Bounded-degree Forbidden Patterns Problems are Constraint Satisfaction Problems

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Abstract. Forbidden Patterns problem (FPP) is a proper generalisation of Constraint Satisfaction Problem (CSP). FPP was introduced in [1] as a combinatorial counterpart of MMSNP, a logic which was in turn introduced in relation to CSP by Feder and Vardi [2]. We prove that *Forbidden Patterns Problems are Constraint Satisfaction Problems* when restricted to *graphs of bounded degree*. This is a generalisation of a result by Häggkvist and Hell who showed that F-moteness of bounded-degree graphs is a CSP (that is, for a given graph F there exists a graph H so that the class of bounded-degree graphs that do not admit a homomorphism from F is exactly the same as the class of bounded-degree graphs that are homomorphic to H). Forbidden-pattern property is a strict generalisation of F-moteness (in fact of F-moteness combined with a CSP) as it involves both vertex- and edge-colourings of the graph F, and thus allows to express $\mathcal{NP}$-complete problems (while F-moteness is always in $\mathcal{P}$). We finally extend our result to arbitrary relational structures, and prove that every problem in MMSNP, restricted to connected inputs of bounded (hyper-graph) degree, is in fact in CSP.
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1 Introduction

Graph Homomorphisms and related problems have received considerable attention in the recent years not only as a topic in Combinatorics and Graph Theory but also in relation with Constraint Satisfaction. A lot of possible directions of research in the area have been opened by different motivations, and have been explored (the very recent monograph [3] serves as a good survey of the area).

The present work was motivated mainly by Constraint Satisfaction Problem (CSP) and its generalisations, Forbidden Patterns Problem (FPP), introduced in [1]. Our results fall into the category of so-called (restricted) duality theorems, and have the same flavour as the results in [4].

To better explain the motivation behind our work, we need to start with MMSNP, a fragment of monadic second order logic, that was introduced in [2], and was motivated by the search of a logical characterisation of Constraint

Satisfaction Problem. Though computationally very closely related to CSP, the logic MMSNP is too strong and captures problems which are not constraint satisfaction problems [2, 5] (or, in a more general setting, at least not CSPs with a finite template [6]). The combinatorial counterpart of MMSNP is the class of Forbidden Patterns Problems. FPPs were studied at length in [7] where an exact characterisation of FPPs that are not CSPs was proved. This characterisation subsumes the characterisation of duality pairs obtained by Tardif and Nešetřil in [4].

In [8], Häggkvist and Hell showed a new kind of (restricted) duality result for Graph Homomorphism (or CSP). They built a universal graphs H for the class of F-mote graphs, i.e. the class of graphs that do not admit a homomorphism from F, of bounded degree b. Thus, any degree-d graph G is F-mote if, and only if, there is a homomorphism from G to H. The result can be seen as a restricted form of *duality* and provides a "good characterisation" (in the sense of Edmonds) for F-moteness. In the last decade, a series of papers built upon the Häggkvist and Hell's technique. In particular, a number of results by Nešetřil and various coauthors are relevant to our work. First, the complexity of H-colouring of bounded-degree graphs was investigated in more details in [9]. It was conjectured that if H is a triangle-free graph with chromatic number three then the H-colouring problem for 3-bounded graphs is $\mathcal{NP}$-complete. This conjectured was disproved in a very recent paper [10], where the authors investigated more generally the existence of universal graphs for the class of graphs that are simultaneously F-mote and H-colorable.

The main contribution of the present paper is a proof that Forbidden Patterns Problems, restricted to inputs of bounded-degree, become Constraint Satisfaction Problems. More precisely, given a forbidden patterns problem Ω and a bound b on the degree of the input graph, we explicitly construct a graph H, such that any graph G of bounded degree b is a yes-instance of Ω if, and only if, G is homomorphic to H. We say that H is a universal graph for Ω (and b) as it does not depend on the input G.

This result is a significant jump in the (descriptive) complexity from the result of Häggkvist and Hell as their result corresponds to problems, which are captured by fragments of first-order logic, and therefore are in $\mathcal{P}$. In our case we retain problems which are $\mathcal{N}P$-complete even on bounded-degree graphs.

Finally, we investigate the fragment of existential MSO logic that corresponds to the original FPPs as defines in [1], Feder and Vardi's logic MMSNP. It is perhaps worth mentioning that in the present paper we assume a definition of FPP that is more general than the original one in that the new definition allows colourings not only of the vertices but also of the edges of the input graph.

The rest of paper is organised as follows. In section 2 we give the necessary background as well as some motivating examples. Then we prove the main result in section 3. Finally, in section 4, we briefly discuss the logic MMSNP and extend our result to arbitrary relational structures. In 5, we conclude by a few questions and open problems.

2 Preliminaries

2.1 Definitions

To simplify the exposition, we shall assume mostly that the input is a graph G. We use standard graph notation. We denote by $V(G)$ the vertex set of G and by $E(G)$ the edge set of G. Let G and H be two graphs. A *graph homomorphism* is a vertex mapping $h : V(G) \to V(H)$ such that for every vertices x and y in $V(G)$, if (x, y) belongs to $E(G)$ then $(h(x), h(y))$ belongs to $E(H)$. Let F and G be two graphs. We say that G is *F-mote* if, and only if, there is no homomorphism from F to G. The *F-moteness* is the property of being F-mote, a term that we also use to refer to the corresponding decision problem.

