Symmetric Normalisation for Intuitionistic Logic

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Abstract

We present two proof systems for implication-only intuitionistic logic in the calculus of structures. The first is a direct adaptation of the standard sequent calculus to the deep inference setting, and we describe a procedure for cut elimination, similar to the one from the sequent calculus, but using a non-local rewriting. The second system is the symmetric completion of the first, as normally given in deep inference for logics with a DeMorgan duality: all inference rules have duals, as cut is dual to the identity axiom. We prove a generalisation of cut elimination, that we call symmetric normalisation, where all rules dual to standard ones are permuted up in the derivation. The result is a decomposition theorem having cut elimination and interpolation as corollaries.

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1. Introduction

The *calculus of structures* [GS01, Gug07] is a formalism for proof theory following the deep inference methodology, where inference rules do not only operate on the outer connective of formulas, but also arbitrarily deep inside formulas. As a consequence, there is no distinction between the object-level and the meta-level in the deductive system, and there is no "comma" or "branching" as in the sequent calculus.

On one side, this provides a greater freedom when designing new deductive systems such as completely local proof systems, as done in classical [Brü03a] and linear logic [Str03b, Str03a], or for proving new results as decomposition theorems [Str03b, SG11]. On the other side, this means that the traditional techniques applied in normalisation proofs cannot directly be used. As a consequence, methods for proving cut elimination in the calculus of structures have been developed, from the usual semantic arguments [BT01] or rule permutations [Str03b] to the use of graphical representations of proofs such as atomic flows [GGS11]. One of the most standard techniques involves using a *splitting lemma* [Gug07, GS11] that performs a complex, non-local transformation.

Unfortunately, all of these methods were developed for logics with a De Morgan duality. Although recently, several systems for intuitionistic logic in the deep inference setting have been proposed [Tiu06, BM08, GHP13a] and their relation to the λ-calculus has been investigated, no normalisation procedure has been given yet that would be entirely *internal to the calculus of structures*. Our goal is to give such a procedure for a basic intuitionistic system.

First, we define in Section 2 a proof system $\mathsf{JS} \cup \{\mathsf{i}{\uparrow}\}$, which has a shape typical for systems in the calculus of structures but is also close to the standard sequent calculus LJ, which is used for proving soundness and completeness in Section 3. The $\mathsf{i}{\uparrow}$ rule in this system corresponds to the cut rule of LJ, and it can be eliminated from any proof using the transformation described in Section 4, based on an upwards permutation of cuts. However, it relies on the fact that all cuts can be reduced to apply on atoms before starting to permute them. This is not possible in the sequent calculus, and it requires a non-local rewriting of the whole proof.

In order to devise a purely local procedure for normalisation, we move to the second step, where the JS system is completed into the fully symmetric system SJS, in which each *down* rule $\mathsf{r}{\downarrow}$ of JS is given a dual *up* rule $\mathsf{r}{\uparrow}$ — for example, cut is dual to the identity axiom. In this setting, described in Section 5, the two subsystems $\mathsf{JS}{\downarrow}$ (down rules) and $\mathsf{JS}{\uparrow}$ (up rules) can be separated to obtain a normal form, not only for proofs but for open derivations in general:

$$
\begin{array}{c} A \\ \mathsf{SJS}\|\\ B \end{array}
\;\;\rightarrow\;\;
\begin{array}{c} A \\ \mathsf{JS}{\uparrow}\| \\ I \\ \mathsf{JS}{\downarrow}\| \\ B \end{array}
\quad
\left(\;\equiv\; \begin{array}{c}\mathsf{JS}\| \\ B\end{array} \quad \text{when } A \text{ is truth} \right)
\qquad (1)
$$

so that no up rule $\mathsf{r}_1{\uparrow}$ can appear below a down rule $\mathsf{r}_2{\downarrow}$. This is a generalisation of the standard cut elimination procedure, since in the case of a proof, where A is the truth unit, we can see by inspection of the rules of $\mathsf{JS}{\uparrow}$ that the upper part of the derivation must be empty, resulting in a normal proof in the JS system. Furthermore, the formula I in (1) is an interpolant for A and B. This gives us an alternative proof of the interpolation theorem for intuitionistic logic (see for example [Min01]).

The termination argument for normalisation relies on a complex measure based on flow-graphs, as found in [Str03b, SG11]. There are simpler normalisation proofs for deep inference systems using reducibility sets, as given for example in [GHP13b, GM13], but these did not achieve full decomposition. We also think that the system presented here can be the basis for a systematic treatment of intermediate logics in the calculus of structures, as done in [CR13] for display calculi, and in [CGT08, CST09] for hypersequents.

2. An Intuitionistic Calculus of Structures

We consider first a system for implication-only intuitionistic logic presented in the calculus of structures [Gug07], a formalism based on *deep inference*, merging the object-level (usually formulas) and

the meta-level (usually sequents) in a single level of *structures*. To define these objects, we assume given a countable set $\mathcal{A}$ of *atoms*, denoted by small latin letters such as a, b, c, the connective $\supset$ for intuitionistic implication, and the constant t for intuitionistic truth, that will be left unit of the implication. Formally, we define the set of *formulas* by the grammar:

$$A, B ::= a \mid \mathsf{t} \mid A \supset B \qquad (2)$$

As usual in deep inference, we consider equivalence classes of formulas, similar to sequents, where comma is commutative and associative in most systems.

Definition 1. The *structures* of our system are equivalence classes of formulas generated by the smallest congruence relation $\equiv$ that is determined by the following equations:

$$\mathsf{t} \supset A \equiv A \qquad A \supset (B \supset C) \equiv B \supset (A \supset C) \qquad (3)$$

The first equation makes t the left unit of $\supset$ and the second allows the exchange of formulas on the left, corresponding to the exchange of formulas in the antecedent of a sequent. We sometimes make explicit the rewriting steps corresponding to the congruence, by using an inference rule $\equiv$ which can be applied with premise A and conclusion B whenever $A \equiv B$. Note that implication is right associative, so that we can write $A \supset B \supset C$ without ambiguity.

In the setting of deep inference, we have the ability to apply any inference rule inside some "context", which means rewriting a structure located inside another structure. In the intuitionistic framework, there are two kinds of contexts, the *positive* and the *negative* ones, on the left-hand side of an even (respectively odd) number of implications.

Definition 2. The *positive* and *negative contexts*, denoted by π and η respectively, are structures with a hole $-$, defined as:

$$\pi ::= - \mid \eta \supset A \qquad \eta ::= \pi \supset A$$

Note that for contexts the same equational theory (3) applies as for structures. In particular, the context $- \supset a \supset b$ can be written $a \supset - \supset b$. A structure is obtained by replacing the hole with some structure, written $\pi\{A\}$ or $\eta\{A\}$, respectively. Notice that positive contexts preserve the polarity of a context plugged inside, and negative contexts invert it. A substructure occurrence A of some B is in *positive position* if $B = \pi\{A\}$ for a positive context π, and in *negative position* if $B = \eta\{A\}$ for some negative η. We write $\Delta \supset B$, where Δ is a multiset $\{A_1, \ldots, A_n\}$ of structures, to denote $A_1 \supset \cdots \supset A_n \supset B$, and by extension we will write $\pi\{\Delta\}$ or $\eta\{\Delta\}$ when a context is of the shape $\eta_0\{- \supset B\}$ or $\pi_0\{- \supset B\}$, respectively. This notation is *overloaded*, so that $\pi\{\Delta\}$ is $\pi\{A\}$ when Δ is the singleton $\{A\}$, and similarly $\eta\{\Delta\}$ is $\eta\{A\}$.

Example 3. Given two contexts $\pi\{-\} = a \supset (- \supset b) \supset c$ and $\eta\{-\} = - \supset d$, and multisets $\Gamma = \{F\}$ and $\Delta = \{a, G\}$, we have $\pi\{F\} = a \supset (F \supset b) \supset c$, and $\eta\{\Delta\} = a \supset G \supset d$ as well as $\pi\{\eta\{\Gamma\}\} = a \supset ((F \supset d) \supset b) \supset c$. ∎

In the calculus of structures, an inference rule must always have exactly one premise. A *rule instance* is obtained by applying the scheme of a rule on particular structures, inside a context. Then, a *derivation* $\mathcal{D}$ has one premise A and a conclusion B, and a *proof* $\mathcal{D}'$ of B is some derivation with premise t and conclusion B. In this context, we use the standard notations of the calculus of structures for such objects:

$$\begin{array}{c} A \\ \mathcal{D}\| \\ B \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathcal{D}'\| \\ B \end{array}$$

The specific rules used in the system $\mathsf{JS} \cup \{\mathsf{i}{\uparrow}\}$ are shown in Figure 1. The *cut-free* system JS is obtained by removing the $\mathsf{i}{\uparrow}$

$$\mathsf{i}{\downarrow} \frac{\pi\{\Delta\}}{\pi\{(\Delta \supset A) \supset A\}} \qquad \mathsf{i}{\uparrow} \frac{\eta\{(\Delta \supset A) \supset A\}}{\eta\{\Delta\}} \qquad \mathsf{w}{\downarrow} \frac{\pi\{B\}}{\pi\{A \supset B\}}$$

$$\mathsf{s}{\downarrow} \frac{\pi\{((A \supset B) \supset C) \supset D\}}{\pi\{A \supset (B \supset C) \supset D\}} \qquad \mathsf{c}{\downarrow} \frac{\pi\{A \supset A \supset B\}}{\pi\{A \supset B\}}$$

Figure 1. Inference rules for the system $\mathsf{JS} \cup \{\mathsf{i}{\uparrow}\}$

rule of cut. In an instance of $\mathsf{i}{\downarrow}$ or $\mathsf{i}{\uparrow}$, the structure A introduced in the conclusion or premise respectively, is called *principal structure*. This system has a typical shape for deep inference, and it is related to the classical system KS [Brü03a]. Apart from the *switch* rule $\mathsf{s}{\downarrow}$, rules are similar to those of the sequent calculus, with a variation on the identity rule. There, the structures in Δ are carried over, as they have to be proved for the conclusion to hold — note that an equivalent rule has been proposed for the sequent calculus [SH11].

