Universal Structures and the logic of Forbidden Patterns.

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Abstract. Forbidden Patterns Problems (FPPs) are a proper generalisation of Constraint Satisfaction Problems (CSPs). However, we show that when the input belongs to a proper minor closed class, a FPP becomes a CSP. This result can also be rephrased in terms of expressiveness of the logic MMSNP, introduced by Feder and Vardi in relation with CSPs. Our proof generalises that of a recent paper by Nešetřil and Ossona de Mendez. Note that our result holds in the general setting of problems over arbitrary relational structures (not just for graphs).
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1 Introduction

Graph homomorphisms and related problems have received considerable attention in the recent years not only as a topic in combinatorics and graph theory but also in relation with constraint satisfaction. A lot of possible directions of research in the area have been opened by different motivations, and have been explored (the very recent monograph [1] serves as a good survey of the area).

The present work was motivated mainly by constraint satisfaction problems (CSPs) and their generalisations, forbidden patterns problems (FPPs), introduced in [2]. Our main result falls into the category of so-called (restricted) duality theorems, and has the same flavour as results from combinatorics and graph theory such as [3–7], (some of which will be explained shortly). Our main result can also be presented in terms of the expressiveness of the logic MMSNP, a fragment of monadic second order logic, that is closely related to FPPs (every sentence of MMSNP expresses a finite union of FPPs). For arbitrary structures, MMSNP captures problems that are not CSPs. However, when the input is restricted, it may be that MMSNP collapses to the class of CSPs (with the same input restriction): for example, when the input is connected and of bounded degree [6]. The main result of this paper states that this is also the case when the input is connected and belongs to some *proper minor closed class* (that is, a class that is defined by forbidding a finite set of graphs as minors, *e.g.* planar graphs). In the rest of this introduction, we will need to elaborate on work and ideas from two different areas, descriptive complexity and combinatorics.

Let us start with MMSNP, a fragment of monadic second order logic that was introduced in [8], motivated by the search of a logic that exhibits a dichotomy (every problem is either in $\mathcal{P}$ or $\mathcal{NP}$-complete). It is still open whether MMSNP has a dichotomy but Feder and Vardi proved that every problem Ω in MMSNP is equivalent to a CSP Ω' (recall that, for a fixed structure H, the *Constraint Satisfaction Problem* with *template* H is the class of structures G such that there exists a homomorphism from G to H). Feder and Vardi's reduction from Ω' to Ω was randomised but has been recently derandomised by Kun [9]. Hence, MMSNP has a dichotomy if, and only if, the class of CSPs has a dichotomy. The dichotomy conjecture for CSPs is still open but is supported by numerous results (see *e.g.* [10–13]). So, one could argue that MMSNP and the class of CSPs are essentially the same. However, it is known that the logic MMSNP is too strong and captures problems which are not CSPs [8, 14]. Note also that in general the signature of Ω' is exponentially larger than the signature of Ω and that Kun's derandomised reduction involves gadgets that are built from graph-expanders which are themselves obtained via the random method (and as such their size is some unknown constant). Moreover, Bodirsky *et al.* [15, 16] showed that problems in MMSNP are in fact examples of CSPs with a countable template. So, in the context of descriptive complexity, we argue that MMSNP and CSPs are rather different. The combinatorial counterpart of MMSNP is directly related to *Forbidden Patterns Problems* (FPPs): every sentence of MMSNP captures a finite union of FPPs. In [17, 2], an exact characterisation of FPPs that are CSPs was proved: given a forbidden patterns problem, we can decide whether it is a CSP with a finite or an infinite template and in the former case, we can compute effectively this finite template. Since the transformation of a sentence of MMSNP into a finite union of FPPs is also effective, as a corollary, we can decide whether a given sentence of MMSNP captures a finite union of (finite) CSPs, or captures a finite union of infinite CSPs. The characterisation of FPPs that are finite CSPs subsumes the characterisation of duality pairs obtained by Tardif and Nešetřil [3]. A *duality pair* is a pair (F, H) of structures such that for every structure G, there is no homomorphism from F to G (we say that G is F-mote) if, and only if, there is a homomorphism from G to H (we say that G is homomorphic to H). The word *duality* is used in this context to describe the inversion of the direction of homomorphisms. Forbidden patterns problems generalise F-moteness by involving coloured structures (see Section 2 for definition and examples) and the property of F-moteness corresponds to FPPs with a single colour and a single forbidden pattern (namely F). So a characterisation of duality pairs corresponds to a characterisation of those rather restricted FPPs which are also CSPs. Using a similar language, we could describe the characterisation of those FPPs that are CSPs as a *coloured duality theorem*.

Häggkvist and Hell [5] showed a different kind of duality result for graph homomorphism (or CSPs for graphs). They built a universal graph H for the class of F-mote graphs of bounded degree b. Thus, any degree-d graph G is F-mote if, and only if, there is a homomorphism from G to H. Note that the usual notion of universal graph, as used by Fraïssé [18] is for induced substructures, not homomorphism, and that the universal graph is usually infinite. In our context, the word universal graph refers to a *finite* graph and is universal with respect to the existence of homomorphisms. The result of Häggkvist and Hell can be seen as a restricted form of *duality*, as they quantify only over bounded degree graphs, rather than all graphs. This type of result is called a *restricted duality theorem*. In the last decade, a series of papers built upon Häggkvist and Hell's technique. In particular, in [4], where the authors investigated more generally the existence of universal graphs for the class of graphs that are simultaneously F-mote and H-colourable. In [6], we recently extended this last result by proving that FPPs restricted to connected inputs of bounded-degree become CSPs. This result could be described as a *restricted coloured duality theorem*.

A well-known extension of monadic second order logic consists in allowing monadic predicates to range over tuples of elements, rather than just elements of a structure. This extension is denoted by MSO_2 whereas the more standard logic with monadic predicates ranging over elements only is denoted by MSO_1. In general, MSO_2 is strictly more expressive than MSO_1. However Courcelle [19] proved that MSO_2 collapses to MSO_1, for some restriction of the input: for graphs of bounded degree, for partial k-trees (for fixed k), for planar graphs and, more generally for graphs belonging to a *proper minor closed class* (a class $\mathcal{K}$ such that: a minor of a graph G belongs to $\mathcal{K}$ provided that G belongs to $\mathcal{K}$; and, $\mathcal{K}$ does not contain all graphs).

It is perhaps worth mentioning that in [6] and the present paper we assume a definition of FPPs that is more general than the original one in [2] in that the new definition allows colourings not only of the vertices but also of the edges of the input graph. Essentially, this means that we are now considering problems related to MMSNP_2, the extension of MMSNP with edge quantification (for clarity, from now on, we denote by MMSNP_1 the original logic, i.e. without edge quantification). Thus, in [6] we obtain as a corollary that MMSNP_1 and MMSNP_2 collapse to CSP when restricted to connected inputs of bounded degree.

Nešetřil and Ossona de Mendez [7] introduced the notion of *tree-depth* of a graph and, using a result of De Vos *et al.* [20] on low tree-width decomposition for proper minor closed class, they proved a similar result using tree-depth rather than tree-width. This allowed them to prove a restricted duality theorem for proper minor closed classes of graphs. In this paper, we prove a *restricted coloured duality theorem for proper minor closed classes*: we extend the notion of tree-depth to arbitrary structures and prove this theorem for arbitrary structures, not only for graphs. As a corollary, we get the main result of this paper: we prove that MMSNP_1 and MMSNP_2 collapse to CSP when restricted to connected structures whose Gaifman graphs belong to a proper minor closed class. Note that the proof of our previous result [6] uses a different technique: for bounded degree d input, the construction of the universal graph relies on the fact that every subgraph induced by the vertices at distance at most p from a given vertex in a bounded degree graph is finite and its size depends only of d and p (p is a constant that is fixed according to the size of the forbidden patterns). This is not the case when the input comes from a proper minor closed class.