Constraint Satisfaction Problems. The *constraint satisfaction problem* with *template* H is the decision problem with,

- input: a graph G; and,
- question: does there exist a homomorphism from G to H?

Example 1. The constraint satisfaction problem with template K_3 (the clique with three elements, *i.e.* a triangle) is nothing else than the 3 colorability problem from graph theory. Hence, the restriction to (undirected) graphs is also known as the *H-colouring problem*.

We denote by CSP the class of constraint satisfaction problems (not necessarily graph problems).

Edge and Vertex Coloured Forbidden Patterns Problems. Let $\mathcal{V}$ (respectively, $\mathcal{E}$) be a finite set of *vertex colours* (respectively, *edge colours*). A *pattern* is a connected graph whose vertices are coloured with $\mathcal{V}$ and whose edges are coloured with $\mathcal{E}$. Formally, it is a triple $(F, f^{\mathcal{V}}, f^{\mathcal{E}})$, where F is a finite graph, $f^{\mathcal{V}}$ is a mapping from $V(F)$ to $\mathcal{V}$ and $f^{\mathcal{E}}$ is a mapping from $E(F)$ to $\mathcal{E}$.

Let G be a graph, $g^{\mathcal{V}} : V(G) \to \mathcal{V}$ and $g^{\mathcal{E}} : E(G) \to \mathcal{E}$. Let H be a graph, $h^{\mathcal{V}} : V(H) \to \mathcal{V}$ and $h^{\mathcal{E}} : E(H) \to \mathcal{E}$. We say that a homomorphism h from G to H *preserve the colours* (given by $g^{\mathcal{V}}, g^{\mathcal{E}}, h^{\mathcal{V}}$ and $h^{\mathcal{E}}$) if, for every x in $V(G)$, $g^{\mathcal{V}}(x) = h^{\mathcal{V}} \circ h(x)$ holds; and, for every $e = (x, y)$ in $E(G)$, $g^{\mathcal{E}}(e) = h^{\mathcal{E}}(e')$ holds, where $e' = (h(x), h(y))$ belongs to $E(H)$. When the colours are clear from the context, we simply write that *h preserves colours*. Note that the composition of two homomorphism that preserve colours is also a homomorphism that preserve colours.

In this paper, patterns are used to model constraints in a negative fashion and consequently, we refer to them as *forbidden* patterns. Let $\mathfrak{F}$ be a finite set of forbidden patterns. Let G be a graph, $g^{\mathcal{V}} : V(G) \to \mathcal{V}$ and $g^{\mathcal{E}} : E(G) \to \mathcal{E}$. We say that $(G, g^{\mathcal{V}}, g^{\mathcal{E}})$ is *valid* with respect to $\mathfrak{F}$ if, and only if, for any pattern $(F, f^{\mathcal{V}}, f^{\mathcal{E}})$ there does not exist any homomorphism h from F to G that preserves these colours

The *problem with forbidden patterns* $\mathfrak{F}$ is the decision problem with,

– input: a graph G
– question: does there exist any mapping $g^{\mathcal{V}} : V(G) \to \mathcal{V}$ and $g^{\mathcal{E}} : E(G) \to \mathcal{E}$ such that $(G, g^{\mathcal{V}}, g^{\mathcal{E}})$ is valid with respect to $\mathfrak{F}$.

Remark 2. The class of forbidden patterns problems with colours over the vertices only, corresponds to the problems that can be expressed by a formula in Feder and Vardi's MMSNP (Monotone Monadic SNP without inequalities, see [2, 1]). Note that allowing colours over the edges does not amount to drop the hypothesis of monadicity. Rather, it corresponds to a logic, let's call it MMSNP_2, which is similar to MMSNP but allows *first-order* variables over edges (just like Courcelle's MSO (Monadic Second Order logic) and MSO_2, see [11]).

2.2 Motivating Examples

In this section, we motivate this paper through the study of the following concrete graph problems.

1. Vertex-No-Mono-Tri: consists of the graphs for which there exists a partition of the vertex set in two sets such that no triangle has its three vertices occurring in a single partition. It was proved in [2, 5] that this problem is not in CSP and in [12] that it is $\mathcal{NP}$-complete.
2. Tri-Free-Tri: consists of the graphs that are both three colourable (tripartite) and in which there is no triangle. It was proved in [5] that this problem is not in CSP.
3. Edge-No-Mono-Tri: consists of the graphs for which there exists a partition of the edge set in two sets such that no triangle has its three edges occurring in a single partition. It is known to be $\mathcal{NP}$-complete (see [13]).

