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Some Observations on the Proof Theory of Second Order Propositional Multiplicative Linear Logic

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Abstract. We investigate the question of what constitutes a proof when quantifiers and multiplicative units are both present. On the technical level this paper provides two new aspects of the proof theory of MLL2 with units. First, we give a novel proof system in the framework of the calculus of structures. The main feature of the new system is the consequent use of deep inference, which allows us to observe a decomposition which is a version of Herbrand's theorem that is not visible in the sequent calculus. Second, we show a new notion of proof nets which is independent from any deductive system. We have "sequentialisation" into the calculus of structures as well as into the sequent calculus. Since cut elimination is terminating and confluent, we have a category of MLL2 proof nets. The treatment of the units is such that this category is star-autonomous.

1 Introduction

The question of when two proofs are the same is important for proof theory and its applications. It comes down to the question of which information contained in a proof is essential, and which information is purely bureaucratic, due to the chosen deductive system. One of the first results in that direction is Herbrand's theorem which allows a separation between the quantifiers and the propositional fragment of first order classical predicate logic. The work on expansion trees by Miller [19] shows how Herbrand's result can be generalized to higher order. In this paper we present a similar result for linear logic. Our work is motivated by the desire to find eventually a general treatment for the quantifiers, independent from the propositional fragment of the logic (see the related work by McKinley [18]).

The first contribution of this paper is a presentation of MLL2 in the calculus of structures, which is a new deductive formalism using *deep inference*. That means that inferences are allowed anywhere deep inside a formula, very similar to what happens in term rewriting. As a consequence of this freedom we can show a decomposition theorem, which is not possible in the sequent calculus, and which can be seen as a version of Herbrand's Theorem for MLL2. Secondly, we give a combinatorial presentation of MLL2 proofs that we call here *proof nets* (following the tradition) and that quotient away irrelevant rule permutations in the deductive systems (sequent calculus and calculus of structures). The identifications made by these proof nets are consistent with ones for MLL (with units) made by star-autonomous categories [1, 16, 17]. The main

$$\mathsf{id}\;\frac{}{\vdash a^{\perp}, a} \qquad \perp\;\frac{\vdash \Gamma}{\vdash \perp, \Gamma} \qquad 1\;\frac{}{\vdash 1} \qquad \mathsf{exch}\;\frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}$$

$$\parr\;\frac{\vdash A, B, \Gamma}{\vdash [A \parr B], \Gamma} \qquad \otimes\;\frac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, (A \otimes B), \Delta} \qquad \exists\;\frac{\vdash A\langle a \backslash B\rangle, \Gamma}{\vdash \exists a.A, \Gamma} \qquad \forall\;\frac{\vdash A, \Gamma}{\vdash \forall a.A, \Gamma}\;\begin{array}{l}a \text{ not}\\ \text{free}\\ \text{in } \Gamma\end{array}$$

Fig. 1. Sequent calculus system for MLL2

motivation for these proof nets is to exhibit the precise relation between deep inference and the existing presentations of MLL2-proofs: sequent calculus, Girard's proof nets with boxes [9], and Girard's proof nets with jumps [10]. Our proof nets are the first to accomodate the quantifiers and the multiplicative units together without boxes. Furthermore, the proof nets proposed here are independent from the deductive system, i.e., we do not have the strong connection between links in the proof net and rule applications in the sequent calculus. However, we have "sequentialization" into the sequent calculus as well as into the calculus of structures. As expected, there is a confluent and terminating cut elimination procedure, and thus, the two conclusion proof nets form a category.

2 MLL2 in the sequent calculus

Let us recall how MLL2 is presented in the sequent calculus. Let $\mathscr{A} = \{a, b, c, \ldots\}$ be a countable set of *propositional variables*. Then the set $\mathscr{F}$ of *formulas* is generated by

$$\mathscr{F} ::= \perp \mid 1 \mid \mathscr{A} \mid \mathscr{A}^{\perp} \mid [\mathscr{F} \parr \mathscr{F}] \mid (\mathscr{F} \otimes \mathscr{F}) \mid \forall \mathscr{A}.\mathscr{F} \mid \exists \mathscr{A}.\mathscr{F}$$

Formulas are denoted by capital Latin letters ($A, B, C, \ldots$). Linear negation $(-)^{\perp}$ is defined for all formulas by the De Morgan laws. *Sequents* are finite lists of formulas, separated by comma, and are denoted by capital Greek letters ($\Gamma, \Delta, \ldots$). The notions of *free* and *bound variable* are defined in the usual way, and we can always rename bound variables. In view of the later parts of the paper, and in order to avoid changing syntax all the time, we use the following syntactic conventions:

(i) We always put parentheses around binary connectives. For readability we use $[\ldots]$ for $\parr$ and $(\ldots)$ for $\otimes$.

(ii) We omit parentheses if they are superfluous under the assumption that $\parr$ and $\otimes$ associate to the left, e.g., $[A \parr B \parr C \parr D]$ abbreviates $[[[A \parr B] \parr C] \parr D]$.

(iii) The scope of a quantifier ends at the earliest possible place (and not at the latest possible place as usual). This helps saving unnecessary parentheses. For example, in $[\forall a.(a \otimes b) \parr \exists c.c \parr a]$, the scope of $\forall a$ is $(a \otimes b)$, and the scope of $\exists c$ is just c. In particular, the a at the end is free.

The inference rules for MLL2 are shown in Figure 1. In the following, we will call this system $\mathsf{MLL2}_{\mathsf{Seq}}$. As shown in [9], it has the cut elimination property:

2.1 Theorem *The cut rule* $\mathsf{cut}\;\dfrac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}$ *is admissible for* $\mathsf{MLL2}_{\mathsf{Seq}}$.

$$
\begin{array}{llll}
\mathsf{ai}{\downarrow}\,\dfrac{S\{1\}}{S[a^{\perp} \parr a]} &
{\perp}{\downarrow}\,\dfrac{S\{A\}}{S[\perp \parr A]} &
1{\downarrow}\,\dfrac{S\{A\}}{S(1 \otimes A)} &
\mathsf{e}{\downarrow}\,\dfrac{S\{1\}}{S\{\forall a.1\}} \\[2ex]
\alpha{\downarrow}\,\dfrac{S[[A \parr B] \parr C]}{S[A \parr [B \parr C]]} &
\sigma{\downarrow}\,\dfrac{S[A \parr B]}{S[B \parr A]} &
\mathsf{ls}\,\dfrac{S([A \parr B] \otimes C)}{S[A \parr (B \otimes C)]} &
\mathsf{rs}\,\dfrac{S(A \otimes [B \parr C])}{S[(A \otimes B) \parr C]} \\[2ex]
\mathsf{u}{\downarrow}\,\dfrac{S\{\forall a.[A \parr B]\}}{S[\forall a.A \parr \exists a.B]} &
\mathsf{n}{\downarrow}\,\dfrac{S\{A\langle a \backslash B \rangle\}}{S\{\exists a.A\}} &
\mathsf{f}{\downarrow}\,\dfrac{S\{\exists a.A\}}{S\{A\}}\ \text{a not free in A.} &
\end{array}
$$

