A Local System for Linear Logic

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Abstract

In this paper I will present a deductive system for linear logic, in which all rules are *local*. In particular, the contraction rule is reduced to an atomic version, and there is no global promotion rule. In order to achieve this, it is necessary to depart from the sequent calculus to the *calculus of structures*, which is a generalization of the one-sided sequent calculus: in a rule, premise and conclusion are not sequents, but *structures*, which are expressions that share properties for formulae and sequents.

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1 Introduction

The calculus of structures [5, 6, 1, 9, 7] is a new proof-theoretical framework, like the sequent calculus [2, 3], natural deduction [2, 8] and proof nets [4], for specifying logical systems. In contrast to the sequent calculus, the calculus of structures does not depend on the notion of main connective and the distinction between sequents and formulae. Further, it permits the application of inferences anywhere deep inside formulae. This means that all rules have only one premise. Hence, derivations are not trees as in the sequent calculus but chains of instances of inference rules. This gives the calculus of structures a new top-down symmetry that is not present in the sequent calculus.

There are two main reasons to study logical systems in the calculus of structures. First, there exist systems, which cannot be presented in the sequent calculus [5, 10] but which have a very natural presentation in the calculus of structures, and second, for logical systems that have a well-known presentation in the sequent calculus, there exist equivalent systems in the calculus of structures that have new and unexpected properties, that cannot be observed in the sequent calculus [1, 9].

For example, in [1], a system for classical logic is presented, in which all rules are *local*. There, a rule is called local if it does not require a global view of formulae of unbounded size. In other words, the computational resources (time and space) that are needed for applying a local rule are bounded. This is a very important property from the viewpoint of proof search, in particular, if it is implemented on a distributed system.

Consider, for instance, the par rule of linear logic [4]:

$$\parr \frac{\vdash A, B, \Phi}{\vdash A \parr B, \Phi} \quad .$$

Only the main connective $\parr$ of the formula $A \parr B$ has to be considered. The contents of A, B and Φ does not play a rôle. Hence, this rule is local. Consider now the contraction rule.

$$?\mathsf{c} \frac{\vdash ?A, ?A, \Phi}{\vdash ?A, \Phi} \quad .$$

Here, the whole formula $?A$ has to be copied. This means that it is not sufficient to look only at the main connective $?$, but is is necessary to consider all of A. Hence, this rule is not local. The same is true for the contraction rule for classical logic. In [1], it has been shown that in the calculus of structures it is possible, to reduce the general contraction rule of classical logic to an atomic version, which is local because the computational resources needed to copy an atom are bounded.

The idea of reducing rules to their atomic version is not entirely new. It is well-known that in many systems in the sequent calculus, the general identity rule can be reduced to an atomic version. For example in linear logic, we have that

$$\mathsf{id} \frac{}{\vdash A, A^\perp} \quad \text{is admissible for} \quad \mathsf{id} \frac{}{\vdash a, a^\perp} \quad ,$$

by using an inductive argument on the formula A. For example, if $A = B \otimes C$ we can replace

$$\mathsf{id}\,\frac{}{\vdash B \otimes C, B^\perp \parr C^\perp} \qquad \text{by} \qquad \parr\,\frac{\otimes\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash B, B^\perp} \quad \mathsf{id}\,\dfrac{}{\vdash C, C^\perp}}{\vdash B \otimes C, B^\perp, C^\perp}}{\vdash B \otimes C, B^\perp \parr C^\perp} \quad .$$

But it is also well-known that for the general cut rule

$$\mathsf{cut}\,\frac{\vdash A, \Phi \quad \vdash A^\perp, \Psi}{\vdash \Phi, \Psi} \quad ,$$

such an argument is impossible. But contrarily to this observation in the sequent calculus, in the calculus of structures, it is possible to reduce the cut rule to its atomic version, by the use of the new top-down symmetry.

Let me now mention the other non-local rules in the sequent calculus system for linear logic. The most prominent is probably the promotion rule:

$$!\,\frac{\vdash A, ?B_1, \ldots, ?B_n}{\vdash !A, ?B_1, \ldots, ?B_n} \quad .$$

In order to apply it, it is necessary to check for each formula in the context of $!A$, whether it has the form $?B$. In [9, 6] it has already been shown, how this rule can be made local in the calculus of structures by using the linear implication

$$!(A \parr B) \multimap !A \parr ?B \quad .$$

Another non-local rule is the with rule:

$$\&\,\frac{\vdash A, \Phi \quad \vdash B, \Phi}{\vdash A \mathbin{\&} B, \Phi} \quad .$$

When this rule is applied, the whole context Φ has to be copied. In order to make this rule local, we have to do two things. First, the copying has to be made explicit, and second, this explicit copying has to be reduced to an atomic form. The first step is done by employing the linear implication $(A \parr B) \mathbin{\&} (C \parr D) \multimap (A \mathbin{\&} C) \parr (B \oplus D)$ to produce a purely multiplicative rule for the additives. The additive behaviour is regained by employing $A \oplus A \multimap A$ in a contraction rule for the additives. This new contraction rule is in the second step reduced to an atomic version by the same method that has been employed in [1].

Let me now sketch the outline of this paper. In the next section, I will introduce the basic notions of the calculus of structures and present the standard system for linear logic in the calculus of structures. In Section 3, I will show that it is equivalent to the system for linear logic in the sequent calculus, and in Section 4, I will show cut elimination for it. In the next two sections I will then present a local system for linear logic. For doing so, I will start from the multiplicative additive fragment in Section 5, before dealing with full propositional linear logic in Section 6. Then, in Section 7, I will show some decomposition theorems.

2 Linear Logic in the Calculus of Structures

A system in the calculus of structures requires a language of *structures*, which are intermediate expressions between formulae and sequents. I will now define the language LS for the systems presented in this paper. Intuitively, $[R_1, \dots, R_h]$ corresponds to a sequent $\vdash R_1, \dots, R_h$ in linear logic, whose formulae are essentially connected by pars, subject to commutativity (and associativity). The structure $(R_1, \dots, R_h)$ corresponds to the associative and commutative times connection of $R_1, \dots, R_h$. The structures $\{\!\!\{ R_1, \dots, R_h \}\!\!\}$ and $\langle\!\langle R_1, \dots, R_h \rangle\!\rangle$, which are also associative and commutative, correspond to the additive disjunction and conjunction, respectively, of $R_1, \dots, R_h$.

2.1 Definition There are countably many *positive* and *negative propositional variables*, denoted with a, b, c, ... There are four *constants*, called *bottom, one, zero* and *top*, denoted with $\bot$, $\mathsf{1}$, $\mathsf{0}$ and $\top$, respectively. An *atom* is either a propositional variable or a constant. The *structures* of the language LS are denoted with P, Q, R, S, ..., and are generated by

$$R ::= a \mid [\underbrace{R, \dots, R}_{>0}] \mid (\underbrace{R, \dots, R}_{>0}) \mid \{\!\!\{\underbrace{R, \dots, R}_{>0}\}\!\!\} \mid \langle\!\langle\underbrace{R, \dots, R}_{>0}\rangle\!\rangle \mid\, !R \mid\, ?R \mid\, \bar{R} \ ,$$

where a stands for any atom (positive or negative propositional variable or constant). A structure $[R_1, \dots, R_h]$ is called a *par structure*, $(R_1, \dots, R_h)$ is called a *times structure*, $\{\!\!\{ R_1, \dots, R_h \}\!\!\}$ is called a *plus structure*, $\langle\!\langle R_1, \dots, R_h \rangle\!\rangle$ is called a *with structure*, $!R$ is called an *of-course structure*, and $?R$ is called a *why-not structure*; $\bar{R}$ is the *negation* of the structure R. Structures are considered to be equivalent modulo the relation $=$, which is the smallest congruence relation induced by the equations shown in Figure 1, where $\vec{R}$ and $\vec{T}$ stand for finite, non-empty sequences of structures.

2.2 Definition In the same setting, we can define *structure contexts*, which are structures with a hole. Formally, they are generated by

$$S \ ::= \ \{\ \} \mid \ [R, S] \ \mid \ (R, S) \ \mid \ \{\!\!\{R, S\}\!\!\} \ \mid \ \langle\!\langle R, S\rangle\!\rangle \mid\, !S \mid\, ?S \quad .$$

Because of the De Morgan laws there is no need to include the negation into the definition of the context, which means that the structure that is plugged into the hole of a context will always be positive. Structure contexts will be denoted with $R\{\ \}$, $S\{\ \}$, $T\{\ \}$, ... Then, $S\{R\}$ denotes the structure that is obtained by replacing the hole $\{\ \}$ in the context $S\{\ \}$ by the structure R. The structure R is a *substructure* of $S\{R\}$ and $S\{\ \}$ is its *context*. For a better readability, I will omit the context braces if no ambiguity is possible, e.g. I will write $S[R, T]$ instead of $S\{[R, T]\}$.

2.3 Example Let $S\{\ \} = [(a, ![\{\ \}, ?a], \bar{b}), b]$ and $R = c$ and $T = (\bar{b}, \bar{c})$ then

$$S[R, T] = [(a, ![c, (\bar{b}, \bar{c}), ?a], \bar{b}), b] \quad .$$

Associativity

$$
\begin{aligned}
[\vec{R},[\vec{T}]] &= [\vec{R},\vec{T}] \\
(\vec{R},(\vec{T})) &= (\vec{R},\vec{T}) \\
\{\!\{\vec{R},\{\!\{\vec{T}\}\!\}\}\!\} &= \{\!\{\vec{R},\vec{T}\}\!\} \\
\langle\vec{R},\langle\vec{T}\rangle\rangle &= \langle\vec{R},\vec{T}\rangle
\end{aligned}
$$

Commutativity

$$
\begin{aligned}
[\vec{R},\vec{T}] &= [\vec{T},\vec{R}] \\
(\vec{R},\vec{T}) &= (\vec{T},\vec{R}) \\
\{\!\{\vec{R},\vec{T}\}\!\} &= \{\!\{\vec{T},\vec{R}\}\!\} \\
\langle\vec{R},\vec{T}\rangle &= \langle\vec{T},\vec{R}\rangle
\end{aligned}
$$

Units

$$
\begin{aligned}
[\bot,\vec{R}] &= [\vec{R}] \\
(\mathsf{1},\vec{R}) &= (\vec{R}) \\
\{\!\{\mathsf{0},\vec{R}\}\!\} &= \{\!\{\vec{R}\}\!\} \\
\langle\top,\vec{R}\rangle &= \langle\vec{R}\rangle
\end{aligned}
$$

$$
\begin{aligned}
\{\!\{\bot,\bot\}\!\} &= \bot = ?\bot \\
\langle\mathsf{1},\mathsf{1}\rangle &= \mathsf{1} = !\mathsf{1}
\end{aligned}
$$

Singleton

$$
\begin{aligned}
[R] &= R = (R) \\
\{\!\{R\}\!\} &= R = \langle R\rangle
\end{aligned}
$$

Exponentials

$$
\begin{aligned}
??R &= ?R \\
!!R &= !R
\end{aligned}
$$

Negation

$$
\begin{aligned}
\overline{\bot} &= \mathsf{1} \\
\overline{\mathsf{1}} &= \bot \\
\overline{\mathsf{0}} &= \top \\
\overline{\top} &= \mathsf{0} \\
\overline{[R_1,\dots,R_h]} &= (\bar{R}_1,\dots,\bar{R}_h) \\
\overline{(R_1,\dots,R_h)} &= [\bar{R}_1,\dots,\bar{R}_h] \\
\overline{\{\!\{R_1,\dots,R_h\}\!\}} &= \langle\bar{R}_1,\dots,\bar{R}_h\rangle \\
\overline{\langle R_1,\dots,R_h\rangle} &= \{\!\{\bar{R}_1,\dots,\bar{R}_h\}\!\} \\
\overline{?R} &= !\bar{R} \\
\overline{!R} &= ?\bar{R} \\
\bar{\bar{R}} &= R
\end{aligned}
$$

Figure 1: Basic equations for the syntactic congruence =

2.4 Definition In the calculus of structures, an *inference rule* is a scheme of the kind

$$\rho\,\frac{T}{R}\quad,$$

where ρ is the *name* of the rule, T is its *premise* and R is its *conclusion*. An inference rule is called an *axiom* if its premise is empty, i.e. the rule is of the shape

$$\rho\,\frac{}{R}\quad.$$

A typical rule has shape $\rho\,\dfrac{S\{T\}}{S\{R\}}$ and specifies a step of rewriting, by the implication $T \Rightarrow R$, inside a generic context $S\{\ \}$. Rules with empty contexts correspond to the case of the sequent calculus.

2.5 Definition A (*formal*) *system* $\mathscr{S}$ is a set of inference rules.

2.6 Definition A *derivation* Δ in a certain formal system is a finite chain of instances of inference rules in the system:

$$
\rho\,\frac{R}{R'}\qquad
\begin{array}{l}
\rho' \, \overline{} \\
\quad\vdots \\
\rho'' \, \dfrac{}{R''}
\end{array}
\quad .
$$

A derivation can consist of just one structure. The topmost structure in a derivation, if present, is called the *premise* of the derivation, and the bottommost structure is called its *conclusion*. A derivation Δ whose premise is T, whose conclusion is R, and whose inference rules are in $\mathscr{S}$ will be indicated with $\begin{array}{c}T\\ \Delta \Vert \mathscr{S}\\ R\end{array}$. A *proof* Π in the calculus of structures is a finite derivation whose topmost inference rule is an axiom. It will be denoted by $\begin{array}{c}\Pi \Vert \mathscr{S}\\ R\end{array}$.

In the calculus of structures, rules come in pairs, a down-version $\rho{\downarrow}\,\dfrac{S\{T\}}{S\{R\}}$ and an up-version $\rho{\uparrow}\,\dfrac{S\{\bar{R}\}}{S\{\bar{T}\}}$. This duality derives from the duality between $T \Rightarrow R$ and $\bar{R} \Rightarrow \bar{T}$, where $\Rightarrow$ is the implication modelled in the system. In our case it is linear implication.