Remark 4. The identity rule $\mathsf{i}{\downarrow}$ can only be applied when the notation $\pi\{\Delta\}$ is meaningful, which means that either $\pi\{-\}$ is not $-$, so that the structures carried over appear on the left of an implication, or Δ is reduced to a single structure B. The standard identity rule can be recovered by considering that Δ is $\{\mathsf{t}\}$, since a proof is a derivation with premise t. ∎

Example 5. Here are two valid derivations in $\mathsf{JS} \cup \{\mathsf{i}{\uparrow}\}$:

$$\mathsf{i}{\uparrow} \cfrac{\mathsf{i}{\downarrow} \cfrac{\mathsf{w}{\downarrow} \cfrac{\equiv \cfrac{\mathsf{t} \supset A}{A}}{B \supset A}}{((B \supset A) \supset A) \supset A}}{B \supset A} \qquad \equiv \cfrac{\mathsf{s}{\downarrow} \cfrac{\mathsf{i}{\downarrow} \cfrac{\mathsf{c}{\downarrow} \cfrac{(C \supset D \supset B) \supset (C \supset D \supset B) \supset B}{(C \supset D \supset B) \supset B}}{(((C \supset D \supset A) \supset A) \supset B) \supset B}}{(C \supset D \supset A) \supset (A \supset B) \supset B}}{(A \supset B) \supset (C \supset D \supset A) \supset B} \quad ∎$$

Because of the choices in its design, $\mathsf{JS} \cup \{\mathsf{i}{\uparrow}\}$ is quite different from other intuitionistic systems in deep inference [Tiu06, BM08]. Nonetheless, it has the properties that one expects of a system in this setting — concerning the identity rule, cut and its elimination: we do not investigate atomic contraction here, although it usually possible to obtain it in deep inference [Tiu06]. We can, for example, always reduce the $\mathsf{i}{\downarrow}$ rule to the form where the principal structure is an atom:

$$\mathsf{ai}{\downarrow} \frac{\pi\{\Delta\}}{\pi\{(\Delta \supset a) \supset a\}} \qquad (4)$$

and we call this rule *atomic identity*. The reduction of any instance of the general identity to several instances of the atomic identity rule is straightforward, as in the sequent calculus.

Proposition 6. *The rule* $\mathsf{i}{\downarrow}$ *is derivable for* $\{\mathsf{s}{\downarrow}, \mathsf{ai}{\downarrow}\}$.

Proof. Given an instance r of $\mathsf{i}{\downarrow}$, we proceed by induction on the number of atoms in the principal structure A of r. If A is simply an atom a, we are done. In general, A has the shape $B \supset C$, we use the rewriting from:

$$\mathsf{i}{\downarrow} \frac{\pi\{\Delta\}}{\pi\{(\Delta \supset (B \supset C)) \supset (B \supset C)\}}$$

to the derivation below, and conclude by induction hypothesis.

$$\mathsf{s}{\downarrow} \cfrac{\mathsf{i}{\downarrow} \cfrac{\equiv \cfrac{\mathsf{i}{\downarrow} \cfrac{\pi\{\Delta\}}{\pi\{(\Delta \supset C) \supset C\}}}{\pi\{(\mathsf{t} \supset \Delta \supset C) \supset C\}}}{\pi\{(((\mathsf{t} \supset B) \supset B) \supset \Delta \supset C) \supset C\}}}{\equiv \cfrac{\pi\{(\mathsf{t} \supset B) \supset (B \supset \Delta \supset C) \supset C\}}{\pi\{(\Delta \supset (B \supset C)) \supset (B \supset C)\}}} \qquad \square$$

Furthermore, we have the following property of the switch rule, that can be used to push structures inside contexts, based on the idea that inside a formula, a t unit can be created anywhere using the congruence, as a placeholder where a formula can be plugged.

Lemma 7. *For π and ρ positive contexts, there is a derivation in* $\{s\downarrow\}$ *from* $\pi\{\rho\{A\} \supset B\}$ *to* $\pi\{A \supset \rho\{t\} \supset B\}$.

Proof. We proceed by induction on $\rho\{-\}$. In the base case, we use the congruence. In the inductive case, for some C and D we have $\rho\{-\} = (\rho'\{-\} \supset C) \supset D$. We use the following derivation:

$$
\begin{array}{c}
\pi\{((\rho'\{A\} \supset C) \supset D) \supset B\} \\
\mathcal{D}\| \\
\hline
\pi\{((A \supset \rho'\{t\} \supset C) \supset D) \supset B\} \\
s\downarrow \frac{}{\pi\{A \supset ((\rho'\{t\} \supset C) \supset D) \supset B\}}
\end{array}
$$

where $\mathcal{D}$ exists by induction hypothesis. $\qquad\square$

3. Relation to the Sequent Calculus

The simplest way to prove $\mathsf{JS} \cup \{i\uparrow\}$ suitable for intuitionistic logic is to prove soundness and completeness with respect to the sequent calculus shown in Figure 2, based on the definition of formulas given in (2). This will require translations between structures and sequents: for any structure, pick a representation A and translate it into $\vdash A$. Conversely, given any sequent, we proceed by induction on the length of the antecedent to find its representation:

$$
[\![\Gamma, A \vdash B]\!]_{\mathsf{S}} = [\![\Gamma \vdash A \supset B]\!]_{\mathsf{S}} \qquad [\![\vdash A]\!]_{\mathsf{S}} = A
$$

Theorem 8 (Soundness of $\mathsf{JS} \cup \{i\uparrow\}$). *If A is provable in $\mathsf{JS} \cup \{i\uparrow\}$, then the sequent $\vdash A$ is provable in $\mathsf{LJ} \cup \{cut\}$.*

Proof. We proceed by structural induction on any proof $\mathcal{D}$ of A in $\mathsf{JS} \cup \{i\uparrow\}$. In the base case, when $\mathcal{D}$ is just the structure t, we can use the t_R rule. In the general case, we consider the bottommost instance r in $\mathcal{D}$, as shown below on the left:

$$
\begin{array}{c}
\mathcal{D}'\| \\
\pi\{C\} \\
r \frac{}{\pi\{B\}}
\end{array}
\quad \rightarrow \quad
\begin{array}{c}
\overbrace{\Pi'}^{} \qquad \overbrace{\Pi''}^{} \\
cut \frac{\vdash \pi\{C\} \quad \pi\{C\} \vdash \pi\{B\}}{\pi\{B\}}
\end{array}
$$

Any rule instance r can be considered to follow the scheme shown above, including the ones defined in a negative context, because such a context must be contained in a positive context. By case analysis on r one can show that $C \vdash B$ is provable in $\mathsf{LJ} \cup \{cut\}$ — note that whenever $C \equiv B$ then $B \vdash C$ and $C \vdash B$ are provable. From a proof of $C \vdash B$ in $\mathsf{LJ} \cup \{cut\}$ we obtain the proof Π'' of $\pi\{C\} \vdash \pi\{B\}$, by induction on π. Therefore, we can build the proof shown above on the right, where Π' exists by induction hypothesis, applied on $\mathcal{D}'$. $\qquad\square$

Theorem 9 (Completeness of $\mathsf{JS} \cup \{i\uparrow\}$). *If a sequent $\Gamma \vdash A$ is provable in $\mathsf{LJ} \cup \{cut\}$, then $[\![\Gamma \vdash A]\!]_{\mathsf{S}}$ is provable in $\mathsf{JS} \cup \{i\uparrow\}$.*

Proof. We proceed by structural induction on some proof Π of $\Gamma \vdash A$ in $\mathsf{LJ} \cup \{cut\}$, using a case analysis on the bottommost instance r in Π. We only show the $\supset_L$ case below.

$$
\begin{array}{c}
\overbrace{\Pi_1}^{} \quad \overbrace{\Pi_2}^{} \\
\supset_L \frac{\Delta \vdash B \quad \Psi, C \vdash A}{\Delta, \Psi, B \supset C \vdash A}
\end{array}
\rightarrow
\begin{array}{c}
\mathcal{D}_2\| \\
\Psi \supset C \supset A \\
\equiv \overline{\Psi \supset (t \supset C) \supset A} \\
\mathcal{D}'_1\| \\
\Psi \supset ((\Delta \supset B) \supset C) \supset A \\
s\downarrow^* \overline{\overline{\Delta \supset \Psi \supset (B \supset C) \supset A}}
\end{array}
$$

$$
\begin{array}{cc}
ax \frac{}{A \vdash A} &
cut \frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B} &
t_R \frac{}{\vdash t}
\end{array}
$$

$$
\begin{array}{cc}
wk \frac{\Gamma \vdash B}{\Gamma, A \vdash B} &
ct \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B} &
t_L \frac{\Gamma \vdash A}{\Gamma, t \vdash A}
\end{array}
$$

$$
\begin{array}{cc}
\supset_R \frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B} &
\supset_L \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \supset B \vdash C}
\end{array}
$$

Figure 2. Inference rules for system $\mathsf{LJ} \cup \{cut\}$

By induction hypothesis we produce proofs $\mathcal{D}_1$ of $[\![\Delta \vdash B]\!]_{\mathsf{S}}$ and $\mathcal{D}_2$ of $[\![\Psi, C \vdash A]\!]_{\mathsf{S}}$, and $\mathcal{D}'_1$ is obtained by plugging $\mathcal{D}_1$ inside the context $\Psi \supset (- \supset C) \supset A$. In the translation, $s\downarrow^*$ denotes the sequence of $s\downarrow$ needed to move all structures from Δ to the left of B (as done in Lemma 7). The case for cut is similar but it uses $i\uparrow$, and all other cases are straightforward. $\qquad\square$

As the cut rule in $\mathsf{LJ} \cup \{cut\}$, the rule $i\uparrow$ is admissible in the $\mathsf{JS} \cup \{i\uparrow\}$ system. One proof of this can be obtained directly from the translation above: the completeness proof of $\mathsf{JS} \cup \{i\uparrow\}$ uses $i\uparrow$ only for translating a cut instance. Thus, cut elimination for LJ entails cut elimination for JS. However, in the following, we give another proof, using an internal cut elimination procedure described in terms of permutation of rule instances.

4. Cut Elimination

The procedure we will present here is based on the permutation of inference rules, similar to what can be seen in the sequent calculus. However, permuting rule instances in a system with deep inference can be more involved than in the traditional, shallow formalisms, because of the freedom offered when composing or modifying the structures. As a consequence, the induction measure used to show that the permutation process terminates needs to keep track of the relationship between occurrences of structures and substructures in a given derivation. For this, we will use a variant of the notion of *logical flow-graphs* [Bus91].

Definition 10. An occurrence of t in a structure is *non-trivial* if it occurs on the right-most position in a substructure not equal to t via equations (3), otherwise it is called *trivial*.

In other words, non-trivial occurrences of t are those that cannot be removed using equations (3), except when consider t as a whole structure. Conversely, every structure has a representation with no trivial t occurrence.

Definition 11. A *particle* in a derivation $\mathcal{D}$ is an occurrence of an atom or a non-trivial occurrence of t in a structure in $\mathcal{D}$.

Definition 12. The *head* of a structure A is the rightmost particle in A; a particle is *positive* (respectively, *negative*) if it is in positive (respectively, negative) position.