Fig. 2. Deep inference system for MLL2

3 MLL2 in the calculus of structures

We now present a deductive system for MLL2 based on deep inference. We use the calculus of structures, in which the distinction between formulas and sequents disappears. This is the reason for the syntactic conventions introduced above.[1]

The inference rules work directly (as rewriting rules) on the formulas. The system for MLL2 is shown in Figure 2. There, $S\{\ \}$ stands for an arbitrary (positive) formula context. We omit the braces if the structural parentheses fill the hole. E.g., $S[A \parr B]$ abbreviates $S\{[A \parr B]\}$. The system in Figure 2 is called $\mathsf{MLL2}_{\mathsf{DI}\downarrow}$. We consider here only the so-called *down fragment* of the system, which corresponds to the cut-free system in the sequent calculus.[2] Note that the $\forall$-rule of $\mathsf{MLL2}_{\mathsf{Seq}}$ is in $\mathsf{MLL2}_{\mathsf{DI}\downarrow}$ decomposed into three pieces, namely, $\mathsf{e}{\downarrow}$, $\mathsf{u}{\downarrow}$, and $\mathsf{f}{\downarrow}$. We also need an explicit rule for associativity which is "built in" the sequent calculus. The relation between the $\otimes$-rule and the rules ls and rs (called *left switch* and *right switch*) has already in detail been investigated by several authors [20, 3, 8, 11]. The following theorem ensures that $\mathsf{MLL2}_{\mathsf{DI}\downarrow}$ is indeed a deductive system for MLL2.

3.1 Theorem *For every proof of* $\vdash A_1, \ldots, A_n$ *in* $\mathsf{MLL2}_{\mathsf{Seq}}$, *there is a proof of* $[A_1 \parr \cdots \parr A_n]$ *in* $\mathsf{MLL2}_{\mathsf{DI}\downarrow}$, *and vice versa.*

As for $\mathsf{MLL2}_{\mathsf{Seq}}$, we also have for $\mathsf{MLL2}_{\mathsf{DI}\downarrow}$ the cut elimination property, which can be stated as follows:

3.2 Theorem *The cut rule* $\mathsf{i}{\uparrow}\,\dfrac{S(A \otimes A^{\perp})}{S\{\perp\}}$ *is admissible for* $\mathsf{MLL2}_{\mathsf{DI}\downarrow}$.

[1] In the literature on deep inference, e.g., [5, 11], the formula $(a \otimes [b \parr (a^{\perp} \otimes c)])$ would be written as $(a, [b, (a^{\perp}, c)])$, while without our convention it would be written as $a \otimes (b \parr (a^{\perp} \otimes c))$. Our convention can therefore be seen as an attempt to please both communities. In particular, note that the motivation for the syntactic convention (iii) above is the collapse of the $\parr$ on the formula level and the comma on the sequent level, e.g., $[\forall a.(a \otimes b) \parr \exists c.c \parr a]$ is the same as $[\forall a.(a, b), \exists c.c, a]$.

[2] The *up fragment* (which corresponds to the cut in the sequent calculus) is obtained by dualizing the rules in the down fragment, i.e., by negating and exchanging premise and conclusion. See, e.g., [21, 4, 5, 13] for details.

$$
\mathsf{x}\,\frac{S\{\exists a.\forall b.A\}}{S\{\forall b.\exists a.A\}}
\qquad
\mathsf{y}{\downarrow}\,\frac{S\{\exists a.\exists b.A\}}{S\{\exists b.\exists a.A\}}
\qquad
\mathsf{v}{\downarrow}\,\frac{S\{\exists a.[A \parr B]\}}{S[\exists a.A \parr \exists a.B]}
\qquad
\mathsf{w}{\downarrow}\,\frac{S\{\exists a.(A \otimes B)\}}{S(\exists a.A \otimes \exists a.B)}
$$

$$
\mathsf{1f}{\downarrow}\,\frac{S\{\exists a.1\}}{S\{1\}}
\qquad
\bot\mathsf{f}{\downarrow}\,\frac{S\{\exists a.\bot\}}{S\{\bot\}}
\qquad
\mathsf{af}{\downarrow}\,\frac{S\{\exists a.b\}}{S\{b\}}
\qquad
\hat{\mathsf{a}}\mathsf{f}{\downarrow}\,\frac{S\{\exists a.b^{\perp}\}}{S\{b^{\perp}\}}
\qquad
\text{in } \mathsf{af}{\downarrow} \text{ and } \hat{\mathsf{a}}\mathsf{f}{\downarrow}\text{,}\ a \text{ is different from } b
$$

Fig. 3. Towards a local system for MLL2

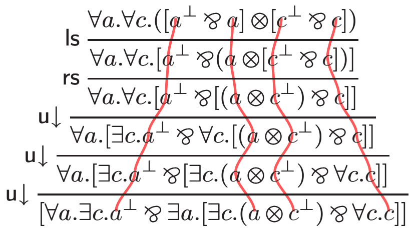
We write $\mathsf{MLL2_{DI\downarrow}}\ \begin{array}{c} A \\ \Vert\,\mathscr{D} \\ B \end{array}$ for denoting a derivation $\mathscr{D}$ in $\mathsf{MLL2_{DI\downarrow}}$ with premise A and conclusion B. The following decomposition theorem for $\mathsf{MLL2_{DI\downarrow}}$ can be seen as a version of Herbrand's theorem for MLL2 and has no counterpart in the sequent calculus.

3.3 Theorem

Every derivation $\mathsf{MLL2_{DI\downarrow}}\ \begin{array}{c} 1 \\ \Vert\,\mathscr{D} \\ C \end{array}$ *can be transformed into* $\begin{array}{rcl} & 1 & \\ \{\mathsf{ai}{\downarrow}, \bot{\downarrow}, 1{\downarrow}, \mathsf{e}{\downarrow}\} & \Vert & \mathscr{D}_1 \\ & A & \\ \{\alpha{\downarrow}, \sigma{\downarrow}, \mathsf{ls}, \mathsf{rs}, \mathsf{u}{\downarrow}\} & \Vert & \mathscr{D}_2 \\ & B & \\ \{\mathsf{n}{\downarrow}, \mathsf{f}{\downarrow}\} & \Vert & \mathscr{D}_3 \\ & C & \end{array}$.

This decomposition is obtained by permuting all instances of $\mathsf{ai}{\downarrow}, \bot{\downarrow}, 1{\downarrow}, \mathsf{e}{\downarrow}$ up and permuting all instances of $\mathsf{n}{\downarrow}, \mathsf{f}{\downarrow}$ down. There are two versions of the "switch" in $\mathsf{MLL2_{DI\downarrow}}$, the *left switch* ls, and the *right switch* rs. For Thm. 3.1, the ls-rule would be sufficient, but for obtaining the decomposition in Thm. 3.3 we also need the rs-rule.

