that not all the variables $x_{i_1}, x_{i_2}, \dots, x_{i_{2k+1}}$ are mapped on the same vertex.*

Correctness of the reduction Assume that φ is a yes-instance. We build a winning strategy for proponent accordingly, such that she plays w on the vertex 0 in a minimal listing of W. For every vertex z_i in a variable gadget that corresponds to a universal variable v_i of φ, no matter where z_i is played by Opponent in W, by definition of ω, there exists an ω-walk from this vertex to one of the vertices k or $k+1$. Thus, Proponent may play x_i on k or $k+1$ and ensure that the correspondence with boolean assignment still holds. However, by Lemma 12, by choosing carefully the vertex on which he plays, Opponent can force $h(v_i)$ to be false or true, at his will. Similarly, if a variable is quantified existentially then it follows from Lemma 12 that Proponent can play z_i on a vertex such that x_i is played on k or $k+1$, at her will. This extends the correspondence between boolean assignments and homomorphisms to take the quantifiers into account. Assume a homomorphism h has been played by Opponent and Proponent, where Proponent mimics the assignment induced by the fact that φ is a yes-instance. Since h satisfies φ, for every clause, not all variables have the same value and by Lemma 14, we can extend h to the whole graph.

Conversely, assume that $\mathscr{G} \xrightarrow{alt} W$. We have already observed earlier that Proponent must play w on 0. We build a winning assignment h for φ accordingly. This time we use the converse implication of Lemma 14.

Theorem 15. *Let G be a connected graph that has a unique cycle, which is odd. Then the quantified H-coloring problem is Pspace-complete.*

Proof. The problem *quantified not-all-equal 3-sat* is Pspace-complete [4] and reduces trivially to the problem not-all-equal $(2k+1)$-sat. Together with the reduction presented above, this proves that if W is an odd Catherine wheel then the quantified W-coloring problem is Pspace complete.

Let H be a connected graph with a unique odd cycle that is odd. H induces an odd Catherine wheel W as follows. For every tree T_i attached to the odd cycle, replace T_i by a path P_i of the same length as the height of T_i. If φ is an instance of quantified not-all-equal $(2k+1)$-sat, we build $\mathscr{G}$ as above, and we claim that $\mathscr{G} \xrightarrow{alt} H$ if, and only if $\mathscr{G} \xrightarrow{alt} W$. This is due to the fact that in $\mathscr{G}$ any vertex z_i in a universal partition lies on the tip of a path that lies in the next existential partition. Thus, if Opponent plays z_i in a path P'_i of a tree T_i, at distance d of the cycle of H then Proponent can forget every other path and act as if Opponent had played in the vertex at distance d of the cycle in the path P_i of W (by construction $P'_i \subseteq P_i$). The converse holds trivially and the result follows. □

Note that the reduction used for odd Catherine wheels can probably be extended to graphs with more than one cycle, provided that they are similar to an odd Catherine wheel. Extending this class of Pspace complete graphs could provide us with a large base case, from which using indicator constructionsx or similar constructions, suitably adapted to our purpose, as in the proof of Hell and Nešetřil [11], we could hope to complete the classification via a similar indirect approach.

5 Conclusion

Combining the results of these papers in a single statement, we get a trichotomy for the quantified H-coloring problem, when H has at most one cycle.

Theorem 16 (main result). *Let H be a graph with at most one cycle. The problem quantified H-coloring exhibits a trichotomy.*

- *If H is bipartite then the quantified H-coloring problem is tractable.*
- *If H is not bipartite and not connected then the quantified H-coloring problem is NP-complete.*
- *If H is not bipartite and connected then the quantified H-coloring problem is Pspace-complete.*

Proof. The first case is a corollary of Theorem 9. The second case is a corollary of Theorem 11. Finally, the statement of the third case is exactly that of Theorem 15. □

The observations made on the end of the last section lead us to state the following conjecture.

Conjecture 17. If H is connected and not bipartite then the quantified H-coloring problem is Pspace-complete.

We also think that the methods used in this paper can be used for directed graphs. Though the coloring problem is much more difficult to classify for directed graphs, partial results are known: *e.g.* for tournaments [1]. Such results, together with the method used in this paper should yield partial results towards a classification of the quantified coloring problem for directed graphs.