Definition 13. The *down-flow-graph* $G^{\downarrow\downarrow}(\mathcal{D})$ of a derivation $\mathcal{D}$ is the directed graph whose vertices are the negative particles of $\mathcal{D}$, such that there is an edge from a negative particle p to a negative particle q, if and only if p occurs in the premise of a rule instance r in $\mathcal{D}$ and q occurs in the conclusion of the same instance r, and one of the following conditions holds:

1. p and q are the same occurrence in the context $\pi\{-\}$ or $\eta\{-\}$ of r (as shown in Figure 1), or
2. p and q are the same occurrence in one of B, C, D, Δ in an instance of $w\downarrow$, $c\downarrow$, $s\downarrow$, $i\downarrow$, or $i\uparrow$ (see Figure 1), or

$$r_1{\downarrow}\ \overset{\pi_1\{\Psi\}}{\underset{\displaystyle ai{\uparrow}\ \dfrac{\pi_1\{F\}}{\pi_0\{F\}}}{\rule{0pt}{0pt}}} \qquad \overset{(1)}{\longrightarrow} \qquad ai{\uparrow}\ \dfrac{\pi_1\{\Psi\}}{\underset{r_1{\downarrow}\ \dfrac{\pi_0\{\Psi\}}{\pi_0\{F\}}}{\rule{0pt}{0pt}}} \qquad\qquad ai{\downarrow}\ \dfrac{\pi\{\Delta \supset a\}}{\underset{ai{\uparrow}\ \dfrac{\pi\{((\Delta \supset a)\supset a)\supset a\}}{\pi\{\Delta \supset a\}}}{\rule{0pt}{0pt}}} \qquad \overset{(3)}{\longrightarrow} \qquad \pi\{\Delta \supset a\} \qquad\qquad w{\downarrow}\ \dfrac{\pi\{B\}}{\underset{ai{\uparrow}\ \dfrac{\pi\{F \supset B\}}{\pi\{\Delta \supset B\}}}{\rule{0pt}{0pt}}} \qquad \overset{(6)}{\longrightarrow} \qquad w{\downarrow}^*\ \dfrac{\pi\{B\}}{\pi\{\Delta \supset B\}}$$

$$r_1{\downarrow}\ \dfrac{\eta_1\{F\}}{\underset{ai{\uparrow}\ \dfrac{\eta_0\{F\}}{\eta_0\{\Delta\}}}{\rule{0pt}{0pt}}} \qquad \overset{(2)}{\longrightarrow} \qquad ai{\uparrow}\ \dfrac{\eta_1\{F\}}{\underset{r_1{\downarrow}\ \dfrac{\eta_1\{\Delta\}}{\eta_0\{\Delta\}}}{\rule{0pt}{0pt}}} \qquad\qquad ai{\downarrow}\ \dfrac{\eta\{a\}}{\underset{ai{\uparrow}\ \dfrac{\eta\{(a\supset a)\supset a\}}{\eta\{a\}}}{\rule{0pt}{0pt}}} \qquad \overset{(4)}{\longrightarrow} \qquad \eta\{a\} \qquad\qquad c{\downarrow}\ \dfrac{\pi\{F \supset F \supset B\}}{\underset{ai{\uparrow}\ \dfrac{\pi\{F \supset B\}}{\pi\{\Delta \supset B\}}}{\rule{0pt}{0pt}}} \qquad \overset{(7)}{\longrightarrow} \qquad \dfrac{ai{\uparrow}\ \dfrac{\pi\{F \supset F \supset B\}}{ai{\uparrow}\ \dfrac{\pi\{F \supset \Delta \supset B\}}{\pi\{\Delta \supset \Delta \supset B\}}}}{c{\downarrow}^*\ \dfrac{}{\pi\{\Delta \supset B\}}}$$

$$s{\downarrow}\ \dfrac{\pi\{((B \supset \Delta \supset a)\supset a)\supset C\}}{\underset{ai{\uparrow}\ \dfrac{\pi\{B \supset ((\Delta \supset a)\supset a)\supset C\}}{\pi\{B \supset \Delta \supset C\}}}{\rule{0pt}{0pt}}} \qquad \overset{(5)}{\longrightarrow} \qquad ai{\uparrow}\ \dfrac{\pi\{((B \supset \Delta \supset a)\supset a)\supset C\}}{\pi\{B \supset \Delta \supset C\}}$$

Figure 3. Reduction steps for atomic cut elimination in $\mathsf{JS} \cup \{ai{\uparrow}\}$, where $F = (\Delta \supset a) \supset a$

---

3. p and q are the same occurrence in the A in an instance of $s{\downarrow}$ (see Figure 1), or
4. p occurs inside one of the two A in the premise of a $c{\downarrow}$, and q is the same occurrence in the A in the conclusion of the $c{\downarrow}$ (see Figure 1).

An edge in $G^{\|}(\mathcal{D})$ is also called a *down edge*.

Definition 14. The *multiplicity* of a (negative) particle p in a derivation $\mathcal{D}$ is the number of particles q in $\mathcal{D}$, such that there is a path from q to p in $G^{\|}(\mathcal{D})$ and no incoming edges to q.

Example 15. In the derivation below we show all down edges. The multiplicity of a in the conclusion is 3, the one of c is 2.

$$c{\downarrow}\ \dfrac{(((((a \supset a \supset b)\supset c)\supset a \supset b)\supset c)\supset d}{\underset{c{\downarrow}\ \dfrac{s{\downarrow}\ \dfrac{((((a \supset b)\supset c)\supset a \supset b)\supset c)\supset d}{((a \supset b)\supset c)\supset ((a \supset b)\supset c)\supset d}}{((a \supset b)\supset c)\supset d}}{\rule{0pt}{0pt}}} \qquad \blacksquare$$

Definition 16. The *height* of a derivation $\mathcal{D}$, denoted by $h(\mathcal{D})$, is the number of rule instances in $\mathcal{D}$.

Lemma 17. *For any positive context π and a structure A, if $\pi\{A\}$ is provable in JS then $\pi\{t\}$ is also provable in JS.*

The proof of this lemma uses a straighforward induction on the height of the derivation. We are now ready to see our cut elimination, which is done in two stages. First, we reduce cuts to their *atomic form*, which is dual to atomic identity (4):

$$ai{\uparrow}\ \dfrac{\eta\{(\Delta \supset a)\supset a\}}{\eta\{\Delta\}} \qquad\qquad (5)$$

In $\mathsf{JS} \cup \{i{\uparrow}\}$, the reduction from general to atomic cuts is achieved via the non-local rewriting called *merging* described in Lemma 18 below. For stating it, we need contexts with more than one hole. The notation $\pi\{A_1\}\ldots\{A_n\}$ denotes a structure with substructures A_1 to A_n in positive positions. Replacing one A_i with a hole $-$ produces an ordinary positive context. The object $\pi\{-\}\ldots\{-\}$, which is obtained by replacing each of $A_1,\ldots,A_n$ with a hole $-$, is called an *n-ary positive context*.

Lemma 18 (Merging)**.** *Let η be a negative context, let π be an n-ary positive context, and let A and B be structures. For every proof $\mathcal{D}$ of $\eta\{A \supset \pi\{t\}\ldots\{t\} \supset B\}$ in $\mathsf{JS} \cup \{ai{\uparrow}\}$, there is also a proof $\mathcal{D}'$ of $\eta\{\pi\{A\}\ldots\{A\} \supset B\}$ in $\mathsf{JS} \cup \{ai{\uparrow}\}$.*

Proof. Let $m_1,\ldots,m_n$ be the multiplicities of the n distinguished occurrences of t in $\pi\{t\}\ldots\{t\}$ in the conclusion of $\mathcal{D}$, and let $m = \sum_{i=1}^{n} m_i$. We proceed by induction on the pair $(m, h(\mathcal{D}))$,

under lexicographic ordering, using a case analysis on the instance r at the bottom of $\mathcal{D}$:

1. If r acts inside A, we can use the induction hypothesis on the proof $\mathcal{D}_1$ above r, and apply n instances of r, one inside each hole in π:

$$r\ \dfrac{\eta\{A' \supset \pi\{t\}\ldots\{t\} \supset B\}}{\eta\{A \supset \pi\{t\}\ldots\{t\} \supset B\}}$$

$$\longrightarrow \qquad r^*\ \dfrac{\eta\{\pi\{A'\}\ldots\{A'\} \supset B\}}{\eta\{\pi\{A\}\ldots\{A\} \supset B\}}$$

and if A' is just t, we need no induction hypothesis. Notice that since we start with a proof, this case must occur eventually.
2. If r acts inside B or inside η, we can proceed immediately by induction hypothesis.
3. If r acts inside $\pi\{t\}\ldots\{t\}$, we have some subcases:
   - If r is weakening, we proceed immediately by induction hypothesis. (If r deletes all the "holes" of π, then we also need Lemma 17 for building $\mathcal{D}'$.)
   - If r is a contraction, possibly duplicating some "holes" of π, observe that even though n increases through instances of $c{\downarrow}$, the sum of the m_i cannot. Thus, we can proceed by induction hypothesis.
   - If r is a switch, note that it is a purely linear rule, moving substructures around but keeping their polarity. Thus, m is preserved, and we can we proceed by induction hypothesis.
   - If r is an identity removing a pair of substructures, there are three possibilities:
     - None of the distinguished occurrences of t is affected. We proceed by induction hypothesis.
     - Some of the t are inside the removed substructures. As in the weakening case, we can proceed by induction hypothesis.
     - We have a situation where r is an $i{\downarrow}$, and one (or more) of the distinguished occurrences of t is inside the Δ that goes through the identity. More precisely, we have

$$\pi\{t\}\ldots\{t\} = \rho\{(t \supset \cdots \supset t \supset \Delta \supset F) \supset F\} \quad (6)$$

for some (unary) positive context ρ, such that all the t on the right in (6) are in one of the holes on the left (but not all holes on the left need to be as shown on the right). Then we have the following situation:

$$i{\downarrow}\ \dfrac{\eta\{A \supset \rho\{t,\ldots,t,\Delta\} \supset B\}}{\eta\{A \supset \rho\{(t \supset \cdots \supset t \supset \Delta \supset F) \supset F\} \supset B\}} \quad (7)$$

where $\rho\{t,\ldots,t,\Delta\} \supset B$ is

$$\begin{cases} t \supset \cdots \supset t \supset \Delta \supset B & \text{if } \rho\{-\} = \{-\} \\ \nu\{t \supset \cdots \supset t \supset \Delta \supset X\} \supset B & \text{if } \rho\{-\} = \nu\{-\supset X\} \end{cases}$$

for some negative context ν and some structure X. We can now replace (7) above by

$$\text{i}{\downarrow} \frac{\eta\{\rho'\{A,\ldots,A,\Delta'\} \supset B\}}{\eta\{\rho'\{(A \supset \cdots \supset A \supset \Delta' \supset F) \supset F\} \supset B\}}$$

and proceed by induction hypothesis. Here, ρ' and Δ' are obtained from ρ and Δ, respectively, by replacing the distinguished occurrences of t in π by A, so that $\rho'\{(A \supset \cdots \supset A \supset \Delta' \supset F) \supset F\} = \pi\{A\}\ldots\{A\}$.