If a derivation $\mathscr{D}$ uses only the rules $\alpha{\downarrow}, \sigma{\downarrow}, \mathsf{ls}, \mathsf{rs}, \mathsf{u}{\downarrow}$, then premise and conclusion of $\mathscr{D}$ (and every formula in between the two) must contain the same atom occurrences. Hence, the *atomic flow-graph* [6, 12] of the derivation $\mathscr{D}$ defines a bijection between the atom occurrences of premise and conclusion of $\mathscr{D}$. Here is an example of a derivation together with its flow-graph. (We left some some applications of $\alpha{\downarrow}$ and $\sigma{\downarrow}$ implicit.)



$$
\mathsf{u}{\downarrow}\,\dfrac{\mathsf{u}{\downarrow}\,\dfrac{\mathsf{u}{\downarrow}\,\dfrac{\mathsf{rs}\,\dfrac{\mathsf{ls}\,\dfrac{\forall a.\forall c.([a^{\perp} \parr a] \otimes [c^{\perp} \parr c])}{\forall a.\forall c.[a^{\perp} \parr (a \otimes [c^{\perp} \parr c])]}}{\forall a.\forall c.[a^{\perp} \parr [(a \otimes c^{\perp}) \parr c]]}}{\forall a.[\exists c.a^{\perp} \parr \forall c.[(a \otimes c^{\perp}) \parr c]]}}{\forall a.[\exists c.a^{\perp} \parr [\exists c.(a \otimes c^{\perp}) \parr \forall c.c]]}}{[\forall a.\exists c.a^{\perp} \parr \exists a.[\exists c.(a \otimes c^{\perp}) \parr \forall c.c]]}
\tag{1}
$$

In the sequent calculus the $\forall$-rule has a non-local behavior, in the sense that for applying the rule we need some global knowledge about the context Γ, namely, that the variable a does not appear freely in it. This is the reason for the boxes in [9] and the jumps in [10]. In the calculus of structures this "checking" whether a variable appears freely is done in the rule $\mathsf{f}{\downarrow}$, which is as non-local as the $\forall$-rule in the sequent calculus. However, with deep inference, this rule can be made local, i.e., reduced to an atomic version (in the same sense as the identity axiom can be reduced to an atomic version). For this, we need an additional set of rules which is shown in Figure 3 (again, we show only the down fragment), and which is called $\mathsf{Lf}\!\downarrow$. Clearly, all rules are sound, i.e., proper implications of MLL2. Now we have the following:

3.4 Theorem *Every derivation* $\{\mathsf{n}\!\downarrow,\mathsf{f}\!\downarrow\}\begin{array}{c}B\\ \Big\Vert\,\mathscr{D}\\ C\end{array}$ *can be transformed into* $\{\mathsf{n}\!\downarrow\}\cup\mathsf{Lf}\!\downarrow\begin{array}{c}B\\ \Big\Vert\,\mathscr{D}'\\ C\end{array}$ *, and vice versa.*

4 Proof nets for MLL2

For defining proof nets for MLL2, we follow the ideas presented in [23, 17] where the axiom linking of multiplicative proof nets has been replaced by a *linking formula* to accommodate the units 1 and $\bot$. In such a linking formula, the ordinary axiom links are replaced by $\otimes$-nodes, which are then connected by $\parr$s. A unit can then be attached to a sublinking by another $\otimes$, and so on. Here we extend the syntax for the linking formula by an additional construct to accommodate the quantifiers. Now, the set $\mathscr{L}$ of *linking formulas* is generated by the grammar

$$\mathscr{L} ::= \bot \mid (\mathscr{A}\otimes\mathscr{A}^\bot) \mid (1\otimes\mathscr{L}) \mid [\mathscr{L}\parr\mathscr{L}] \mid \exists\mathscr{A}.\mathscr{L}$$

In [23, 17] a proof net consists of the sequent forest and the linking formula. The presence of the quantifiers, in particular, the presence of instantiation and substitution, makes it necessary to expand the structure of the sequent in the proof net. The set $\mathscr{E}$ of *expanded formulas*[3] is generated by

$$\mathscr{E} ::= \bot \mid 1 \mid \mathscr{A} \mid \mathscr{A}^\bot \mid [\mathscr{E}\parr\mathscr{E}] \mid (\mathscr{E}\otimes\mathscr{E}) \mid \forall\mathscr{A}.\mathscr{E} \mid \exists\mathscr{A}.\mathscr{E} \mid \reflectbox{$\mathbb{E}$}\mathscr{A}.\mathscr{E} \mid \boldsymbol{\exists}\mathscr{A}.\mathscr{E}$$

There are only two additional syntactic primitives: the $\reflectbox{$\mathbb{E}$}$, called *virtual existential quantifier*, and the $\boldsymbol{\exists}$, called *bold existential quantifier*. An *expanded sequent* is a finite list of expanded formulas, separated by comma. We denote expanded sequents by capital Greek letters (Γ, Δ, ...). For disambiguation, the formulas/sequents introduced in Section 2 (i.e., those without $\reflectbox{$\mathbb{E}$}$ and $\boldsymbol{\exists}$) will also be called *simple formulas/sequents*.

In the following we will identify formulas with their syntax trees, where the leaves are decorated by elements of $\mathscr{A}\cup\mathscr{A}^\bot\cup\{1,\bot\}$. We can think of the inner nodes as decorated either with the connectives/quantifiers $\otimes$, $\parr$, $\forall a$, $\exists a$, $\boldsymbol{\exists}a$, $\reflectbox{$\mathbb{E}$}a$, or with the whole subformula rooted at that node. For this reason we will use capital Latin letters (A, B, C, ...) to denote nodes in a formula tree. We write $A \preccurlyeq B$ if A is a (not necessarily proper) ancestor of B, i.e., B is a subformula occurrence in A. We write $\leaf\Gamma$ (resp. $\leaf A$) for denoting the set of leaves of a sequent Γ (resp. formula A).

4.1 Definition A *stretching σ for a sequent Γ* consists of two binary relations $\overset{\sigma}{\underset{+}{\curvearrowright}}$ and $\overset{\sigma}{\underset{-}{\curvearrowright}}$ on the set of nodes of Γ (i.e., its subformula occurrences) such that $\overset{\sigma}{\underset{+}{\curvearrowright}}$ and $\overset{\sigma}{\underset{-}{\curvearrowright}}$ are disjoint, and whenever $A\overset{\sigma}{\underset{+}{\curvearrowright}}B$ or $A\overset{\sigma}{\underset{-}{\curvearrowright}}B$ then $A = \boldsymbol{\exists}a.A'$ with $A' \preccurlyeq B$ in Γ.

[3] This is almost the same structure as Miller's *expansion trees* [19]. The idea is to code a formula and its "expansion" together in the same syntactic object. But our case is simpler than in [19] because we do not have to deal with duplication.