The paper is organised as follows. In Section 2, we define CSPs, FPPs and give some examples. Then, we define MMSNP_1 and recall some known results and extend them to MMSNP_2. Next, we prove that problems in MMSNP_2 are also infinite CSP in the sense of Bodirsky. We conclude Section 2 by stating our main result, the collapse to CSPs of MMSNP_1 and MMSNP_2 when restricted to connected structures from a proper minor closed class. We illustrate this result by a few examples. We also give a brief overview of the proof structure to guide the reader in the following technical sections. In Section 3, we introduce the notion of tree-depth for structures. Then, we show that coloured structures of bounded tree-depth have bounded cores. Finally, we show that structures that belong to a (fixed) proper minor closed class have a low tree-depth decomposition. In Section 4, we build a finite universal coloured structure for forbidden patterns for problems that are restricted to a class of structure that have a low tree-depth decomposition. We conclude this section by a proof of our main result. Omitted proofs are available in the appendix.

2 Preliminaries

2.1 CSP and FPP

Let σ be a signature that consists of finitely many relation symbols. From now on, unless otherwise stated, every structure considered will be a σ-structure. Let S and T be two structures. A *homomorphism* h from S to T is a mapping from $|S|$ (the domain of S) to $|T|$ such that for every r-ary relation symbol R in σ and every elements $x_1, x_2, \ldots, x_r$ of S, if $R(x_1, x_2, \ldots, x_r)$ holds in S then $R(h(x_1), h(x_2), \ldots, h(x_r))$ holds in T.

Constraint Satisfaction Problems. The *constraint satisfaction problem* with *template* T is the decision problem with,

- input: a finite structure S; and,
- question: does there exist a homomorphism from S to T?

We denote by CSP the class of constraint satisfaction problems with a *finite* template.

Example 1. The constraint satisfaction problem with template K_3 (the clique with three elements, *i.e.* a triangle) is nothing else than the 3-colourability problem from graph theory.

Let $\mathcal{V}$ (respectively, $\mathcal{E}$) be a finite set of *vertex colours* (respectively, *edge colours*). A *coloured structure* is a triple $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$, where S is a structure, $s^{\mathcal{V}}$ is a mapping from $|S|$ to $\mathcal{V}$ and $s^{\mathcal{E}}$ is a mapping from $E(S)$ to $\mathcal{E}$ where,

$$E(S) := \bigcup_{R \in \sigma} \big\{ (R, x_1, x_2, \ldots, x_r) \text{ s.t. } R(x_1, x_2, \ldots, x_r) \text{ holds in } S \big\}.$$

Let $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ and $(S', {s'}^{\mathcal{V}}, {s'}^{\mathcal{E}})$ be two coloured structures. A *colour-preserving homomorphism* h from S to S' is a homomorphism from S to S' such that ${s'}^{\mathcal{V}} \circ h = s$ and for every tuple $t = (R, x_1, x_2, \ldots, x_r)$ in $E(S)$, ${s'}^{\mathcal{E}}(t') = s^{\mathcal{E}}(t)$, where $t' := \big(R, h(x_1), h(x_2), \ldots, h(x_r)\big)$. When the colours are clear from the context, we simply write that h *preserves colours*. Note that the composition of two homomorphism that preserve colours is also a homomorphism that preserves colours.

A structure S is *connected* if it can not be partitioned into two disjoint induced substructures. A *pattern* is a finite coloured structure $(F, f^{\mathcal{V}}, f^{\mathcal{E}})$ such that F is connected. In this paper, patterns are used to model constraints in a negative fashion and consequently, we refer to them as *forbidden* patterns. Let $\mathfrak{F}$ be a finite set of forbidden patterns. We say that a coloured structure $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ is *valid* with respect to $\mathfrak{F}$ if, and only if, for every forbidden pattern $(F, f^{\mathcal{V}}, f^{\mathcal{E}})$ in $\mathfrak{F}$, there does not exist any colour-preserving homomorphism h from F to S.

Forbidden Patterns Problems. The *problem with forbidden patterns* $\mathfrak{F}$ is the decision problem with,

- input: a finite structure S
- question: does there exist $s^{\mathcal{V}} : |S| \rightarrow \mathcal{V}$ and $s^{\mathcal{E}} : E(S) \rightarrow \mathcal{E}$ such that $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ is valid with respect to $\mathfrak{F}$?

We denote by FPP_1 the class of forbidden patterns problem with vertex colour only (that is for which $\mathcal{E}$ has size one) and by FPP_2 the class of forbidden patterns problems.

Example 2. Let G be an undirected graph. It is usual to represent G as a relational structure with a single binary relation E that is symmetric. However, the logics considered in this paper are monotone and we can not express that E is symmetric and we use a different representation to encode graphs. We say that a structure S with one binary relation symbol E *encodes* G, if $|S| = V(G)$ and for any x and y in $V(G)$, x and y are adjacent in G if, and only if, $E(x, y)$ or $E(y, x)$ holds in S. Note that this encoding is not bijective. Modulo this encoding, the following graph problems are Forbidden Patterns Problems.

1. Vertex-No-Mono-Tri: consists of the graphs for which there exists a partition of the vertex set in two sets such that no triangle has its three vertices occurring in a single partition. It was proved in [8, 14] that this problem is not in CSP and in [21] that it is $\mathcal{NP}$-complete.
2. Tri-Free-Tri: consists of the graphs that are both three colourable (tripartite) and in which there is no triangle. It was proved in [14] that this problem is not in CSP.
3. Edge-No-Mono-Tri: consists of the graphs for which there exists a partition of the edge set in two sets such that no triangle has its three edges occurring in a single partition. It is known to be $\mathcal{NP}$-complete (see [22]).

2.2 $\mathbf{MMSNP_1}$ and $\mathbf{MMSNP_2}$

The class of forbidden patterns problems with colours over the vertices only, corresponds to the problems that can be expressed by a formula in Feder and Vardi's MMSNP (Monotone Monadic SNP without inequalities, see [8, 2]). Note that allowing colours over the edges does not amount to drop the hypothesis of monadicity altogether. Rather, it corresponds to a logic, let's call it MMSNP_2, which is similar to MMSNP but allows *first-order* variables over edges (just like Courcelle's MSO (Monadic Second Order logic) and MSO_2, see [19]).

This section is organised as follows: first, we introduce formally MMSNP_1 and recall some known results; and, secondly, we introduce MMSNP_2 and prove that it captures finite union of problems in FPP_2. Finally, we prove that problems in FPP_2 are infinite CSP in the sense of Bodirsky [15].

Definition 3. Monotone Monadic SNP without inequality*, $MMSNP_1$, is the fragment of ESO consisting of those formulae Φ of the following form:*

$$\exists \mathbf{M} \forall \mathbf{t} \bigwedge_i \neg\big(\alpha_i(\sigma, \mathbf{t}) \wedge \beta_i(\mathbf{M}, \mathbf{t})\big),$$

where $\mathbf{M}$ is a tuple of monadic relation symbols (not in σ), $\mathbf{t}$ is a tuple of (first-order) variables and for every negated conjunct $\neg(\alpha_i \wedge \beta_i)$:

- *α_i consists of a conjunction of positive atoms involving relation symbols from σ and variables from $\mathbf{t}$; and*
- *β_i consists of a conjunction of atoms or negated atoms involving relation symbols from $\mathbf{M}$ and variables from $\mathbf{t}$.*

(Notice that the equality symbol does not occur *in Φ.)*

The negated conjuncts $\neg(\alpha \wedge \beta)$ correspond to (partially coloured) forbidden structures (and this is the reason why we use such a notation in the definition rather than using implications or clausal form). To get forbidden patterns problems, we need to restrict sentences so that negated conjuncts correspond precisely to coloured connected structures. Such a restriction was introduced in [2] as follows.