4. If r is a s$\downarrow$ moving a structure A' from the context ν to the left of A, we apply the induction hypothesis to the proof above this s$\downarrow$ and proceed as follows, with $\eta\{-\} = \rho\{- \supset D\}$:

$$\text{s}{\downarrow} \frac{\rho\{((A' \supset A) \supset \pi\{t\}\ldots\{t\} \supset B) \supset D\}}{\rho\{A' \supset (A \supset \pi\{t\}\ldots\{t\} \supset B) \supset D\}} \quad \longrightarrow$$

$$\rho\{(\pi\{A' \supset A\}\ldots\{A' \supset A\} \supset B) \supset D\}$$
$$\tilde{\mathcal{D}}\|\text{s}{\downarrow}$$
$$\text{c}{\downarrow}^* \frac{\rho\{A' \supset \cdots \supset A' \supset (\pi\{A\}\ldots\{A\} \supset B) \supset D\}}{\rho\{A' \supset (\pi\{A\}\ldots\{A\} \supset B) \supset D\}}$$

where we use $n-1$ instances of c$\downarrow$ to get n copies of A', and where the derivation $\tilde{\mathcal{D}}$ consisting of s$\downarrow$ is obtained by applying n times Lemma 7.

5. If r is a w$\downarrow$ erasing $A \supset \pi\{t\}\ldots\{t\} \supset B$, we use w$\downarrow$ to obtain $\mathcal{D}'$ without using the induction hypothesis.

6. If r is a c$\downarrow$ duplicating $A \supset \pi\{t\}\ldots\{t\} \supset B$, we rewrite premise and conclusion, and apply the induction hypothesis twice, as the multiplicity of the copies is less than the original, which included all of the m_i. $\qquad\square$

The second stage of cut elimation is a permutation argument moving all the ai$\uparrow$ instances towards the top of the derivation, where they must eventually disappear. The key observation is that these permutations will be simpler than expected because atomic cuts interact less with other rules than general cuts.

Lemma 19. *For any given proof $\mathcal{D}$ in* JS $\cup$ {i$\uparrow$}*, there is a proof in* JS $\cup$ {ai$\uparrow$} *with the same conclusion.*

Proof. Let the *rank* of a cut be the size of its principal structure — the A in i$\uparrow$ in Figure 1. We proceed by induction, using the multiset of the ranks of all cuts in $\mathcal{D}$ as induction measure. If all cuts are atomic we are done and $\mathcal{D}$ is a proof in JS $\cup$ {ai$\uparrow$}. Otherwise, we consider the topmost i$\uparrow$ instance in $\mathcal{D}$ with a non-atomic principal structure $B \supset C$ and rewrite $\mathcal{D}$ as:

$$\mathcal{D}_1\|$$
$$\equiv \frac{\eta\{B \supset (\Delta \supset (B \supset C)) \supset C\}}{\text{i}{\uparrow} \frac{\eta\{(\Delta \supset (B \supset C)) \supset (B \supset C)\}}{\eta\{\Delta\}}}$$

$$\mathcal{D}_1'\|$$
$$\longrightarrow \quad \text{i}{\uparrow} \frac{\eta\{(\Delta \supset (B \supset B)) \supset C) \supset C\}}{\text{i}{\uparrow} \frac{\eta\{(\Delta \supset C) \supset C\}}{\eta\{\Delta\}}}$$

where $\mathcal{D}_1'$ is obtained from $\mathcal{D}_1$ by Lemma 18, after observing that the premise of the cut can be rewritten via $\equiv$ to have the shape of the structure considered in that lemma. $\qquad\square$

Lemma 20 (Atomic cut elimination). *Any proof in* JS $\cup$ {ai$\uparrow$} *without non-atomic* i$\downarrow$ *instances can be turned into a* JS *proof.*

Proof. By induction on the number of ai$\uparrow$ instances in the proof; if there is any, consider the topmost one. Let $\mathcal{D}$ be the cut-free derivation above, and let m be the multiplicity in $\mathcal{D}$ of the head of $F = (\Delta \supset a) \supset a$ in the premise of this ai$\uparrow$ instance. We proceed by a second induction on the pair $(m, \mathsf{h}(\mathcal{D}))$, and show that this ai$\uparrow$ can be eliminated, using a case analysis on the bottommost instance r_1 in $\mathcal{D}$ — see the list of cases in Figure 3:

1-2. If our ai$\uparrow$ and r_1 operate on unrelated structures, we can trivially permute ai$\uparrow$ above r_1, and proceed by induction hypothesis, as $\mathsf{h}(\mathcal{D})$ has decreased.

3-4. If r_1 is an identity ai$\downarrow$ matching our ai$\uparrow$, then both instances can disappear and we conclude by the main induction hypothesis, since one cut was erased.

5. If r_1 is a s$\downarrow$ moving some structure to the left of the positive a introduced in the cut, we assimilate it in the ai$\uparrow$ instance and go on by induction hypothesis, as $\mathsf{h}(\mathcal{D})$ decreased.

6. If r_1 is a w$\downarrow$ erasing both principal atoms of our ai$\uparrow$, we can remove the cut, keep the w$\downarrow$ and then conclude by the main induction hypothesis, as one cut was erased.

7. If r_1 is a c$\downarrow$ duplicating the two principal atoms of our ai$\uparrow$, we copy the cut, compose the two copies and apply the induction hypothesis twice. This is possible because both new ai$\uparrow$ have smaller multiplicity, and no transformation from Figure 3 can increase the multiplicity of any particle introduced in a cut. $\quad\square$

Theorem 21 (Cut elimination). *Any proof of A in* JS $\cup$ {i$\uparrow$} *can be transformed into a proof of A in* JS.

Proof. By Lemma 19 we make cut instances atomic, then we apply Proposition 6 to make identity instances atomic. Finally, we can conclude by applying Lemma 20. $\qquad\square$

This cut elimination procedure is not local, because *merging* in Lemma 18 rewrites the whole proof, when making all cuts atomic by Lemma 19. To solve this we need a derivation or inference rule to perform the merging transformation of:

$$\equiv \frac{\eta\{B \supset (\Delta \supset B \supset C) \supset C\}}{\text{i}{\uparrow} \frac{\eta\{(\Delta \supset (B \supset C)) \supset (B \supset C)\}}{\eta\{\Delta\}}} \quad (8)$$

into the derivation:

$$\frac{\eta\{B \supset (\Delta \supset B \supset C) \supset C\}}{\text{i}{\uparrow} \frac{\eta\{(\Delta \supset (B \supset B) \supset C) \supset C\}}{\text{i}{\uparrow} \frac{\eta\{(\Delta \supset C) \supset C\}}{\eta\{\Delta\}}}} \quad (9)$$

This is the motivation for the symmetric system introduced in the next section.

5. Symmetric Normalisation

The missing piece to decompose cuts to their atomic form as shown in (9) is the rule s$\uparrow$, dual to the switch s$\downarrow$ rule, which allows to move a substructure in positive position outside of some deep context — the opposite of moving a structure deeper inside other structures, as done by the switch rule in Lemma 7. This logically leads us to the completion of JS $\cup$ {i$\uparrow$} into the symmetric system SJS, following the usual scheme in deep inference [Brü03a, Str03b]. In the intuitionistic setting, where DeMorgan duality does not exist, the dual of a rule is obtained by inverting premise and conclusion, and then using the result in the opposite context.

$$\mathsf{i}{\downarrow}\ \frac{\pi\{\Delta\}}{\pi\{(\Delta \supset A) \supset A\}} \qquad\qquad \mathsf{i}{\uparrow}\ \frac{\eta\{(\Delta \supset A) \supset A\}}{\eta\{\Delta\}}$$

$$\mathsf{s}{\downarrow}\ \frac{\pi\{((A \supset B) \supset C) \supset D\}}{\pi\{A \supset (B \supset C) \supset D\}} \qquad \mathsf{s}{\uparrow}\ \frac{\eta\{A \supset (B \supset C) \supset D\}}{\eta\{((A \supset B) \supset C) \supset D\}}$$

$$\mathsf{c}{\downarrow}\ \frac{\pi\{A \supset A \supset B\}}{\pi\{A \supset B\}} \qquad\qquad \mathsf{c}{\uparrow}\ \frac{\eta\{A \supset B\}}{\eta\{A \supset A \supset B\}}$$

$$\mathsf{w}{\downarrow}\ \frac{\pi\{B\}}{\pi\{A \supset B\}} \qquad\qquad \mathsf{w}{\uparrow}\ \frac{\eta\{A \supset B\}}{\eta\{B\}}$$

Figure 4. Inference rules for the symmetric system SJS

Starting from the same definitions, as used in $\mathsf{JS} \cup \{\mathsf{i}{\uparrow}\}$, we add the whole *up fragment* $\{\mathsf{i}{\uparrow}, \mathsf{c}{\uparrow}, \mathsf{w}{\uparrow}, \mathsf{s}{\uparrow}\}$ to JS, and we call the resulting system SJS, for which the set of inference rules is shown in Figure 4. In this extension, the basic JS system is called the *down fragment*, also denoted by $\mathsf{JS}{\downarrow}$, and we will write $\mathsf{JS}{\uparrow}$ to denote the up fragment. Many properties of JS are kept in SJS, and in particular the rule $\mathsf{i}{\downarrow}$ can be reduced to atomic form, using the same transformation as for JS in Proposition 6. But due to the new duality, the rule $\mathsf{i}{\uparrow}$ can also be reduced to its atomic form, without applying Lemma 18 or any other non-local transformation.

Proposition 22. *The rule* $\mathsf{i}{\uparrow}$ *is derivable for* $\{\mathsf{s}{\uparrow}, \mathsf{ai}{\uparrow}\}$.

Proof. This is dual to Proposition 6: we proceed by induction on the principal structure of the cut. At each step, the cut is decomposed into two cuts on smaller structures by using a $\mathsf{s}{\uparrow}$ instance, as shown in (9). □

Another standard property of any such symmetric system in the calculus of structures is that the dual of any rule can be derived by using cut and identity.