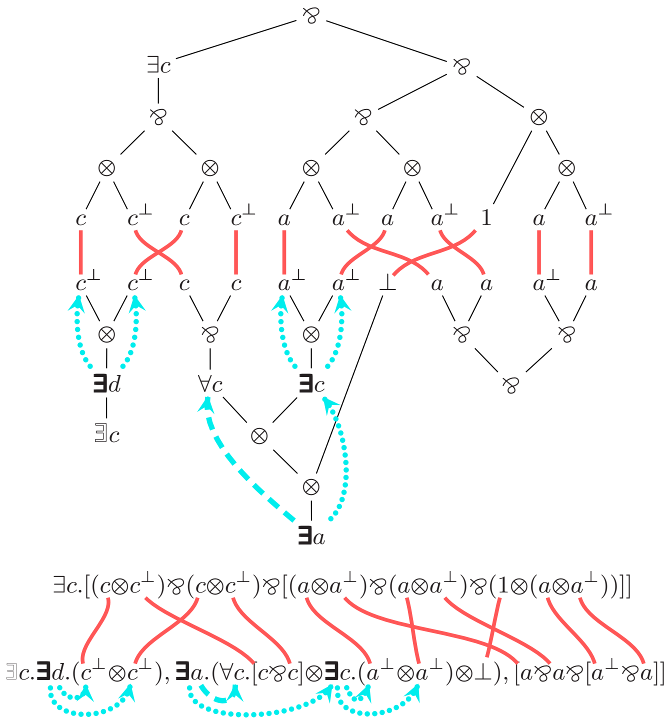

∃c.[(c⊗c⊥)⅋(c⊗c⊥)⅋[(a⊗a⊥)⅋(a⊗a⊥)⅋(1⊗(a⊗a⊥))]]
∃c.∃d.(c⊥⊗c⊥), ∃a.(∀c.[c⅋c]⊗∃c.(a⊥⊗a⊥)⊗⊥), [a⅋a⅋[a⊥⅋a]]

Fig. 4. Two ways of writing a proof graph

A stretching consists of edges connecting $\boldsymbol{\exists}$-nodes with some of its subformulas, and these edges can be positive or negative. Their purpose is to mark the places of the substitution of the atoms quantified by the $\boldsymbol{\exists}$. When writing an expanded sequent Γ with a stretching σ, denoted by $\Gamma \blacktriangleleft \sigma$, we will draw these edges either inside Γ when it is written as a tree, or below Γ when it is written as string. The positive edges are dotted and the negative ones are dashed. Examples are shown in Figures 4, 6 and 7 below.

4.2 Definition A *pre-proof graph*[4] is a quadruple, denoted by $P \overset{\nu}{\vartriangleright} \Gamma \blacktriangleleft \sigma$, where P a linking formula, Γ is an expanded sequent, σ is a stretching for Γ, and ν is a bijection $\oslash\Gamma \xrightarrow{\nu} \oslash P$ such that only dual atoms/units are paired up. If Γ is simple, we say that the pre-proof graph is *simple*. In this case σ is empty, and we can simply write $P \overset{\nu}{\vartriangleright} \Gamma$.

For $B \in \oslash\Gamma$ we write B^ν for its image under ν in $\oslash P$. When we draw a pre-proof graph $P \overset{\nu}{\vartriangleright} \Gamma \blacktriangleleft \sigma$, then we write P above Γ (as trees or as strings) and the leaves are connected by edges according to ν. Figure 4 shows an example written in both ways.

4.3 Definition A *switching* s of a pre-proof graph $P \overset{\nu}{\vartriangleright} \Gamma \blacktriangleleft \sigma$ is the graph that is obtained by removing all stretching edges and by removing for each $\parr$-node one of the two edges connecting it to its children. A pre-proof graph $P \overset{\nu}{\vartriangleright} \Gamma \blacktriangleleft \sigma$ is *multiplicatively correct* if all its switchings are acyclic and connected [7].

[4] The “pre-” means that we do not yet know whether it really comes from an actual proof. The concept of a “not yet proof” is in the literature (e.g., [7]) also called “proof structure”.

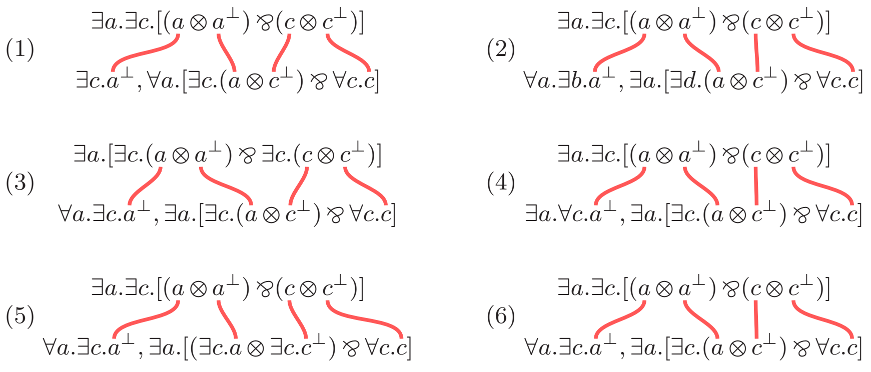

(1)
$\exists a.\exists c.[(a\otimes a^\perp)\parr(c\otimes c^\perp)]$
$\exists c.a^\perp, \forall a.[\exists c.(a\otimes c^\perp)\parr\forall c.c]$
(2)
$\exists a.\exists c.[(a\otimes a^\perp)\parr(c\otimes c^\perp)]$
$\forall a.\exists b.a^\perp, \exists a.[\exists d.(a\otimes c^\perp)\parr\forall c.c]$
(3)
$\exists a.[\exists c.(a\otimes a^\perp)\parr\exists c.(c\otimes c^\perp)]$
$\forall a.\exists c.a^\perp, \exists a.[\exists c.(a\otimes c^\perp)\parr\forall c.c]$
(4)
$\exists a.\exists c.[(a\otimes a^\perp)\parr(c\otimes c^\perp)]$
$\exists a.\forall c.a^\perp, \exists a.[\exists c.(a\otimes c^\perp)\parr\forall c.c]$
(5)
$\exists a.\exists c.[(a\otimes a^\perp)\parr(c\otimes c^\perp)]$
$\forall a.\exists c.a^\perp, \exists a.[(\exists c.a\otimes\exists c.c^\perp)\parr\forall c.c]$
(6)
$\exists a.\exists c.[(a\otimes a^\perp)\parr(c\otimes c^\perp)]$
$\forall a.\exists c.a^\perp, \exists a.[\exists c.(a\otimes c^\perp)\parr\forall c.c]$

Fig. 5. Examples (1)–(5) are not well-nested, only (6) is well-nested

For multiplicative correctness the quantifiers are treated as unary connectives and are therefore completely irrelevant. The example in Figure 4 is multiplicatively correct. For involving the quantifiers into a correctness criterion, we need some more conditions.

Let s be a switching for $P \overset{\nu}{\triangleright} \Gamma$, and let A and B be two nodes in Γ. We write $A \smile\!\circledS\!\smile B$ if there is a path in s from A to B, starting from A by going down to its parent and coming into B from below. Similarly, one can define the notations $A \frown\!\circledS\!\frown B$ and $A \frown\!\circledS\!\smile B$ and $A \smile\!\circledS\!\frown B$.