2.7 Definition The structural rules

$$
\mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]}\;,\qquad
\mathsf{d}{\downarrow}\,\frac{S(\bullet[R,U],[T,V]\bullet)}{S[(\bullet R,T\bullet),[\bullet U,V\bullet]]}\;,\qquad
\mathsf{d}{\uparrow}\,\frac{S([\bullet R,U\bullet],(\bullet T,V\bullet))}{S[\bullet(R,T),(U,V)\bullet]}\;,
$$

$$
\mathsf{w}{\downarrow}\,\frac{S\{0\}}{S\{R\}}\;,\qquad
\mathsf{w}{\uparrow}\,\frac{S\{R\}}{S\{\top\}}\;,\qquad
\mathsf{c}{\downarrow}\,\frac{S[\bullet R,R\bullet]}{S\{R\}}\;,\qquad
\mathsf{c}{\uparrow}\,\frac{S\{R\}}{S(\bullet R,R\bullet)}\;,
$$

$$
\mathsf{p}{\downarrow}\,\frac{S\{![R,T]\}}{S[!R,?T]}\;,\qquad
\mathsf{p}{\uparrow}\,\frac{S(?R,!T)}{S\{?(R,T)\}}\;,
$$

$$
\mathsf{v}{\downarrow}\,\frac{S\{\bot\}}{S\{?R\}}\;,\qquad
\mathsf{v}{\uparrow}\,\frac{S\{!R\}}{S\{1\}}\;,\qquad
\mathsf{b}{\downarrow}\,\frac{S[?R,R]}{S\{?R\}}\qquad\text{and}\qquad
\mathsf{b}{\uparrow}\,\frac{S\{!R\}}{S(!R,R)}
$$

are called *switch* (s), *additive* ($\mathsf{d}{\downarrow}$), *coadditive* ($\mathsf{d}{\uparrow}$), *weakening* ($\mathsf{w}{\downarrow}$), *coweakening* ($\mathsf{w}{\uparrow}$), *contraction* ($\mathsf{c}{\downarrow}$), *cocontraction* ($\mathsf{c}{\uparrow}$), *promotion* ($\mathsf{p}{\downarrow}$), *copromotion* ($\mathsf{p}{\uparrow}$), *vanishing* ($\mathsf{v}{\downarrow}$), *covanishing* ($\mathsf{v}{\uparrow}$), *absorption* ($\mathsf{b}{\downarrow}$) and *coabsorption* ($\mathsf{b}{\uparrow}$), respectively.

Observe that the switch rule is self-dual, i.e. if premise and conclusion are negated and exchanged, we obtain again an instance of switch, whereas all other rules have a dual corule.

2.8 Definition The rules

$$
\mathsf{i}{\downarrow}\,\frac{S\{1\}}{S[R,\bar{R}]}\qquad\text{and}\qquad
\mathsf{i}{\uparrow}\,\frac{S(R,\bar{R})}{S\{\bot\}}
$$

are called *interaction* and *cut* (or *cointeraction*), respectively.

Observe that these rules correspond to the identity and cut rule in the sequent calculus (the exact correspondence is shown in the proof of Theorem 3.7), with the difference that the duality between identity and cut is more vivid.

2.9 Definition The rules

$$\mathsf{ai}{\downarrow}\,\frac{S\{1\}}{S[a,\bar{a}]} \qquad \text{and} \qquad \mathsf{ai}{\uparrow}\,\frac{S(a,\bar{a})}{S\{\bot\}}$$

are called *atomic interaction* and *atomic cut* (or *atomic cointeraction*), respectively.

The rules $\mathsf{ai}{\downarrow}$ and $\mathsf{ai}{\uparrow}$ are obviously instances of the rules $\mathsf{i}{\downarrow}$ and $\mathsf{i}{\uparrow}$ above. It is well known that in many systems in the sequent calculus, the identity rule can be reduced to its atomic version. In the calculus of structures we can do the same. But furthermore, by duality, we can do the same to the cut rule, which is not possible in the sequent calculus.

2.10 Definition A rule ρ is *strongly admissible* for a system $\mathscr{S}$ if $\rho \notin \mathscr{S}$ and for every application of $\rho\,\dfrac{T}{R}$ there is a derivation $\begin{array}{c}T\\ \Delta\Vert\mathscr{S}\\ R\end{array}$.

2.11 Proposition *The rule* $\mathsf{i}{\downarrow}$ *is strongly admissible for the system* $\{\mathsf{ai}{\downarrow},\mathsf{s},\mathsf{d}{\downarrow},\mathsf{p}{\downarrow}\}$. *Dually, the rule* $\mathsf{i}{\uparrow}$ *is strongly admissible for* $\{\mathsf{ai}{\uparrow},\mathsf{s},\mathsf{d}{\uparrow},\mathsf{p}{\uparrow}\}$.

Proof: For a given application of $\mathsf{i}{\downarrow}\,\dfrac{S\{1\}}{S[R,\bar{R}]}$, by structural induction on R, we will construct an equivalent derivation that contains only $\mathsf{ai}{\downarrow}$, s, $\mathsf{d}{\downarrow}$ and $\mathsf{p}{\downarrow}$.

- R is an atom: Then the given instance of $\mathsf{i}{\downarrow}$ is an instance of $\mathsf{ai}{\downarrow}$.
- $R=[P,Q]$, where $P\neq\bot\neq Q$: Apply the induction hypothesis on

$$\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{i}{\downarrow}\,\dfrac{\mathsf{i}{\downarrow}\,\dfrac{S\{1\}}{S[Q,\bar{Q}]}}{S([P,\bar{P}],[Q,\bar{Q}])}}{S[Q,([P,\bar{P}],\bar{Q})]}}{S[P,Q,(\bar{P},\bar{Q})]} \quad .$$

- $R=(P,Q)$, where $P\neq 1\neq Q$: Similar to the previous case.
- $R=\{\!\{P,Q\}\!\}$, where $P\neq 0\neq Q$: Apply the induction hypothesis on

$$\mathsf{d}{\downarrow}\,\dfrac{\mathsf{i}{\downarrow}\,\dfrac{\mathsf{i}{\downarrow}\,\dfrac{S\langle\!\langle 1,1\rangle\!\rangle}{S\langle\!\langle 1,[Q,\bar{Q}]\rangle\!\rangle}}{S\langle\!\langle [P,\bar{P}],[Q,\bar{Q}]\rangle\!\rangle}}{S[\{\!\{P,Q\}\!\},\langle\!\langle\bar{P},\bar{Q}\rangle\!\rangle]} \quad .$$

(Note that $S\langle\!\langle 1,1\rangle\!\rangle = S\{1\}$.)

- $R = \langle P, Q \rangle$, where $P \neq \top \neq Q$: Similar to the previous case.
- $R = ?P$, where $P \neq \bot$: Apply the induction hypothesis on

$$\mathsf{i}\downarrow \frac{S\{!1\}}{\mathsf{p}\downarrow \frac{S\{![P, \bar{P}]\}}{S[?P, !\bar{P}]}} \quad .$$

  (Note that $S\{!1\} = S\{1\}$.)

- $R = !P$, where $P \neq 1$: Similar to the previous case.

The second statement is dual to the first. □

2.12 Definition The system

$$\{\mathsf{ai}\downarrow, \mathsf{ai}\uparrow, \mathsf{s}, \mathsf{d}\downarrow, \mathsf{d}\uparrow, \mathsf{w}\downarrow, \mathsf{w}\uparrow, \mathsf{c}\downarrow, \mathsf{c}\uparrow, \mathsf{p}\downarrow, \mathsf{p}\uparrow, \mathsf{v}\downarrow, \mathsf{v}\uparrow, \mathsf{b}\downarrow, \mathsf{b}\uparrow\} \quad ,$$

shown in Figure 2 is called *symmetric (or self-dual) linear logic in the calculus of structures*, or system SLS. The set $\{\mathsf{ai}\downarrow, \mathsf{d}\downarrow, \mathsf{w}\downarrow, \mathsf{c}\downarrow, \mathsf{p}\downarrow, \mathsf{v}\downarrow, \mathsf{b}\downarrow\}$ is called the *down-fragment* and $\{\mathsf{ai}\uparrow, \mathsf{d}\uparrow, \mathsf{w}\uparrow, \mathsf{c}\uparrow, \mathsf{p}\uparrow, \mathsf{v}\uparrow, \mathsf{b}\uparrow\}$ is called the *up-fragment*. Further, the subsystem

$$\{\mathsf{ai}\downarrow, \mathsf{ai}\uparrow, \mathsf{s}, \mathsf{d}\downarrow, \mathsf{d}\uparrow, \mathsf{w}\downarrow, \mathsf{w}\uparrow, \mathsf{c}\downarrow, \mathsf{c}\uparrow\}$$

is called the *multiplicative additive fragment*, or system SALS. Similarly, the system

$$\{\mathsf{ai}\downarrow, \mathsf{ai}\uparrow, \mathsf{s}, \mathsf{p}\downarrow, \mathsf{p}\uparrow, \mathsf{v}\downarrow, \mathsf{v}\uparrow, \mathsf{b}\downarrow, \mathsf{b}\uparrow\}$$

is called the *multiplicative exponential fragment*, or system SELS.

System SELS has already been studied in [9].

There is another strong admissibility result involved here, that has already been observed in [5]. If the rules $\mathsf{i}\downarrow$, $\mathsf{i}\uparrow$ and s are in a system, then any other rule ρ makes its corule ρ', i.e. the rule obtained from ρ by exchanging and negating premise and conclusion, be strongly admissible. Let $\rho \dfrac{S\{P\}}{S\{Q\}}$ be given. Then any instance of $\rho' \dfrac{S\{\bar{Q}\}}{S\{\bar{P}\}}$ can be replaced by the following derivation:

$$\mathsf{i}\downarrow \frac{S\{\bar{Q}\}}{\mathsf{s} \frac{S(\bar{Q}, [P, \bar{P}])}{\rho \frac{S[(\bar{Q}, P), \bar{P}]}{\mathsf{i}\uparrow \frac{S[(\bar{Q}, Q), \bar{P}]}{S\{\bar{P}\}}}}} \quad .$$

2.13 Proposition *Every rule $\rho\uparrow$ is strongly admissible for $\{\mathsf{i}\downarrow, \mathsf{i}\uparrow, \mathsf{s}, \rho\downarrow\}$.*

Propositions 2.11 and 2.13 together say, that the general cut rule $\mathsf{i}\uparrow$ is as powerful as the whole up-fragment of the system and vice versa.

$\mathsf{ai}\downarrow \dfrac{S\{1\}}{S[a,\bar{a}]}$	$\mathsf{ai}\uparrow \dfrac{S(a,\bar{a})}{S\{\bot\}}$

$$\mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]}$$

$\mathsf{d}\downarrow \dfrac{S\bullet([R,U],[T,V])\bullet}{S[\bullet(R,T)\bullet,\bullet\{U,V\}\bullet]}$	$\mathsf{d}\uparrow \dfrac{S(\bullet\{R,U\}\bullet,\bullet(T,V)\bullet)}{S\bullet\{(R,T),(U,V)\}\bullet}$
$\mathsf{w}\downarrow \dfrac{S\{0\}}{S\{R\}}$	$\mathsf{w}\uparrow \dfrac{S\{R\}}{S\{\top\}}$
$\mathsf{c}\downarrow \dfrac{S\bullet\{R,R\}\bullet}{S\{R\}}$	$\mathsf{c}\uparrow \dfrac{S\{R\}}{S\bullet(R,R)\bullet}$
$\mathsf{p}\downarrow \dfrac{S\{![R,T]\}}{S[!R,?T]}$	$\mathsf{p}\uparrow \dfrac{S(?R,!T)}{S\{?(R,T)\}}$
$\mathsf{v}\downarrow \dfrac{S\{\bot\}}{S\{?R\}}$	$\mathsf{v}\uparrow \dfrac{S\{!R\}}{S\{1\}}$
$\mathsf{b}\downarrow \dfrac{S[?R,R]}{S\{?R\}}$	$\mathsf{b}\uparrow \dfrac{S\{!R\}}{S(!R,R)}$

Figure 2: System SLS

So far we are only able to describe derivations. In order to describe proofs, we need an axiom.

2.14 Definition The following rule is called *one*:

$$1\downarrow\,\frac{}{1}\quad .$$

In the language of linear logic it simply says that $\vdash 1$ is provable. I will put this rule to the down-fragment of system SLS and by this break the top-down symmetry of derivations and observe proofs.

2.15 Definition The system $\{1\downarrow, \mathsf{ai}\downarrow, \mathsf{s}, \mathsf{d}\downarrow, \mathsf{w}\downarrow, \mathsf{c}\downarrow, \mathsf{p}\downarrow, \mathsf{v}\downarrow, \mathsf{b}\downarrow\}$, shown in Figure 3, that is obtained from the down-fragment of system SLS together with the switch rule and the axiom, is called *linear logic in the calculus of structures*, or system LS. The restriction to the multiplicative additive fragment $\{1\downarrow, \mathsf{ai}\downarrow, \mathsf{s}, \mathsf{d}\downarrow, \mathsf{w}\downarrow, \mathsf{c}\downarrow\}$ is called system ALS

$$
1{\downarrow}\,\frac{}{\mathsf{1}} \qquad \mathsf{ai}{\downarrow}\,\frac{S\{\mathsf{1}\}}{S[a,\bar{a}]} \qquad \mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]}
$$

$$
\mathsf{d}{\downarrow}\,\frac{S{\bullet\!(}[R,U],[T,V]{)\!\bullet}}{S[{\bullet\!(}R,T{)\!\bullet},{\bullet\![}U,V{]\!\bullet}]} \qquad \mathsf{w}{\downarrow}\,\frac{S\{\mathsf{0}\}}{S\{R\}} \qquad \mathsf{c}{\downarrow}\,\frac{S{\bullet\![}R,R{]\!\bullet}}{S\{R\}}
$$

$$
\mathsf{p}{\downarrow}\,\frac{S\{![R,T]\}}{S[!R,?T]} \qquad \mathsf{v}{\downarrow}\,\frac{S\{\bot\}}{S\{?R\}} \qquad \mathsf{b}{\downarrow}\,\frac{S[?R,R]}{S\{?R\}}
$$

Figure 3: System LS

and the restriction to the multiplicative exponential fragment $\{\mathsf{1}{\downarrow},\mathsf{ai}{\downarrow},\mathsf{s},\mathsf{p}{\downarrow},\mathsf{v}{\downarrow},\mathsf{b}{\downarrow}\}$ is called system ELS.

Observe that in every proof in system LS, the rule $\mathsf{1}{\downarrow}$ occurs exactly once, namely as the topmost rule of the proof.

2.16 Definition Two systems $\mathscr{S}$ and $\mathscr{S}'$ are *strongly equivalent* if for every derivation $\begin{array}{c}T\\ \Delta\|\mathscr{S}\\ R\end{array}$ there is a derivation $\begin{array}{c}T\\ \Delta'\|\mathscr{S}'\\ R\end{array}$, and vice versa.

2.17 Theorem *The systems* $\mathsf{SLS}\cup\{\mathsf{1}{\downarrow}\}$ *and* $\mathsf{LS}\cup\{\mathsf{i}{\uparrow}\}$ *are strongly equivalent.*

Proof: Immediate consequence of Propositions 2.11 and 2.13. □

3 Equivalence to Linear Logic in the Sequent Calculus

In this section I will first recall the well-known sequent calculus system for linear logic [4], and then show that it is equivalent to the systems defined in the previous section. More precisely, system SLS corresponds to linear logic in the sequent calculus with cut, and system LS corresponds to linear logic in the sequent calculus without cut.

Let me start with the definitions of formulae, sequents, derivations and proofs in the sequent calculus.