Proposition 23. *Any up rule* $\mathsf{r}{\uparrow}$ *is derivable for* $\{\mathsf{i}{\downarrow}, \mathsf{i}{\uparrow}, \mathsf{r}{\downarrow}\}$, *and dually any down rule* $\mathsf{r}{\downarrow}$ *is derivable for* $\{\mathsf{i}{\downarrow}, \mathsf{i}{\uparrow}, \mathsf{r}{\uparrow}\}$.

Proof. From any given $\mathsf{r}{\downarrow}$ we can derive $\mathsf{r}{\uparrow}$ and dually $\mathsf{r}{\downarrow}$ from $\mathsf{r}{\uparrow}$, as follows:

$$\mathsf{r}{\downarrow}\ \frac{\pi\{B\}}{\pi\{A\}} \quad,\quad \mathsf{r}{\uparrow}\ \frac{\eta\{A\}}{\eta\{B\}} \quad\longrightarrow\quad \mathsf{i}{\uparrow}\ \cfrac{\mathsf{r}{\downarrow}\ \cfrac{\mathsf{i}{\downarrow}\ \cfrac{\eta\{A\}}{\eta\{(B \supset B) \supset A\}}}{\eta\{(B \supset A) \supset A\}}}{\eta\{B\}} \qquad \square$$

This definition of SJS as a symmetric extension of the basic JS system leads to a simple proof of adequacy with respect to the standard sequent calculus LJ for intuitionistic logic, even though the language of proofs in SJS is much richer.

Theorem 24. *A structure A is provable in the* SJS *system if and only if the sequent* $\vdash A$ *is provable in* $\mathsf{LJ} \cup \{\mathsf{cut}\}$.

Proof. Completeness is immediate from Theorem 9, since $\mathsf{JS} \cup \{\mathsf{i}{\uparrow}\}$ is a subset of the SJS system. Then, soundness can be proved by using Proposition 23 and Theorem 8. □

As mentioned before, the new up-down symmetry appearing in the inference rules of SJS allows us to prove cut elimination with local rule permutations only. Furthermore, we are able to state and prove a strong decomposition theorem inside SJS that generalises cut elimination by separating more groups of rules than the up and down fragments.

Theorem 25 (Decomposition). *Any derivation can be decomposed in* SJS *as shown on the left below, and any proof can be turned into one of the shape shown on the right below:*

$$\mathsf{SJS}\left\|\begin{array}{c}P\\Q\end{array}\right. \quad\rightarrow\quad \begin{array}{c}P\\ {}_{\{\mathsf{c}{\uparrow}\}}\|\\ P'\\ {}_{\{\mathsf{w}{\uparrow}\}}\|\\ P''\\ {}_{\{\mathsf{s}{\uparrow}\}}\|\\ P'''\\ {}_{\{\mathsf{i}{\uparrow}\}}\|\\ I\\ {}_{\{\mathsf{i}{\downarrow}\}}\|\\ Q'''\\ {}_{\{\mathsf{s}{\downarrow}\}}\|\\ Q''\\ {}_{\{\mathsf{w}{\downarrow}\}}\|\\ Q'\\ {}_{\{\mathsf{c}{\downarrow}\}}\|\\ Q\end{array} \qquad \mathsf{SJS}\left\|\begin{array}{c}\\Q\end{array}\right. \quad\rightarrow\quad \begin{array}{c}{}_{\{\mathsf{i}{\downarrow}\}}\overline{\overline{\|}}\\ Q'''\\ {}_{\{\mathsf{s}{\downarrow}\}}\|\\ Q''\\ {}_{\{\mathsf{w}{\downarrow}\}}\|\\ Q'\\ {}_{\{\mathsf{c}{\downarrow}\}}\|\\ Q\end{array} \qquad (10)$$

Before proving this in the remainder of this section, we observe that it entails cut elimination but also gives a new proof for Craig's interpolation theorem in intuitionistic logic [Min01]. For any given provable formula $P \supset Q$, there is a derivation from P to Q in SJS, and the formula I obtained by decomposition (10) is an interpolant of P and Q. Such a strong decomposition theorem, immediately entailing interpolation, has to our knowledge only be obtained for classical propositional logic [Brü03b] and for the multiplicative and exponential fragment of linear logic [Str03a]. Proving Theorem 25 requires a generalisation of Definition 13.

Definition 26. For a derivation $\mathcal{D}$, its *(full) flow-graph* $G^{\leftrightarrow\rho}(\mathcal{D})$ is the directed graph whose vertices are all the particles of $\mathcal{D}$ and whose set of edges is the union of the sets of *down edges*, *up edges*, *down-left edges*, *up-left edges* and *pure left edges*:

- A *down edge* is defined as an edge between negative particles p and q in $\mathcal{D}$ as in Definition 13, with the difference that in 2) we include the rules $\mathsf{w}{\uparrow}$, $\mathsf{c}{\uparrow}$, and $\mathsf{s}{\uparrow}$, and in 3) we include the rule $\mathsf{s}{\uparrow}$. We also add:
  5. p is an occurrence in the A in the premise of a $\mathsf{c}{\uparrow}$, and q is the same occurrence in one of the two A in the conclusion of the $\mathsf{c}{\uparrow}$.
- An *up edge* is defined dually between positive particles p and q going upwards, so for having an edge from p to q, the positive particle p must occur in the conclusion of a rule instance r and q must occur in the premise of the same rule instance r.
- There is an *down-left edge* from some negative particle p to a negative particle q, if and only if p is the head of C in the premise of some instance of $\mathsf{s}{\downarrow}$ and q is the head of A in the conclusion of that $\mathsf{s}{\downarrow}$.
- Dually, there is an *up-left edge* from some positive particle p to a positive particle q, if and only if p is the head of C in the conclusion of some instance of $\mathsf{s}{\uparrow}$ and q is the head of A in the premise of that $\mathsf{s}{\uparrow}$.
- Finally, there is a *pure left edge* from p to q, if and only if either p is the head of B and q is the head of A in the premise of an instance of $\mathsf{s}{\downarrow}$, or dually, p is the head of B and q the head of A in the conclusion of a $\mathsf{s}{\uparrow}$ instance.

Observe that the pure left edges are the only ones connecting particles of different polarities in a flow-graph.

$$
c{\uparrow}\,\frac{r\,\dfrac{\eta\{E'\supset D\}}{\eta\{E\supset D\}}}{\eta\{E\supset E\supset D\}}
\quad\xrightarrow{(1)}\quad
\frac{r\,\dfrac{r\,\dfrac{c{\uparrow}\,\dfrac{\eta\{E'\supset D\}}{\eta\{E'\supset E'\supset D\}}}{\eta\{E'\supset E\supset D\}}}{\eta\{E\supset E\supset D\}}}{}
$$

$$
c{\uparrow}\,\frac{r\,\dfrac{\eta'\{E\supset D\}}{\eta\{E\supset D\}}}{\eta\{E\supset E\supset D\}}
\quad\xrightarrow{(2)}\quad
r\,\frac{c{\uparrow}\,\dfrac{\eta'\{E\supset D\}}{\eta'\{E\supset E\supset D\}}}{\eta\{E\supset E\supset D\}}
$$

$$
c{\uparrow}\,\frac{c{\downarrow}\,\dfrac{\pi\{C'\supset C'\supset D\}}{\pi\{C'\supset D\}}}{\pi\{C\supset D\}}
\quad\xrightarrow{(3)}\quad
c{\downarrow}\,\frac{c{\uparrow}\,\dfrac{c{\uparrow}\,\dfrac{\pi\{C'\supset C'\supset D\}}{\pi\{C\supset C'\supset D\}}}{\pi\{C\supset C\supset D\}}}{\pi\{C\supset D\}}
$$

$$
c{\uparrow}\,\frac{w{\downarrow}\,\dfrac{\pi\{D\}}{\pi\{C'\supset D\}}}{\pi\{C\supset D\}}
\quad\xrightarrow{(4)}\quad
w{\downarrow}\,\frac{\pi\{D\}}{\pi\{C\supset D\}}
$$

$$
c{\uparrow}\,\frac{s{\downarrow}\,\dfrac{\pi\{((A\supset D)\supset E)\supset F\}}{\pi\{A\supset(D\supset E)\supset F\}}}{\pi\{A\supset(D\supset D\supset E)\supset F\}}
\quad\xrightarrow{(5)}\quad
c{\downarrow}\,\frac{s{\downarrow}\,\dfrac{s{\downarrow}\,\dfrac{c{\uparrow}\,\dfrac{\pi\{((A\supset D)\supset E)\supset F\}}{\pi\{((A\supset D)\supset(A\supset D)\supset E)\supset F\}}}{\pi\{A\supset(D\supset(A\supset D)\supset E)\supset F\}}}{\pi\{A\supset A\supset(D\supset D\supset E)\supset F\}}}{\pi\{A\supset(D\supset D\supset E)\supset F\}}
$$

$$
c{\uparrow}\,\frac{s{\uparrow}\,\dfrac{\eta\{A\supset(B\supset C)\supset D\}}{\eta\{((A\supset B)\supset C)\supset D\}}}{\eta\{((A\supset A\supset B)\supset C)\supset D\}}
\quad\xrightarrow{(6)}\quad
s{\uparrow}\,\frac{s{\uparrow}\,\dfrac{c{\uparrow}\,\dfrac{\eta\{A\supset(B\supset C)\supset D\}}{\eta\{A\supset A\supset(B\supset C)\supset D\}}}{\eta\{A\supset((A\supset B)\supset C)\supset D\}}}{\eta\{((A\supset A\supset B)\supset C)\supset D\}}
$$

$$
c{\uparrow}\,\frac{s{\uparrow}\,\dfrac{\eta\{A\supset(B\supset C)\supset D\}}{\eta\{((A\supset B)\supset C)\supset D\}}}{\eta\{((A\supset B)\supset C)\supset((A\supset B)\supset C)\supset D\}}
\quad\xrightarrow{(7)}\quad
s{\uparrow}\,\frac{s{\uparrow}\,\dfrac{\equiv\,\dfrac{c{\uparrow}\,\dfrac{c{\uparrow}\,\dfrac{\eta\{A\supset(B\supset C)\supset D\}}{\eta\{A\supset(B\supset C)\supset(B\supset C)\supset D\}}}{\eta\{A\supset A\supset(B\supset C)\supset(B\supset C)\supset D\}}}{\eta\{A\supset(B\supset C)\supset A\supset(B\supset C)\supset D\}}}{\eta\{A\supset(B\supset C)\supset((A\supset B)\supset C)\supset D\}}}{\eta\{((A\supset B)\supset C)\supset((A\supset B)\supset C)\supset D\}}
$$