The above examples can be formulated as Forbidden Patterns Problems. The corresponding set of forbidden patterns are depicted on Figure 1. In the case of Edge-No-Mono-Tri, the two type of colours for edges are depicted with dashed and full line respectively. In [8], Häggvist and Hell proved a result that

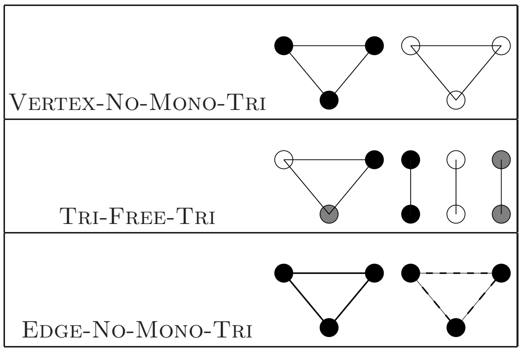


Fig. 1. Some forbidden patterns problems

can be rephrased as follows.

Theorem 3. *Let F be a connected graph and b a positive integer. There exists a graph U such that there is no homomorphism from F to U; and, for every graph G of bounded degree b, there is no homomorphism from F to G if, and only if, there is a homomorphism from G to U.*

Using this theorem, we prove the following.

Corollary 4. *Let b be a positive integer. The problem* Vertex-No-Mono-Tri, *restricted to graph of bounded degree b, is a constraint satisfaction problem with input restricted to graph of bounded degree b.*

Proof. Let b be a positive integer. Let U be the universal graph for the class of graph of bounded degree b given by Theorem 3 for $F := K_3$. Let U' be the graph that consists of two disjoint copies U_1 and U_2, together with all the edges (x_1, x_2), where $x_1 \in V(U_1)$ and $x_2 \in V(U_2)$. Let G be a graph of bounded degree b. We model Vertex-No-Mono-Tri as a forbidden patterns problem with $\mathcal{V} := \{1, 2\}$ and two forbidden patterns (K_3, c_1) and (K_3, c_2) where c_1 is the constant 1 and c_2 the constant 2. We drop altogether the notation for edge colours since there is a single edge colour. Assume that G is a yes-instance of Vertex-No-Mono-Tri. Let $g^{\mathcal{V}} : V(G) \to \mathcal{V}$ such that $(G, g^{\mathcal{V}})$ is valid. Let G_1, respectively G_2, be the subgraph of G induced by the vertices of colour 1, respectively 2. It follows that there is neither a homomorphism from K_3 to G_1, nor from K_3 to G_2. Thus, there exist a homomorphism h_1 from G_1 to U_1 and a homomorphism h_2 from G_2 to U_2. Let h be the mapping induced by the union of h_1 and h_2. It is a homomorphism from G to U' by construction of U'. Conversely, if there is a h homomorphism from G to U', it induces a colour map $g^{\mathcal{V}}$ from G to $\mathcal{V}$, according to which copy of U a vertex of G is mapped to. Similarly to the above, it follows that $(G, g^{\mathcal{V}})$ is valid. Thus, Vertex-No-Mono-Tri, restricted to graph of bounded degree b, is the constraint satisfaction problem with template U' and input restricted to graph of bounded degree b.

It does not seem to be straightforward to come up with a similar graph construction to prove the same for the problem Tri-Free-Tri. However, Theorem 3 has been generalised in [10] and can be reformulated as follows. Here, we give a shorter proof than in [10].

Theorem 5. *Let F and H be connected graphs, b a positive integer. Then there exists a graph U with the following properties:*

1. *there exists a homomorphism u from U to H;*
2. *For every graph G of bounded degree b such that there is no homomorphism from F to G and there exists a homomorphism g from G to H, there exist a homomorphism $G \xrightarrow{a} U$ such that $g = u \circ a$; and,*
3. *there is no homomorphism from F to U.*

Proof. By Häggvist an Hell's theorem (Theorem 3), there exists a finite graph $U_{F,b}$ such that for any graph G of bounded degree b, there is no homomorphism from F to G if, and only if, there is a homomorphism from G to $U_{F,b}$.

Choose for U the (categorical) product of $U_{F,b}$ and H and denote by π_1 and π_2 the two projections (see [3] for details). Choose for u the projection π_2.

Let G be a graph of bounded degree b such that there is no homomorphism from F to G and there is a homomorphism g from G to H. It follows that there exists a homomorphism h from G to $U_{F,b}$. We choose for a the homomorphism (h, g) and by definition of the product we have $g = u \circ a$.

Finally there can not be a homomorphism f from F to U as otherwise, by composition with π_1, there would be a homomorphism from F to $U_{F,b}$ in contradiction with Theorem 3.

The above result involves problems that corresponds to the intersection of a constraint satisfaction problem with template H, with a very simple forbidden patterns problem that has a single vertex colour, a single edge colour and a single forbidden pattern F. Applying this theorem with $F = H = K_3$, we get the problem Tri-Free-Tri (intersection of 3-colorability with triangle free, see previous examples) and the following result.

Corollary 6. *Let b be a positive integer. The problem* Tri-Free-Tri*, restricted to graph of bounded degree b, is a constraint satisfaction problem with input restricted to graph of bounded degree b.*

Note that, it is unlikely that the problems corresponding to Theorem 5 are as general as FPP, even when restricted to colours over vertices only.