3.1 Definition An *atom* is a propositional variable or its dual, denoted with $a,b,c,\ldots$ or $a^{\bot},b^{\bot},c^{\bot},\ldots$, respectively. *Formulae* (denoted with $A,B,C,\ldots$) are built from atoms and the constants $\bot,\mathsf{1},\mathsf{0},\top$ (called *bottom, one, zero, top*, respectively) by means of the (binary) connectives $⅋,\otimes,\oplus,\&$ (called *par, times, plus, with*, respectively) and the modalities $!,?$ (called *of-course* and *why-not*, respectively). *Lin-*

$$\mathsf{id}\ \frac{}{\vdash A, A^{\perp}} \qquad \mathsf{cut}\ \frac{\vdash A, \Phi \quad \vdash A^{\perp}, \Psi}{\vdash \Phi, \Psi}$$

$$\otimes\ \frac{\vdash A, \Phi \quad \vdash B, \Psi}{\vdash A \otimes B, \Phi, \Psi} \qquad \parr\ \frac{\vdash A, B, \Phi}{\vdash A \parr B, \Phi} \qquad \perp\ \frac{\vdash \Phi}{\vdash \perp, \Phi} \qquad \mathsf{1}\ \frac{}{\vdash \mathsf{1}}$$

$$\&\ \frac{\vdash A, \Phi \quad \vdash B, \Phi}{\vdash A \,\&\, B, \Phi} \qquad \oplus_1\ \frac{\vdash A, \Phi}{\vdash A \oplus B, \Phi} \qquad \oplus_2\ \frac{\vdash B, \Phi}{\vdash A \oplus B, \Phi} \qquad \top\ \frac{}{\vdash \top, \Phi}$$

$$?\mathsf{d}\ \frac{\vdash A, \Phi}{\vdash ?A, \Phi} \qquad ?\mathsf{c}\ \frac{\vdash ?A, ?A, \Phi}{\vdash ?A, \Phi} \qquad ?\mathsf{w}\ \frac{\vdash \Phi}{\vdash ?A, \Phi} \qquad !\ \frac{\vdash A, ?B_1, \ldots, ?B_n}{\vdash !A, ?B_1, \ldots, ?B_n}\ (n \geqslant 0)$$

Figure 4: System LL in the sequent calculus

ear negation $(\cdot)^{\perp}$ is defined on formulae by De Morgan equations:

$$\begin{aligned} \mathsf{1}^{\perp} &:= \perp & \perp^{\perp} &:= \mathsf{1} \\ \top^{\perp} &:= \mathsf{0} & \mathsf{0}^{\perp} &:= \top \\ (a)^{\perp} &:= a^{\perp} & (a^{\perp})^{\perp} &:= a \\ (A \otimes B)^{\perp} &:= A^{\perp} \parr B^{\perp} & (A \parr B)^{\perp} &:= A^{\perp} \otimes B^{\perp} \\ (A \,\&\, B)^{\perp} &:= A^{\perp} \oplus B^{\perp} & (A \oplus B)^{\perp} &:= A^{\perp} \,\&\, B^{\perp} \\ (!A)^{\perp} &:= ?A^{\perp} & (?A)^{\perp} &:= !A^{\perp} \end{aligned}$$

Linear implication $\multimap$ is defined by $A \multimap B = A^{\perp} \parr B$.

It follows immediately from the definition that $A = A^{\perp\perp}$ for each formula A.

3.2 Definition A *sequent* is an expression of the kind

$$\vdash A_1, \ldots, A_h \quad ,$$

where $h \geqslant 0$ and the comma between the formulae $A_1, \ldots, A_h$ stands for multiset union. Multisets of formulae are denoted with Φ and Ψ.

3.3 Definition *Derivations*, are trees where the nodes are sequents to which a finite number (possibly zero) of instances of the inference rules shown in Figure 4 are applied. The sequents in the leaves are called *premises*, and the sequent in the root is the *conclusion*. A derivation with no premises is a *proof*, denoted with Π. A sequent Σ is *provable* if there is a proof Π with conclusion Σ.

3.4 Definition The function $\underline{\cdot}_{\mathsf{s}}$ defines the obvious translation from LL formulae into LS structures:

$$
\begin{array}{rcll}
\underline{a}_{\mathsf{S}} & = & a & , \\
\underline{\bot}_{\mathsf{S}} & = & \bot & , \\
\underline{\mathsf{1}}_{\mathsf{S}} & = & \mathsf{1} & , \\
\underline{\mathsf{0}}_{\mathsf{S}} & = & \mathsf{0} & , \\
\underline{\top}_{\mathsf{S}} & = & \top & , \\
\underline{A \parr B}_{\mathsf{S}} & = & [\underline{A}_{\mathsf{S}}, \underline{B}_{\mathsf{S}}] & , \\
\underline{A \otimes B}_{\mathsf{S}} & = & (\underline{A}_{\mathsf{S}}, \underline{B}_{\mathsf{S}}) & , \\
\underline{A \oplus B}_{\mathsf{S}} & = & [\!\bullet \underline{A}_{\mathsf{S}}, \underline{B}_{\mathsf{S}} \bullet\!] & , \\
\underline{A \mathbin{\&} B}_{\mathsf{S}} & = & (\!\bullet \underline{A}_{\mathsf{S}}, \underline{B}_{\mathsf{S}} \bullet\!) & , \\
\underline{?A}_{\mathsf{S}} & = & ?\underline{A}_{\mathsf{S}} & , \\
\underline{!A}_{\mathsf{S}} & = & !\underline{A}_{\mathsf{S}} & , \\
\underline{A^{\bot}}_{\mathsf{S}} & = & \overline{\underline{A}_{\mathsf{S}}} & .
\end{array}
$$

The domain of $\underline{\cdot}_{\mathsf{S}}$ is extended to sequents by

$$
\begin{array}{rcll}
\underline{\vdash}_{\mathsf{S}} & = & \bot & \text{and} \\
\underline{\vdash A_1, \dots, A_h}_{\mathsf{S}} & = & [\underline{A_1}_{\mathsf{S}}, \dots, \underline{A_h}_{\mathsf{S}}] & , \text{ for } h \geqslant 0 \quad .
\end{array}
$$

3.5 Definition The translation from LS structures into LL formulae is realized by the function $\underline{\cdot}_{\mathsf{L}}$:

$$
\begin{array}{rcll}
\underline{a}_{\mathsf{L}} & = & a & , \\
\underline{\bot}_{\mathsf{L}} & = & \bot & , \\
\underline{\mathsf{1}}_{\mathsf{L}} & = & \mathsf{1} & , \\
\underline{\mathsf{0}}_{\mathsf{L}} & = & \mathsf{0} & , \\
\underline{\top}_{\mathsf{L}} & = & \top & , \\
\underline{[R_1, \dots, R_h]}_{\mathsf{L}} & = & \underline{R_1}_{\mathsf{L}} \parr \cdots \parr \underline{R_h}_{\mathsf{L}} & , \\
\underline{(R_1, \dots, R_h)}_{\mathsf{L}} & = & \underline{R_1}_{\mathsf{L}} \otimes \cdots \otimes \underline{R_h}_{\mathsf{L}} & , \\
\underline{[\!\bullet R_1, \dots, R_h \bullet\!]}_{\mathsf{L}} & = & \underline{R_1}_{\mathsf{L}} \oplus \cdots \oplus \underline{R_h}_{\mathsf{L}} & , \\
\underline{(\!\bullet R_1, \dots, R_h \bullet\!)}_{\mathsf{L}} & = & \underline{R_1}_{\mathsf{L}} \mathbin{\&} \cdots \mathbin{\&} \underline{R_h}_{\mathsf{L}} & , \\
\underline{?R}_{\mathsf{L}} & = & ?\underline{R}_{\mathsf{L}} & , \\
\underline{!R}_{\mathsf{L}} & = & !\underline{R}_{\mathsf{L}} & , \\
\underline{\bar{R}}_{\mathsf{L}} & = & (\underline{R}_{\mathsf{L}})^{\bot} & .
\end{array}
$$

3.6 Theorem *If a given structure R is provable in system $\mathsf{SLS} \cup \{\mathsf{1}{\downarrow}\}$, then its translation $\vdash \underline{R}_{\mathsf{L}}$ is provable in LL (with cut).*

Proof: Suppose, we have a proof Π of R in system $\mathsf{SLS} \cup \{\mathsf{1}{\downarrow}\}$. By induction on the length of Π, let us build a proof $\underline{\Pi}_{\mathsf{L}}$ of $\vdash \underline{R}_{\mathsf{L}}$ in LL.

Base case: Π is $1{\downarrow}\,\dfrac{}{1}$: Let $\underline{\Pi}_{\mathsf{L}}$ be the proof $\mathsf{1}\,\dfrac{}{\vdash \mathsf{1}}$.

Inductive case: Suppose Π is $\begin{array}{c}\Pi' \Big\Vert \mathsf{SLS} \cup \{1{\downarrow}\} \\ \rho\,\dfrac{S\{R\}}{S\{T\}}\end{array}$, i.e. $\rho\,\dfrac{S\{R\}}{S\{T\}}$ is the last rule to be applied in Π. The following LL-proofs show that $\vdash (\underline{R}_{\mathsf{L}})^{\perp}, \underline{T}_{\mathsf{L}}$ is provable in LL for every rule $\rho\,\dfrac{S\{R\}}{S\{T\}}$ in SLS, i.e. $\underline{R}_{\mathsf{L}} \multimap \underline{T}_{\mathsf{L}}$ is a theorem in LL:

$$\parr\,\dfrac{\parr\,\dfrac{\otimes\,\dfrac{\otimes\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash R^{\perp}, R} \qquad \mathsf{id}\,\dfrac{}{\vdash U^{\perp}, U}}{\vdash R^{\perp}, U^{\perp}, R \otimes U} \qquad \mathsf{id}\,\dfrac{}{\vdash T^{\perp}, T}}{\vdash R^{\perp} \otimes T^{\perp}, U^{\perp}, R \otimes U, T}}{\vdash R^{\perp} \otimes T^{\perp}, U^{\perp}, (R \otimes U) \parr T}}{\vdash (R^{\perp} \otimes T^{\perp}) \parr U^{\perp}, (R \otimes U) \parr T} \quad , \qquad \parr\,\dfrac{!\,\dfrac{?\mathsf{d}\,\dfrac{?\mathsf{d}\,\dfrac{\otimes\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash R^{\perp}, R} \qquad \mathsf{id}\,\dfrac{}{\vdash T^{\perp}, T}}{\vdash R^{\perp} \otimes T^{\perp}, R, T}}{\vdash ?(R^{\perp} \otimes T^{\perp}), R, T}}{\vdash ?(R^{\perp} \otimes T^{\perp}), ?R, T}}{\vdash ?(R^{\perp} \otimes T^{\perp}), ?R, !T}}{\vdash ?(R^{\perp} \otimes T^{\perp}), ?R \parr !T} \quad ,$$

$$\bot\,\dfrac{\parr\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash a, a^{\perp}}}{\vdash a \parr a^{\perp}}}{\vdash \bot, a \parr a^{\perp}} \quad , \qquad \top\,\dfrac{}{\vdash \top, R} \quad , \qquad ?\mathsf{w}\,\dfrac{\mathsf{1}\,\dfrac{}{\vdash \mathsf{1}}}{\vdash \mathsf{1}, ?R} \quad ,$$

$$\parr\,\dfrac{\&\,\dfrac{\oplus\,\dfrac{\oplus\,\dfrac{\otimes\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash R^{\perp}, R} \qquad \mathsf{id}\,\dfrac{}{\vdash T^{\perp}, T}}{\vdash R^{\perp} \otimes T^{\perp}, R, T}}{\vdash R^{\perp} \otimes T^{\perp}, R, T \oplus V}}{\vdash (R^{\perp} \otimes T^{\perp}) \oplus (U^{\perp} \otimes V^{\perp}), R, T \oplus V} \qquad \oplus\,\dfrac{\oplus\,\dfrac{\otimes\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash U^{\perp}, U} \qquad \mathsf{id}\,\dfrac{}{\vdash V^{\perp}, V}}{\vdash U^{\perp} \otimes V^{\perp}, U, V}}{\vdash U^{\perp} \otimes V^{\perp}, U, T \oplus V}}{\vdash (R^{\perp} \otimes T^{\perp}) \oplus (U^{\perp} \otimes V^{\perp}), U, T \oplus V}}{\vdash (R^{\perp} \otimes T^{\perp}) \oplus (U^{\perp} \otimes V^{\perp}), R \,\&\, U, T \oplus V}}{\vdash (R^{\perp} \otimes T^{\perp}) \oplus (U^{\perp} \otimes V^{\perp}), (R \,\&\, U) \parr (T \oplus V)} \quad ,$$

$$\&\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash R^{\perp}, R} \qquad \mathsf{id}\,\dfrac{}{\vdash R^{\perp}, R}}{\vdash R^{\perp} \,\&\, R^{\perp}, R} \quad , \qquad ?\mathsf{c}\,\dfrac{\otimes\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash !R^{\perp}, ?R} \qquad ?\mathsf{d}\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash R^{\perp}, R}}{\vdash R^{\perp}, ?R}}{\vdash (!R^{\perp} \otimes R^{\perp}), ?R, ?R}}{\vdash (!R^{\perp} \otimes R^{\perp}), ?R} \quad .$$

This means that for any context $S\{\ \}$, we also have that $\underline{S\{R\}}_{\mathsf{L}} \multimap \underline{S\{T\}}_{\mathsf{L}}$ is a theorem in LL, i.e. $\vdash (\underline{S\{R\}}_{\mathsf{L}})^{\perp}, \underline{S\{T\}}_{\mathsf{L}}$ is provable in LL. By induction hypothesis we have a proof $\underline{\Pi'}_{\mathsf{L}}$ of $\vdash \underline{S\{R\}}_{\mathsf{L}}$ in LL. Now we can get a proof $\underline{\Pi}_{\mathsf{L}}$ of $\vdash \underline{S\{T\}}_{\mathsf{L}}$ by applying the cut rule:

$$\mathsf{cut}\,\dfrac{\vdash \underline{S\{R\}}_{\mathsf{L}} \qquad \vdash (\underline{S\{R\}}_{\mathsf{L}})^{\perp}, \underline{S\{T\}}_{\mathsf{L}}}{\vdash \underline{S\{T\}}_{\mathsf{L}}} \quad .$$

□

3.7 Theorem (a) *If a given sequent $\vdash \Phi$ is provable in* LL *(with cut), then the structure* $\underline{\vdash \Phi}_{\mathsf{s}}$ *is provable in system* $\mathsf{SLS} \cup \{1\!\downarrow\}$. (b) *If a given sequent* $\vdash \Phi$ *is cut-free provable in* LL*, then the structure* $\underline{\vdash \Phi}_{\mathsf{s}}$ *is provable in system* LS.

Proof: Let Π be the proof of $\vdash \Phi$ in LL. By structural induction on Π, we will construct a proof $\underline{\Pi}_{\mathsf{s}}$ of $\underline{\vdash \Phi}_{\mathsf{s}}$ in system $\mathsf{SLS} \cup \{1\!\downarrow\}$ (or system LS if Π is cut-free).