Figure 5. Permutation cases for $c{\uparrow}$

Example 27. Here is a derivation and its complete flow-graph:

$$
c{\uparrow}\,\frac{s{\downarrow}\,\dfrac{((a\supset d)\supset e)\supset f}{a\supset(d\supset e)\supset f}}{a\supset(d\supset d\supset e)\supset f}
$$
∎

Lemma 28. *Any derivation in the* SJS *system can be transformed in the following two ways:*

$$
\begin{array}{c}
P\\
\text{SJS}\big\|\\
Q
\end{array}
\;\rightarrow\;
\begin{array}{c}
P\\
\{c{\uparrow}\}\big\|\\
P'\\
\text{SJS}\backslash\{c{\uparrow}\}\big\|\\
Q
\end{array}
\qquad\text{and}\qquad
\begin{array}{c}
P\\
\text{SJS}\big\|\\
Q
\end{array}
\;\rightarrow\;
\begin{array}{c}
P\\
\text{SJS}\backslash\{c{\downarrow}\}\big\|\\
Q'\\
\{c{\downarrow}\}\big\|\\
Q
\end{array}
$$

Proof sketch. The first transformation is obtained using local transformations shown in Figure 5, and the second one is obtained dually. To prove termination, we need an induction measure making heavy use of the flow-graphs defined above. The measure is more complex than one might expect, because of non-trivial interferences in the reductions of Figure 5. For example, case (7) duplicates a $c{\uparrow}$, but the second $c{\uparrow}$ can be duplicated via case (3) by an instance of $c{\downarrow}$ that is introduced by case (5) moving up the first $c{\uparrow}$ of (7), and so on. The precise definition of the induction measure is beyond the space limitations of this paper, but it is very close to the one given in [Str03b] (and simplified in [SG11]) for a system for multiplicative exponential linear logic. Here, the situation is slightly simpler because we are in an intuitionistic setting. $\square$

Definition 29. An *n-flip path* inside $G^{\leftrightarrow p}(\mathcal{D})$ is a path with at most n pure left edges. The *flipping number* of a path is the number of pure left edges in it, and the *flipping number* of a particle p is the maximum of the flipping numbers of all paths starting in p.

Lemma 30. *Every derivation $\mathcal{D}$ from P to Q in* SJS *can be transformed as shown below:*

$$
\begin{array}{c}
P\\
\text{SJS}\big\|\\
Q
\end{array}
\;\rightarrow\;
\begin{array}{c}
P\\
\{c{\uparrow}\}\big\|\\
P'\\
\text{SJS}\backslash\{c{\downarrow},c{\uparrow}\}\big\|\\
Q'\\
\{c{\downarrow}\}\big\|\\
Q
\end{array}
$$

Proof. We repeatedly use the transformations of the previous lemma, permuting all instances of $c{\downarrow}$ down, and all instances of $c{\uparrow}$ up. Let us define the *flipping number* of a $c{\downarrow}$ instance to be the flipping number of the head of A in its conclusion, and similarly, the flipping number of a $c{\uparrow}$ to be the flipping number of the head of the A in its premise. Then, the *flipping value* of a derivation $\mathcal{D}$, denoted $fl(\mathcal{D})$, is the multiset of all the flipping numbers of all $c{\downarrow}$ and $c{\uparrow}$ in $\mathcal{D}$. By inspecting the cases in Figure 5 used in the proof of Lemma 28, one can see that the flipping value of $\mathcal{D}$ decreases after each run, because the flipping number of the new $c{\downarrow}$ in case (5) in Figure 5 is smaller than the flipping number of the $c{\uparrow}$ before. This is enough to ensure termination of the whole process, provided that there is no cycle in the flow-graph of $\mathcal{D}$, because no transformation in Figure 5 increases the flipping number of a path. The acyclicity of $G^{\leftrightarrow p}(\mathcal{D})$ is ensured by Lemmas 31 and 32 below. $\square$

$$w\uparrow\frac{r\dfrac{\eta\{E' \supset D\}}{\eta\{E \supset D\}}}{\eta\{D\}} \longrightarrow w\uparrow\frac{\eta\{E' \supset D\}}{\eta\{D\}}$$

$$w\uparrow\frac{r\dfrac{\eta'\{E \supset D\}}{\eta\{E \supset D\}}}{\eta\{D\}} \longrightarrow w\uparrow\frac{\eta'\{E \supset D\}}{r\dfrac{\eta'\{D\}}{\eta\{D\}}}$$

$$w\uparrow\frac{w\downarrow\dfrac{\pi\{D\}}{\pi\{C' \supset D\}}}{\pi\{C \supset D\}} \longrightarrow w\downarrow\frac{\pi\{D\}}{\pi\{C \supset D\}}$$

$$w\uparrow\frac{s\downarrow\dfrac{\pi\{((C \supset D) \supset E) \supset F\}}{\pi\{C \supset (D \supset E) \supset F\}}}{\pi\{C \supset E \supset F\}} \longrightarrow w\uparrow\frac{\pi\{((C \supset D) \supset E) \supset F\}}{w\downarrow\dfrac{\pi\{E \supset F\}}{\pi\{C \supset E \supset F\}}}$$

$$w\uparrow\frac{s\uparrow\dfrac{\eta\{A \supset (B \supset C) \supset D\}}{\eta\{((A \supset B) \supset C) \supset D\}}}{\eta\{(B \supset C) \supset D\}} \longrightarrow w\uparrow\frac{\eta\{A \supset (B \supset C) \supset D\}}{\eta\{(B \supset C) \supset D\}}$$

$$w\uparrow\frac{s\uparrow\dfrac{\eta\{A \supset (B \supset C) \supset D\}}{\eta\{((A \supset B) \supset C) \supset D\}}}{\eta\{D\}} \longrightarrow w\uparrow\frac{\eta\{A \supset (B \supset C) \supset D\}}{\eta\{(B \supset C) \supset D\}}$$

Figure 6. Permutation cases for w↑

Lemma 31. *For any derivation $\mathcal{D}$ in SJS obtained by moving all c↓ down and all c↑ up, using Lemma 28 twice, if $G^{\leftrightarrow p}(\mathcal{D})$ contains a cycle, then there is a derivation $\mathcal{D}^{\circ}$ in the subsystem $\{i\downarrow, i\uparrow, s\downarrow, s\uparrow\}$ whose flow-graph also contains a cycle.*

Proof. Note that $\mathcal{D}$ consists of two parts: an upper part consisting only of c↑, and a lower part that does not contain any c↑. In that lower part of $\mathcal{D}$, all c↓ are in a position as in case (5) in Figure 5. If $G^{\leftrightarrow p}(\mathcal{D})$ contains a cycle, then this cycle must be present in the lower part. So, without loss of generality, we can discard the upper part, and assume $\mathcal{D}$ to be c↑-free. We now transform $\mathcal{D}$ into $\mathcal{D}^{-}$ by eliminating all instances of c↓, without destroying the cycle, but changing premise and conclusion of the derivation. We proceed by induction on the number of instances of c↓ in $\mathcal{D}$, using a case analysis on the bottommost such instance:

1. The cycle does not pass through A in the conclusion of the c↓. Then it also does not pass through any of the two A in the premise. We can therefore eliminate this instance of c↓ as shown below, and go on by induction.

$$c\downarrow\frac{\overset{P}{\overset{\mathcal{D}_1\|}{\pi\{A \supset A \supset B\}}}}{\underset{\underset{Q}{\mathcal{D}_2\|}}{\pi\{A \supset B\}}} \quad\to\quad \begin{array}{c} P \\ \mathcal{D}_1\| \\ \pi\{A \supset A \supset B\} \\ \tilde{\mathcal{D}}\|\{s\downarrow\} \\ A \supset \pi\{A \supset B\} \\ {}^{A \supset}\mathcal{D}_2\| \\ A \supset Q \end{array} \quad (11)$$

where $\tilde{\mathcal{D}}$ exists by Lemma 7. The resulting derivation $\mathcal{D}'$ has one instance of c↓ less, but the cyle is still present in $G^{\leftrightarrow p}(\mathcal{D}')$ since only edges connecting one of the A in the premise to the A in the conclusion of c↓ are removed, and all paths not using those edges are preserved.

2. The cycle goes through A in the conclusion and through only one A in the premise, as indicated below:

$$(12)$$
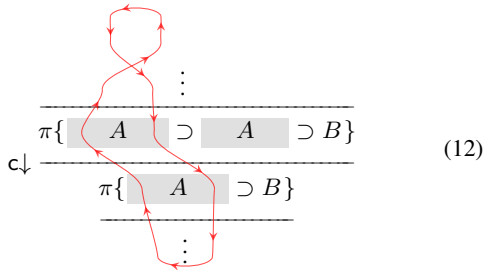
In this case we proceed as in the previous case, using (11) to construct $\mathcal{D}'$ because the cycle does not use the edges that connect the right A in the premise to the A in the conclusion

of the c↓. As before, $\mathcal{D}'$ still contains a cycle and has one c↓ instance less.

3. The cycle contains a path between the two copies of A in the premise of c↓ without going through its conclusion:
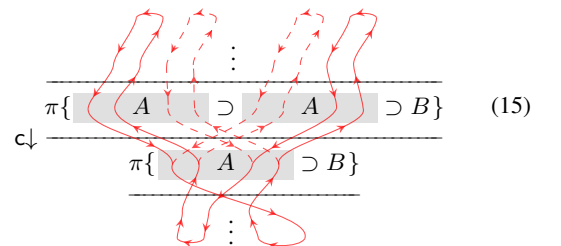

$$(13)$$
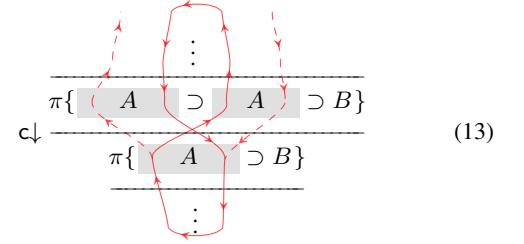
This means there is an instance of s↓ above the c↓ whose pure left edge connects a particle in the left A to one in the right A. But the c↓ is in a position as shown on the right in case (5) of Figure 5, there is no such s↓ in the derivation in that figure, and permuting the c↑ up cannot introduce such a s↓. Thus, this case is impossible.