Proposition 7. *If $\mathcal{P} \neq \mathcal{NP}$ then* Vertex-No-Mono-Tri *can not be represented as a problem of the form of Theorem 5.*

Proof. Let F and H be two graphs such that Vertex-No-Mono-Tri is the intersection of a constraint satisfaction problem with template H with the very simple forbidden patterns problem that has a single vertex colour, a single edge colour and a single forbidden pattern F. By Proposition 13 (in the appendix), H must have a loop and a graph G is in Vertex-No-Mono-Tri if, and only if, there is no homomorphism from F to G. This last property is expressible in first-order logic. However, it was proved in [12] that this problem is $\mathcal{NP}$-complete. Since checking property expressible in first-order logic can be done in polynomial time, the result follows.

We now turn our attention to the problem Edge-No-Mono-Tri. It seems to be difficult to prove the same result for this problem as a direct corollary from Theorem 5 or Theorem 3. We shall deal with this example as a corollary of our main result: this result is a generalisation of Theorem 5 to the class of all (edge and vertex coloured) forbidden patterns problems.

3 Bounded Degree Forbidden Patterns Problems

In this section, we prove that forbidden patterns problems over graphs, when restricted to input of bounded degree, become constraint satisfaction problems. In graph parlance, we show that there is a universal graph for the (yes-instance of) bounded degree forbidden patterns problems.

Theorem 8. *Let b be a positive integer. Let Ω be the problem with forbidden patterns $\mathfrak{F}$. There exists a universal graph U for the graphs of bounded degree b in Ω. That is,*

1. *There exist $u^{\mathcal{V}} : V(U) \to \mathcal{V}$ and $u^{\mathcal{E}} : E(U) \to \mathcal{E}$ such that $(U, u^{\mathcal{V}}, u^{\mathcal{E}})$ is valid w.r.t. $\mathfrak{F}$;*
2. *For every graph G of bounded degree b such that there exist $g^{\mathcal{V}} : V(G) \to \mathcal{V}$ and $g^{\mathcal{E}} : E(G) \to \mathcal{E}$ such that $(G, g^{\mathcal{V}}, g^{\mathcal{E}})$ is valid w.r.t. $\mathfrak{F}$, there exist a homomorphism a from G to U that preserves these colours.*
3. *Conversely, for every graph G of bounded degree b, if there exists a homomorphism $G \xrightarrow{a} U$ then a induces colour maps $g^{\mathcal{V}}$ and $g^{\mathcal{E}}$ such that $(G, g^{\mathcal{V}}, g^{\mathcal{E}})$ is valid w.r.t. $\mathfrak{F}$.*

Remark 9. Using the standard notion of product of graphs, our new proof of Theorem 5 makes this result a direct corollary of Häggvist and Hell's theorem. However the proof in [10] present an idea that is key to our paper: the fact that Hell and Häggvist general technique can be extended to take *coloured* graphs into account. Hence, apart from the fact that we allow also colours on the edges, one could see Theorem 8 as the true statement behind Dreyer, Malon and Nešetřil's proof.

We shall first explain the intuition and ideas behind the proof by giving an informal outline. Let us denote the largest diameter of a forbidden patterns by m. This parameter, together with the degree-bound b, are absolute constants, i.e. they are not part of the input of the FPP. The universal graph U has to contain all possible *small graphs* that are *yes-instances of the FPP*, "small" meaning "of diameter $m + 1$" here and thereafter. The intuition then is that each graph G, which is a yes-instance of the FPP, however big, can be homomorphically mapped to U as such a mapping should be only locally consistent. Given that the degree of G is bounded by b, we need to distinguish among no more than $X \leq b^{m+2}$ vertices in a small neighbourhood, so we can use X many different labels in constructing the vertex set of U. On the other hand, in order to define the adjacency relation of the universal graph, *i.e.* to correctly "glue" all the possible small neighbourhoods, any vertex of U should carry information not only about its label, but also about its neighbourhood. In other words, any such vertex will represent a small graph together with a vertex which is the "centre" (or the root) of the small graph. Thus the vertex set of the universal graph will consists of all such rooted small graphs vertex- and edge-coloured in all possible ways that are yes-instances of the FPP. Two vertices will be adjacent in U if, and only if, the graphs they represent "agree", i.e. have most of their vertices with the same labels and colours and the induced subgraphs of these vertices coincide including the edge colours; for the precise definition of what "agree" means, one should see the formal proof below. It is now intuitively clear why a yes-instance G of the FPP should be homomorphic to the universal graph U: the vertices of G can be labelled so that any two adjacent vertices get different labels, then one can choose a good vertex- and edge-colouring of G, and because of the construction of U now every vertex u in G can be mapped to the vertex of U that represents the small neighbourhood of u rooted at u. It is straightforward to see that the mapping preserves edges.

The universal graph U has a very useful property, namely every small neighbourhood of U rooted at a vertex v is homomorphic to the small graph represented by the vertex u (see Lemma 11 below). This property immediately implies that a no-instance of the FPP cannot be homomorphic to U. Indeed, suppose for the sake of contradiction, there is a no-instance G that is homomorphic to U. Fix the colouring induced by the homomorphism and observe that G there is a homomorphism from the FPP graph F into G. The composition of the two homomorphisms gives a homomorphism from F into U, and by the property above, by another composition, we get a homomorphism from the FPP graph F to some small graph represented by a vertex of U. This gives a contradiction with our construction as we have taken small no-instances only to be represented by the vertices of the universal graph.