Definition 4. *Moreover, Φ is* primitive *if, and only if, for every negated conjunct $\neg(\alpha \wedge \beta)$:*

- *for every first-order variable x that occurs in $\neg(\alpha \wedge \beta)$ and for every monadic symbol C in $\mathbf{M}$, exactly one of $C(x)$ and $\neg C(x)$ occurs in β;*
- *unless x is the only first-order variable that occurs in $\neg(\alpha \wedge \beta)$, an atom of the form $R(\mathbf{t})$, where x occurs in $\mathbf{t}$ and R is a relation symbol from σ, must occur in α; and,*
- *the structure induced by α is connected.*

Theorem 5. *[2] The class of problems captured by the primitive fragment of the logic $MMSNP_1$ is exactly the class FPP_1 of forbidden patterns problems with vertex colours only.*

It is only a technical exercise to relate any sentence of MMSNP_1 with its primitive fragment.

Proposition 6. *[2] Every sentence of* $\mathbf{MMSNP}_1$ *is logically equivalent to a finite disjunction of primitive sentences.*

Consequently, we have the following characterisation.

Corollary 7. *Every sentence Φ in $MMSNP_1$ captures a problem Ω that is the union of finitely many problems in FPP_1.*

Remark 8. We have altered slightly the definitions w.r.t. [2]. We now require a pattern to be connected and we have amended the notion of a primitive sentence accordingly.

The logic MMSNP_2 is the extension of the logic MMSNP_1 where in each negated conjunct $\neg(\alpha \wedge \beta)$, we allow a monadic predicate to range over a tuple of elements of the structure, that is we allow new "literals" of the form $M(R(x_1, x_2, \ldots, x_n))$ in β, where R is n-ary relation symbol from the signature σ. We also insists that whenever such a literal occurs in β then $R(x_1, x_2, \ldots, x_n)$ appears in α. The semantic of a monadic predicate M is extended and is defined as both a subset of the domain and a subset of the set of tuples that occur in some input relation: that is, for a structure S, $M^S \subset |S| \cup E(S)$. We say that a sentence of MMSNP_2 is *primitive* if each negated conjunct $\neg(\alpha \wedge \beta)$ satisfies the same conditions as in Definition 4 and a further condition:

– if $R(x_1, x_2, \ldots, x_n)$ occurs in α then for every (existentially quantified) monadic predicate C exactly one of $C(R(x_1, x_2, \ldots, x_n))$ or $\neg C(R(x_1, x_2, \ldots, x_n))$ occurs in β.

It is only a technical exercise to extend all the previous results of this section concerned with MMSNP_1 and FPP_1 to MMSNP_2 and FPP_2. In particular, we have the following result.

Corollary 9. *Every sentence Φ in $MMSNP_2$ captures a problem Ω that is the union of finitely many problems in FPP_2.*

2.3 Infinite CSP and $\mathbf{MMSNP_2}$

Bodirsky *et al.* have investigated constraint satisfaction problems, where the template is infinite and have proposed restrictions that ensure that the problems are decidable (and in $\mathcal{NP}$): when the template is countable and *homogeneous* in [23], and more recently to a more general case when the template is ω-categorical in [15] (a countable structure Γ is *ω-categorical* if all countable models of its first-order theory of Γ are isomorphic to Γ). Denote by $\mathrm{CSP}^\star$ the set of constraint satisfaction problems that have a ω-categorical countable template and belong to $\mathcal{NP}$. Using a recent result due to Cherlin, Shelah and Chi [24], Bodirsky and Dalmau proved the following result.

Theorem 10. *[16] Every non-empty problem in MMSNP$_1$ that is closed under disjoint union belongs to CSP*.*

It follows directly from its definition that every problem in FPP$_1$ is closed under disjoint union. Hence, we get the following result.

Corollary 11. *Every problem in FPP$_1$ is in CSP*. Consequently, every problem in MMSNP$_1$ is the union of finitely many problems in CSP*.*

Since the ω-categoricity is preserved by first-order interpretation, we can prove the following.

Theorem 12. *Every problem in FPP$_2$ is in CSP*. Consequently, every problem in MMSNP$_2$ is the union of finitely many problems in CSP*.*

Remark 13. As pointed out in [15], there are problems that are in CSP* but not in MMSNP$_1$. For example, the problem over directed graphs with template induced by the linear order over $\mathbb{Q}$. Unfortunately, this problem is not expressible in MMSNP$_2$ either. In fact, we do not know whether MMSNP$_1$ is strictly contained in MMSNP$_2$. Indeed, to the best of our knowledge, none of the problems used to separate MSO$_1$ from MSO$_2$ are expressible in MMSNP$_2$. We suspect that the problem Edge-No-Mono-Tri is not expressible in MMSNP$_1$ and that MMSNP$_1$ is strictly contained in MMSNP$_2$.

2.4 Main Result: Collapse of MMSNP$_1$ and MMSNP$_2$

Let S be a structure. The *Gaifman graph* of S, which we denote by $\mathcal{G}_S$ is the graph with vertices $|S|$ in which two distincts elements x and y of S are adjacent if, and only if, there exists a tuple in a relation of S in which they occur simultaneously. Given a class $\mathcal{K}$ of structures, we denote by $\mathcal{G}_{\mathcal{K}}$ the class of their Gaifman graphs. A class of graphs $\mathcal{G}$ is said to be a *proper minor closed class* if the following holds: first, for any graph G and any minor H of G, if G belongs to $\mathcal{G}$ then so does H; and, secondly $\mathcal{G}$ does not contain all graphs. Alternatively, $\mathcal{G}$ is proper minor closed if it excludes at least one fixed graph as a minor, a so-called *forbidden minor*. We say that a class of structures $\mathcal{K}$ is a *proper minor closed class* if, and only if $\mathcal{G}_{\mathcal{K}}$ is a proper minor closed class.

The restricted duality theorem for proper minor closed classes of graphs proved in [7] can be rephrased as follows.

Theorem 14. *Let $\mathcal{K}$ be a proper minor closed class of graphs. Let $\mathfrak{F}$ be a finite set of connected graphs. There exists a universal graph U such that for any graph G in $\mathcal{K}$, the graph G is F-mote (there is no homomorphism from F to G) for every graph F in $\mathfrak{F}$ if, and only if, there exists a homomorphism from G to U.*

We can use this result and *ad hoc* techniques from [6] to show that two of our examples are CSP when restricted to a proper minor closed class $\mathcal{K}$. Let U be the universal graph for $\mathfrak{F} = \{K_3\}$. Let U' be the product of U with K_3. It is not difficult to check that Tri-Free-Tri, restricted to $\mathcal{K}$ is the CSP with template U'. Similarly Vertex-No-Mono-Tri is the CSP with template U', where U'' is the graph that consists of two copies of U, such that every pair of elements from different copies are adjacent. Note that technically, our problems being defined over structures with a single binary relation E, we have not really expressed them as a CSP. However, according to our encoding, replacing any two adjacent vertices x and y in the above graphs by two arcs $E(x, y)$ and $E(y, x)$ provides us with a suitable template. The last problem EDGE-NO-MONO-TRI is also a CSP when restricted to a proper minor closed class. However, we do not know how to get a template directly from the above result. For this, we need to use the main result of this paper which is a *restricted coloured duality theorem for proper minor closed class* and is formulated as follows.