Let A and B be nodes in Γ with $A \preccurlyeq B$. The *quantifier depth* of B in A, denoted by $\nabla_A B$, is the number of quantifier nodes on the path from A to B (including A if it happens to be an $\forall$ or an $\exists$, but not including B). Similarly we define $\nabla_\Gamma B$. For quantifier nodes A' in P and A in Γ, we say A and A' are *partners*, denoted by $A' \overset{P\ \Gamma}{\leftarrow\!\blacksquare\!\rightarrow} A$, if there is a leaf $B \in \varnothing\Gamma$ with $A \preccurlyeq B$ in Γ, and $A' \preccurlyeq B^\nu$ in P, and $\nabla_A B = \nabla_{A'} B^\nu$.

4.4 Definition We say a simple pre-proof graph $P \overset{\nu}{\triangleright} \Gamma$ is *well-nested* if the following five conditions are satisfied:

1. For every $B \in \varnothing\Gamma$, we have $\nabla_\Gamma B = \nabla_P B^\nu$.
2. If $A' \overset{P\ \Gamma}{\leftarrow\!\blacksquare\!\rightarrow} A$, then A' and A quantify the same variable.
3. For every quantifier node A in Γ there is exactly one $\exists$-node A' in P with $A' \overset{P\ \Gamma}{\leftarrow\!\blacksquare\!\rightarrow} A$.
4. For every $\exists$-node A' in P there is exactly one $\forall$-node A in Γ with $A' \overset{P\ \Gamma}{\leftarrow\!\blacksquare\!\rightarrow} A$.
5. If $A' \overset{P\ \Gamma}{\leftarrow\!\blacksquare\!\rightarrow} A_1$ and $A' \overset{P\ \Gamma}{\leftarrow\!\blacksquare\!\rightarrow} A_2$, then there is no switching s with $A_1 \smile\!\circledS\!\smile A_2$.

Every quantifier node in P must be an $\exists$, and every quantifier node in Γ has exactly one of them as partner. On the other hand, an $\exists$ in P can have many partners in Γ, but exactly one of them has to be an $\forall$. Following Girard [9], we can call an $\exists$ in P together with its partners in Γ the *doors of an* $\forall$*-box* and the sub-graph induced by the nodes that have such a door as ancestor is called the $\forall$*-box* associated to the unique $\forall$-door. Even if the boxes are not really present, we can use the terminology to relate our work to Girard's. In order to help the reader to understand these five conditions, we show in Figure 5 six simple pre-proof graphs, where the first fails Condition 1, the second one fails Condition 2, and so on; only the sixth one is well-nested.

4.5 Definition A simple pre-proof graph $P \overset{\nu}{\triangleright} \Gamma$ is *correct* if it is well-nested and multiplicatively correct. In this case we will also speak of a *simple proof graph*.

Let us now turn our attention towards substitution, which is the *raison d'être* for the expansion with $\boldsymbol{\exists}$ and $\mathbb{\exists}$.

4.6 Definition For an expanded formula E and a stretching σ, we define the *ceiling* and the *floor*[5], denoted by $\lceil E \blacktriangleleft \sigma \rceil$ and $\lfloor E \blacktriangleleft \sigma \rfloor$, respectively, to be simple formulas, which are inductively defined as follows:

$$\begin{array}{rll}
\lceil 1 \blacktriangleleft \emptyset \rceil = 1 & \lceil A \parr B \blacktriangleleft \sigma \rceil = \lceil A \blacktriangleleft \sigma' \rceil \parr \lceil B \blacktriangleleft \sigma'' \rceil & \\
\lceil \bot \blacktriangleleft \emptyset \rceil = \bot & \lceil A \otimes B \blacktriangleleft \sigma \rceil = \lceil A \blacktriangleleft \sigma' \rceil \otimes \lceil B \blacktriangleleft \sigma'' \rceil & \\
\lceil a \blacktriangleleft \emptyset \rceil = a & \lceil \forall a.A \blacktriangleleft \sigma \rceil = \forall a.\lceil A \blacktriangleleft \sigma \rceil & \lceil \mathbb{\exists} a.A \blacktriangleleft \sigma \rceil = \exists a.\lceil A \blacktriangleleft \sigma \rceil \\
\lceil a^\bot \blacktriangleleft \emptyset \rceil = a^\bot & \lceil \exists a.A \blacktriangleleft \sigma \rceil = \exists a.\lceil A \blacktriangleleft \sigma \rceil & \lceil \boldsymbol{\exists} a.A \blacktriangleleft \sigma \rceil = \lceil A \blacktriangleleft \sigma' \rceil
\end{array}$$

$$\begin{array}{rll}
\lfloor 1 \blacktriangleleft \emptyset \rfloor = 1 & \lfloor A \parr B \blacktriangleleft \sigma \rfloor = \lfloor A \blacktriangleleft \sigma' \rfloor \parr \lfloor B \blacktriangleleft \sigma'' \rfloor & \\
\lfloor \bot \blacktriangleleft \emptyset \rfloor = \bot & \lfloor A \otimes B \blacktriangleleft \sigma \rfloor = \lfloor A \blacktriangleleft \sigma' \rfloor \otimes \lfloor B \blacktriangleleft \sigma'' \rfloor & \\
\lfloor a \blacktriangleleft \emptyset \rfloor = a & \lfloor \forall a.A \blacktriangleleft \sigma \rfloor = \forall a.\lfloor A \blacktriangleleft \sigma \rfloor & \lfloor \mathbb{\exists} a.A \blacktriangleleft \sigma \rfloor = \lfloor A \blacktriangleleft \sigma \rfloor \\
\lfloor a^\bot \blacktriangleleft \emptyset \rfloor = a^\bot & \lfloor \exists a.A \blacktriangleleft \sigma \rfloor = \exists a.\lfloor A \blacktriangleleft \sigma \rfloor & \lfloor \boldsymbol{\exists} a.A \blacktriangleleft \sigma \rfloor = \exists a.\lfloor \tilde{A} \blacktriangleleft \tilde{\sigma} \rfloor
\end{array}$$

where σ' is the restriction of σ to A, and σ'' is the restriction of σ to B. The expanded formula $\tilde{A}$ is obtained from A as follows: For every node B with $A \preccurlyeq B$ and $\boldsymbol{\exists} a.A \overset{\sigma}{\underset{+}{\curvearrowright}} B$ remove the whole subtree B and replace it by a, and for every B with $\boldsymbol{\exists} a.A \overset{\sigma}{\underset{-}{\curvearrowright}} B$ replace B by $a^\bot$. The stretching $\tilde{\sigma}$ is the restriction of σ to $\tilde{A}$.

Note that ceiling and floor of an expanded sequent Γ differ from Γ only on $\boldsymbol{\exists}$ and $\mathbb{\exists}$. In the ceiling, the $\mathbb{\exists}$ is treated as ordinary $\exists$, and the $\boldsymbol{\exists}$ is completely ignored. In the floor, the $\mathbb{\exists}$ is ignored, and the $\boldsymbol{\exists}$ uses the information of the stretching to "undo the substitution". To provide this information on the location is the purpose of the stretching. To ensure that we really only "undo the substitution" instead of doing something weird, we need some further constraints, which are given by Definition 4.7 below.