We illustrated the construction of Theorem 8 for our final example EDGE-NO-MONO-TRI on Figure 2 for input of bounded degree $b \geq 3$. Any input G can be labelled by elements from $\{1, 2, \ldots, 10, \ldots, X\}$ such that for every vertex x in $V(G)$, the vertices at distance at most 2 (the diameter of a triangle plus one) of x have a different label.

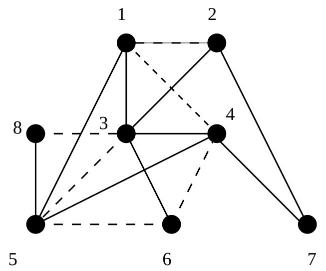
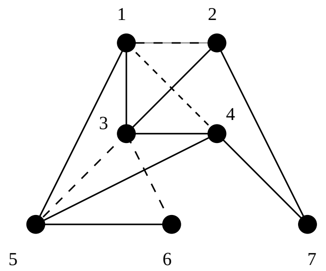
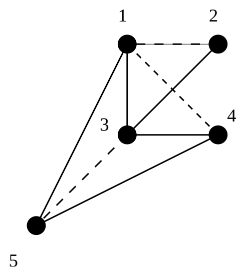
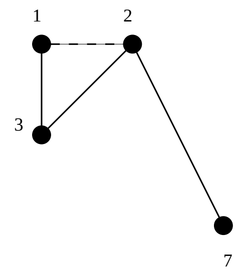
Corollary 10. *Let b be a positive integer. The problem* EDGE-NO-MONO-TRI*, restricted to graph of bounded degree b, is a constraint satisfaction problem with input restricted to graph of bounded degree b.*

The remaining of this section is devoted to the formal proof of Theorem 8.

Let b be a positive integer and $\mathfrak{F}$ be a set of forbidden patterns. We write $\delta(G)$ to denote the *diameter* of a graph G: that is, the minimum for all vertices x in $V(G)$ of the maximum distance from x to any other vertex y in $V(G)$. Let $m := \max\{\delta(F) \text{ such that } (F, f^{\mathcal{V}}, f^{\mathcal{E}}) \in \mathfrak{F}\}$. Let $X := 1 + \Sigma_{j=0}^{m} b(b-1)^j$.

Construction of U. Let $\mathcal{S}$ be the set of connected graphs $(S, s^{\mathcal{V}}, s^{\mathcal{E}})$ that are valid w.r.t. $\mathfrak{F}$, such that $V(S)$ is a subset of $\{1, 2, \ldots, X\}$. Let U be the graph with:

- vertices $(v, (S, s^{\mathcal{V}}, s^{\mathcal{E}}))$, where $(S, s^{\mathcal{V}}, s^{\mathcal{E}}) \in \mathcal{S}$ and $v \in S$; and,
- such that $(v, (S, s^{\mathcal{V}}, s^{\mathcal{E}}))$ is adjacent to $(v', (S', s'^{\mathcal{V}}, s'^{\mathcal{E}}))$ if, and only if, the following holds:
  (*i*) (v, v') belongs to both $E(S)$ and $E(S')$; and,

(Edge) coloured graph S_1	(Edge) coloured graph S_2
	
The coloured graph induced by the vertices at distance at most one from the vertex 1 is identical in both S_1 and S_2	The coloured graph induced by the vertices at distance at most one from the vertex 2 is identical in both S_1 and S_2
	

The template U has the edge

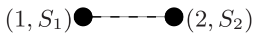


Fig. 2. Illustration of the construction for EDGE-NO-MONO-TRI.

(*ii*) the induced coloured subgraph of S induced by every vertex at distance at most m of v (respectively, v') is *identical* to the induced coloured subgraph of S' induced by every vertex at distance at most m of v (respectively, v').

U is a Yes-Instance. We first define "*universal colours*" for U as follows.

- $u^{\mathcal{V}}(v,(S,s^{\mathcal{V}},s^{\mathcal{E}})) := s^{\mathcal{V}}(v)$, for every vertex $(S,s^{\mathcal{V}},s^{\mathcal{E}})$ in $V(U)$; and,
- $u^{\mathcal{E}}((v,(S,s^{\mathcal{V}},s^{\mathcal{E}}))(v',(S',{s'}^{\mathcal{V}},{s'}^{\mathcal{E}}))) := s^{\mathcal{E}}(v,v')$, for every edge $((v,(S,s^{\mathcal{V}},s^{\mathcal{E}}))(v',(S',{s'}^{\mathcal{V}},{s'}^{\mathcal{E}})))$ in $E(U)$.

We need the following lemma.