Theorem 15. *The logic MMSNP$_1$ and MMSNP$_2$ have the same expressive power when restricted to any proper minor closed class $\mathcal{K}$: they both express finite union of (finite) CSPs. Furthermore, the logic MMSNP$_2$ (and,* a fortiori *MMSNP$_1$) collapses to CSP, when restricted to connected structures that belong to any proper minor closed class.*

We wish to show that FPPs are CSPs when the input belongs to a proper minor closed class (Theorem 15 follows directly from this fact using results of Section 3). To prove this fact, given a forbidden patterns problem, we need to build a structure U such that, for every input structure S (that comes from a fixed proper minor closed class $\mathcal{K}$), S is a yes-instance of the given forbidden patterns problem if, and only if, S is homomorphic to U. So we wish to build a structure U that is *universal* with respect to our forbidden patterns problem. Recall that we use the word *universal structure* with a different meaning to that of Fraïssé, in particular U has to be finite rather than infinite and is universal w.r.t. the existence of homomorphisms rather than induced substructures (*i.e.* existence of embeddings). Assume for now that the forbidden patterns problem in question has a single colour. The key property to build such a finite U is that of *bounded tree-depth*. It turns out that though the size of a structure of bounded tree-depth may be arbitrarily large, the size of its *core* is bounded. The core of a structure S is the smallest structure that is homomorphically equivalent to S. Let Y_p be the disjoint union of all cores of structures of tree-depth at most p that are valid (w.r.t. our fixed forbidden patterns problem). Note that Y_p is finite and for any structure S of tree-depth at most p, S is homomorphic to Y_p if, and only if, S is valid. The reason why we impose that the input comes from a *proper minor closed class* is that any such structure can be decomposed into a finite number of parts, say q, so that any $p \leq q$ parts induce a structure of tree-depth at most p. So, given some input S together with such a decomposition, if the largest forbidden pattern has size p then it suffices to check that for any choice of p parts of the input, the structure induced by these p-parts is valid, or equivalently, that it is homomorphic to Y_p. Finally, we use the key concept of pth *truncated product*. This concept allows us to translate the existence of homomorphisms to a structure T, for any $p-1$ parts of a p partitioned input S, to the existence of a homomorphism to the pth truncated product of T. Hence, by taking a sequence of suitable truncated products of Y_p, we get the desired finite universal structure U. Note that we assumed that the forbidden patterns problem had a single colour. In order to get our result in general, we adapt the above ideas and concepts to coloured structures.

3 Tree-depth, Cores and Low Tree-depth Decomposition.

3.1 Tree-depth and Elimination Tree of a Structure.

Following the theory of tree-depth of graphs introduced in [7], we develop elements of a theory of tree-depth for structures. Let S be a structure. We denote by $\mathcal{H}_S$ the *hypergraph induced by* S, that has the same domain as S and whose hyperedges are the sets that consists of the elements that occur in the tuples of the relations of S. If $\mathcal{H}$ is an hypergraph and r is an element of the domain of $\mathcal{H}$ then we denote by $\mathcal{H} \setminus \{r\}$ the hypergraph obtained from $\mathcal{H}$ by deleting r from the domain of $\mathcal{H}$ and removing r from every hyperedge in which it occurs (*e.g.* $\{a, b, r\}$ is replaced by $\{a, b\}$). A connected component $\mathcal{H}_i$ of $\mathcal{H}_S \setminus \{r\}$ induces a substructure S_i of S in a natural way: S_i is the induced substructure of S with the same domain as $\mathcal{H}_i$. If S is connected, then we say that a rooted tree (r, Y) is an *elimination tree* for S if, and only if, either $|S| = \{r\}$ and $|Y| = r$, or for every component S_i of S $(1 \leq i \leq p)$ induced by the connected components $\mathcal{H}_i$ of $\mathcal{H}_S \setminus \{r\}$, Y is the tree with root r adjacent to subtrees (r_i, Y_i), where (r_i, Y_i) is an elimination tree of S_i. Let F be a rooted forest (disjoint union of rooted trees). We define the *closure of* F $\mathrm{clos}(F, \sigma)$ to be the structure with domain $|F|$ and all tuples $R_i(x_1, x_2, \ldots, x_{r_i})$ such that the elements mentioned in this tuple $\{x_i | 1 \leq i \leq r_i\}$ form a chain w.r.t. $\leq_F$, where $\leq_F$ is the partial order induced by F, *i.e.* $x \leq_F y$ if, and only if, x is an ancestor of y in F. The *tree-depth* of S, denoted by $\mathrm{td}(S)$, is the minimum height of a rooted forest F such that S is a substructure of the closure of F, $\mathrm{clos}(F, \sigma)$.

These notions are closely related.

Lemma 16. *Let S be a connected structure. A rooted tree (r, Y) is an elimination tree for S if, and only if, S is a substructure of* $\mathrm{clos}(Y, \sigma)$. *Consequently, the tree-depth of S is the minimum height of an elimination tree.*

3.2 Tree-depth and Cores.

We show that a coloured structures of bounded tree-depth $\mathcal{K}$ has a core of bounded size. Recall that a *retract* of a structure S is an induced substructure S' of S for which there exists a homomorphism from S to S'. A minimal retract of a structure S is called a *core* of S. Since it is unique up to isomorphism, we may speak of *the* core of a structure [1]. This notion extends naturally to coloured structures.

We say that a (colour-preserving) automorphism μ of a (coloured structure) S has the *fixed-point property* if, for every connected substructure T of S, either $\mu(T) \cap T = \emptyset$ or there exist an element x in T such that $\mu(x) = x$. We say that μ is *involuting* if $\mu \circ \mu$ is the identity.

Theorem 17. *There exists a function $\eta : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ such that, for any coloured structure $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ such that $|S| > \eta(N, \mathrm{td}(S))$ and any mapping $g : |S| \to \{1, 2, \ldots, N\}$, there exists a non-trivial involuting g-preserving automorphism μ of $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ with the fixed-point property.*

Theorem 18. *Let $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ be a coloured structure. If $|S| > \eta(1, \mathrm{td}(S))$ then $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ maps homomorphically into one of its proper induced substructure $(S_0, s_0^{\mathcal{V}}, s_0^{\mathcal{E}})$. Consequently, $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ has a core of size at most $\eta(1, \mathrm{td}(S))$.*

Thus, we get the following result.

Corollary 19. *Let $\mathcal{K}$ be any set of coloured structures of bounded tree-depth k. Then the set $\mathcal{K}'$ of cores of structures from $\mathcal{K}$ (up to isomorphism) is finite.*

3.3 Decomposition of Low Tree-depth.

In [20] De Vos *et al.* proved that for any proper minor closed class $\mathcal{K}$ and any integer $p \geq 1$, there exists an integer q such that for every graph G in $\mathcal{K}$ there exists a vertex partition of G into q parts such that any subgraph of G induced by at most $j \leq p$ parts has tree-width at most $p - 1$. Using this result, Nešetřil and Ossona de Mendez prove a similar result for bounded tree-depth in [7], which can be rephrased as follows.

Theorem 20. *Let $p \geq 1$. Let $\mathcal{K}$ be a proper minor closed class of graphs. There exists an integer q such that any graph in $\mathcal{K}$ has a proper q-colouring in which any p colours induce a graph of tree-depth at most p.*

We extend this result to arbitrary structure using the following lemma.

Lemma 21. *A rooted tree (r, Y) is an elimination tree for a structure S if, and only if, it is an elimination-tree for its Gaifman graph $\mathcal{G}_S$.*

Corollary 22. *Let $p \geq 1$. Let $\mathcal{K}$ be a proper minor closed class of structures. There exists an integer q such that for any structure S in $\mathcal{K}$, there exists a partition of $|S|$ into q sets such that any substructure of S induced by at most p of these sets has tree-depth at most p.*

4 Restricted Coloured Dualities

4.1 Truncated Product

We extend the definition of truncated product and adapt two lemmas from [7] to coloured structures. Let $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ be a coloured structure and $p \geq 2$ be an integer. We define the pth *truncated product* of $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$, to be the coloured structure $(S', {s'}^{\mathcal{V}}, {s'}^{\mathcal{E}})$ defined as follows.

- Its domain is a subset of $\bigcup_{i=1}^{p} W^i$ where,

$$W^i := \{(x_1, x_2, \ldots, x_{i-1}, \star, x_{i+1}, \ldots, x_p) \text{ s.t.} x_k \in |S|, 1 \leq k \leq p, k \neq i\}$$

  ($\star$ denotes a new element, *i.e.* $\star \notin |S|$).

– We restrict further W^i to $\tilde{W}^i$ that consists of elements

$$w^i = (x_1, x_2, \ldots, x_{i-1}, \star, x_{i+1}, \ldots, x_p)$$

of W^i such that there exists $v \in \mathcal{V}$ such that for every $1 \leq k \leq p$ with $k \neq i$ we have $s^{\mathcal{V}}(x_k) = v$ and we set $|S'| := \cup_{i=1}^{p} \tilde{W}^i$ and ${s'}^{\mathcal{V}}(w^i) := v$.