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Appendix

All proofs that are not provided in the paper are included in the first section. In the second section, we provide an example, together with a figure to illustrate the reduction from not-all-equal $2k+1$-sat to the quantified H coloring problem, where H is an odd Catherine wheel.

A Omitted proofs

Proposition 18. *Let $\mathscr{G}$ and $\mathscr{G}'$ be two partitioned graphs. If $\mathscr{G}$ and $\mathscr{G}'$ are rewrite-equivalent then they are problem-equivalent.*

Proof. (aka Proposition 2 in the paper) The rewrite rules preserve the existence of alternating homomorphism. In the first rule, an isolated vertex does not influence the fact that $f_{U_1} \cup f_{X_2} \cup \ldots f_{U_{2n+1}} \cup f_{X_{2n+2}}$ (defined as in the definition of alternating homomorphisms) is a homomorphism, and can be discarded. Similarly, if one of the functions of $f_{U_1} \cup f_{X_2} \cup \ldots f_{U_{2n+1}} \cup f_{X_{2n+2}}$ is void, it does not influence the fact that the union of these functions is a homomorphism. Thus, we may discard the corresponding partition, and relabel the partitions accordingly. $\square$

Lemma 19. *Let W be an odd Catherine wheel with a cycle of size $2k+1$. Let $P^0, P^1, \ldots, P^{2k}$ be a minimal listing of W. There exists t in $V(W)$ such that there is an ω-walk from t to k, but no ω-walk from t to $k+1$, and there exists s in $V(W)$ such that there is an ω-walk from s to $k+1$, but no ω-walk from s to k.*

Proof. (aka Lemma 12 in the paper) Let $0 \leq i \leq 2k$ and λ_i be the length of the path P_i. Let v_i be the extremity of P_i that does not belong to the odd cycle. The vertices $1, 2, \ldots, k$ are nearer to k than to $k+1$ and the vertices $k+1, k+2, \ldots, 2k$ are nearer to $k+1$ than to k. We set, $\alpha_i := d(v_i, k) = \lambda_i + k - i$, for $0 < i \leq k$ and $\alpha_j := d(v_j, k+1) = \lambda_j + j - k - 1$, for $k < j \leq 2k$. Note that, $d(v_i, k+1) = \alpha_i + 1$, for $0 < i \leq k$ and $d(v_j, k) = \alpha_j + 1$, for $k < j \leq 2k$. We also set $\alpha_0 := \lambda_0 + k = d(0, k) = d(0, k+1)$. Note that for every vertex u_i on the path P_i, we have $d(v_i, k) \geq d(u_i, k)$ and $d(v_i, k+1) \geq d(u_i, k+1)$. Consequently,

$$D\{k, k+1\} = max\{min\{d(k, v_i), d(k+1, v_i)\} | 0 \leq i \leq 2k\}.$$

Hence, using the notation just introduced,

$$D\{k, k+1\} = max\{\alpha_i | 0 \leq i \leq 2k\}.$$

We consider three different cases.

1. $D\{k, k+1\} = \alpha_i$, for $0 < i \leq k$. We set t to be v_i and s to be the next vertex on the path P_i if $\lambda_i \neq 0$, otherwise t is the vertex i and we set s to be $i+1$ (the path has length 0 and both elements are in fact on the cycle). By construction, $d(t, k) = d(s, k+1) = \omega$ and $d(t, k+1) > \omega$. This proves the three first statements. The path from i to k, going anti-clockwise on the cycle has length $k+i+1$. Thus, any walk from s to k using this path has length at least $\lambda_i + k + i > \lambda_i + k - i = \alpha_i = \omega$. The path from i to k, going clockwise on the cycle has length $k - i$ and the distance from s to k is $\omega - 1$. Thus, any walk from s to k using this path has length at least $\omega + 1$. This proves the last statement.
2. $D\{k, k+1\} = \alpha_j$, for $k < j \leq 2k$. This case is symmetric to the previous case.
3. Finally, assume that $D\{k, k+1\} = \alpha_0$ and that we are not in any of the two cases treated above. We have:

   $$D\{k+1, k+2\} = \max\{\alpha_1 + 1, \ldots, \alpha_k + 1, \alpha_{k+1}\alpha_{k+2} - 1, \ldots, \alpha_{2k} - 1, \alpha_0 - 1\}.$$