- If Π is $\mathsf{id}\,\dfrac{}{\vdash A, A^{\perp}}$ for some formula A, then let $\underline{\Pi}_{\mathsf{s}}$ be the proof obtained via Proposition 2.11 from

$$\mathsf{i}\!\downarrow \dfrac{1\!\downarrow \dfrac{}{1}}{[\underline{A}_{\mathsf{s}}, \overline{\underline{A}_{\mathsf{s}}}]} \quad .$$

- If $\mathsf{cut}\,\dfrac{\vdash A, \Phi \quad \vdash A^{\perp}, \Psi}{\vdash \Phi, \Psi}$ is the last rule applied in Π, then there are by induction hypothesis two derivations $\begin{array}{c} 1 \\ \Delta_1 \| \mathsf{SLS} \\ [\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}] \end{array}$ and $\begin{array}{c} 1 \\ \Delta_2 \| \mathsf{SLS} \\ [\overline{\underline{A}_{\mathsf{s}}}, \underline{\Psi}_{\mathsf{s}}] \end{array}$. Let $\underline{\Pi}_{\mathsf{s}}$ be the proof obtained via Proposition 2.11 from

$$\mathsf{i}\!\uparrow \dfrac{\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{\begin{array}{c} 1\!\downarrow \dfrac{}{1} \\ \Delta_1' \| \mathsf{SLS} \\ [\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}] \\ \Delta_2' \| \mathsf{SLS} \\ ([\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}], [\overline{\underline{A}_{\mathsf{s}}}, \underline{\Psi}_{\mathsf{s}}]) \end{array}}{[([\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}], \overline{\underline{A}_{\mathsf{s}}}), \underline{\Psi}_{\mathsf{s}}]}}{[\underline{\Phi}_{\mathsf{s}}, \underline{\Psi}_{\mathsf{s}}, (\underline{A}_{\mathsf{s}}, \overline{\underline{A}_{\mathsf{s}}})]}}{[\underline{\Phi}_{\mathsf{s}}, \underline{\Psi}_{\mathsf{s}}]} \quad .$$

- If $\parr\,\dfrac{\vdash A, B, \Phi}{\vdash A \parr B, \Phi}$ is the last rule applied in Π, then let $\underline{\Pi}_{\mathsf{s}}$ be the proof of $[\underline{A}_{\mathsf{s}}, \underline{B}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}]$ that exists by induction hypothesis.

- If $\otimes\,\dfrac{\vdash A, \Phi \quad \vdash B, \Psi}{\vdash A \otimes B, \Phi, \Psi}$ is the last rule applied in Π, then there are by induction hypothesis two derivations $\begin{array}{c} 1 \\ \Delta_1 \Vert \mathsf{SLS} \\ [\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}] \end{array}$ and $\begin{array}{c} 1 \\ \Delta_2 \Vert \mathsf{SLS} \\ [\underline{B}_{\mathsf{s}}, \underline{\Psi}_{\mathsf{s}}] \end{array}$. Let $\underline{\Pi}_{\mathsf{s}}$ be the proof

$$\begin{array}{c} 1\!\downarrow \dfrac{}{1} \\ \Delta_1' \Vert \mathsf{SLS} \\ [\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}] \\ \Delta_2' \Vert \mathsf{SLS} \\ \mathsf{s}\,\dfrac{([\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}], [\underline{B}_{\mathsf{s}}, \underline{\Psi}_{\mathsf{s}}])}{[([\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}], \underline{B}_{\mathsf{s}}), \underline{\Psi}_{\mathsf{s}}]} \\ \mathsf{s}\,\dfrac{}{[(\underline{A}_{\mathsf{s}}, \underline{B}_{\mathsf{s}}), \underline{\Phi}_{\mathsf{s}}, \underline{\Psi}_{\mathsf{s}}]} \end{array} \quad .$$

- If $\bot \dfrac{\vdash \Phi}{\vdash \bot, \Phi}$ is the last rule applied in Π, then let $\underline{\Pi}_{\mathsf{s}}$ be the proof of $\underline{\vdash \Phi}_{\mathsf{s}}$ that exists by induction hypothesis.
- If Π is $1 \dfrac{}{\vdash 1}$, then let $\underline{\Pi}_{\mathsf{s}}$ be $1\!\downarrow \dfrac{}{1}$.
- If $\& \dfrac{\vdash A, \Phi \quad \vdash B, \Phi}{\vdash A \mathbin{\&} B, \Phi}$ is the last rule applied in Π, then there are by induction hypothesis two derivations $\begin{array}{c} 1 \\ \Delta_1 \Vert \mathsf{SLS} \\ [\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}] \end{array}$ and $\begin{array}{c} 1 \\ \Delta_2 \Vert \mathsf{SLS} \\ [\underline{B}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}] \end{array}$. Let $\underline{\Pi}_{\mathsf{s}}$ be the proof

$$\begin{array}{c} 1\!\downarrow \dfrac{}{1} \\ \Delta_1' \Vert \mathsf{SLS} \\ \lBrace [\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}], 1 \rBrace \\ \Delta_2' \Vert \mathsf{SLS} \\ \mathsf{d}\!\downarrow \dfrac{\lBrace [\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}], [\underline{B}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}] \rBrace}{[\lBrace \underline{A}_{\mathsf{s}}, \underline{B}_{\mathsf{s}} \rBrace, \lBrace \underline{\Phi}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}} \rBrace]} \\ \mathsf{c}\!\downarrow \dfrac{}{[\lBrace \underline{A}_{\mathsf{s}}, \underline{B}_{\mathsf{s}} \rBrace, \underline{\Phi}_{\mathsf{s}}]} \end{array} \quad .$$

(Note that $1 = \lBrace 1, 1 \rBrace$.)

- If $\oplus_1 \dfrac{\vdash A, \Phi}{\vdash A \oplus B, \Phi}$ is the last rule applied in Π, then let $\underline{\Pi}_{\mathsf{s}}$ be the proof

$$
\mathsf{w}{\downarrow}\,\dfrac{\begin{array}{c}\Pi' \Big\Vert \mathsf{SLS}\cup\{\mathsf{1}{\downarrow}\} \\ {[\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}]}\end{array}}{[\{\!\bullet\,\underline{A}_{\mathsf{s}}, \underline{B}_{\mathsf{s}}\,\bullet\!\}, \underline{\Phi}_{\mathsf{s}}]} \quad ,
$$

where Π' exists by induction hypothesis.

- The case for the rule $\oplus_2 \dfrac{\vdash B, \Phi}{\vdash A \oplus B, \Phi}$ is similar.

- If Π is $\top \dfrac{}{\vdash \top, \Phi}$, then let $\underline{\Pi}_{\mathsf{s}}$ be the proof

$$
\mathsf{w}{\downarrow}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{\mathsf{1}{\downarrow}\,\dfrac{}{\mathsf{1}}}{[\top, \mathsf{0}]}}{[\top, \underline{\Phi}_{\mathsf{s}}]} \quad .
$$

- If $?\mathsf{d} \dfrac{\vdash A, \Phi}{\vdash ?A, \Phi}$ is the last rule applied in Π, then let $\underline{\Pi}_{\mathsf{s}}$ be the proof

$$
\mathsf{b}{\downarrow}\,\dfrac{\mathsf{v}{\downarrow}\,\dfrac{\begin{array}{c}\Pi' \Big\Vert \mathsf{SLS}\cup\{\mathsf{1}{\downarrow}\} \\ {[\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}]}\end{array}}{[?\underline{A}_{\mathsf{s}}, \underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}]}}{[?\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}]} \quad ,
$$

where Π' exists by induction hypothesis.

- If $?\mathsf{c} \dfrac{\vdash ?A, ?A, \Phi}{\vdash ?A, \Phi}$ is the last rule applied in Π, then let $\underline{\Pi}_{\mathsf{s}}$ be the proof

$$
\mathsf{b}{\downarrow}\,\dfrac{\begin{array}{c}\Pi' \Big\Vert \mathsf{SLS}\cup\{\mathsf{1}{\downarrow}\} \\ {[??\underline{A}_{\mathsf{s}}, ?\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}]}\end{array}}{[??\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}]} \quad ,
$$

where Π' exists by induction hypothesis. (Note that $??\underline{A}_{\mathsf{s}} = ?\underline{A}_{\mathsf{s}}$.)

- If $?\mathsf{w} \dfrac{\vdash \Phi}{\vdash ?A, \Phi}$ is the last rule applied in Π, then let $\underline{\Pi}_{\mathsf{s}}$ be the proof

$$
\mathsf{v}{\downarrow}\,\dfrac{\begin{array}{c}\Pi' \Big\Vert \mathsf{SLS}\cup\{\mathsf{1}{\downarrow}\} \\ \underline{\Phi}_{\mathsf{s}}\end{array}}{[?\underline{A}_{\mathsf{s}}, \underline{\Phi}_{\mathsf{s}}]} \quad ,
$$

where Π' exists by induction hypothesis.

- If $!\,\dfrac{\vdash A, ?B_1, \dots, ?B_n}{\vdash !A, ?B_1, \dots, ?B_n}$ is the last rule applied in Π, then there is by induction hypothesis a derivation $\begin{array}{c} 1 \\ \Delta \Vert \mathsf{SLS} \\ [\underline{A}_{\mathsf{s}}, ?\underline{B_1}_{\mathsf{s}}, \dots, ?\underline{B_n}_{\mathsf{s}}] \end{array}$. Now let $\underline{\Pi}_{\mathsf{s}}$ be the proof

$$
\begin{array}{c}
\mathsf{1}\!\downarrow \dfrac{}{!\mathsf{1}} \\
\Delta' \Vert \mathsf{SLS} \\
\mathsf{p}\!\downarrow \dfrac{![\underline{A}_{\mathsf{s}}, ?\underline{B_1}_{\mathsf{s}}, \dots, ?\underline{B_n}_{\mathsf{s}}]}{\vdots} \\
\mathsf{p}\!\downarrow \dfrac{}{[![\underline{A}_{\mathsf{s}}, ?\underline{B_1}_{\mathsf{s}}], ??\underline{B_2}_{\mathsf{s}}, \dots, ??\underline{B_n}_{\mathsf{s}}]} \\
\mathsf{p}\!\downarrow \dfrac{}{[!\underline{A}_{\mathsf{s}}, ??\underline{B_1}_{\mathsf{s}}, ??\underline{B_2}_{\mathsf{s}}, \dots, ??\underline{B_n}_{\mathsf{s}}]}
\end{array} \quad .
$$

□

4 Cut Elimination

By inspecting the rules of system SLS, one can observe that the only infinitary rules are atomic cut, coweakening and covanishing. This means, that in order to get cut elimination, one could get rid only of the rules $\mathsf{ai}\!\uparrow$, $\mathsf{w}\!\uparrow$ and $\mathsf{v}\!\uparrow$. But we can get more: the whole up-fragment is admissible.

4.1 Definition A rule ρ is *(weakly) admissible* for a system $\mathscr{S}$ if $\rho \notin \mathscr{S}$ and for every proof $\begin{array}{c} \Pi \Vert \mathscr{S} \cup \{\rho\} \\ R \end{array}$ there is a proof $\begin{array}{c} \Pi' \Vert \mathscr{S} \\ R \end{array}$.

4.2 Definition Two systems $\mathscr{S}$ and $\mathscr{S}'$ are *(weakly) equivalent* if for every proof $\begin{array}{c} \Pi \Vert \mathscr{S} \\ R \end{array}$ there is a proof $\begin{array}{c} \Pi' \Vert \mathscr{S}' \\ R \end{array}$, and vice versa.

4.3 Theorem (Cut elimination) *System* LS *is equivalent to every subsystem of* $\mathsf{SLS} \cup \{\mathsf{1}\!\downarrow\}$ *containing* LS.

Proof: Given a proof in $\mathsf{SLS} \cup \{\mathsf{1}\!\downarrow\}$, transform it into a proof in LL (by Theorem 3.6), to which we can apply the cut-elimination procedure in the sequent calculus. The cut-free proof in LL can then be transformed into a proof in system LS by Theorem 3.7. □

4.4 Corollary *The rule* $\mathsf{i}\!\uparrow$ *is admissible for system* LS.

Proof: Immediate consequence of Theorems 2.17 and 4.3. □

The proof of Theorem 4.3 relies on the results of the previous section together with the well-known cut elimination proof for linear logic in the sequent calculus. But it should be mentioned here that it is also possible to prove Theorem 4.3 directly inside

$$
\text{ai}\!\downarrow \frac{S\{1\}}{S[a,\bar{a}]} \qquad \text{ai}\!\uparrow \frac{S(a,\bar{a})}{S\{\bot\}}
$$

$$
\text{s}\,\frac{S([R,U],T)}{S[(R,T),U]}
$$

$$
\text{d}\!\downarrow \frac{S{\bullet\!(}[R,U],[T,V]{)\!\bullet}}{S[{\bullet\!(}R,T{)\!\bullet},{\bullet\![}U,V{]\!\bullet}]} \qquad \text{d}\!\uparrow \frac{S({\bullet\![}R,U{]\!\bullet},{\bullet\!(}T,V{)\!\bullet})}{S{\bullet\![}(R,T),(U,V){]\!\bullet}}
$$

$$
\text{w}\!\downarrow \frac{S\{0\}}{S\{R\}} \qquad \text{w}\!\uparrow \frac{S\{R\}}{S\{\top\}}
$$

$$
\text{c}\!\downarrow \frac{S{\bullet\![}R,R{]\!\bullet}}{S\{R\}} \qquad \text{c}\!\uparrow \frac{S\{R\}}{S{\bullet\!(}R,R{)\!\bullet}}
$$

Figure 5: System SALS

the calculus of structures, without using the sequent calculus. This proof uses the technique of *splitting*, which has first been used in [7]. However, I will not present this proof here because it is rather technical and boring.

5 A Local System for the Multiplicative Additive Fragment

In this section, I will start from system SALS (the multiplicative additive fragment of system SLS, shown in Figure 5) and produce a strongly equivalent system, in which all rules are local. The new system will also admit a cut elimination result.

In system SALS, the only rules which are not local are $\text{w}\!\downarrow$, $\text{w}\!\uparrow$, $\text{c}\!\downarrow$ and $\text{c}\!\uparrow$. For all other rules, it is not necessary to perform an erasing, introducing, copying or comparing of a structure of unknown size when they are applied. In order to obtain a system in which all rules are local, we have to restrict the rules $\text{w}\!\downarrow$, $\text{w}\!\uparrow$, $\text{c}\!\downarrow$ and $\text{c}\!\uparrow$ to atoms.

5.1 Definition The structural rules

$$
\text{aw}\!\downarrow \frac{S\{0\}}{S\{a\}} \quad , \qquad \text{aw}\!\uparrow \frac{S\{a\}}{S\{\top\}} \quad , \qquad \text{ac}\!\downarrow \frac{S{\bullet\![}a,a{]\!\bullet}}{S\{a\}} \qquad \text{and} \qquad \text{ac}\!\uparrow \frac{S\{a\}}{S{\bullet\!(}a,a{)\!\bullet}}
$$

are called *atomic weakening* ($\text{aw}\!\downarrow$), *atomic coweakening* ($\text{aw}\!\uparrow$), *atomic contraction* ($\text{ac}\!\downarrow$) and *atomic cocontraction* ($\text{ac}\!\uparrow$), respectively.