– For every relation symbol R of arity r, and every tuple $(w^{i_1}, w^{i_2}, \ldots, w^{i_r})$ where for every $1 \leq k \leq p$, w^{i_k} belongs to $\tilde{W}^{i_k}$ and

$$w^{i_k} = (x_1^{i_k}, x_2^{i_k}, \ldots, x_{i_k-1}^{i_k}, \star, x_{i_k+1}^{i_k}, \ldots, x_p^{i_k})$$

satisfies,

- for every $1 \leq k < k' \leq r$, we have $i_k \neq i_{k'}$;
- for every $1 \leq i \leq p$, such that $i \notin \{i_1, i_2, \ldots, i_r\}$, $R(t_i)$ holds in S, where $t_i = (x_i^{i_1}, x_i^{i_2}, \ldots, x_i^{i_r})$; and,
- there exists $e \in \mathcal{E}$ such that for every $1 \leq i \leq p$, $s^{\mathcal{E}}(t_i) = e$,

we set $R(w^{i_1}, w^{i_2}, \ldots, w^{i_r})$ to hold in S' and we set ${s'}^{\mathcal{E}}(w^{i_1}, w^{i_2}, \ldots, w^{i_r}) := e$.

We denote the pth truncated product by $(S, s^{\mathcal{V}}, s^{\mathcal{E}})^{\Uparrow p}$.

This product has two important properties: it preserves validity w.r.t. a forbidden patterns problem (for a suitable p); and, the existence of all "partial" colour-preserving homomorphisms is equivalent to the existence of a homomorphism to the truncated product (see Lemma 23 and Lemma 24 below).

Lemma 23. *Let $p \geq 2$. Let $\mathfrak{F}$ be a set of forbidden patterns such that for every $(F, f^{\mathcal{V}}, f^{\mathcal{E}})$ in $\mathfrak{F}$, we have $|F| < p$. If a coloured structure $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ is valid w.r.t $\mathfrak{F}$ then its p-truncated product $(S', {s'}^{\mathcal{V}}, {s'}^{\mathcal{E}})$ is also valid w.r.t. $\mathfrak{F}$.*

Lemma 24. *Let $(U, u^{\mathcal{V}}, u^{\mathcal{E}})$ be a coloured structure and let p be an integer greater than the arity of any symbol in σ. Let $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ be a coloured structure. If there exists a partition $V_1, V_2, \ldots, V_p$ of $|S|$ such that for every substructure $(\tilde{S}_i, \tilde{s}^{\mathcal{V}}, \tilde{s}^{\mathcal{E}})$ of $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ induced by $|S| \setminus V_i$ there exist a colour-preserving homomorphism $\tilde{s}_i$ from $(\tilde{S}_i, \tilde{s}^{\mathcal{V}}, \tilde{s}^{\mathcal{E}})$ to $(U, u^{\mathcal{V}}, u^{\mathcal{E}})$ then there exists a colour-preserving homomorphism $\tilde{s}$ from $(\tilde{S}_i, \tilde{s}^{\mathcal{V}}, \tilde{s}^{\mathcal{E}})$ to $(U, u^{\mathcal{V}}, u^{\mathcal{E}})^{\Uparrow p}$.*

Using the two previous lemma, we get the following result.

Proposition 25. *Let p be an integer greater than the arity of any symbol in σ. Let $\mathfrak{F}$ be a set of forbidden patterns such that for every $(F, f^{\mathcal{V}}, f^{\mathcal{E}})$ in $\mathfrak{F}$, we have $|F| < p$. Let $q \geq p$. Let $(U, u^{\mathcal{V}}, u^{\mathcal{E}})$ be a coloured structure that is valid w.r.t $\mathfrak{F}$. Let $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ be a coloured structure.*

Assume that there exists a partition $V_1, V_2, \ldots, V_q$ of $|S|$ such that for every substructure $(\tilde{S}_i, \tilde{s}_i^{\mathcal{V}}, \tilde{s}_i^{\mathcal{E}})$ of $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ induced by p subsets $V_{i_1}, V_{i_2}, \ldots, V_{i_p}$ there exist a colour-preserving homomorphism $\tilde{s}_i$ from $(\tilde{S}_i, \tilde{s}_i^{\mathcal{V}}, \tilde{s}_i^{\mathcal{E}})$ to $(U, u^{\mathcal{V}}, u^{\mathcal{E}})$.

Then there exists a colour-preserving homomorphism $\tilde{s}$ from $(\tilde{S}_i, \tilde{s}^{\mathcal{V}}, \tilde{s}^{\mathcal{E}})$ to $(U', {u'}^{\mathcal{V}}, {u'}^{\mathcal{E}})$ and $(U', {u'}^{\mathcal{V}}, {u'}^{\mathcal{E}})$ is valid with respect to $\mathfrak{F}$, where

$$(U', {u'}^{\mathcal{V}}, {u'}^{\mathcal{E}}) := (U, u^{\mathcal{V}}, u^{\mathcal{E}})^{\Uparrow (p+1) \Uparrow (p+2) \ldots \Uparrow q}.$$

4.2 Universal Structure

We say that a coloured structure $(U, u^{\mathcal{V}}, u^{\mathcal{E}})$ is *universal* for a forbidden patterns problem restricted to a class $\mathcal{K}$ of structures if for every valid coloured structure there exists a colour-preserving homomorphism to $(U, u^{\mathcal{V}}, u^{\mathcal{E}})$, and *vice versa*.

Theorem 26. *Let p be an integer greater than the arity of any symbol in σ. Let $\mathfrak{F}$ be a set of forbidden patterns such that for every $(F, f^{\mathcal{V}}, f^{\mathcal{E}})$ in $\mathfrak{F}$, we have $|F| < p$. Let $\mathcal{K}$ be a proper minor closed class. There exists a universal coloured structure $(U, u^{\mathcal{V}}, u^{\mathcal{E}})$ for the restriction to $\mathcal{K}$ of the forbidden patterns problems with representation $\mathfrak{F}$.*

Proof. Let $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ be a coloured structure such that S belongs to $\mathcal{K}$. By Corollary 22, there exists an integer N such that every structure S in $\mathcal{K}$ can be partitioned into q parts, such that every p parts induce a substructure of S of tree-depth at most p. By Theorem 18, the core of the coloured structure induced by p parts is bounded. Let $(U, u^{\mathcal{V}}, u^{\mathcal{E}})$ be the disjoint union of all such cores that are valid w.r.t. $\mathfrak{F}$ (there are only finitely many). Since the forbidden patterns have size at most p, it suffices to check every substructure of S of size at most p and *a fortiori* $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ is valid w.r.t. $\mathfrak{F}$ if, and only if, every of its coloured substructure induced by p parts is valid, or equivalently if every such coloured substructure maps homomorphically into $(U, u^{\mathcal{V}}, u^{\mathcal{E}})$.

By Proposition 37, if $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ is valid w.r.t. $\mathfrak{F}$ then it is homomorphic to $(U, u^{\mathcal{V}}, u^{\mathcal{E}})^{\Uparrow(p+1)\Uparrow(p+2)\ldots\Uparrow q}$; and, since $(U, u^{\mathcal{V}}, u^{\mathcal{E}})^{\Uparrow(p+1)\Uparrow(p+2)\ldots\Uparrow q}$ is valid w.r.t. $\mathfrak{F}$ the converse holds as forbidden patterns problems are closed under inverse homomorphism. ◻

We have now all the elements to settle the problem of restricted coloured dualities for proper minor closed class.

Corollary 27. *Every forbidden patterns problem restricted to a proper minor closed class is a CSP.*

Proof. Let $(U, u^{\mathcal{V}}, u^{\mathcal{E}})$ be the universal coloured structure. We may take U as a template. A homomorphism from a structure S to U induces mapping $s^{\mathcal{V}}$ and $s^{\mathcal{E}}$ such that $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ is valid. Conversely, if S is not homomorphic to U then for every colour maps $s^{\mathcal{V}}$ and $s^{\mathcal{E}}$, there is no colour-preserving homomorphism from $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ to $(U, u^{\mathcal{V}}, u^{\mathcal{E}})$ and $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ is not valid. ◻

We are now ready to prove our main result.