   We now prove that $D\{k+1, k+2\} = \alpha_i + 1$ for some $0 < i \leq k$. Indeed, we can not have $D\{k+1, k+2\} = \alpha_j - 1$ for any $k+1 < j \leq 2k$. Otherwise, we would have $\alpha_j - 1 \geq \omega = \alpha_0$ by minimality of $D\{k, k+1\}$, and since by assumption $D\{k, k+1\} = \alpha_0 > \alpha_j$, we would get $\alpha_j > \alpha_j$, which is absurd. Similarly, $D\{k+1, k+2\} = \alpha_{k+1} \geq \alpha_0 > \alpha_{k+1}$ is also absurd. We have proved that $D\{k+1, k+2\} = \alpha_i + 1$ for some $0 < i \leq k$. Since $\alpha_i + 1 \geq \omega = \alpha_0$ but $\alpha_0 > \alpha_i$ this means that $\alpha_i = \omega - 1$. In this case, we set $s := v_i$ and $d(v_i, k+1) = \omega$ and via a similar argument as above, we know that there is no walk from v_i to k of length ω.
   We now consider $D\{k-1, k\}$. Via a symmetric argument, we derive the existence of a path P_j with $k < j \leq 2k$ such that $\alpha_j = \omega - 1$. We chose $t := v_j$ and the result follows.

$\square$

Lemma 20. *Let W be an odd Catherine wheel with a cycle of size $2k+1$. Let x and y be any vertex on this cycle. There is a walk from x to y of length $2k-1$ if, and only if x is distinct from y.*

Proof. (aka Lemma 13 in the paper) Assume that $x = y$. Then, a walk from x to y that does not go once around the cycle must be of even length. Since the cycle has size $2k+1$, there is no closed walk of length $2k-1$ around x.
If $x \neq y$ then since the cycle has size $2k+1$, it induces two paths from x to y, one clockwise, the other anti-clockwise. One of them must have odd length $2k'+1$. If $2k'+1 = 2k-1$ then we are done. Otherwise we can walk in $2k'+1$ steps to y and then move back and forth to z, a neighbour of y, $\frac{2k-1-2k'-1}{2} = k-k'-1$ times. This constitutes a $2k-1$ walk from x to y. □

Lemma 21. *Let $N(v_{i_1}, v_{i_2}, \ldots, v_{i_{2k+1}})$ be a clause of φ. $N(v_{i_1}, v_{i_2}, \ldots, v_{i_{2k+1}})$ is satisfied by h if, and only if, there exists a homomorphism from the clause gadget associated with $N(v_{i_1}, v_{i_2}, \ldots, v_{i_{2k+1}})$, together with the partial gadgets associated with the variables $v_{i_1}, v_{i_2}, \ldots, v_{i_{2k+1}}$ such that not all the variables $x_{i_1}, x_{i_2}, \ldots, x_{i_{2k+1}}$ are mapped on the same vertex.*

Proof. (aka Lemma 14 in the paper) Assume that there is a homomorphism h. We show that not all the variables x_{i_j} are mapped on the same vertex. The cycle of the clause gadget must map to the cycle of W and every path of length $2k-1$ with extremities i_j and x_{i_j} induces a $2k-1$ walk in W. By Lemma 13, i_j must be mapped on a different vertex than x_{i_j}. Since every i_j must be mapped on distinct vertices, it follows by the pigeon-hole principle that it is not possible to map every x_{i_j} to the same vertex of the cycle of W. Hence, h (as a boolean assignment) satisfies the clause $N(v_{i_1}, v_{i_2}, \ldots, v_{i_{2k+1}})$.

Conversely, assume that h is a boolean assignment that satisfies φ. We only need to extend h (as a homomorphism) from the partial variable gadgets to the clause gadget associated to $N(v_{i_1}, v_{i_2}, \ldots, v_{i_{2k+1}})$. At least two variables are assigned a different value by h (as a boolean assignment). Assume w.l.o.g. that $h(v_1)$ is true and $h(v_2)$ is false. This means that x_{i_1} is mapped to k and x_{i_2} is mapped to $k+1$. By Lemma 13, there exists a walk of length $2k-1$ between any two distinct vertices of C_{2k+1}. Thus, we can map i_1 to $k+1$ and the corresponding path of length $2k-1$ suitably. Similarly, we can map i_2 to k. The remaining elements of the cycle of the clause gadget are now fixed: i_3 must map to $k-1$ etc. Since they do not map on either k or $k+1$, by Lemma 13, the homomorphism can be extended to the corresponding paths. □