Obviously, the rules $\text{aw}\!\downarrow, \text{aw}\!\uparrow, \text{ac}\!\downarrow, \text{ac}\!\uparrow$ are instances of the rules $\text{w}\!\downarrow, \text{w}\!\uparrow, \text{c}\!\downarrow, \text{c}\!\uparrow$. But if we replace the rules $\mathsf{w}\!\downarrow, \mathsf{w}\!\uparrow, \mathsf{c}\!\downarrow, \mathsf{c}\!\uparrow$ by their atomic counterparts, we lose equivalence to the original system. In order to regain this eqivalence, we have to add further rules to the system. In particular, the rules

$$\mathsf{l}\!\downarrow \frac{S\{\!\!\{[R,U],[T,V]\}\!\!\}}{S[\{\!\!\{R,T\}\!\!\},\{\!\!\{U,V\}\!\!\}]} \quad , \quad \mathsf{l}\!\uparrow \frac{S((\!\!(R,U)\!\!),(\!\!(T,V)\!\!))}{S(\!\!((R,T),(U,V))\!\!)} \quad ,$$

$$\mathsf{k}\!\downarrow \frac{S\{\!\!\{(R,U),(T,V)\}\!\!\}}{S(\{\!\!\{R,T\}\!\!\},\{\!\!\{U,V\}\!\!\})} \quad , \quad \mathsf{k}\!\uparrow \frac{S[(\!\!(R,U)\!\!),(\!\!(T,V)\!\!)]}{S(\!\!([R,T],[U,V])\!\!)} \quad , \quad \mathsf{m} \frac{S\{\!\!\{(\!\!(R,U)\!\!),(\!\!(T,V)\!\!)\}\!\!\}}{S(\!\!(\{\!\!\{R,T\}\!\!\},\{\!\!\{U,V\}\!\!\})\!\!)} \quad ,$$

which are in the same spirit as the medial rule of [1], are necessary to reduce $\mathsf{c}\!\downarrow$ and $\mathsf{c}\!\uparrow$ to their atomic versions $\mathsf{ac}\!\downarrow$ and $\mathsf{ac}\!\uparrow$, respectively, in the same way as it has been done for cut and interaction in Section 2.

5.2 Proposition *The rule* $\mathsf{c}\!\downarrow$ *is stongly admissible for* $\{\mathsf{ac}\!\downarrow, \mathsf{l}\!\downarrow, \mathsf{k}\!\downarrow, \mathsf{m}\}$. *Dually, the rule* $\mathsf{c}\!\uparrow$ *is stongly admissible for* $\{\mathsf{ac}\!\uparrow, \mathsf{l}\!\uparrow, \mathsf{k}\!\uparrow, \mathsf{m}\}$.

Proof: For a given instance $\mathsf{c}\!\downarrow \frac{S\{\!\!\{R,R\}\!\!\}}{S\{R\}}$, I will construct a derivation $\begin{array}{c} S\{\!\!\{R,R\}\!\!\} \\ \Delta \Vert \{\mathsf{ac}\downarrow,\mathsf{l}\downarrow,\mathsf{k}\downarrow,\mathsf{m}\} \\ S\{R\} \end{array}$ by structural induction on R.

- If R is an atom (or constant) then the given instance of $\mathsf{c}\!\downarrow$ is an instance of $\mathsf{ac}\!\downarrow$.
- If $R = \{\!\!\{P,Q\}\!\!\}$ (where $P \neq 0 \neq Q$), then apply the induction hypothesis to

$$\mathsf{c}\!\downarrow \frac{S\{\!\!\{P,Q,P,Q\}\!\!\}}{\mathsf{c}\!\downarrow \dfrac{S\{\!\!\{P,P,Q\}\!\!\}}{S\{\!\!\{P,Q\}\!\!\}}} \quad .$$

- If $R = [P,Q]$ (where $P \neq \bot \neq Q$), then apply the induction hypothesis to

$$\mathsf{l}\!\downarrow \frac{S\{\!\!\{[P,Q],[P,Q]\}\!\!\}}{\mathsf{c}\!\downarrow \dfrac{S[\{\!\!\{P,P\}\!\!\},\{\!\!\{Q,Q\}\!\!\}]}{\mathsf{c}\!\downarrow \dfrac{S[\{\!\!\{P,P\}\!\!\},Q]}{S[P,Q]}}} \quad .$$

- If $R = (P,Q)$ (where $P \neq 1 \neq Q$), then apply the induction hypothesis to

$$\mathsf{k}\!\downarrow \frac{S\{\!\!\{(P,Q),(P,Q)\}\!\!\}}{\mathsf{c}\!\downarrow \dfrac{S(\{\!\!\{P,P\}\!\!\},\{\!\!\{Q,Q\}\!\!\})}{\mathsf{c}\!\downarrow \dfrac{S(\{\!\!\{P,P\}\!\!\},Q)}{S(P,Q)}}} \quad .$$

- If $R = (\!\!(P,Q)\!\!)$ (where $P \neq \top \neq Q$), then apply the induction hypothesis to

$$\mathsf{m} \frac{S\{\!\!\{(\!\!(P,Q)\!\!),(\!\!(P,Q)\!\!)\}\!\!\}}{\mathsf{c}\!\downarrow \dfrac{S(\!\!(\{\!\!\{P,P\}\!\!\},\{\!\!\{Q,Q\}\!\!\})\!\!)}{\mathsf{c}\!\downarrow \dfrac{S(\!\!(\{\!\!\{P,P\}\!\!\},Q)\!\!)}{S(\!\!(P,Q)\!\!)}}} \quad .$$

The proof for $\mathsf{c}{\uparrow}$ is dual. □

For reducing the rules $\mathsf{w}{\downarrow}$ and $\mathsf{w}{\uparrow}$ to their atomic versions, the following rules are needed:

$$\mathsf{l_0}{\downarrow}\,\frac{S\{\mathsf{0}\}}{S[\mathsf{0},\mathsf{0}]} \quad , \quad \mathsf{k_0}{\downarrow}\,\frac{S\{\mathsf{0}\}}{S(\mathsf{0},\mathsf{0})} \quad , \quad \mathsf{m_0}{\downarrow}\,\frac{S\{\mathsf{0}\}}{S\llparenthesis \mathsf{0},\mathsf{0}\rrparenthesis} \quad ,$$

$$\mathsf{l_0}{\uparrow}\,\frac{S(\top,\top)}{S\{\top\}} \quad , \quad \mathsf{k_0}{\uparrow}\,\frac{S[\top,\top]}{S\{\top\}} \quad , \quad \mathsf{m_0}{\uparrow}\,\frac{S\llbracket \top,\top\rrbracket}{S\{\top\}} \quad .$$

They can be considered to be the nullary versions of the rules $\mathsf{l}{\downarrow}, \mathsf{l}{\uparrow}, \mathsf{k}{\downarrow}, \mathsf{k}{\uparrow}, \mathsf{m}$ above.

5.3 Proposition *The rule* $\mathsf{w}{\downarrow}$ *is stongly admissible for* $\{\mathsf{aw}{\downarrow}, \mathsf{l_0}{\downarrow}, \mathsf{k_0}{\downarrow}, \mathsf{m_0}{\downarrow}\}$*. Dually, the rule* $\mathsf{w}{\uparrow}$ *is stongly admissible for* $\{\mathsf{aw}{\uparrow}, \mathsf{l_0}{\uparrow}, \mathsf{k_0}{\uparrow}, \mathsf{m_0}{\uparrow}\}$*.*

Proof: For a given instance $\mathsf{w}{\downarrow}\,\dfrac{S\{\mathsf{0}\}}{S\{R\}}$, I will construct a derivation $\begin{array}{c} S\{\mathsf{0}\} \\ \Delta \Big\Vert \{\mathsf{aw}{\downarrow},\mathsf{l_0}{\downarrow},\mathsf{k_0}{\downarrow},\mathsf{m_0}{\downarrow}\} \\ S\{R\} \end{array}$ by structural induction on R.

- If R is an atom then the given instance of $\mathsf{w}{\downarrow}$ is an instance of $\mathsf{aw}{\downarrow}$.
- If $R = [P, Q]$ (where $P \neq \bot \neq Q$), then apply the induction hypothesis to

$$\begin{array}{rc} \mathsf{l_0}{\downarrow} & S\{\mathsf{0}\} \\ \cline{2-2} \mathsf{w}{\downarrow} & S[\mathsf{0},\mathsf{0}] \\ \cline{2-2} \mathsf{w}{\downarrow} & S[P,\mathsf{0}] \\ \cline{2-2} & S[P,Q] \end{array} \quad .$$

- If $R = (P, Q)$ (where $P \neq \mathsf{1} \neq Q$), then apply the induction hypothesis to

$$\begin{array}{rc} \mathsf{k_0}{\downarrow} & S\{\mathsf{0}\} \\ \cline{2-2} \mathsf{w}{\downarrow} & S(\mathsf{0},\mathsf{0}) \\ \cline{2-2} \mathsf{w}{\downarrow} & S(P,\mathsf{0}) \\ \cline{2-2} & S(P,Q) \end{array} \quad .$$

- If $R = \llparenthesis P, Q\rrparenthesis$ (where $P \neq \top \neq Q$), then apply the induction hypothesis to

$$\begin{array}{rc} \mathsf{m_0}{\downarrow} & S\{\mathsf{0}\} \\ \cline{2-2} \mathsf{w}{\downarrow} & S\llparenthesis \mathsf{0},\mathsf{0}\rrparenthesis \\ \cline{2-2} \mathsf{w}{\downarrow} & S\llparenthesis P,\mathsf{0}\rrparenthesis \\ \cline{2-2} & S\llparenthesis P,Q\rrparenthesis \end{array} \quad .$$

- If $R = \llbracket P, Q\rrbracket$ (where $P \neq \mathsf{0} \neq Q$), then apply the induction hypothesis to

$$\begin{array}{rc} \mathsf{w}{\downarrow} & S\{\mathsf{0}\} \\ \cline{2-2} \mathsf{w}{\downarrow} & S\llbracket P,\mathsf{0}\rrbracket \\ \cline{2-2} & S\llbracket P,Q\rrbracket \end{array} \quad .$$

  (Note that $\mathsf{0} = \llbracket \mathsf{0},\mathsf{0}\rrbracket$.)

The proof for $\mathsf{w}\!\uparrow$ is dual. □

Of course, all new rules are strongly admissible for system SALS, i.e. represent sound implications in linear logic. More precisely, we have the following:

5.4 Proposition *The rules $\mathsf{l}\!\downarrow, \mathsf{k}\!\downarrow$ and m are strongly admissible for $\{\mathsf{w}\!\downarrow, \mathsf{c}\!\downarrow\}$. Dually, the rules $\mathsf{l}\!\uparrow, \mathsf{k}\!\uparrow$ and m are strongly admissible for $\{\mathsf{w}\!\uparrow, \mathsf{c}\!\uparrow\}$.*

Proof: For the rule $\mathsf{l}\!\downarrow$, use the derivation

$$
\mathsf{c}\!\downarrow\frac{\mathsf{w}\!\downarrow\frac{\mathsf{w}\!\downarrow\frac{\mathsf{w}\!\downarrow\frac{\mathsf{w}\!\downarrow\frac{S\{\!\!\bullet[R,U],[T,V]\bullet\!\!\}}{S\{\!\!\bullet[R,U],[T,\{\!\!\bullet U,V\bullet\!\!\}]\bullet\!\!\}}}{S\{\!\!\bullet[R,U],[\{\!\!\bullet R,T\bullet\!\!\},\{\!\!\bullet U,V\bullet\!\!\}]\bullet\!\!\}}}{S\{\!\!\bullet[R,\{\!\!\bullet U,V\bullet\!\!\}],[\{\!\!\bullet R,T\bullet\!\!\},\{\!\!\bullet U,V\bullet\!\!\}]\bullet\!\!\}}}{S\{\!\!\bullet[\{\!\!\bullet R,T\bullet\!\!\},\{\!\!\bullet U,V\bullet\!\!\}],[\{\!\!\bullet R,T\bullet\!\!\},\{\!\!\bullet U,V\bullet\!\!\}]\bullet\!\!\}}}{S[\{\!\!\bullet R,T\bullet\!\!\},\{\!\!\bullet U,V\bullet\!\!\}]} \quad .
$$

The other cases are similar. □

5.5 Proposition *The rules $\mathsf{l}_0\!\downarrow, \mathsf{k}_0\!\downarrow$ and $\mathsf{m}_0\!\downarrow$ are strongly admissible for $\{\mathsf{w}\!\downarrow\}$. Dually, the rules $\mathsf{l}_0\!\uparrow, \mathsf{k}_0\!\uparrow$ and $\mathsf{m}_0\!\uparrow$ are strongly admissible for $\{\mathsf{w}\!\uparrow\}$.*

Proof: Trivial. □

5.6 Definition The system

$$
\{\mathsf{ai}\!\downarrow, \mathsf{ai}\!\uparrow, \mathsf{s}, \mathsf{d}\!\downarrow, \mathsf{d}\!\uparrow, \mathsf{aw}\!\downarrow, \mathsf{aw}\!\uparrow, \mathsf{ac}\!\downarrow, \mathsf{ac}\!\uparrow, \mathsf{l}\!\downarrow, \mathsf{l}\!\uparrow, \mathsf{k}\!\downarrow, \mathsf{k}\!\uparrow, \mathsf{m}, \mathsf{l}_0\!\downarrow, \mathsf{l}_0\!\uparrow, \mathsf{k}_0\!\downarrow, \mathsf{k}_0\!\uparrow, \mathsf{m}_0\!\downarrow, \mathsf{m}_0\!\uparrow\} \quad ,
$$

shown in Figure 6 is called *system* SALLS.

5.7 Theorem *Systems SALLS and SALS are strongly equivalent.*

Proof: Immediate consequence of Propositions 5.2, 5.3, 5.4 and 5.5. □

5.8 Definition System ALLS, shown in Figure 7, is obained from the down-fragment of System SALLS together with the rules s, m and the axiom $1\!\downarrow$.