Proof. (of Theorem 15). By Corollary 7 (resp. Corollary 9) every problem in MMSNP_1 (resp. MMSNP_2) is equivalent to a finite union of problems from FPP_1 (resp. FPP_2). By the previous theorem, when restricted to a proper minor closed class , every forbidden patterns problem is a restricted CSP. This proves that MMSNP_1 and MMSNP_2 collapse to finite union of CSP.

Moreover, if the problem is restricted to elements of a proper minor closed class that are connected then by taking the disjoint union of the template of each CSP, we get a template for our problem. ◻

5 Conclusion and Open Problems

We have proved that every forbidden patterns (with colours on both edges and vertices) problem is in fact a constraint satisfaction problem, when restricted to inputs that belong to a proper minor closed class. We derived from this result that the logic MMSNP_2 (and MMSNP_1) coincides with the class of CSPs on connected inputs that belong to a proper minor closed class. We proved, using a different method, a similar result for bounded degree graphs in [6]. So, together these results cover the restrictions considered by Courcelle in [19] under which MSO_1 and MSO_2 have the same expressive power. However, we still have not proved that for arbitrary input, MMSNP_2 is more expressive than MMSNP_1.

Another interesting open question is related to $\text{CSP}^\star$, the class of (well-behaved) infinite CSPs. We know that any problem in MMSNP_2 is a finite union of problems from $\text{CSP}^\star$. However, there are problems in $\text{CSP}^\star$ that are not expressible in MMSNP_2, which yields the following question. *Which extension of MMSNP_2 captures precisely $\text{CSP}^\star$?*

Courcelle [25] has recently shown that MSO_1 and MSO_2 have the same expressiveness for *uniformly k-sparse graphs.* This notion subsumes the restrictions from [19] mentioned above, which allows for a unified proof. Thus, it is reasonable to ask ourselves the following question: *Do MMSNP_1 and MMSNP_2 have the same expressiveness over uniformly k-sparse graphs?* And more precisely, *do forbidden patterns problems become constraint satisfaction problems when restricted to uniformly k-sparse graphs?* Nešetřil and Ossona de Mendez [26] have recently unified the proofs of the restricted duality theorems for bounded degree and proper minor closed classes using a more general concept, that of *bounded expansion.* Moreover, it turns out that graphs of bounded expansion are examples of (very well behaved) sparse graphs. So if we cannot settle the previous question in its full generality, we can at least hope to settle it in the case of graphs of bounded expansion.

A final point concerns the notion of a proper minor closed class of *structures.* In the present paper, we use the Gaifman graph to define this concept. However, it would be more natural and perhaps preferable to define a notion of *minor for structures.* The following definition seems reasonable. A minor of a structure S is obtained from S by performing a finite sequence of the following operations: taking a (not necessarily induced) substructure; and, identifying some elements, provided that they all occur in some tuple of some relation. This new definition subsumes the definition used in this paper and provokes our final question: *Do the results of this paper hold under this new definition?*

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Appendix

We provide the proofs that were omitted from the paper due to space restriction. The order follows that of the paper.

Infinite CSP

We give the proof of Theorem 12 which was omitted from the paper due to space restriction. We also make some remarks regarding the definition of CSP* given in this paper which is more general than the definition given in[15].

Before giving its proof, let us recall the statement of Theorem 12.

Theorem 28. *Every problem in FPP$_2$ is in CSP*. Consequently, every problem in MMSNP$_2$ is the union of finitely many problems in CSP*.*

Proof. Note that the second claim follows from the first claim together with Corollary 9. We now prove the first claim in the case of a problem Ω in FPP$_2$. To simplify the notation, we assume that Ω is a problem over digraphs encoded using a single binary relation E. For simplicity, we may also assume that there are only arc colours (vertex colours can be easily encoded using more arc colours and additional forbidden patterns). Let $\mathcal{E}$ be the set of arc colours and let $|\mathcal{E}| = c$.

Consider the problem Ω' in MMSNP$_1$ defined over a structure with one monadic predicate T and one ternary predicates R, given by a sentence Φ with $m = \lceil \log c \rceil$ monadic predicates. We use these monadic predicates to encode the c colours in a natural way, using a conjunction χ_a of m monadic predicates for each colour a (if necessary, Φ may contain negated conjuncts forbidding certain combination of monadic predicates if $2^m > c$). Each forbidden pattern $(F, f^{\mathcal{E}})$ of Ω is encoded as a negated conjunct as follows: for each arc $E(x,y)$ of colour a, add the literals $R(x,e,y)$, $T(e)$ and $\chi_a(e)$.

The following formula provides an interpretation Π of the signature $< E >$ in the signature $< T, R >$: $\psi_E(x,y) = \exists e T(e) \wedge R(x,e,y)$. By construction of Ω', this interpretation Π is a first-order reduction from Ω' to Ω. Note also that Ω' is closed under disjunction (since every forbidden pattern of Ω is connected by definition of FPP$_2$). By Theorem 10, there exists an ω-categorical structure Γ' such that $\Omega' = CSP(\Gamma')$. Let Γ be $\Pi(\Gamma')$. By Proposition 2.7 of [15] (page 27), it follows that Γ is also ω-categorical. Moreover, it is not difficult to check that $\Omega = CSP*(\Gamma)$.

Remark 29. In [15], the definition of CSP* is more restricted than ours. A template is required to be both ω-categorical and *finitely constrained* (see below for a precise definition). Though, the latter property is only used in order to enforce that the problem belongs to $\mathcal{NP}$. As observed in [15], the ω-categorical template from Theorem 10 is also finitely constrained. It is not clear whether the template we build in the proof of Theorem 12 has this property but (as any problem in MMSNP$_2$ is in $\mathcal{NP}$) the corresponding problem is an infinite CSP according to our more general definition of CSP*.

Let Γ be a countable structure. The *age of* Γ (denoted $Age(\Gamma)$) is the set of all finite induced substructures of Γ. Let $\mathcal{N}$ be a finite set of finite structures. We denote by $Forb(\mathcal{N})$ the set of all finite structures that do not admit any of the structures from $\mathcal{N}$ as an induced substructure. The structure Γ is *finitely constrained* if there is a first-order expansion Γ' of Γ over some expanded signature τ' and a finite set $\mathcal{N}'$ of finite τ'-structures such that $Age(\Gamma') = Forb(\mathcal{N}')$.

Tree-depth and Elimination Tree

Before giving its proof, let us recall the statement of Lemma 16.

Lemma 30. *Let S be a connected structure. A rooted tree (r, Y) is an elimination tree for S if, and only if, S is a substructure of* $\operatorname{clos}(Y, \sigma)$. *Consequently, the tree-depth of S is the minimum height of an elimination tree.*

Proof. We prove this result by induction on $|S|$. If S has a single vertex r then the result holds trivially. Assume that the above equivalence holds for every connected structure of size at most $n-1$ and assume that S has size n.

Let (r, Y) be a tree that consists of a root r adjacent to rooted subtrees (r_i, Y_i). The tree (r, Y) is an elimination tree for S if, and only if, the subtrees (r_i, Y_i) are elimination trees for the components S_i induced by the connected component $\mathcal{H}_i$ of $\mathcal{H}_S \setminus \{r\}$. By induction S_i is a substructure of $\operatorname{clos}(Y_i, \sigma)$. Moreover every tuple t in a relation of S either does not mention r and occurs in a single component S_i or t mentions r and apart from r only elements from a single structure S_i. Hence, equivalently we have that S is a substructure of $\operatorname{clos}(Y, \sigma)$. □

Tree-depth and cores

Statement and proof of Theorem 17.

Theorem 31. *There exists a function $\eta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that, for any coloured structure $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ such that $|S| > \eta(N, \operatorname{td}(S))$ and any mapping $g : |S| \to \{1, 2, \ldots, N\}$, there exists a non-trivial involuting g-preserving automorphism μ of $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ with the fixed-point property.*

Proof. Let $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ be a coloured structure with $p \geq 1$ connected components $(S_i, s_i^{\mathcal{V}}, s_i^{\mathcal{E}})$ where $1 \leq i \leq p$ and let $g : |S| \to \{1, 2, \ldots, N\}$. Let F be a rooted forest that consists of rooted trees (r_i, Y_i), where $1 \leq i \leq p$, such that (r_i, Y_i) is an elimination tree for S_i of height at most $\operatorname{td}(S)$. We prove this result by induction on $\operatorname{td}(S)$.