B An example of a reduction for Catherine wheels

Let W be an odd Catherine wheel, with a cycle of size 5. Consider the following instance of quantified not-all-equal 5-sat:

$$\exists v_1, \exists v_2, \forall v_3, \exists v_4 N(v_1, v_1, v_1, v_2, v_4) \wedge N(v_2, v_3, v_4.v_4, v_4).$$

The underlying graph of the corresponding partitioned graph is depicted on 1. On the top of the figure, each pentagon represents a variable. On the bottom, each pentagon represents a clause. The partitions can be described as follows.

- U_1 is empty;
- X_2 contains w, the rest of the variable gadget associated with v_1 (on the top of the figure, first pentagon from the left, together with z_1 and the path from z_1 to y_1) and the whole variable gadget associated with v_2 (top, second pentagon from the left, together with z_2 and the path from z_2 to y_2);
- U_3 consists of the vertex z_3 from the variable gadget associated with v_3; and,

- X_4 contains the variable gadget associated with v_3 except for z_3 and w, the variable gadget associated with v_4 except for w, and all the clause variables (the two pentagons at the bottom of the figure, together with the paths depicted by dotted lines, with the exception of the other extremities, which belong to variable gadgets).

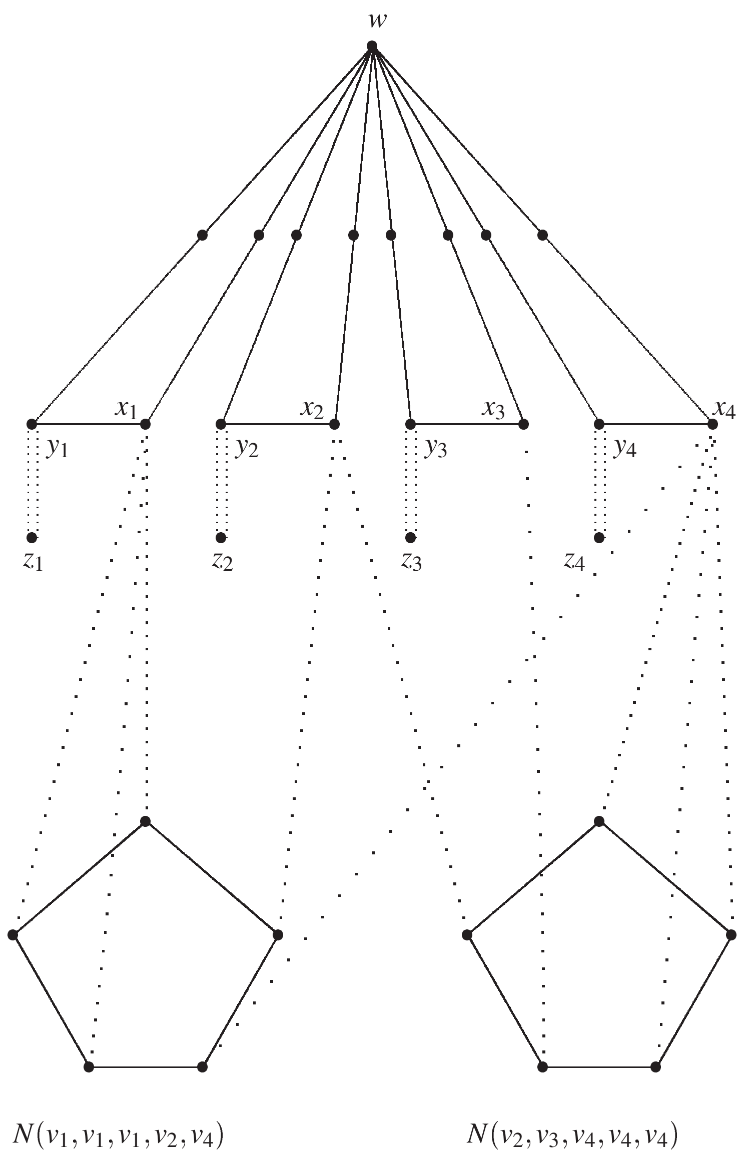

Figure 1: Reduction from quantified not-all-equal 5-sat; dotted lines are $(2k-1) = 3$-paths and double dotted lines are ω-paths.