5.9 Theorem *Systems ALLS and ALS are strongly equivalent.*

5.10 Corollary (Cut elimination) *System ALLS is equivalent to every subsystem of $\mathsf{SALLS} \cup \{1\!\downarrow\}$ containing ALLS.*

5.11 Corollary *The rule $\mathsf{i}\!\uparrow$ is admissible for system ALLS.*

6 A Local System for Full Linear Logic

In this section, I will extend the systems SALLS and ALLS to full propositional linear logic. This means that the rules $\mathsf{v}\!\downarrow, \mathsf{v}\!\uparrow, \mathsf{b}\!\downarrow, \mathsf{b}\!\uparrow$ have to be made local. However, I was not able to reduce the rules $\mathsf{v}\!\downarrow, \mathsf{v}\!\uparrow, \mathsf{b}\!\downarrow, \mathsf{b}\!\uparrow$ directly to their atomic versions. This is not really a surprise since the basic idea of the the exponentials is to guard whole subformulas such that no arbitrary weakening and contraction is possible. The idea that is applied in this section is to reduce the rules $\mathsf{v}\!\downarrow, \mathsf{v}\!\uparrow, \mathsf{b}\!\downarrow, \mathsf{b}\!\uparrow$ to the rules

$$\mathsf{ai}{\downarrow}\,\frac{S\{1\}}{S[a,\bar{a}]} \qquad \mathsf{ai}{\uparrow}\,\frac{S(a,\bar{a})}{S\{\bot\}}$$

$$\mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{d}{\downarrow}\,\frac{S(\!\mid[R,U],[T,V]\mid\!)}{S[(\!\mid R,T\mid\!),\{\!\mid U,V\mid\!\}]} \qquad \mathsf{d}{\uparrow}\,\frac{S(\{\!\mid R,U\mid\!\},(\!\mid T,V\mid\!))}{S\{\!\mid(R,T),(U,V)\mid\!\}}$$

$$\mathsf{aw}{\downarrow}\,\frac{S\{0\}}{S\{a\}} \qquad \mathsf{ac}{\downarrow}\,\frac{S\{\!\mid a,a\mid\!\}}{S\{a\}} \qquad \mathsf{ac}{\uparrow}\,\frac{S\{a\}}{S(\!\mid a,a\mid\!)} \qquad \mathsf{aw}{\uparrow}\,\frac{S\{a\}}{S\{\top\}}$$

$$\mathsf{l}_0{\downarrow}\,\frac{S\{0\}}{S[0,0]} \qquad \mathsf{l}{\downarrow}\,\frac{S\{\!\mid[R,U],[T,V]\mid\!\}}{S[\{\!\mid R,T\mid\!\},\{\!\mid U,V\mid\!\}]} \qquad \mathsf{l}{\uparrow}\,\frac{S((\!\mid R,U\mid\!),(\!\mid T,V\mid\!))}{S(\!\mid(R,T),(U,V)\mid\!)} \qquad \mathsf{l}_0{\uparrow}\,\frac{S(\top,\top)}{S\{\top\}}$$

$$\mathsf{k}_0{\downarrow}\,\frac{S\{0\}}{S(0,0)} \qquad \mathsf{k}{\downarrow}\,\frac{S\{\!\mid(R,U),(T,V)\mid\!\}}{S(\{\!\mid R,T\mid\!\},\{\!\mid U,V\mid\!\})} \qquad \mathsf{k}{\uparrow}\,\frac{S[(\!\mid R,U\mid\!),(\!\mid T,V\mid\!)]}{S(\!\mid[R,T],[U,V]\mid\!)} \qquad \mathsf{k}_0{\uparrow}\,\frac{S[\top,\top]}{S\{\top\}}$$

$$\mathsf{m}_0{\downarrow}\,\frac{S\{0\}}{S(\!\mid 0,0\mid\!)} \qquad \mathsf{m}\,\frac{S\{\!\mid(\!\mid R,U\mid\!),(\!\mid T,V\mid\!)\mid\!\}}{S(\!\mid\{\!\mid R,T\mid\!\},\{\!\mid U,V\mid\!\}\mid\!)} \qquad \mathsf{m}_0{\uparrow}\,\frac{S\{\!\mid\top,\top\mid\!\}}{S\{\top\}}$$

Figure 6: System SALLS

$\mathsf{w}{\downarrow}, \mathsf{w}{\uparrow}, \mathsf{c}{\downarrow}, \mathsf{c}{\uparrow}$, by using the well-known eqivalence

$$!(A \mathbin{\&} B) \;\equiv\; !A \otimes !B \quad .$$

This is done by using the following rules:

$$\mathsf{z}_0{\downarrow}\,\frac{S\{\bot\}}{S\{?0\}} \quad , \quad \mathsf{z}_0{\uparrow}\,\frac{S\{!\top\}}{S\{1\}} \quad , \quad \mathsf{z}{\downarrow}\,\frac{S[?R,T]}{S\{?\{\!\mid R,T\mid\!\}\}} \quad , \quad \mathsf{z}{\uparrow}\,\frac{S\{!(\!\mid R,T\mid\!)\}}{S(!R,T)} \quad .$$

6.1 Proposition *The rule* $\mathsf{v}{\downarrow}$ *is strongly admissible for* $\{\mathsf{z}_0{\downarrow}, \mathsf{w}{\downarrow}\}$. *Dually, the rule* $\mathsf{v}{\uparrow}$ *is strongly admissible for* $\{\mathsf{z}_0{\uparrow}, \mathsf{w}{\uparrow}\}$.

Proof: Replace every instance of

$$\mathsf{v}{\downarrow}\,\frac{S\{\bot\}}{S\{?R\}} \qquad \text{by} \qquad \mathsf{w}{\downarrow}\,\frac{\mathsf{z}_0{\downarrow}\,\dfrac{S\{\bot\}}{S\{?0\}}}{S\{?R\}} \quad .$$

The case of $\mathsf{v}{\uparrow}$ is dual. □

$$\mathsf{1}\!\downarrow \frac{}{\mathsf{1}} \qquad \mathsf{aw}\!\downarrow \frac{S\{\mathsf{0}\}}{S\{a\}} \qquad \mathsf{ac}\!\downarrow \frac{S\{\!\{a,a\}\!\}}{S\{a\}}$$

$$\mathsf{ai}\!\downarrow \frac{S\{\mathsf{1}\}}{S[a,\bar{a}]} \qquad \mathsf{l_0}\!\downarrow \frac{S\{\mathsf{0}\}}{S[\mathsf{0},\mathsf{0}]} \qquad \mathsf{l}\!\downarrow \frac{S\{\!\{[R,U],[T,V]\}\!\}}{S[\{\!\{R,T\}\!\},\{\!\{U,V\}\!\}]}$$

$$\mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]} \qquad \mathsf{k_0}\!\downarrow \frac{S\{\mathsf{0}\}}{S(\mathsf{0},\mathsf{0})} \qquad \mathsf{k}\!\downarrow \frac{S\{\!\{(R,U),(T,V)\}\!\}}{S(\{\!\{R,T\}\!\},\{\!\{U,V\}\!\})}$$

$$\mathsf{d}\!\downarrow \frac{S(\!([R,U],[T,V])\!)}{S[(\!(R,T)\!),\{\!\{U,V\}\!\}]} \qquad \mathsf{m_0}\!\downarrow \frac{S\{\mathsf{0}\}}{S(\!(\mathsf{0},\mathsf{0})\!)} \qquad \mathsf{m}\,\frac{S\{\!\{(\!(R,U)\!),(\!(T,V)\!)\}\!\}}{S(\!(\{\!\{R,T\}\!\},\{\!\{U,V\}\!\})\!)}$$

Figure 7: System ALLS

6.2 Proposition *The rule* $\mathsf{b}\!\downarrow$ *is strongly admissible for* $\{\mathsf{z}\!\downarrow,\mathsf{c}\!\downarrow\}$. *Dually, the rule* $\mathsf{b}\!\uparrow$ *is strongly admissible for* $\{\mathsf{z}\!\uparrow,\mathsf{c}\!\uparrow\}$.

Proof: Replace every instance of

$$\mathsf{b}\!\downarrow \frac{S[?R,R]}{S\{?R\}} \qquad \text{by} \qquad \mathsf{c}\!\downarrow \frac{\mathsf{z}\!\downarrow \dfrac{S[?R,R]}{S\{?\{\!\{R,R\}\!\}\}}}{S\{?R\}} \quad .$$

The case of $\mathsf{b}\!\uparrow$ is dual. □

Further, we have to ensure the admissibility of the rules $\mathsf{w}\!\downarrow,\mathsf{w}\!\uparrow,\mathsf{c}\!\downarrow,\mathsf{c}\!\uparrow$ also in the presence of the exponentials. This is done by adding the rules

$$\mathsf{x}\!\downarrow \frac{S\{\!\{?R,?T\}\!\}}{S\{?\{\!\{R,T\}\!\}\}} \quad , \quad \mathsf{x}\!\uparrow \frac{S\{!(\!(R,T)\!)\}}{S(\!(!R,!T)\!)} \quad , \quad \mathsf{y}\!\downarrow \frac{S\{\!\{!R,!T\}\!\}}{S\{!\{\!\{R,T\}\!\}\}} \quad , \quad \mathsf{y}\!\uparrow \frac{S\{?(\!(R,T)\!)\}}{S(\!(?R,?T)\!)} \quad ,$$

as well as their nullary versions:

$$\mathsf{x_0}\!\downarrow \frac{S\{\mathsf{0}\}}{S\{?\mathsf{0}\}} \quad , \quad \mathsf{x_0}\!\uparrow \frac{S\{!\top\}}{S\{\top\}} \quad , \quad \mathsf{y_0}\!\downarrow \frac{S\{\mathsf{0}\}}{S\{!\mathsf{0}\}} \quad , \quad \mathsf{y_0}\!\uparrow \frac{S\{?\top\}}{S\{\top\}} \quad .$$

6.3 Proposition *The rule* $\mathsf{w}\!\downarrow$ *is stongly admissible for* $\{\mathsf{aw}\!\downarrow,\mathsf{l_0}\!\downarrow,\mathsf{k_0}\!\downarrow,\mathsf{m_0}\!\downarrow,\mathsf{x_0}\!\downarrow,\mathsf{y_0}\!\downarrow\}$. *Dually, the rule* $\mathsf{w}\!\uparrow$ *is stongly admissible for* $\{\mathsf{aw}\!\uparrow,\mathsf{l_0}\!\uparrow,\mathsf{k_0}\!\uparrow,\mathsf{m_0}\!\uparrow,\mathsf{x_0}\!\uparrow,\mathsf{y_0}\!\uparrow\}$.

Proof: Similar as for Proposition 5.3. □

6.4 Proposition *The rule* $\mathsf{c}\!\downarrow$ *is stongly admissible for* $\{\mathsf{ac}\!\downarrow,\mathsf{l}\!\downarrow,\mathsf{k}\!\downarrow,\mathsf{m},\mathsf{x}\!\downarrow,\mathsf{y}\!\downarrow\}$. *Dually, the rule* $\mathsf{c}\!\uparrow$ *is stongly admissible for* $\{\mathsf{ac}\!\uparrow,\mathsf{l}\!\uparrow,\mathsf{k}\!\uparrow,\mathsf{m},\mathsf{x}\!\uparrow,\mathsf{y}\!\uparrow\}$.

Proof: The proof is the same as for Proposition 5.2. The remaining cases are:

- If $R = ?P$ (where $P \neq \bot$), then apply the induction hypothesis to

$$\mathsf{x}{\downarrow}\frac{S\{\!\{?P, ?P\}\!\}}{\mathsf{c}{\downarrow}\dfrac{S\{?\{\!\{P, P\}\!\}\}}{S\{?P\}}} \quad .$$

- If $R = !P$ (where $P \neq 1$), then apply the induction hypothesis to

$$\mathsf{y}{\downarrow}\frac{S\{\!\{!P, !P\}\!\}}{\mathsf{c}{\downarrow}\dfrac{S\{!\{\!\{P, P\}\!\}\}}{S\{!P\}}} \quad .$$

The proof for $\mathsf{c}\uparrow$ is dual. □

All new rules in this section are sound, i.e. correspond to linear implications. Hence, they are strongly admissible for system SLS. More precisely, we have:

6.5 Proposition *The rules* $\mathsf{l}\downarrow, \mathsf{k}\downarrow, \mathsf{m}, \mathsf{x}\downarrow, \mathsf{y}\downarrow$ *are strongly admissible for* $\{\mathsf{w}\downarrow, \mathsf{c}\downarrow\}$*, the rule* $\mathsf{z}\downarrow$ *is strongly admissible for* $\{\mathsf{w}\downarrow, \mathsf{b}\downarrow\}$*, the rules* $\mathsf{l}_0\downarrow, \mathsf{k}_0\downarrow, \mathsf{m}_0\downarrow, \mathsf{x}_0\downarrow, \mathsf{y}_0\downarrow$ *are strongly admissible for* $\{\mathsf{w}\downarrow\}$*, and the rule* $\mathsf{z}_0\downarrow$ *is strongly admissible for* $\{\mathsf{v}\downarrow\}$*. Dually, the rules* $\mathsf{l}\uparrow, \mathsf{k}\uparrow, \mathsf{m}, \mathsf{x}\uparrow, \mathsf{y}\uparrow$ *are strongly admissible for* $\{\mathsf{w}\uparrow, \mathsf{c}\uparrow\}$*, the rule* $\mathsf{z}\uparrow$ *is strongly admissible for* $\{\mathsf{w}\uparrow, \mathsf{b}\uparrow\}$*, the rules* $\mathsf{l}_0\uparrow, \mathsf{k}_0\uparrow, \mathsf{m}_0\uparrow, \mathsf{x}_0\uparrow, \mathsf{y}_0\uparrow$ *are strongly admissible for* $\{\mathsf{w}\uparrow\}$*, and the rule* $\mathsf{z}_0\uparrow$ *is strongly admissible for* $\{\mathsf{v}\uparrow\}$*.*

Proof: Trivial. □

6.6 Definition The system shown in Figure 8 is called *system* SLLS.

System SLLS can be regarded as the local version of system SLS. Both systems are strongly equivalent. Further, the local system does admit a cut elimination result.

6.7 Theorem *Systems* SLLS *and* SLS *are strongly equivalent.*

Proof: Immediate consequence of Propositions 6.1–6.5. □

6.8 Definition System LLS (*local linear logic in the calculus of structures*), shown in Figure 9, is obained from the down-fragment of System SLLS together with the rules s, m and the axiom $1\downarrow$.