If $\operatorname{td}(S) = 1$ then every (r_i, Y_i) has height 1 and every $(S_i, s_i^{\mathcal{V}}, s_i^{\mathcal{E}})$ is a coloured self-loops (a self-loop is a structure with a single element). Let $|\sigma|$ be the number of relation symbols in σ. There are at most $N \times |\mathcal{V}| \times |\mathcal{E}| \times 2^{|\sigma|}$ g-valued coloured self-loops. Hence, we set $\eta(N, 1) := N \times |\mathcal{V}| \times |\mathcal{E}| \times 2^{|\sigma|}$ and for any $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ satisfying $|S| \geq \eta(N, 1)$ and any mapping $g : |S| \to \{1, 2 \ldots, N\}$, there exists a non-trivial involuting g-preserving automorphism μ of $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ that has the fixed point property (simply choose for μ an automorphism that permutes two identical coloured self-loops with the same g value). In the rest of the proof, we shall write that μ is a "good" g-preserving automorphism for short.

Assume that the results holds for every coloured structure of tree depth at most n and let S be a structure with tree-depth $n+1$. Assume first that $p=1$. That is, F consists of a single rooted tree (r,Y). Let (r_j,Y_j), $1\leq j\leq q$ be the subtrees of Y where r_j is a son of r in Y. We wish to define a mapping g' over $S\setminus\{r\}$ such that $g'(x)$ describes both $g(x)$ and the relationship of x and r in S. Let $N(x,r)$ be the coloured structure (together with the restriction of g) that is the substructure of S induced by the following set of elements:

$$\bigcup\{e \text{ hyperedge of } \mathcal{H}_S \text{ s.t. } x\in e \text{ and } r\in e\}.$$

If this set is empty, we set $N(x,r)$ to be $\emptyset$. Let Θ be the set of such neighbourhood $N(x,r)$ with two constants x and r considered up to isomorphisms. We also assume that $\emptyset$ is an element of Θ. Since σ, $\mathcal{E}$ and $\mathcal{V}$ are fixed, it follows that Θ is finite. We define g' as the mapping from $|S|\setminus\{r\}$ to $N\times\Theta$ such that $g'(x):=(g(x),\tilde{N}(x,r))$, where $\tilde{N}(x,r)$ belongs to Θ and is isomorphic to $N(x,r)$. Hence, by induction if $|S\setminus\{r\}|>\eta(N|\Theta|,n)$, there exists a "good" g'-preserving automorphism μ' of $S\setminus\{r\}$. Let μ' be the extension of μ' to S such that $\mu(r):=r$. By construction, μ is a "good" g-preserving automorphism.

Assume now that $p>1$. If one of the component $(S_i,s_i^{\mathcal{V}},s_i^{\mathcal{E}})$ with elimination tree (r_i,Y_i) has size strictly greater than $\eta(N|\Theta|,n)+1$ then from the previous case, it has a "good" g-preserving automorphism μ_i which we can extend by the identity into a "good" g-preserving automorphism μ of the whole structure. Assume now that every component $(S_i,s_i^{\mathcal{V}},s_i^{\mathcal{E}})$ has size at most $\eta(N|\Theta|,n)+1$. The number $\zeta(N,k)$ of coloured structures (together with a mapping g to a set of size N) of size at most k considered up to isomorphism depends only on N and k (and the constants $\sigma,\mathcal{V}$ and $\mathcal{E}$). Hence, if $p>\zeta\big(N,\eta(N(|\Theta|,n)+1\big)$ then there must be at least two components that are isomorphic (also w.r.t. both colours and g-values). Hence, μ is a "good" g-preserving automorphism, where μ is the automorphism that exchanges these two components and leaves the element of any other component fixed. Thus, we may set,

$$\eta(N,n+1):=\big(\eta(N|\Theta|,n)+1\big)\times\zeta\big(N,\eta(N|\Theta|,n)+1\big).$$

The result follows by induction on $\mathrm{td}(S)$. $\square$

Statement and proof of Theorem 18.

Theorem 32. *Let $(S,s^{\mathcal{V}},s^{\mathcal{E}})$ be a coloured structure. If $|S|>\eta(1,\mathrm{td}(S))$ then $(S,s^{\mathcal{V}},s^{\mathcal{E}})$ maps homomorphically into one of its proper induced substructure $(S_0,s_0^{\mathcal{V}},s_0^{\mathcal{E}})$. Consequently, $(S,s^{\mathcal{V}},s^{\mathcal{E}})$ has a core of size at most $\eta(1,\mathrm{td}(S))$.*

Proof. By the previous theorem, if $|S|>\eta(1,\mathrm{td}(S))$ then there exists a non-trivial involuting automorphism μ of $(S,s^{\mathcal{V}},s^{\mathcal{E}})$ with the fixed-point property. Let F be the set of fixed-points of μ. Let S' be the substructure of S induced by

$|S| \setminus |F|$. Let S'' be a connected substructure of S'. Since μ has the fixed-point property and S'' has no fixed point by definition, we know that $\mu(S'') \cap S'' = \emptyset$. Note that there can not be any tuple t in a relation of S' that involves simultaneously elements of S'' and $\mu(S'')$ (Otherwise, the substructure S''_t of S' induced by $|S''|$ and the element of t would be connected and $\mu(S''_t) \cap S''_t \neq \emptyset$). There could be however a tuple that involves simultaneously elements of F of S'' and $\mu(S'')$. We need a slightly stronger result than the previous theorem. Note that in the previous proof, given a structure S we can fix a tree decomposition (r, Y) of height $\mathrm{td}(S)$ with subtrees (r_i, Y_i) that corresponds to substructures S_i of S and subsequently at the induction stage, rather than using any tree-decomposition of the structures S_i, we can assume that (r_i, Y_i) is used. With this further assumption, we ensure that the automorphism μ we build has still the required properties and moreover that there are no tuple that involves simultaneously elements of F, S'' and $\mu(S'')$, for any connected substructure S'' of S'.

Hence, starting with two empty sets A and B, we can inductively pick a connected component S'' of S', add S'' to A and $\mu(S'')$ to B until S' is partitioned into A and B. By construction, there is no tuple involving elements of both A and B(and possibly some element of F). Let h be the mapping such that $h(x) := \mu(x)$, if $x \in B$ and $h(x) := x$, otherwise. By construction of μ, A and B, it follows that h is a homomorphism from $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ to $(S_0, s_0^{\mathcal{V}}, s_0^{\mathcal{E}})$, the substructure of $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ induced by $|F| \cup A$. Since μ is non-trivial, $B \neq \emptyset$ and $(S_0, s_0^{\mathcal{V}}, s_0^{\mathcal{E}})$ is a proper induced substructure of $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$. This proves the first claim.

For the second claim, we apply inductively the first claim until we get a proper induced substructure $(S_\star, s_\star^{\mathcal{V}}, s_\star^{\mathcal{E}})$ of size at most $\eta(1, \mathrm{td}(S))$. Thus, $(S_\star, s_\star^{\mathcal{V}}, s_\star^{\mathcal{E}})$ is an induced substructure of $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ in which $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ maps homomorphically into (composing the homomorphisms, we get a homomorphism). The core of $(S_\star, s_\star^{\mathcal{V}}, s_\star^{\mathcal{E}})$ is also the core of $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ (since the two structures are homomorphically equivalent) and its size is at most that of $(S_\star, s_\star^{\mathcal{V}}, s_\star^{\mathcal{E}})$, which is bounded by $\eta(1, \mathrm{td}(S))$. □

Decomposition of Low Tree-depth

Recall the statement of Lemma 21.