Lemma 11. *Let $(v,(S,s^{\mathcal{V}},s^{\mathcal{E}}))$ in $V(U)$. Let $(B,c^{\mathcal{B}},s^{\mathcal{E}})$ be the coloured graph induced by all vertices at distance at most m from $(v,(S,s^{\mathcal{V}},s^{\mathcal{E}}))$. Then, there exists a homomorphism from $(B,c^{\mathcal{B}},s^{\mathcal{E}})$ to $(S,s^{\mathcal{V}},s^{\mathcal{E}})$ that preserves the colours.*

Proof. Let $(v',(S',{s'}^{\mathcal{V}},{s'}^{\mathcal{E}}))$ in $V(B)$. We set $h(v',(S',{s'}^{\mathcal{V}},{s'}^{\mathcal{E}})) := v'$. By induction on the distance from $(v,(S,s^{\mathcal{V}},s^{\mathcal{E}}))$ to $(v',(S',{s'}^{\mathcal{V}},{s'}^{\mathcal{E}}))$, using ($i$), it follows that the vertex v' belongs to $V(S)$ and that h is homomorphism. Similarly, using (ii) it follows that h preserves colours.

We now prove that $(U,u^{\mathcal{V}},u^{\mathcal{E}})$ is valid with respect to the forbidden patterns $\mathfrak{F}$. Assume for contradiction that $(U,u^{\mathcal{V}},u^{\mathcal{E}})$ is not valid and that there exists some forbidden pattern $(F,f^{\mathcal{V}},f^{\mathcal{E}})$ in $\mathfrak{F}$ and some homomorphism f' from F to U that preserves these colours. Since the diameter of F is at most m, there exists a vertex x in $V(F)$ such that every vertex of F is at distance at most m. Hence, it is also the case for $f'(x)$ in the homomorphic image of F via f'. By lemma 11, there exists a homomorphism h from F to S, where $f'(F) = (v,(S,s^{\mathcal{V}},s^{\mathcal{E}}))$. Both homomorphisms are colour preserving, thus composing these two homomorphisms, we get that $h \circ f'$ is a colour-preserving homomorphism from F to S. However, by definition of $\mathcal{S}$, the coloured graph $(S,s^{\mathcal{V}},s^{\mathcal{E}})$ is valid with respect to the forbidden patterns $\mathfrak{F}$. We reach a contradiction and the result follows.

Yes-Instances Are Homomorphic to U. Let G be a graph of bounded degree b for which there exist $g^{\mathcal{V}} : V(G) \to \mathcal{V}$ and $g^{\mathcal{E}} : E(G) \to \mathcal{E}$ such that $(G,g^{\mathcal{V}},g^{\mathcal{E}})$ is valid w.r.t. $\mathfrak{F}$. Since G has bounded degree b, for every vertex x in $V(G)$, there are at most $X-1$ vertices y at distance at most m from x. Therefore, there exists a map χ from $V(G)$ to $\{1,2,\ldots,X\}$ such that every two distinct vertices within distance m or less take a different colour via χ. Thus, for every vertex x in G, the subgraph of G induced by the vertices at distance at most m of x can be identified (via the labelling χ) to a graph S_x with domain $\{1,2,\ldots,X\}$. Similarly, the restriction of $g^{\mathcal{V}}$ and $g^{\mathcal{E}}$ to this subgraph induce colour maps $s_x^{\mathcal{V}}$ and $s_x^{\mathcal{E}}$ of S. We set $a(x) := (\chi(x),(S_x,s_x^{\mathcal{V}},s_x^{\mathcal{E}}))$. It follows directly from the definition of U that a is homomorphism that preserve colours.

No-instances are not homomorphic to U. Let G be a graph of bounded degree b that is a no instance of the forbidden patterns problem represented by $\mathfrak{F}$ and assume for contradiction that $G \xrightarrow{a} U$. The homomorphism a together with the universal colouring $(U, u^{\mathcal{V}}, u^{\mathcal{E}})$ induces colourings $g^{\mathcal{V}}$ and $g^{\mathcal{E}}$ as follows: For every vertex x in $V(G)$, set $g^{\mathcal{V}}(x) := u^{\mathcal{V}}(a(x))$; and, for every edge (x, y) in $E(G)$, set $g^{\mathcal{E}}(x, y) := u^{\mathcal{E}}(a(x), a(y))$. Since G is a no instance, there exists a forbidden pattern $(F, f^{\mathcal{V}}, f^{\mathcal{E}})$ in $\mathfrak{F}$ and a homomorphism f' from F to G that preserve these colours. Composing the two homomorphisms, we get that $a \circ f'$ is a homomorphism from F to U that preserves the colours of F and U. This contradicts the fact that $(U, u^{\mathcal{V}}, u^{\mathcal{E}})$ is valid w.r.t. $\mathfrak{F}$. This concludes the proof of Theorem 8.

4 Logic

In [2], a fragment of MSO was defined in the search of a logic to capture CSP by Feder and Vardi: the logic MMSNP (Monotone Monadic SNP without inequalities). They proved that though CSP is strictly included in MMSNP, for every problem Ω in MMSNP, there exists a problem Ω' in CSP such that Ω reduces to Ω' by a polynomial-time Karp reduction and Ω' reduces to Ω by a randomised polynomial-time Turing reduction. While in [1], we attached ourselves to understand the difference between CSP and MMSNP and characterised precisely the problems in MMSNP that are not in CSP, we adopt the opposite stance in the present work and show that the two classes coincide once restricted to input of bounded degree.