6.9 Theorem *Systems* LLS *and* LS *are strongly equivalent.*

6.10 Corollary (Cut elimination) *System* LLS *is equivalent to every subsystem of* $\mathsf{SLLS} \cup \{1\downarrow\}$ *containing* LLS*.*

6.11 Corollary *The rule* $\mathsf{i}\uparrow$ *is admissible for system* LLS*.*

7 Decomposition of Derivations

In this section, I will state and prove several so-called decomposition theorems. All of them say that any derivation in system SLLS can be rearranged in a certain way. The first is Theorem 7.2, which states that any derivation can be decomposed into three parts, which can be called *creation*, *merging* and *destruction*. The merging

$$\mathsf{ai}\!\downarrow\frac{S\{1\}}{S[a,\bar{a}]}\qquad\mathsf{ai}\!\uparrow\frac{S(a,\bar{a})}{S\{\bot\}}$$

$$\mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{d}\!\downarrow\frac{S\bullet([R,U],[T,V])\bullet}{S[\bullet(R,T)\bullet,\bullet[U,V]\bullet]}\qquad\mathsf{d}\!\uparrow\frac{S(\bullet[R,U]\bullet,\bullet(T,V)\bullet)}{S\bullet[(R,T),(U,V)]\bullet}$$

$$\mathsf{p}\!\downarrow\frac{S\{![R,T]\}}{S[!R,?T]}\qquad\mathsf{p}\!\uparrow\frac{S(?R,!T)}{S\{?(R,T)\}}$$

$$\mathsf{aw}\!\downarrow\frac{S\{0\}}{S\{a\}}\qquad\mathsf{ac}\!\downarrow\frac{S\bullet[a,a]\bullet}{S\{a\}}\qquad\mathsf{ac}\!\uparrow\frac{S\{a\}}{S\bullet(a,a)\bullet}\qquad\mathsf{aw}\!\uparrow\frac{S\{a\}}{S\{\top\}}$$

$$\mathsf{l_0}\!\downarrow\frac{S\{0\}}{S[0,0]}\qquad\mathsf{l}\!\downarrow\frac{S\bullet[[R,U],[T,V]]\bullet}{S[\bullet[R,T]\bullet,\bullet[U,V]\bullet]}\qquad\mathsf{l}\!\uparrow\frac{S(\bullet(R,U)\bullet,\bullet(T,V)\bullet)}{S\bullet((R,T),(U,V))\bullet}\qquad\mathsf{l_0}\!\uparrow\frac{S(\top,\top)}{S\{\top\}}$$

$$\mathsf{k_0}\!\downarrow\frac{S\{0\}}{S(0,0)}\qquad\mathsf{k}\!\downarrow\frac{S\bullet[(R,U),(T,V)]\bullet}{S(\bullet[R,T]\bullet,\bullet[U,V]\bullet)}\qquad\mathsf{k}\!\uparrow\frac{S[\bullet(R,U)\bullet,\bullet(T,V)\bullet]}{S\bullet([R,T],[U,V])\bullet}\qquad\mathsf{k_0}\!\uparrow\frac{S[\top,\top]}{S\{\top\}}$$

$$\mathsf{m_0}\!\downarrow\frac{S\{0\}}{S\bullet(0,0)\bullet}\qquad\mathsf{m}\,\frac{S\bullet[\bullet(R,U)\bullet,\bullet(T,V)\bullet]\bullet}{S\bullet(\bullet[R,T]\bullet,\bullet[U,V]\bullet)\bullet}\qquad\mathsf{m_0}\!\uparrow\frac{S\bullet[\top,\top]\bullet}{S\{\top\}}$$

$$\mathsf{x_0}\!\downarrow\frac{S\{0\}}{S\{?0\}}\qquad\mathsf{x}\!\downarrow\frac{S\bullet[?R,?T]\bullet}{S\{?\bullet[R,T]\bullet\}}\qquad\mathsf{x}\!\uparrow\frac{S\{!\bullet(R,T)\bullet\}}{S\bullet(!R,!T)\bullet}\qquad\mathsf{x_0}\!\uparrow\frac{S\{!\top\}}{S\{\top\}}$$

$$\mathsf{y_0}\!\downarrow\frac{S\{0\}}{S\{!0\}}\qquad\mathsf{y}\!\downarrow\frac{S\bullet[!R,!T]\bullet}{S\{!\bullet[R,T]\bullet\}}\qquad\mathsf{y}\!\uparrow\frac{S\{?\bullet(R,T)\bullet\}}{S\bullet(?R,?T)\bullet}\qquad\mathsf{y_0}\!\uparrow\frac{S\{?\top\}}{S\{\top\}}$$

$$\mathsf{z_0}\!\downarrow\frac{S\{\bot\}}{S\{?0\}}\qquad\mathsf{z}\!\downarrow\frac{S[?R,T]}{S\{?\bullet[R,T]\bullet\}}\qquad\mathsf{z}\!\uparrow\frac{S\{!\bullet(R,T)\bullet\}}{S(!R,T)}\qquad\mathsf{z_0}\!\uparrow\frac{S\{!\top\}}{S\{1\}}$$

Figure 8: System SLLS

part is in the middle of the derivation, and (depending on your preferred reading of a derivation) the creation and destruction are at the top and at the bottom, as depicted in Figure 10. In system SLLS the merging part contains the rules $\mathsf{s}, \mathsf{d}\!\downarrow, \mathsf{d}\!\uparrow, \mathsf{p}\!\downarrow, \mathsf{p}\!\uparrow, \mathsf{l}\!\downarrow, \mathsf{l}\!\uparrow, \mathsf{k}\!\downarrow, \mathsf{k}\!\uparrow, \mathsf{m}\mathsf{x}\!\downarrow, \mathsf{x}\!\uparrow, \mathsf{y}\!\downarrow, \mathsf{y}\!\uparrow, \mathsf{z}\!\downarrow, \mathsf{z}\!\uparrow$, which do not change the size of the structure.

$$1\!\downarrow \frac{}{1} \qquad \mathsf{ai}\!\downarrow \frac{S\{1\}}{S[a,\bar{a}]} \qquad \mathsf{l_0}\!\downarrow \frac{S\{0\}}{S[0,0]} \qquad \mathsf{l}\!\downarrow \frac{S\{\!\!\{[R,U],[T,V]\}\!\!\}}{S[\{\!\!\{R,T\}\!\!\},\{\!\!\{U,V\}\!\!\}]}$$

$$\mathsf{s} \frac{S([R,U],T)}{S[(R,T),U]} \qquad \mathsf{k_0}\!\downarrow \frac{S\{0\}}{S(0,0)} \qquad \mathsf{k}\!\downarrow \frac{S\{\!\!\{(R,U),(T,V)\}\!\!\}}{S(\{\!\!\{R,T\}\!\!\},\{\!\!\{U,V\}\!\!\})}$$

$$\mathsf{d}\!\downarrow \frac{S\langle[R,U],[T,V]\rangle}{S[\langle R,T\rangle,\{\!\!\{U,V\}\!\!\}]} \qquad \mathsf{m_0}\!\downarrow \frac{S\{0\}}{S\langle 0,0\rangle} \qquad \mathsf{m} \frac{S\{\!\!\{\langle R,U\rangle,\langle T,V\rangle\}\!\!\}}{S\langle\{\!\!\{R,T\}\!\!\},\{\!\!\{U,V\}\!\!\}\rangle}$$

$$\mathsf{p}\!\downarrow \frac{S\{![R,T]\}}{S[!R,?T]} \qquad \mathsf{x_0}\!\downarrow \frac{S\{0\}}{S\{?0\}} \qquad \mathsf{x}\!\downarrow \frac{S\{\!\!\{?R,?T\}\!\!\}}{S\{?\{\!\!\{R,T\}\!\!\}\}}$$

$$\mathsf{aw}\!\downarrow \frac{S\{0\}}{S\{a\}} \qquad \mathsf{y_0}\!\downarrow \frac{S\{0\}}{S\{!0\}} \qquad \mathsf{y}\!\downarrow \frac{S\{\!\!\{!R,!T\}\!\!\}}{S\{!\{\!\!\{R,T\}\!\!\}\}}$$

$$\mathsf{ac}\!\downarrow \frac{S\{\!\!\{a,a\}\!\!\}}{S\{a\}} \qquad \mathsf{z_0}\!\downarrow \frac{S\{\bot\}}{S\{?0\}} \qquad \mathsf{z}\!\downarrow \frac{S[?R,T]}{S\{?\{\!\!\{R,T\}\!\!\}\}}$$

Figure 9: System LLS

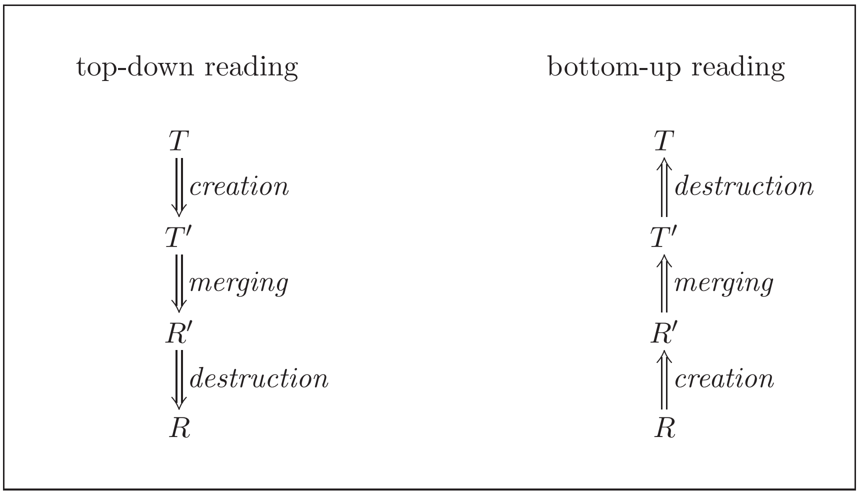

Figure 10: Reading of the decomposition theorem

7.1 Definition Let *system* SLLSm$=\{\mathsf{s}, \mathsf{d}\!\downarrow, \mathsf{d}\!\uparrow, \mathsf{p}\!\downarrow, \mathsf{p}\!\uparrow, \mathsf{l}\!\downarrow, \mathsf{l}\!\uparrow, \mathsf{k}\!\downarrow, \mathsf{k}\!\uparrow, \mathsf{mx}\!\downarrow, \mathsf{x}\!\uparrow, \mathsf{y}\!\downarrow, \mathsf{y}\!\uparrow, \mathsf{z}\!\downarrow, \mathsf{z}\!\uparrow\}$.

7.2 Theorem *For every derivation* $\begin{array}{c} T \\ \Delta\Vert\mathsf{SLLS} \\ R \end{array}$ *there exist* T' *and* R', *such that*

$$\begin{array}{l} T \\ \Vert\{\mathsf{ai}\downarrow,\mathsf{at}\downarrow,\mathsf{l_0}\downarrow,\mathsf{k_0}\downarrow,\mathsf{m_0}\downarrow,\mathsf{x_0}\downarrow,\mathsf{y_0}\downarrow,\mathsf{z_0}\downarrow,\mathsf{ac}\uparrow\} \\ T' \\ \Vert\mathsf{SLLSm} \\ R' \\ \Vert\{\mathsf{ai}\uparrow,\mathsf{at}\uparrow,\mathsf{l_0}\uparrow,\mathsf{k_0}\uparrow,\mathsf{m_0}\uparrow,\mathsf{x_0}\uparrow,\mathsf{y_0}\uparrow,\mathsf{z_0}\uparrow,\mathsf{ac}\downarrow\} \\ R \end{array} \quad .$$

Proof: Every derivation $\rho\,\dfrac{Q}{\pi\,\dfrac{P}{R}}$, where $\rho \in \mathsf{SLLSm}$ and $\pi \in \{\mathsf{ai}\downarrow, \mathsf{at}\downarrow, \mathsf{l_0}\downarrow, \mathsf{k_0}\downarrow, \mathsf{m_0}\downarrow, \mathsf{x_0}\downarrow, \mathsf{y_0}\downarrow, \mathsf{z_0}\downarrow, \mathsf{ac}\uparrow\}$ can be replaced by a derivation

$$\begin{array}{c} \pi\,\dfrac{Q}{P'} \\ \Vert\mathsf{SLLSm} \\ R \end{array} \quad .$$

From this and the dual statement, the theorem follows by an inductive argument. □

This decomposition is not restricted to system SLLS. A similar theorem has already been proved for other systems in the calculus of structures, namely for system SBV (a non-commutative logic) in [5], for system SELS in [9] and for system SKS (classical logic) in [1]. It also holds for system SLLS, a local system for full linear logic, that will be discussed in the next section.

In the following, I will show some further possible decompositions. The first deals with contraction. The intuition behind contraction is to copy parts of a structure to make them usable more than once in a proof or derivation. This intuition tells us that it should be possible to copy first everything as much as it is needed and then go on with the proof without copying anything anymore. The following theorem makes this explicit. Furthermore, because of the top-down symmetry of the calculus of structures, the theorem can be stated for derivations.

7.3 Theorem *For every derivation* $\begin{array}{c} T \\ \Delta\Vert\mathsf{SLLS} \\ R \end{array}$ *there is a derivation*

$$\begin{array}{l} T \\ \Vert\{\mathsf{ac}\uparrow\} \\ T' \\ \Vert\mathsf{SLLS}\setminus\{\mathsf{ac}\uparrow,\mathsf{ac}\downarrow\} \\ R' \\ \Vert\{\mathsf{ac}\downarrow\} \\ R \end{array}$$

for some structures T' *and* R'.

Proof: The proof is an adaption of the proof of a similar theorem for classical logic, presented in [1]. Let $\begin{array}{c} T \\ \Delta \Vert \mathsf{SLLS} \\ R \end{array}$ be given. From this we can obtain a derivation $\begin{array}{c} [T,\bar{T}] \\ \Delta' \Vert \mathsf{SLLS} \\ [R,\bar{T}] \end{array}$ (by replacing each structure S that occurs as premise or conclusion inside Δ by $[S,\bar{T}]$), which can be transformed into a proof by adding $\mathsf{i}{\downarrow}\dfrac{\mathsf{1}{\downarrow}\,\dfrac{}{1}}{[T,\bar{T}]}$. By applying Proposition 2.11 and Theorem 6.9, this proof can be transformed into a proof in system LLS. The rule $\mathsf{ac}{\downarrow}$ can be easily permuted under all other rules in system LLS. By this, we obtain a proof

$$\begin{array}{c} \Vert \mathsf{LLS}\setminus\{\mathsf{ac}{\downarrow}\} \\ U \\ \tilde{\Delta} \Vert \{\mathsf{ac}{\downarrow}\} \\ [R,\bar{T}] \end{array} \qquad (1)$$

for some structure U. Furthermore, we have that $U = [R',\bar{T}']$ for some structures R' and $\bar{T}'$ such that there are derivations $\begin{array}{c} R' \\ \tilde{\Delta}_R \Vert \{\mathsf{ac}{\downarrow}\} \\ R \end{array}$ and $\begin{array}{c} \bar{T}' \\ \tilde{\Delta}_{\bar{T}} \Vert \{\mathsf{ac}{\downarrow}\} \\ \bar{T} \end{array}$. Now, perform the following transformations to the proof in (1):

$$\begin{array}{c} \Vert \mathsf{LLS}\setminus\{\mathsf{ac}{\downarrow}\} \\ [R',\bar{T}'] \\ \tilde{\Delta} \Vert \{\mathsf{ac}{\downarrow}\} \\ [R,\bar{T}] \end{array} \overset{1}{\rightsquigarrow} \begin{array}{c} (1,T) \\ \Vert \mathsf{LLS}\setminus\{\mathsf{ac}{\downarrow}\} \\ ([R',\bar{T}'],T) \\ \tilde{\Delta} \Vert \{\mathsf{ac}{\downarrow}\} \\ ([R,\bar{T}],T) \end{array} \overset{2}{\rightsquigarrow} \begin{array}{c} T \\ \Vert \mathsf{LLS}\setminus\{\mathsf{ac}{\downarrow}\} \\ ([R',\bar{T}'],T) \\ \tilde{\Delta}_{\bar{T}} \Vert \{\mathsf{ac}{\downarrow}\} \\ ([R',\bar{T}],T) \\ \tilde{\Delta}_R \Vert \{\mathsf{ac}{\downarrow}\} \\ \mathsf{s}\,\dfrac{([R,\bar{T}],T)}{\mathsf{i}{\uparrow}\,\dfrac{[R,(\bar{T},T)]}{R}} \end{array} \overset{3}{\rightsquigarrow} \begin{array}{c} T \\ \Vert \mathsf{LLS}\setminus\{\mathsf{ac}{\downarrow}\} \\ \mathsf{s}\,\dfrac{([R',\bar{T}'],T)}{[R',(\bar{T}',T)]} \\ \tilde{\Delta}_{\bar{T}} \Vert \{\mathsf{ac}{\downarrow}\} \\ \mathsf{i}{\uparrow}\,\dfrac{[R',(\bar{T},T)]}{R'} \\ \tilde{\Delta}_R \Vert \{\mathsf{ac}{\downarrow}\} \\ R \end{array}$$

In the first step remove the axiom and replace every structure S occuring as premise or conclusion of a rule by (S,T). In the second step separate the instances of $\mathsf{ac}{\downarrow}$ in $\tilde{\Delta}$ into the two derivations $\tilde{\Delta}_{\bar{T}}$ and $\tilde{\Delta}_R$. Further, add an instance of s and and instance of $\mathsf{i}{\uparrow}$ at the bottom of the derivation to obtain a derivation from T to R. In the last step, all instances of $\mathsf{ac}{\downarrow}$ in $\tilde{\Delta}_R$ are permuted down to the bottom of the derivation and all instances of $\mathsf{ac}{\downarrow}$ in $\tilde{\Delta}_{\bar{T}}$ are permuted under the switch rule.