We write $A \curvearrowright B$ if A is a $\boldsymbol{\exists}$-node and there is a stretching edge from A to B, or A is an ordinary quantifier node and B is the variable (or its negation) that is bound in A and $A \preccurlyeq B$.

4.7 Definition A pair $\Gamma \blacktriangleleft \sigma$ is *appropriate*, if the following three conditions hold:

1. If $A \overset{\sigma}{\underset{+}{\curvearrowright}} B_1$ and $A \overset{\sigma}{\underset{+}{\curvearrowright}} B_2$, then $\lfloor B_1 \blacktriangleleft \sigma_1 \rfloor = \lfloor B_2 \blacktriangleleft \sigma_2 \rfloor$,
   if $A \overset{\sigma}{\underset{-}{\curvearrowright}} B_1$ and $A \overset{\sigma}{\underset{-}{\curvearrowright}} B_2$, then $\lfloor B_1 \blacktriangleleft \sigma_1 \rfloor = \lfloor B_2 \blacktriangleleft \sigma_2 \rfloor$,
   if $A \overset{\sigma}{\underset{+}{\curvearrowright}} B_1$ and $A \overset{\sigma}{\underset{-}{\curvearrowright}} B_2$, then $\lfloor B_1 \blacktriangleleft \sigma_1 \rfloor = \lfloor B_2 \blacktriangleleft \sigma_2 \rfloor^\bot$, (where σ_1 and σ_2 are the restrictions of σ to B_1 and B_2, respectively).
2. If $A_1 \curvearrowright B_1$ and $A_2 \curvearrowright B_2$ and $A_1 \preccurlyeq A_2$ and $B_1 \preccurlyeq B_2$, then $B_1 \preccurlyeq A_2$.
3. For all $\mathbb{\exists} a.A$, the variable a must not occur free in the formula $\lfloor A \blacktriangleleft \sigma' \rfloor$ (where σ' is the restriction of σ to A).

The first condition above says that in a substitution a variable is instantiated everywhere by the same formula B. The second condition ensures that there is no variable capturing in such a substitution step. The third condition is exactly the side condition of the rule $\mathsf{f}\!\downarrow$ in Figure 2. For better explaining the three conditions above, we show in

[5] Note the close correspondece to Miller's expansion trees [19], where these two functions are called *Deep* and *Shallow*, respectively.

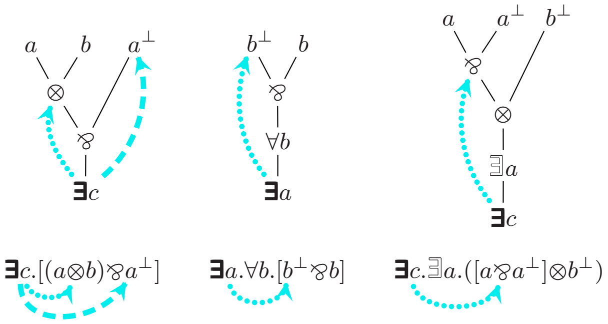


Fig. 6. Examples of expanded sequents with stretchings that are not appropriate

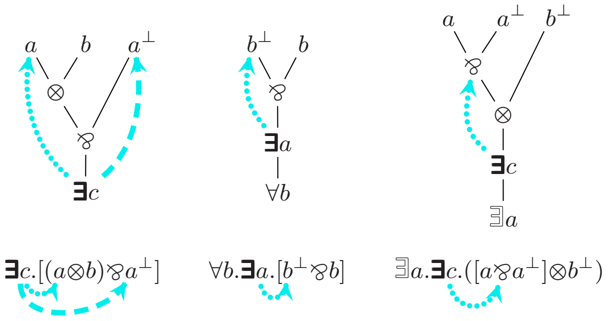


Fig. 7. Appropriate examples of expanded sequents with stretchings

Figure 6 three examples of pairs $\Gamma \blacktriangleleft \sigma$ that are not appropriate: the first fails Condition 1, the second fails Condition 2, and the third fails Condition 3. In Figure 7 all three examples are appropriate. The example in Figure 4 is also appropriate.

In [9] and [10], the first two conditions of Definition 4.7 appear only implicitly without being mentioned in the treatment of the $\exists$-rule. But for capturing the essence of a proof independently of a deductive system, we have to make everything explicit.

4.8 Definition We say that a pre-proof graph $P \overset{\nu}{\vartriangleright} \Gamma \blacktriangleleft \sigma$ is *correct* if the simple pre-proof graph $P \overset{\nu}{\vartriangleright} \lceil \Gamma \blacktriangleleft \sigma \rceil$ is correct and the pair $\Gamma \blacktriangleleft \sigma$ is appropriate. In this case we say that $P \overset{\nu}{\vartriangleright} \Gamma \blacktriangleleft \sigma$ is a *proof graph* and $\lfloor \Gamma \blacktriangleleft \sigma \rfloor$ is its *conclusion*.

The example in Figure 4 is correct. There we have that $\lceil \Gamma \blacktriangleleft \sigma \rceil$ is the simple sequent $\vdash \exists c.(c^\perp \otimes c^\perp), (\forall c.[c \parr c] \otimes (a^\perp \otimes a^\perp) \otimes \perp), [a \parr a \parr [a^\perp \parr a]]$ and the conclusion $\lfloor \Gamma \blacktriangleleft \sigma \rfloor$ is $\vdash \exists d.(d \otimes d), \exists a.(a^\perp \otimes a \otimes \perp), [a \parr a \parr [a^\perp \parr a]]$.

Due to the presence of the multiplicative units (see [23, 17]), we need to enforce an equivalence relation on proof graphs.

4.9 Definition Let $\sim$ be the smallest equivalence on proof graphs satisfying

$$\begin{array}{rcl}
P[Q \parr R] \overset{\nu}{\vartriangleright} \Gamma \blacktriangleleft \sigma & \sim & P[R \parr Q] \overset{\nu}{\vartriangleright} \Gamma \blacktriangleleft \sigma \\
P[[Q \parr R] \parr S] \overset{\nu}{\vartriangleright} \Gamma \blacktriangleleft \sigma & \sim & P[Q \parr [R \parr S]] \overset{\nu}{\vartriangleright} \Gamma \blacktriangleleft \sigma \\
P(1 \otimes (1 \otimes Q)) \overset{\nu}{\vartriangleright} \Gamma \blacktriangleleft \sigma & \sim & P(1 \otimes (1 \otimes Q)) \overset{\nu'}{\vartriangleright} \Gamma \blacktriangleleft \sigma \\
P(1 \otimes [Q \parr R]) \overset{\nu}{\vartriangleright} \Gamma \blacktriangleleft \sigma & \sim & P[(1 \otimes Q) \parr R] \overset{\nu}{\vartriangleright} \Gamma \blacktriangleleft \sigma \\
P(1 \otimes \exists a.Q) \overset{\nu}{\vartriangleright} \Gamma\{\perp\} \blacktriangleleft \sigma & \sim & P\{\exists a.(1 \otimes Q)\} \overset{\nu}{\vartriangleright} \Gamma\{\mathbb{\exists} a.\perp\} \blacktriangleleft \sigma
\end{array}$$