The proof technique of Theorem 8 generalises to arbitrary structures and since every problem in MMSNP is a disjunction of problems that are forbidden patterns problems without edge colours (*i.e.* with a single edge colour to be precise), we get the following result (the proof has been omitted due to space restriction and can be found in the appendix).

Theorem 12. *On connected input of bounded degree, MMSNP coincides with CSP.*

5 Conclusion and Open Problems

We have proved that every forbidden patterns (with colours on both edges and vertices) problem is in fact a constraint satisfaction problem, when restricted to input of bounded degree. We derived from this result that the logic MMSNP coincides with CSP on connected input of bounded degree (whereas CSP is strictly included in MMSNP for unrestricted inputs).

Theorem 12 can be easily adapted to the extension MMSNP_2 of MMSNP with first order quantification over edges as well as vertices. Since CSP is subsumed by MMSNP, this proves that MMSNP and MMSNP_2 have the same expressive power for connected input of bounded degree. This phenomenon can be related to a result due to Courcelle [11], who proved that the same properties of graphs of degree at most b, where b is a fixed integer, can be expressed by monadic second-order formulae using edge and vertex quantifications (MSO_2) as well as by monadic second-order formulae using vertex quantifications only (MSO_1). Courcelle also proved that MSO_2 is more expressive than MSO_1 for unrestricted input. This leads us to ask the following question. *Is MMSNP$_2$ more restrictive than MMSNP$_1$ for unrestricted input?* Proving, for example, that the problem EDGE-NO-MONO-TRI is not expressible in MMSNP_1 would answer this question affirmatively.

Courcelle has presented other restrictions over graphs and digraphs that ensure that MSO_1 and MSO_2 have the same expressive power, such as planarity or bounded tree-width. This provokes the following questions. *Is it the case that MMSNP$_2$ coincides with MMSNP$_1$ for planar graphs or graphs of bounded tree width?* and, *if it is the case, do they also collapse to CSP?*

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Appendix

Vertex-No-Mono-Tri vs. F-mote and H-colourable.

In an attempt to prove that FPP are more general than the problems that can be characterised as simultaneously F-mote and homomorphic to H, we prove that if it was the case then Vertex-No-Mono-Tri could be characterised as F-mote only. We derive some properties of F.

Proposition 13. *Let F and H be two graphs such that* Vertex-No-Mono-Tri *is the intersection of a constraint satisfaction problem with template H with the very simple forbidden pattern problem that has a single vertex colour, a single edge colour and a single forbidden pattern F. Then, the following holds:*

- *H has a loop;*
- *The core of F is not a tree;*
- *F has girth 3; and,*
- *F has chromatic number 5.*

Proof. Let F and H be two graphs such that Vertex-No-Mono-Tri is the intersection of a constraint satisfaction problem with template H, with the very simple forbidden pattern problem that has a single vertex colour, a single edge colour and a single forbidden pattern F. Assume that H is loopless. The graph K_3 is in Vertex-No-Mono-Tri, therefore H can not be bipartite. We deal similarly with higher chromatic numbers: by Erdös Theorem on high girth and chromatic number, there exists a triangle free graph G of chromatic number strictly greater than that of H. Since G is triangle free, it follows that it is in Vertex-No-Mono-Tri. Thus, G is homomorphic to H and by composition G has chromatic number at least that of H, a contradiction. This proves that H must contain a loop. Thus, every graph G is homomorphic to H. Consequently, Vertex-No-Mono-Tri is the class of graph G such that there is no homomorphism from F to G.

If the core of F is a tree then by Tardif and Nešetřil's characterisation of duality pair [4] there exists a graph H' such that the following holds: for any graph G, there is no homomorphism from F to G if, and only if, there exists a homomorphism from G to H'. However, it was proved in [2, 5] that Vertex-No-Mono-Tri is not in CSP. This proves that the core of F is not a tree.

If F is not a tree then its girth (length of its shortest cycle) must be three. Otherwise, F has no triangle and would be a yes-instance of Vertex-No-Mono-Tri and yet there is a trivial homomorphism from F to F, which is absurd.

The complete loopless graph with five vertices, K_5, is a no-instance of Vertex-No-Mono-Tri. Hence, there exists a homomorphism from F to K_5. However, K_4 is a yes-instance and there can not exist a homomorphism from F to K_4. In other words, F has chromatic number 5.

Proof of Theorem 12.

MMSNP (monotone monadic SNP without inequalities) was defined by Feder and Vardi in [2].

Definition 14. Monotone Monadic **SNP** without inequality*,* **MMSNP***, is the fragment of* **ESO** *consisting of those formulae Φ of the following form:*

$$\exists \mathbf{M} \forall \mathbf{t} \bigwedge_i \neg\big(\alpha_i(\sigma, \mathbf{t}) \wedge \beta_i(\mathbf{M}, \mathbf{t})\big),$$

where $\mathbf{M}$ *is a tuple of monadic relation symbols (not in σ),* $\mathbf{t}$ *is a tuple of (first-order) variables and for every negated conjunct $\neg(\alpha_i \wedge \beta_i)$:*

- *α_i consists of a conjunction of positive atoms involving relation symbols from σ and variables from* $\mathbf{t}$*; and*
- *β_i consists of a conjunction of atoms or negated atoms involving relation symbols from* $\mathbf{M}$ *and variables from* $\mathbf{t}$.