Consider now the bottommost instance of $\mathsf{ac}{\downarrow}\,\dfrac{S\{\!\bullet a,a\bullet\!\}}{S\{a\}}$ in $\tilde{\Delta}_{\bar{T}}$. It has its redex inside the structure $\bar{T}$. Hence $\bar{T} = U\{a\}$ for some context $U\{\ \}$. With a little abuse of notation, we can write $T = \bar{U}\{\bar{a}\}$. Now, we can replace

$$\text{ac}\downarrow \frac{\text{i}\uparrow \dfrac{[R', (U\{a, a\}, \bar{U}\{\bar{a}\})]}{[R', (U\{a\}, \bar{U}\{\bar{a}\})]}}{R'} \quad \text{by} \quad \text{ac}\uparrow \frac{\text{i}\uparrow \dfrac{[R', (U\{a, a\}, \bar{U}\{\bar{a}\})]}{[R', (U\{a, a\}, \bar{U}(\bar{a}, \bar{a}))]}}{R'} \quad .$$

The new instance of ac↑ can be permuted up to the top of the derivation because there is no other rule above it that changes T. Repeat this for all instances of ac↓ in $\tilde{\Delta}_{\bar{T}}$. The resulting derivation has the desired shape, except for the remaining instance of i↑, which can be replaced by a derivation containing only ai↑, s and d↑ (by Proposition 2.11). □

The next theorem deals with interaction and cut. In the sequent calculus, the instances of the identity rule are always at the top of a derivation or proof. In the calculus of structures, it is also possible to move the identity to the top of a derivation. Moreover, by duality, it is also possible, to push all cuts town to the bottom, which is certainly not possible in the sequent calculus.

7.4 Theorem *For every derivation* $\begin{array}{c} T \\ \Delta \| \text{SLLS} \\ R \end{array}$ *there is a derivation*

$$\begin{array}{l} T \\ \| \{\text{ai}\downarrow\} \\ T' \\ \| \text{SLLS} \setminus \{\text{ai}\downarrow, \text{ai}\uparrow\} \\ R' \\ \| \{\text{ai}\uparrow\} \\ R \end{array}$$

for some structures T' and R'.

Proof: I will show that every subderivation $\text{ai}\downarrow \dfrac{\rho \dfrac{Q}{S\{1\}}}{S[a, \bar{a}]}$ of Δ, where $\rho \in \text{SLLS} \setminus \{\text{ai}\downarrow\}$, can be replaced by a derivation

$$\begin{array}{l} \text{ai}\downarrow \dfrac{Q'}{Q} \\ \quad \| \text{SLLS} \setminus \{\text{ai}\downarrow, \text{ai}\uparrow\} \\ S[a, \bar{a}] \end{array} \quad .$$

From this and the dual statement, the theorem follows by an easy inductive argument.

Consider now $\text{ai}\downarrow \dfrac{\rho \dfrac{Q}{S\{1\}}}{S[a, \bar{a}]}$, where ρ is not trivial (otherwise the statement follows trivially). It follows an exhaustive list of all possible cases:

(1) $\rho \in \{\text{ai}\uparrow, \text{at}\uparrow\}$: Then there is only one possibility:

(i) The redex $\bot$ of ρ is inside the context $S\{\ \}$. Then $Q = S'\{1\}$ for some context $S'\{\ \}$ and we can replace

$$\mathsf{ai}{\downarrow}\,\dfrac{\rho\,\dfrac{S'\{1\}}{S\{1\}}}{S[a,\bar{a}]} \qquad \text{by} \qquad \rho\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{S'\{1\}}{S'[a,\bar{a}]}}{S[a,\bar{a}]} \quad .$$

(2) $\rho = \mathsf{s}$: There are two subcases:

(i) The redex $[(R,T),U]$ of s is inside the context $S\{\ \}$. Similar to the previous case.

(ii) The contractum 1 of $\mathsf{ai}{\downarrow}$ is inside R, T or U. If it is inside R, we have that $R = R'\{1\}$ for some context $R'\{\ \}$ and $Q = S'([R'\{1\},U],T)$ for some context $S'\{\ \}$. Then we can replace

$$\mathsf{ai}{\downarrow}\,\dfrac{\mathsf{s}\,\dfrac{S'([R'\{1\},U],T)}{S'[(R'\{1\},T),U]}}{S'[(R'[a,\bar{a}],T),U]} \qquad \text{by} \qquad \mathsf{s}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{S'([R'\{1\},U],T)}{S'([R'[a,\bar{a}],U],T)}}{S'[(R'[a,\bar{a}],T),U]} \quad .$$

If the 1 is inside T or U, the situation is similar.

(3) $\rho = \mathsf{d}{\downarrow}$: There are four subcases.

(i) The redex $[\langle R,T\rangle,\{\!\mid U,V\mid\!\}]$ of $\mathsf{d}{\downarrow}$ is inside the context $S\{\ \}$. Similar to (1.i).

(ii) The contractum 1 of $\mathsf{ai}{\downarrow}$ is inside R, T, U or V. Similar to (2.ii).

(iii) $S\{1\} = S'[(\langle R,T\rangle,1),\{\!\mid U,V\mid\!\}]$. Then replace

$$\mathsf{ai}{\downarrow}\,\dfrac{\mathsf{d}{\downarrow}\,\dfrac{S'\langle [R,U],[T,V]\rangle}{S'[\langle R,T\rangle,\{\!\mid U,V\mid\!\}]}}{S'[(\langle R,T\rangle,[a,\bar{a}]),\{\!\mid U,V\mid\!\}]} \qquad \text{by} \qquad \mathsf{s}\,\dfrac{\mathsf{d}{\downarrow}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{S'\langle [R,U],[T,V]\rangle}{S'(\langle [R,U],[T,V]\rangle,[a,\bar{a}])}}{S'([\langle R,T\rangle,\{\!\mid U,V\mid\!\}],[a,\bar{a}])}}{S'[(\langle R,T\rangle,[a,\bar{a}]),\{\!\mid U,V\mid\!\}]} \quad .$$

(iv) $S\{1\} = S'[\langle R,T\rangle,(\{\!\mid U,V\mid\!\},1)]$. Similar to (iii).

(4) $\rho = \mathsf{l}{\downarrow}$: Similar to (3).

(5) $\rho = \mathsf{k}{\downarrow}$: Similar to (2). Observe that a situation as in (3.iii) or (3.iv) cannot occur because $S'((\{\!\mid R,T\mid\!\},1),\{\!\mid U,V\mid\!\}) = S'((\{\!\mid R,T\mid\!\},\{\!\mid U,V\mid\!\}),1)$ and therefore is handled by subcase (i).

(6) $\rho = \mathsf{d}{\uparrow}$: Similar to (2). Observe that a situation as in (3.iii) or (3.iv) cannot occur because $S'\{\!\mid((R,T),1),(U,V)\mid\!\} = S'\{\!\mid((R,1),T),(U,V)\mid\!\}$ and therefore is handled by subcase (ii).

(7) $\rho = \mathsf{l}{\uparrow}$: Similar to (6).

(8) $\rho = \mathsf{k}{\uparrow}$: Again, there are four subcases.

(i) The redex $\llparenthesis [R,T],[U,V] \rrparenthesis$ of $\mathsf{k}\!\uparrow$ is inside the context $S\{\ \}$. Similar to (1.i).

(ii) The contractum $\mathsf{1}$ of $\mathsf{ai}\!\downarrow$ is inside R, T, U or V. Similar to (2.ii).

(iii) $S\{\mathsf{1}\} = S'\llparenthesis ([R,T],\mathsf{1}),[U,V] \rrparenthesis$. Then replace

$$\mathsf{ai}\!\downarrow \dfrac{\mathsf{k}\!\uparrow \dfrac{S'[\llparenthesis R,U \rrparenthesis, \llparenthesis T,V \rrparenthesis]}{S'\llparenthesis [R,T],[U,V] \rrparenthesis}}{S'\llparenthesis ([R,T],[a,\bar{a}]),[U,V] \rrparenthesis} \qquad \text{by} \qquad \mathsf{s}\dfrac{\mathsf{k}\!\uparrow \dfrac{\mathsf{ai}\!\downarrow \dfrac{S'[\llparenthesis R,U \rrparenthesis, \llparenthesis T,V \rrparenthesis]}{S'[\llparenthesis R,U \rrparenthesis, \llparenthesis (T,[a,\bar{a}]),V \rrparenthesis]}}{S'\llparenthesis [R,(T,[a,\bar{a}])],[U,V] \rrparenthesis}}{S'\llparenthesis ([R,T],[a,\bar{a}]),[U,V] \rrparenthesis} \quad .$$

(iv) $S\{\mathsf{1}\} = S'\llparenthesis [R,T],([U,V],\mathsf{1}) \rrparenthesis$. Similar to (iii).

(9) $\rho = \mathsf{m}$: Again, there are four subcases.

(i) The redex $\llparenthesis \{\!\!\{R,T\}\!\!\},\{\!\!\{U,V\}\!\!\} \rrparenthesis$ of m is inside the context $S\{\ \}$. Similar to (1.i).

(ii) The contractum $\mathsf{1}$ of $\mathsf{ai}\!\downarrow$ is inside R, T, U or V. Similar to (2.ii).

(iii) $S\{\mathsf{1}\} = S'\llparenthesis (\{\!\!\{R,T\}\!\!\},\mathsf{1}),\{\!\!\{U,V\}\!\!\} \rrparenthesis$. Then replace

$$\mathsf{ai}\!\downarrow \dfrac{\mathsf{m} \dfrac{S'\{\!\!\{\llparenthesis R,U \rrparenthesis, \llparenthesis T,V \rrparenthesis\}\!\!\}}{S'\llparenthesis \{\!\!\{R,T\}\!\!\},\{\!\!\{U,V\}\!\!\} \rrparenthesis}}{S'\llparenthesis (\{\!\!\{R,T\}\!\!\},[a,\bar{a}]),\{\!\!\{U,V\}\!\!\} \rrparenthesis} \qquad \text{by} \qquad \mathsf{l}\!\uparrow \dfrac{\mathsf{m} \dfrac{\mathsf{ai}\!\downarrow \dfrac{S'\{\!\!\{\llparenthesis R,U \rrparenthesis, \llparenthesis T,V \rrparenthesis\}\!\!\}}{S'(\{\!\!\{\llparenthesis R,U \rrparenthesis, \llparenthesis T,V \rrparenthesis\}\!\!\}, \llparenthesis [a,\bar{a}],\mathsf{1} \rrparenthesis)}}{S'(\llparenthesis \{\!\!\{R,T\}\!\!\},\{\!\!\{U,V\}\!\!\} \rrparenthesis, \llparenthesis [a,\bar{a}],\mathsf{1} \rrparenthesis)}}{S'\llparenthesis (\{\!\!\{R,T\}\!\!\},[a,\bar{a}]),\{\!\!\{U,V\}\!\!\} \rrparenthesis} \quad .$$

(iv) $S\{\mathsf{1}\} = S'\llparenthesis \{\!\!\{R,T\}\!\!\},(\{\!\!\{U,V\}\!\!\},\mathsf{1}) \rrparenthesis$. Similar to (iii).

(10) $\rho = \mathsf{at}\!\downarrow$: Similar to (3).

(11) $\rho = \mathsf{ac}\!\downarrow$: Similar to (1). (Note that $S\{\mathsf{1}\} = S(\mathsf{1},\mathsf{1})$.)

(12) $\rho = \mathsf{ac}\!\uparrow$: Similar to (9). (Note that $\llparenthesis \mathsf{1},\mathsf{1} \rrparenthesis = \mathsf{1}$.) □

8 Conclusions and Future Work

The results presented in this paper are twofold. First, they extend the work of [6, 9] by showing that also full linear logic can benefit from its presentation in the calculus of structures. In particular, the rules for the additives are splitted into two parts, namely, a purely multiplicative part (the rules $\mathsf{d}\!\downarrow$ and $\mathsf{d}\!\uparrow$) and an explicit contraction (the rules $\mathsf{c}\!\downarrow$ and $\mathsf{c}\!\uparrow$), whereas in the sequent calculus, contraction is contained implicitly in the rules for the additives.

The second achievement of this paper is to show that in the calculus of structures it is possible to reduce this contraction for the additives to its atomic version. This is of course not possible in the sequent calculus. By using the logical equivalence $!(A \& B) \equiv\ !A \otimes\ !B$, it is possible to reduce also the contraction rule for the exponentials to an atomic version, by reducing it to the contraction for the additives. The only drawback is that for systen SLLS the separation property is lost, i.e. it is no longer possible to consider the multiplicative exponential fragment independent from the additives.

The calculus of structures is also able to deal with the quantifiers. The rules for the first order predicative case have already been shown in [1]. For the second order propositional case the situation is similar. If we add the quantifiers to the language in the obvious way, together with the equations for the De Morgan laws and

$$\forall a.R = R = \exists a.R \quad , \quad \text{if } a \text{ is not free in } R,$$

we can extend system SLS by the following rules:

$$\mathsf{u}\!\downarrow \frac{S\{\forall a.[R,T]\}}{S[\forall a.R, \exists a.T]} \quad , \qquad \mathsf{u}\!\uparrow \frac{S(\exists a.R, \forall a.T)}{S\{\exists a.(R,T)\}} \quad ,$$

$$\mathsf{n}\!\downarrow \frac{S\{R\{T/a\}\}}{S\{\exists a.R\}} \quad , \qquad \mathsf{n}\!\uparrow \frac{S\{\forall a.R\}}{S\{R\{T/a\}\}} \quad .$$

Observe that there is no proviso saying that the variable a is not allowed to be free in the context, as it is the case in the sequent calculus.

As already observed in [1], it is even under the presence of the quantifiers possible to reduce contraction to it atomic version, by adding the rules

$$\mathsf{ca}\!\downarrow \frac{S\{\!\!\{\forall a.R, \forall a.T\}\!\!\}}{S\{\forall a.\{\!\!\{R,T\}\!\!\}\}} \quad , \qquad \mathsf{ca}\!\uparrow \frac{S\{\exists a.(\!(R,T)\!)\}}{S(\!(\exists a.R, \exists a.T)\!)} \quad ,$$

$$\mathsf{ce}\!\downarrow \frac{S\{\!\!\{\exists a.R, \exists a.T\}\!\!\}}{S\{\exists a.\{\!\!\{R,T\}\!\!\}\}} \quad , \qquad \mathsf{ce}\!\uparrow \frac{S\{\forall a.(\!(R,T)\!)\}}{S(\!(\forall a.R, \forall a.T)\!)} \quad .$$

However, even if contraction is atomic, the new system can no longer be called local because in the rule $\mathsf{n}\!\downarrow$ a structure of arbitrary size is introduced.

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