4. The last possibility is that the cycle passes through both A in the premise of the c↓ without a direct path between the two A, as follows:

$$(14)$$
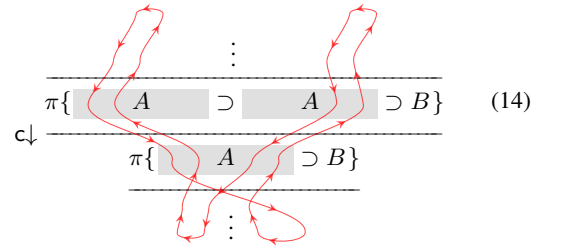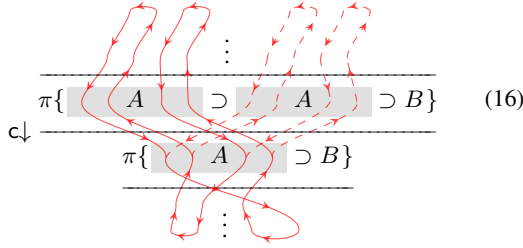
By construction, $\mathcal{D}$ contains no instances of c↑, and all c↓ in $\mathcal{D}$ are as on the right of case (5) in Figure 5. But the c↑ that created the c↓ and will be moved up acts the same on both A in the premise of c↓. Thus, all paths existing on one side also exist on the other, as follows:

$$(15)$$

Hence, the derivation contains another cycle that passes only through one A in the premise of the c↓, as below:

$$\text{c}\downarrow \cfrac{\pi\{\;\boxed{A}\; \supset \;\boxed{A}\; \supset B\}}{\pi\{\;\boxed{A}\; \supset B\}} \qquad (16)$$

This is the same situation as in case 2 above, so that we can proceed as before.

We have obtained a derivation $\mathcal{D}^-$ containing no instances of c↓ and c↑, but $G^{\leftrightarrow p}(\mathcal{D}^-)$ has a cycle. In $\mathcal{D}^-$ we now permute all instances of w↓ upwards in the derivation and all instances of w↑ downwards, and call $\mathcal{D}^\circ$ the part in between.

$$\text{SJS}\setminus\{\text{c}\downarrow,\text{c}\uparrow\}\left\|\mathcal{D}^- \atop Q\right. \begin{array}{c} P \\ \end{array} \quad \longrightarrow \quad \begin{array}{c} P \\ \{\text{w}\downarrow\}\| \\ P^\circ \\ \{\text{i}\downarrow,\text{i}\uparrow,\text{s}\downarrow,\text{s}\uparrow\}\|\mathcal{D}^\circ \\ Q^\circ \\ \{\text{w}\uparrow\}\| \\ Q \end{array} \qquad (17)$$

Transformation (17) is obtained by simple rule permutations. Most cases are trivial, except when w↓ meets i↓ as follows:

$$\text{w}\downarrow \cfrac{\text{i}\downarrow \cfrac{\eta\{\Delta \supset E\}}{\eta\{((\Delta \supset C) \supset C) \supset E\}}}{\eta\{((\Delta \supset C) \supset A \supset C) \supset E\}} \qquad (18)$$

which can be replaced by the following, where $\Delta = D_1, \Delta'$:

$$\equiv \cfrac{\text{s}\downarrow \cfrac{\text{i}\downarrow \cfrac{\text{w}\downarrow \cfrac{\eta\{D_1 \supset \Delta' \supset E\}}{\eta\{(A \supset D_1) \supset \Delta' \supset E\}}}{\eta\{(((A \supset D_1) \supset \Delta' \supset C) \supset C) \supset E\}}}{\eta\{(A \supset (D_1 \supset \Delta' \supset C) \supset C) \supset E\}}}{\eta\{((\Delta \supset C) \supset A \supset C) \supset E\}} \qquad (19)$$

and dually for w↑. Since the cycle cannot go through particles occurring in the structure A introduced by an instance of w↑ or w↓, and the transformation in (17) preserves all paths, our cycle must still be present in $\mathcal{D}^\circ$. □

Lemma 32. *The flow-graph of a derivation in the subsystem* $\{\text{i}\downarrow, \text{i}\uparrow, \text{s}\downarrow, \text{s}\uparrow\}$ *does not contain any cycle.*

Proof. This set of rules is the ⊸-only fragment of intuitionistic multiplicative linear logic. Therefore, the statement we make here follows immediately from the acyclicity condition of proof nets for intuitionistic multiplicative linear logic. For a direct proof of this, see [Str03a, Str03b, SG11]. □

We still need to prove that other parts of the decomposition can be obtained by permuting the other rule instances in any derivation without any c↑ and c↓ instances. We start with weakenings, that can be easily moved, and then decompose the fragment containing only switches and the interactions rules of identity and cut.

Lemma 33. *Every derivation $\mathcal{D}$ from P' to Q' in* $\text{SJS} \setminus \{\text{c}\downarrow, \text{c}\downarrow\}$ *can be transformed as shown below:*

$$\text{SJS}\setminus\{\text{c}\downarrow,\text{c}\uparrow\}\left\| \atop Q'\right. \begin{array}{c} P' \\ \end{array} \quad \longrightarrow \quad \begin{array}{c} P' \\ \{\text{w}\uparrow\}\| \\ P'' \\ \{\text{i}\downarrow,\text{i}\uparrow,\text{s}\downarrow,\text{s}\uparrow\}\| \\ Q'' \\ \{\text{w}\downarrow\}\| \\ Q' \end{array}$$

Proof. We can permute all w↑ up as shown in Figure 6, and all w↓ down by the dual permutations, and repeat this until all instances of w↑ and w↓ are separated. It terminates because no permutation step increases the length of the derivation, each permutation step moves an instance of weakening closer to its destination, and whenever a new weakening is introduced, one instance of switch disappears. □

Lemma 34. *Every derivation in the subsystem* $\{\text{i}\downarrow, \text{i}\uparrow, \text{s}\downarrow, \text{s}\uparrow\}$ *can be transformed as shown below:*

$$\{\text{i}\downarrow,\text{i}\uparrow,\text{s}\downarrow,\text{s}\uparrow\}\left\| \atop Q''\right. \begin{array}{c} P'' \\ \end{array} \quad \longrightarrow \quad \begin{array}{c} P'' \\ \{\text{ai}\uparrow,\text{s}\uparrow\}\| \\ I \\ \{\text{ai}\downarrow,\text{s}\downarrow\}\| \\ Q'' \end{array} \qquad (20)$$

Proof sketch. First, we use Propositions 6 and 22 to reduce instances of i↑ and i↓ to their atomic versions ai↑ and ai↓, respectively. Then, the transformation (20) is obtained by permuting all instances of ai↑ and s↑ up, using an induction on the number of up instances below a down instance in the derivation. If the topmost such instance is an ai↑, we proceed as for Lemma 20, having to consider only the cases (1)–(5) in Figure 3. If it is a s↑, we have three non-trivial cases: the dual of (5) in Figure 3, where the s↑ disappears, and the two cases in Figure 7. Due to space restrictions, we cannot show the details of the induction measure here, but the basic idea is to group s↑ instances into a compound coswitch rule. This is a standard technique for the calculus of structures (see, e.g., [Str03a]). □

Lemma 35. *Any derivation in* $\{i\downarrow, s\downarrow\}$ *(respectively* $\{i\uparrow, s\uparrow\}$*) can be transformed as below on the left (respectively right).*

$$
\begin{array}{c} P'' \\ {\scriptstyle\{i\uparrow,s\uparrow\}}\| \\ I \end{array}
\quad\rightarrow\quad
\begin{array}{c} P'' \\ {\scriptstyle\{s\uparrow\}}\| \\ P''' \\ {\scriptstyle\{i\uparrow\}}\| \\ I \end{array}
\qquad
\begin{array}{c} I \\ {\scriptstyle\{i\downarrow,s\downarrow\}}\| \\ Q'' \end{array}
\quad\rightarrow\quad
\begin{array}{c} I \\ {\scriptstyle\{i\downarrow\}}\| \\ Q''' \\ {\scriptstyle\{s\downarrow\}}\| \\ Q'' \end{array}
$$

Proof. The permutations of the $s\uparrow$ instances over $i\uparrow$ instances are always trivial, so that we can use a straightforward induction, and proceed dually for $s\downarrow$ and $i\downarrow$. □

All of these lemmas together allow us to complete the proof of Theorem 25, but successively applying each transformation to a given SJS derivation.

Proof of Theorem 25. The decomposition of the given derivation is obtained by applying Lemmas 30, 33, 34, and 35. The second statement then follows immediately from the first since $P \equiv \mathsf{t}$ implies $P \equiv P' \equiv P'' \equiv P''' \equiv I \equiv \mathsf{t}$. □

6. Conclusion and Future Work

We presented the system called JS in the calculus of structures, implementing in a quite direct way the usual rules of LJ while obtaining as a benefit the incremental nature of inference rules in deep inference. Although the cut elimination result for JS is not very surprising, the internal procedure in the calculus of structure is interesting, because intuitionistic logic (which is the original source of computational interpretations of proof systems) is the least understood of all standard logics in the deep inference setting.

The extension of JS into the symmetric SJS system is a natural generalisation in setting of the calculus of structures, where the notion of normal form of proofs takes its full strength in terms of symmetry, and where the decomposition theorem obtained is a strong statement on the relation between inference rules.

Such a symmetric system also has a certain potential in terms of computation: all of the up rules in SJS may be interpreted as typing rules for some terms based on λ-terms or combinators. As a consequence, the language of programs, under a Curry-Howard interpretation, would be much richer than what is obtained from proofs in natural deduction or from sequent calculi. For example, this would introduce in the calculus an operator representing the co-contraction rule, which is known to relate to a form of sharing, and this operator would interact with other ones in an intricate way — for example, with the representation of the switch, which can be used to implement a form of internal communication [Gue13]. Moreover, refining SJS with its normalisation procedure into a system with atomic contraction would provide a setting where a local treatment of cuts (usually associated to some form of explicit substitution in computational interpretations) could be successfully mixed with an incremental treatment of duplications, as found in the atomic λ-calculus [GHP13a].

The results established here are also an important step towards the development of the proof theory of intermediate logics in the calculus of structures. The expressivity of formalisms based on the deep inference methodology allows to apply powerful techniques from substructural proof theory. We expect that the techniques developed in the intuitionistic setting will be versatile enough to apply to many other logics without De Morgan duality.

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Appendix

This appendix is not part of the published paper. It contains straightforward but tedious details of the two sketched proofs (Lemma 28 and Lemma 34) in the main text. For the proof of Lemma 28, we need to define the induction measure first.

Definition 36. Given an instance of $\mathsf{c}{\uparrow}$, we define its *head* as the head of A in its premise, and the *head* of a $\mathsf{c}{\downarrow}$ instance is the head of A in its conclusion. A particle p is *n-flip entangled* with an instance of $\mathsf{c}{\uparrow}$ (resp. $\mathsf{c}{\downarrow}$) if there is an n-flip path from the head of that $\mathsf{c}{\uparrow}$ (resp. $\mathsf{c}{\downarrow}$) to p. The *n-flip c-number* of a particle p, denoted by $\#_n\mathsf{c}(p)$, is the number of $\mathsf{c}{\uparrow}$- and $\mathsf{c}{\downarrow}$-instances p is n-flip entangled with, increased by 1.