(Notice that the equality symbol does not occur *in Φ.)*

Moreover, Φ is primitive *if, and only if, for every negated conjunct $\neg(\alpha \wedge \beta)$:*

- *for every first-order variable x that occurs in $\neg(\alpha \wedge \beta)$ and for every monadic symbol C in* $\mathbf{M}$*, exactly one of $C(x)$ and $\neg C(x)$ occurs in β; and*
- *unless x is the only first-order variable that occurs in $\neg(\alpha \wedge \beta)$, an atom of the form $R(\mathbf{t})$, where x occurs in* $\mathbf{t}$ *and R is a relation symbol from σ, must occur in α.*

The following can be easily proved, by adapting the proof of the reduction from an arbitrary problem Ω in MMSNP to a problem in CSP in [2].

Proposition 15. *[1] Every sentence of* **MMSNP** *is logically equivalent to a finite disjunction of primitive sentences.*

Proof. (sketch) Let Φ in MMSNP. We enforce the first condition of the definition on Φ as follows: expand each negated conjunct in all possible ways until for every variable x in the negated conjunct and every (existentially quantified) monadic predicate C, either $C(x)$ or $\neg C(x)$. Simply replace each negated conjunct by the conjunction of these negated conjuncts. Let Φ' be the formula hence obtained. It is logically equivalent to Φ.

For the second condition, note that if there It may be the case that the second property does not hold for Φ' because of a negated conjunct of the form $\neg\big(\alpha(\sigma, \mathbf{t}) \wedge \beta_0(\mathbf{M}, \mathbf{t}) \wedge \beta_1(\mathbf{M}, x)\big)$, where x does not occur in $\mathbf{t}$, where $\alpha(\sigma, \mathbf{t})$ may be empty and where β_1 is the conjunction of all atoms and negated atoms of β involving symbols from $\mathbf{M}$ and the variable x (β_0 is a conjunction of the remaining atoms and negated atoms of β). Let $\Phi'' = \Phi'_1 \vee \Phi'_2$, where:

- Φ'_1 is obtained from Φ' by replacing $\neg(\alpha \wedge (\beta_0 \wedge \beta_1))$ in Φ' by $\neg(\alpha \wedge \beta_0)$; and
- Φ'_2 is obtained from Φ' by replacing $\neg(\alpha \wedge (\beta_0 \wedge \beta_1))$ in Φ' by $\neg\beta_1$.

First, note that Φ_1' and Φ_2' are both in **MMSNP**. Secondly, it is not hard to check that Φ'' is logically equivalent to Φ'.

We need to adapt the proof from [1] as in this paper, patterns are not required to be connected.

Theorem 16. *[1] The class of problems captured by the primitive fragment of the logic* **MMSNP** *is exactly the class* **FPP** *of forbidden patterns problems.*

Proof. (sketch) We use a similar method to the last case of 15 to ensure that each negated conjunct is in fact "connected". By this we mean that the structure induced by α is connected in each negated conjunct $\neg(\alpha \wedge \beta)$.

Next, for each sentence in the disjunction, we build a set of forbidden patterns, one for each negated conjunct (if there are n monadic predicates, we use 2^n colours). The fact that the logic has no inequality corresponds to the fact that we check for forbidden patterns w.r.t. homomorphisms.

We can now prove that MMSNP coincides with CSP on connected input of bounded degree.

Proof. (of Theorem 12) Let σ be a signature that consists of a finite set of relation symbols $\{R_1, R_2, \ldots, R_m\}$ of respective arities $r_1, r_2, \ldots, r_m$. We have proven Theorem 8, for graphs only, but it holds for arbitrary relational structures (or hyper-graphs), too. Indeed, consider a structure $\mathcal{H}$ on a vertex set V over the signature σ. In order to make the technique work, the only notion we have to redefine is that of a distance: the distance between vertices u and v in $\mathcal{H}$, $d_{\mathcal{H}}(u, v)$ is the minimal number of hyper-edges (regardless of their type) that connect u and v. We also need to make precise the notion of bounded degree. We say that the degree of a vertex is the number of hyper-edges it is incident to. Except for a choice of a suitable X, the rest of the proof remains unchanged.

Let Ω be a problem in MMSNP. It follows from Theorem 16 and Proposition 15 (see Appendix) that Ω is in fact the intersection of finitely many forbidden patterns problems with respective representations $\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n$. Hence, for input of bounded degree $b > 0$, each forbidden patterns problem with representation $\mathfrak{F}_i$ (where $1 \leq i \leq n$) is in fact a constraint satisfaction with template U_i. Let U be the disjoint union of these templates U_i. If the input G is also connected and that there exists a homomorphism h from G to U then the homomorphic image of G must be contained in a single U_i. Therefore it must be a yes-instance of the problem with forbidden patterns $\mathfrak{F}_\mathrm{i}$. Conversely, if there is no homomorphism from G to U then there is no homomorphism from G to any of the template U_i and G must be a no-instance. This proves that Ω, when restricted to input of bounded degree is in fact a constraint satisfaction problem with template U.