Lemma 33. *A rooted tree (r, Y) is an elimination tree for a structure S if, and only if, it is an elimination-tree for its Gaifman graph $\mathcal{G}_S$.*

Proof. The direct implication holds since every edge between two elements x and y in the Gaifman graph is induced by at least one tuple in some relation of S in which both x and y occurs. Thus, in particular x and y occur on the same branch in an elimination tree of S.

Conversely, every tuple induces a clique in the Gaifman graph and all elements of a clique must occur on the same branch of an elimination tree. Thus, an elimination tree for $\mathcal{G}_S$ is also an elimination tree for S. □

Statement and proof of Corollary 22

Corollary 34. *Let $p \geq 1$. Let $\mathcal{K}$ be a proper minor closed class of structures. There exists an integer q such that for any structure S in $\mathcal{K}$, there exists a partition of $|S|$ into q sets such that any substructure of S induced by at most p of these sets has tree-depth at most p.*

Proof. Let S be a structure in $\mathcal{K}$. We apply Theorem 20 to $\mathcal{G}_S$, the Gaifman graph of S and get a vertex partition such that every subgraph of $\mathcal{G}_S$ induced by at most p colours has a tree-decomposition of height at most p. We use the same partition for S. For every substructure S' of S induced by at most p parts, the tree-decomposition of $\mathcal{G}_{S'}$ is also a tree-decomposition of S' (by Lemma 21) and the result follows. ◻

Truncated Product

Statement and proof of Lemma23.

Lemma 35. *Let $p \geq 2$. Let $\mathfrak{F}$ be a set of forbidden patterns such that for every $(F, f^{\mathcal{V}}, f^{\mathcal{E}})$ in $\mathfrak{F}$, we have $|F| < p$. If a coloured structure $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ is valid w.r.t $\mathfrak{F}$ then its p-truncated product $(S', {s'}^{\mathcal{V}}, {s'}^{\mathcal{E}})$ is also valid w.r.t. $\mathfrak{F}$.*

Proof. We prove the contrapositive. Assume that $(S', {s'}^{\mathcal{V}}, {s'}^{\mathcal{E}})$ is not valid w.r.t. $\mathfrak{F}$, that is for some $(F, f^{\mathcal{V}}, f^{\mathcal{E}})$ in $\mathfrak{F}$, there exist a colour-preserving homomorphism f' from F to S'. Since $|F| < p$, by the pigeon-hole principle there exists $1 \leq i_0 \leq p$ such that $f'(F) \cap \tilde{W}^{i_0} = \emptyset$.

Let π_{i_0} be the mapping from $|S'| \setminus \tilde{W}^{i_0}$ to $|S|$ defined as $\pi_{i_0}(w^i) = x^i_{i_0}$ (with the same notation as above). It follows directly from the definition of S' that π_{i_0} is a homomorphism from $S' \setminus \tilde{W}^{i_0}$ to S, where $S' \setminus \tilde{W}^{i_0}$ is the substructure of S' induced by $|S'| \setminus \tilde{W}^{i_0}$.

Hence, by composition $f := \pi_{i_0} \circ f'$ is a colour-preserving homomorphism from F to S and we have proved that $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ is not valid w.r.t. $\mathfrak{F}$. ◻

Statement and proof of Lemma 24.

Lemma 36. *Let $(U, u^{\mathcal{V}}, u^{\mathcal{E}})$ be a coloured structure and let p be an integer greater than the arity of any symbol in σ. Let $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ be a coloured structure. If there exists a partition $V_1, V_2, \ldots, V_p$ of $|S|$ such that for every substructure $(\tilde{S}_i, \tilde{s}^{\mathcal{V}}, \tilde{s}^{\mathcal{E}})$ of $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ induced by $|S| \setminus V_i$ there exist a colour-preserving homomorphism $\tilde{s}_i$ from $(\tilde{S}_i, \tilde{s}^{\mathcal{V}}, \tilde{s}^{\mathcal{E}})$ to $(U, u^{\mathcal{V}}, u^{\mathcal{E}})$ then there exists a colour-preserving homomorphism $\tilde{s}$ from $(\tilde{S}_i, \tilde{s}^{\mathcal{V}}, \tilde{s}^{\mathcal{E}})$ to $(U, u^{\mathcal{V}}, u^{\mathcal{E}})^{\Uparrow p}$.*

Proof. Let $(U', {u'}^{\mathcal{V}}, {u'}^{\mathcal{E}}) := (U, u^{\mathcal{V}}, u^{\mathcal{E}})^{\Uparrow p}$. Let x be an element of S such that x belongs to V_i. Then x is an element of $\tilde{S}_k$ for every $1 \leq k \leq p$ such that $k \neq i$ and $\tilde{s}_k(x)$ is defined and we set:

$$\tilde{s}(x) := (\tilde{s}_1(x), \tilde{s}_2(x), \ldots, \tilde{s}_{i-1}(x), \star, \tilde{s}_{i+1}(x), \ldots, \tilde{s}_p(x)).$$

Since $\tilde{s}_k$ is colour-preserving, $u^{\mathcal{V}}(\tilde{s}_k(x)) = \tilde{s}^{\mathcal{V}}(x)$ and $\tilde{s}(x)$ is indeed in $|U'|$ by definition of the truncated product.

Let $x_1, x_2, \ldots, x_r$ be elements of S that belong to the sets $V_{i_1}, V_{i_2}, \ldots, V_{i_r}$, respectively. Let R be a r-ary relation symbol from σ such that $R(x_1, x_2, \ldots, x_r)$ holds in S. Then, for every $1 \leq k \leq p$ such that $k \notin \{i_1, i_2, \ldots, i_r\}$, we have that $R(\tilde{s}_k(x_1), \tilde{s}_k(x_2), \ldots, \tilde{s}_r(x_r))$ holds in U. Moreover, since $\tilde{s}_k$ is colour-preserving, we have that $u^{\mathcal{E}}(\tilde{s}_k(x_1), \tilde{s}_k(x_2), \ldots, \tilde{s}_r(x_r)) = \tilde{s}^{\mathcal{E}}(x_1, x_2, \ldots, x_r)$ and it follows by definition of the truncated product that $R(\tilde{s}(x_1), \tilde{s}(x_2), \ldots, \tilde{s}(x_r))$ holds in U' and that $\tilde{s}$ is a colour-preserving homomorphism. $\square$

Using the two previous lemma, we get the following result.

Proposition 37. *Let p be an integer greater than the arity of any symbol in σ. Let $\mathfrak{F}$ be a set of forbidden patterns such that for every $(F, f^{\mathcal{V}}, f^{\mathcal{E}})$ in $\mathfrak{F}$, we have $|F| < p$. Let $q \geq p$. Let $(U, u^{\mathcal{V}}, u^{\mathcal{E}})$ be a coloured structure that is valid w.r.t $\mathfrak{F}$. Let $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ be a coloured structure.*

Assume that there exists a partition $V_1, V_2, \ldots, V_q$ of $|S|$ such that for every substructure $(\tilde{S}_i, \tilde{s}_i^{\mathcal{V}}, \tilde{s}_i^{\mathcal{E}})$ of $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ induced by p subsets $V_{i_1}, V_{i_2}, \ldots, V_{i_p}$ there exist a colour-preserving homomorphism $\tilde{s}_i$ from $(\tilde{S}_i, \tilde{s}_i^{\mathcal{V}}, \tilde{s}_i^{\mathcal{E}})$ to $(U, u^{\mathcal{V}}, u^{\mathcal{E}})$.

Then there exists a colour-preserving homomorphism $\tilde{s}$ from $(\tilde{S}_i, \tilde{s}^{\mathcal{V}}, \tilde{s}^{\mathcal{E}})$ to $(U', {u'}^{\mathcal{V}}, {u'}^{\mathcal{E}})$ and $(U', {u'}^{\mathcal{V}}, {u'}^{\mathcal{E}})$ is valid with respect to $\mathfrak{F}$, where

$$(U', {u'}^{\mathcal{V}}, {u'}^{\mathcal{E}}) := (U, u^{\mathcal{V}}, u^{\mathcal{E}})^{\Uparrow(p+1)\Uparrow(p+2)\ldots\Uparrow q}.$$