Definition 37. Let $\mathcal{D}$ be a derivation. A particle p in $\mathcal{D}$ is *in the scope* of a particle q, iff p and q occur in the same line in $\mathcal{D}$, and that line contains a subformula occurrence A, such that p is inside A, and q is the head of A. The *onion* of a positive (resp. negative) particle p, denoted by $\circledcirc(p)$, is the set of negative (resp. positive) particles q, such that p is in the scope of q. Now, the *contraction potential* of a particle p, denoted by $\overset{\mathsf{c}}{\curlyvee}(p)$, is defined as the product of the 1-flip c-numbers of its onion, i.e.,

$$\overset{\mathsf{c}}{\curlyvee}(p) = \prod_{q \in \circledcirc(p)} \#_1\mathsf{c}(q)$$

Remark 38. The contraction potential plays essentially the same role as the multiplicity in the previous section. However, dure to the flip and the scope change in case (5) in Figure 5, the multiplicity is not enough for our measure.

Definition 39. For a given instance of a $\mathsf{s}{\uparrow}$ (resp. $\mathsf{s}{\downarrow}$), we define its *front head* to be the head of C and its *rear head* to be the head of A in the conclusion of that $\mathsf{s}{\uparrow}$ (resp. premise of that $\mathsf{s}{\downarrow}$). Then, an instance of a $\mathsf{s}{\uparrow}$ or $\mathsf{s}{\downarrow}$ is on a path in the flow-graph, if its front head is on that path. The *explosiveness* of of an $\mathsf{s}{\uparrow}$- or $\mathsf{s}{\downarrow}$-instance is the 0-flip c-number of its rear head. The s-number of a path is the sum of the explosivenesses of the $\mathsf{s}{\uparrow}$- and $\mathsf{s}{\downarrow}$-instances on that path, increased by 1. The *n-flip s-number* of a particle p, denoted by $\#_n\mathsf{s}(p)$, is the maximum of the s-numbers of the paths starting in p. Now, the *switch-contraction potential* of a particle p, denoted by $\overset{\mathsf{cs}}{\curlyvee}(p)$, is defined as the product of its 0-flip s-number and its contraction potential, i.e,

$$\overset{\mathsf{cs}}{\curlyvee}(p) = \#_0\mathsf{s}(p) \cdot \prod_{q \in \circledcirc(p)} \#_1\mathsf{c}(q)$$

Finally, for a given $\mathsf{c}{\uparrow}$ instance in a derivation $\mathcal{D}$, we define its *status* to be 0 if all rule instances above that $\mathsf{c}{\uparrow}$ in $\mathcal{D}$ are also $\mathsf{c}{\uparrow}$, and 1 otherwise. I.e., a $\mathsf{c}{\uparrow}$ instance with status 0 is "already at the top of the derivation".

Proof of Lemma 28. We define the rank of a $\mathsf{c}{\uparrow}$ instance to be the pair $\langle s, p \rangle$ (under lexicographic ordering), where s is its status and p the switch-contraction potential of its head. Then the $\mathsf{c}{\uparrow}$-rank of a derivation $\mathcal{D}$, denoted by $rk_{\mathsf{c}{\uparrow}}(\mathcal{D})$, is the multiset of the ranks of its $\mathsf{c}{\uparrow}$ instances. we now use as induction measure the pair $(rk_{\mathsf{c}{\uparrow}}(\mathcal{D}), h)$, under lexicographic ordering, where for $rk_{\mathsf{c}{\uparrow}}(\mathcal{D})$ we use the multiset ordering and where h is the number of of non-$\mathsf{c}{\uparrow}$ instance above the topmost $\mathsf{c}{\uparrow}$ that did not yet reach the top. In Figure 8 we repeat the reductions of Figure 8, with added flow-graph edges for the heads of the concerned substructures. A crucial observation is that whenever we have two particles p and q in premise and/or conclusion of a rewriting fragment, there is an n-flip path from p to q on the left, iff there is such a path on the right. It can happen—in cases (5) and (6)—that such a path is duplicated on the right. But this does not affect the rank of $\mathsf{c}{\uparrow}$-instances below. Now, there are three kinds of cases:

First, case (4): the $\mathsf{c}{\uparrow}$ disappears, and our measure clearly goes down.

Second, cases (1), (2), (5), and (6): the $\mathsf{c}{\uparrow}$ moves up and no other $\mathsf{c}{\uparrow}$ is added to the derivation. Then the distance of this $\mathsf{c}{\uparrow}$ to the top has been reduced, and none of the other $\mathsf{c}{\uparrow}$ below in the derivation has changed its rank. This is obvious for cases (1) and (2). For case (6), notice that the sum of the explosivenesses of the two new $\mathsf{s}{\uparrow}$ is smaller than the explosiveness of the old $\mathsf{s}{\uparrow}$. In case (5), note that the contraction potential of a particle in C in the conclusion does not change in the transformation. The new instances of $\mathsf{s}{\downarrow}$ do not play a role in the 0-flip s-number of a $\mathsf{c}{\uparrow}$.

Third, cases (3) and (7): a second $\mathsf{c}{\uparrow}$ is introduced. In case (3), the two new $\mathsf{c}{\uparrow}$ have smaller contraction potential, and therefore also smaller rank than the original one. In case (7), the heads of the two new $\mathsf{c}{\uparrow}$ have a smaller 0-flip s-number, and the two new $\mathsf{s}{\uparrow}$ have both a smaller explosiveness than the one on the left. $\qquad\square$

Complements on proof of Lemma 34. Handling permutations of switches requires to define a compound switch rule $\mathsf{ss}{\uparrow}$ as follows:

$$\mathsf{ss}{\uparrow}\,\frac{\eta\{A \supset \nu\{\mathsf{t}\}\}}{\eta\{\nu\{A\}\}}$$

equivalent to a sequence of $\mathsf{s}{\uparrow}$ instances (dual to Lemma 7). We then proceed by induction on the height of the derivation above the considered $\mathsf{r}{\uparrow}$ instance of this rule, using a case analysis on $\mathsf{r}_1{\downarrow}$, the rule instance directly above it:

1. If $\mathsf{r}_1{\downarrow}$ is not affecting A or the hole of ν, then the permutation is immediate and we can go on by applying the induction hypothesis.
2. If $\mathsf{r}_1{\downarrow}$ is an instance of $\mathsf{ai}{\downarrow}$ interacting with one of the $\mathsf{s}{\uparrow}$ in $\mathsf{r}{\uparrow}$, it assimilates one of these $\mathsf{s}{\uparrow}$ as shown below, where $\nu = \eta\{(\rho\{-\} \supset d) \supset d\}$ for a negative context η and positive context ρ:

$$\mathsf{ss}{\uparrow}\,\frac{\mathsf{ai}{\downarrow}\,\dfrac{\pi\{(A \supset \eta\{\rho\{\mathsf{t}\}\}) \supset C\}}{\pi\{(A \supset \eta\{(\rho\{\mathsf{t}\} \supset d) \supset d\}) \supset C\}}}{\pi\{\eta\{(\rho\{A\} \supset d) \supset d\} \supset C\}} \quad \longrightarrow$$

$$\mathsf{ai}{\uparrow}\,\frac{\mathsf{ss}{\uparrow}\,\dfrac{\pi\{(A \supset \eta\{\rho\{\mathsf{t}\}\}) \supset C\}}{\pi\{\eta\{\rho\{A\}\} \supset C\}}}{\pi\{\eta\{(\rho\{A\} \supset d) \supset d\} \supset C\}}$$

   and we conclude by induction hypothesis — except if $\mathsf{r}{\uparrow}$ was a simple $\mathsf{s}{\uparrow}$ instance, in which case it disappeared.
3. If $\mathsf{r}_1{\downarrow}$ is an instance of $\mathsf{s}{\downarrow}$ moving some B inside A, it is permuted by introducing a sequence of $\mathsf{s}{\downarrow}$, and then we proceed by induction hypothesis:

$$\mathsf{ss}{\uparrow}\,\frac{\mathsf{s}{\downarrow}\,\dfrac{\pi\{((B \supset A) \supset \nu\{\mathsf{t}\}) \supset C\}}{\pi\{B \supset (A \supset \nu\{\mathsf{t}\}) \supset C\}}}{\pi\{B \supset \nu\{A\} \supset C\}} \quad \longrightarrow$$

$$\mathsf{s}{\downarrow}^*\,\frac{\mathsf{ss}{\uparrow}\,\dfrac{\pi\{((B \supset A) \supset \nu\{\mathsf{t}\}) \supset C\}}{\pi\{\nu\{B \supset A\} \supset C\}}}{\pi\{B \supset \nu\{A\} \supset C\}}$$

   This sequence of $\mathsf{s}{\downarrow}$ exists by Lemmma 7.
4. If $\mathsf{r}_1{\downarrow}$ is an instance of $\mathsf{s}{\downarrow}$ moving the hole of one of the $\mathsf{s}{\uparrow}$ composing $\mathsf{r}{\uparrow}$, then one more $\mathsf{s}{\uparrow}$ is needed inside $\mathsf{r}{\uparrow}$ after permutation (cf. Figure 7), so that the permutation is as follows, where $\nu = \eta\{\eta'\{-\} \supset (D \supset E) \supset F\}$ for negative contexts

Figure 8. Permutation cases for c↑ with some flow-graph edges

η and η':

$$\mathsf{ss}\uparrow\frac{\mathsf{s}\downarrow\dfrac{\pi\{(A\supset\eta\{((\eta'\{\mathsf{t}\}\supset D)\supset E)\supset F\})\supset C\}}{\pi\{A\supset\eta\{\eta'\{\mathsf{t}\}\supset(D\supset E)\supset F\}\supset C\}}}{\pi\{\eta\{\eta'\{A\}\supset(D\supset E)\supset F\}\supset C\}}\quad\longrightarrow$$

$$\mathsf{s}\downarrow\frac{\mathsf{ss}\uparrow\dfrac{\pi\{(A\supset\eta\{((\eta'\{\mathsf{t}\}\supset D)\supset E)\supset F\})\supset C\}}{\pi\{\eta\{((\eta'\{A\}\supset D)\supset E)\supset F\}\supset C\}}}{\pi\{\eta\{\eta'\{A\}\supset(D\supset E)\supset F\}\supset C\}}$$

and we conclude by induction hypothesis. $\square$