$$\mathsf{id}\,\frac{}{a \otimes a^{\perp} \overset{\nu_0}{\triangleright} a^{\perp}, a \blacktriangleleft \emptyset} \qquad \perp\,\frac{P \overset{\nu}{\triangleright} \Gamma \blacktriangleleft \sigma}{(1 \otimes P) \overset{\nu_{\perp}}{\triangleright} \Gamma, \perp \blacktriangleleft \sigma} \qquad 1\,\frac{}{\perp \overset{\nu_1}{\triangleright} 1 \blacktriangleleft \emptyset}$$

$$\mathsf{exch}\,\frac{P \overset{\nu}{\triangleright} \Gamma, A, B, \Delta \blacktriangleleft \sigma}{P \overset{\nu}{\triangleright} \Gamma, B, A, \Delta \blacktriangleleft \sigma} \qquad \parr\,\frac{P \overset{\nu}{\triangleright} A, B, \Gamma \blacktriangleleft \sigma}{P \overset{\nu}{\triangleright} [A \parr B], \Gamma \blacktriangleleft \sigma} \qquad \otimes\,\frac{P \overset{\nu}{\triangleright} \Gamma, A \blacktriangleleft \sigma \qquad Q \overset{\nu'}{\triangleright} B, \Delta \blacktriangleleft \tau}{[P \parr Q] \overset{\nu\cup\nu'}{\triangleright} \Gamma, (A \otimes B), \Delta \blacktriangleleft \sigma \cup \tau}$$

$$\forall\,\frac{P \overset{\nu}{\triangleright} A, B_1, \ldots, B_n \blacktriangleleft \sigma}{\exists a.P \overset{\nu}{\triangleright} \forall a.A, \mathbb{\exists} a.B_1, \ldots, \mathbb{\exists} a.B_n \blacktriangleleft \sigma} \qquad \exists\,\frac{P \overset{\nu}{\triangleright} \Gamma, A\langle a \backslash B\rangle \blacktriangleleft \sigma}{P \overset{\nu}{\triangleright} \Gamma, \boldsymbol{\exists} a.A\langle a \backslash B\rangle \blacktriangleleft \sigma'} \qquad \mathsf{cut}\,\frac{P \overset{\nu}{\triangleright} \Gamma, A \blacktriangleleft \sigma \qquad Q \overset{\nu'}{\triangleright} A^{\perp}, \Delta \blacktriangleleft \tau}{[P \parr Q] \overset{\nu\cup\nu'}{\triangleright} \Gamma, (A \varobar A^{\perp}), \Delta \blacktriangleleft \sigma \cup \tau}$$

Fig. 8. Translating sequent calculus proofs into proof nets

where in the third line ν' is obtained from ν by exchanging the preimages of the two 1s. In all other equations the bijection ν does not change. In the last line ν must match the 1 and $\perp$. A *proof net* is an equivalence class of $\sim$.

The first two equations in Definition 4.9 are simply associativity and commutativity of $\parr$ inside the linking. The third is a version of associativity of $\otimes$. The fourth equation could destroy multiplicative correctness, but since we defined $\sim$ only on proof graphs we do not need to worry about that.[6] The last equation says that a $\perp$ can freely tunnel through the borders of a box. Let us emphasize that this quotienting via an equivalence is due to the multiplicative units. If one wishes to use a system without units, one could completely dispose the equivalence by using n-ary $\parr$s in the linking.

5 Sequentialisation

In this section we will discuss how we can translate proofs in the sequent calculus and the calculus of structures into proof nets and back.

Let us begin with the sequent calculus. The translation from $\mathsf{MLL2}_{\mathsf{Seq}}$ proofs into proof graphs is done inductively on the structure of the sequent proof as shown in Figure 8. For the rules id and 1, this is trivial (ν_0 and ν_1 are uniquely determined and the stretching is empty). In the rule $\perp$, the $\nu_{\perp}$ is obtained from ν by adding an edge between the new 1 and $\perp$. The exch and $\parr$-rules are also rather trivial (P, ν, and σ remain unchanged). For the $\otimes$ rule, the two linkings are connected by a new $\parr$-node, and the two principal formulas are connected by a $\otimes$ in the sequent forest. The same is done for the cut rule, where we use a special cut connective $\varobar$. The two interesting rules are the ones for $\forall$ and $\exists$. In the $\forall$-rule, to every root node of the proof graph for the premise a quantifier node is attached. This is what ensures the well-nestedness condition. It can be compared to Girard's putting a box around a proof net. The purpose of the $\mathbb{\exists}$ can be interpreted as simulating the border of the box. The $\exists$-rule is the only one where

[6] In [23, 17] the relation $\sim$ is defined on pre-proof graphs, and therefore a side condition had to be given to that equation (see also [14]).

the stretching σ is changed. As shown in Figure 1, in the conclusion of that rule, the subformula B of A is replaced by the quantified variable a. When translating this rule into proof graphs, we keep the B, but to every place where it has to be substituted we add a positive stretching edge from the new $\boldsymbol{\exists}a$. Similarly, whenever a $B^\perp$ should be replaced by $a^\perp$, we add a negative stretching edge. The new stretching is σ'.

A pre-proof graph is *SC-sequentializable* if it can be obtained from a sequent proof as described above. If a pre-proof graph $P \overset{\nu}{\triangleright} \Gamma \blacktriangleleft \sigma$ is obtained this way then the simple sequent $\lfloor \Gamma \blacktriangleleft \sigma \rfloor$ is exactly the conclusion of the sequent proof we started from.

5.1 Theorem *Every SC-sequentializable pre-proof graph is a proof graph.*

For the other direction, i.e, for going from proof graphs to $\mathsf{MLL2}_{\mathsf{Seq}}$ proofs we need to consider two linking formulas P_1 and P_2 to be equivalent modulo associativity and commutativity of $⅋$. We write this as $P_1 \overset{⅋}{\sim} P_2$. Then, we have to remove all $\exists$-nodes from Γ in order to get a sequentialization theorem because the translation shown in Figure 8 never introduces an $\exists$-node in Γ. For this we replace in Γ every $\exists a.A$ with $\exists a.\boldsymbol{\exists}a.A$ and by adding a stretching edge between the new $\boldsymbol{\exists}a$ and every a and $a^\perp$ that was previously bound by $\exists a$ (i.e, is free in A). Let us write $\widehat{\Gamma \blacktriangleleft \sigma}$ for the result of this modification applied to $\Gamma \blacktriangleleft \sigma$.

5.2 Theorem *If $P \overset{\nu}{\triangleright} \Gamma \blacktriangleleft \sigma$ is correct, then there is a $P' \overset{⅋}{\sim} P$, such that $P' \overset{\nu}{\triangleright} \widehat{\Gamma \blacktriangleleft \sigma}$ is SC-sequentializable.*

The proof works in the usual way by induction on the size of $P \overset{\nu}{\triangleright} \Gamma \blacktriangleleft \sigma$. It is a combination of the sequentialization proofs in [17] and [9], and it makes crucial use of the "splitting tensor lemma" which in our case also needs well-nestedness.

Let us now discuss the translation between proof nets and derivations in the calculus of structures. This can be done in a more modular way than for the sequent calculus.

5.3 Proposition *An* $\mathsf{MLL2}$ *formula P is a linking formula if and only if there is a derivation*

$$\begin{array}{c} 1 \\ \{\mathsf{ai}\!\downarrow, \bot\!\downarrow, 1\!\downarrow, \mathsf{e}\!\downarrow\} \Big\Vert \mathscr{D} \\ P^\perp \end{array} \quad . \tag{2}$$

5.4 Lemma *Let P_1 and P_2 be two linkings. Then there is a derivation*

$$\begin{array}{c} P_1 \\ \{\alpha\!\downarrow, \sigma\!\downarrow, \mathsf{rs}\} \Big\Vert \mathscr{D} \\ P_2 \end{array}$$

if and only if the simple pre-proof graph $P_2 \triangleright P_1^\perp$ is correct.

If P_1 and P_2 have this property, we say that P_1 *is weaker than* P_2, and denote it as $P_1 \lesssim P_2$. We can now characterize simple proof graphs in terms of deep inference:

5.5 Proposition *A simple pre-proof graph $P \overset{\nu}{\triangleright} \Gamma$ is correct if and only if there is a linking P' with $P' \lesssim P$ and a derivation*

$$\begin{array}{c} P'^\perp \\ \{\alpha\!\downarrow, \sigma\!\downarrow, \mathsf{ls}, \mathsf{rs}, \mathsf{u}\!\downarrow\} \Big\Vert \mathscr{D} \\ \Gamma \end{array} \quad , \tag{3}$$

such that ν coincides with the bijection induced by the flow graph of $\mathscr{D}$.

As an example, consider the derivation in (1) which corresponds to (6) in Figure 5.

Finally, we characterize appropriate pairs $\Gamma \blacktriangleleft \sigma$ in terms of deep inference.

5.6 Proposition *For every derivation*

$$\begin{array}{c} D \\ \{\mathsf{n}\!\downarrow, \mathsf{f}\!\downarrow\} \ \Big\Vert\ \mathscr{D} \\ C \end{array} \tag{4}$$

there is an appropriate pair $\Gamma \blacktriangleleft \sigma$ with

$$D = \lceil \Gamma \blacktriangleleft \sigma \rceil \quad \textit{and} \quad C = \lfloor \Gamma \blacktriangleleft \sigma \rfloor \quad . \tag{5}$$

Conversely, if $\Gamma \blacktriangleleft \sigma$ is appropriate, then there is a derivation (4) *with* (5)*.*

We can explain the idea of this proposition by considering again the examples in Figures 6 and 7. To the non-appropriate examples in Figure 6 would correspond the following **incorrect** derivations:

$$\mathsf{n}\!\downarrow \frac{[(a \otimes b) \parr a^{\perp}]}{\exists c.[c \parr c^{\perp}]} \qquad \mathsf{n}\!\downarrow \frac{\forall b.[b^{\perp} \parr b]}{\exists a.\forall b.[a \parr b]} \qquad \begin{array}{r} \mathsf{f}\!\downarrow \dfrac{\exists a.([a \parr a^{\perp}] \otimes b^{\perp})}{([a \parr a^{\perp}] \otimes b^{\perp})} \\ \mathsf{n}\!\downarrow \dfrac{\phantom{([a \parr a^{\perp}] \otimes b^{\perp})}}{\exists c.(c \otimes b^{\perp})} \end{array}$$

And to the appropriate examples in Figure 7 correspond the following **correct** derivations:

$$\mathsf{n}\!\downarrow \frac{[(a \otimes b) \parr a^{\perp}]}{\exists c.[(c \otimes b) \parr c^{\perp}]} \qquad \mathsf{n}\!\downarrow \frac{\forall b.[b^{\perp} \parr b]}{\forall b.\exists a.[a \parr b]} \qquad \begin{array}{r} \mathsf{n}\!\downarrow \dfrac{\exists a.([a \parr a^{\perp}] \otimes b^{\perp})}{\exists a.\exists c.(c \otimes b^{\perp})} \\ \mathsf{f}\!\downarrow \dfrac{\phantom{\exists a.\exists c.(c \otimes b^{\perp})}}{\exists c.(c \otimes b^{\perp})} \end{array}$$

We can now easily translate a $\mathsf{MLL2}_{\mathsf{DI}\downarrow}$ proof into a pre-proof graph by first decomposing it via Theorem 3.3 and then applying Propositions 5.3, 5.5, and 5.6. Let us call a pre-proof graph *DI-sequentializable* if is obtained in this way from a $\mathsf{MLL2}_{\mathsf{DI}\downarrow}$ proof.

5.7 Theorem *Every DI-sequentializable pre-proof graph is a proof graph.*

By the method presented in [22], it is also possible to translate a $\mathsf{MLL2}_{\mathsf{DI}\downarrow}$ directly into a proof graph without prior decomposition. However, the decomposition is the key for the translation from proof graphs into $\mathsf{MLL2}_{\mathsf{DI}\downarrow}$ proofs (i.e., "sequentialization" into the calculus of structures). Propositions 5.3, 5.5, and 5.6 give us the following:

5.8 Theorem *If $P \overset{\nu}{\triangleright} \Gamma \blacktriangleleft \sigma$ is correct, then there is a $P' \lesssim P$, such that $P' \overset{\nu}{\triangleright} \Gamma \blacktriangleleft \sigma$ is DI-sequentializable.*

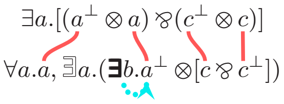
There is an important difference between the two sequentializations. While for the sequent calculus we have a monolithic procedure reducing the proof graph node by node, we have for the calculus of structures a modular procedure that treats the different parts of the proof graph (which correspond to the three different aspects of the logic) separately. The core is Proposition 5.5 which deals with the purely multiplicative part. Then comes Proposition 5.6 which only deals with instantiation and substitution, i.e, the second-order aspect. Finally, Proposition 5.3 takes care of the linking, whose task is to describe the role of the units in the proof. Therefore the equivalence in 4.9, which is due to the mobility if the $\perp$, only deals with the linkings. This modularity in the sequentialization is possible because of the decomposition in Theorem 3.3. Because of this modularity we treated the units via the linking formulas [23, 17] instead of a linking function as done by Hughes in [15, 14].

6 Comparison to Girard's proof nets for MLL2

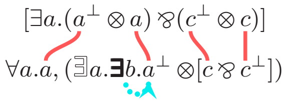
Such a comparison can only make sense for $\mathsf{MLL2}^-$, i.e., the logic without the units 1 and $\bot$. In [10] the units are not considered, and in [9] the units are treated in a way that is completely different from the one suggested here. Consequently, in this section we consider only proof nets without any occurrences of 1 and $\bot$. For simplicity, we will allow n-ary $\parr$s in the linkings, so that we can discard the equivalence relation of Definition 4.9 and identify proof graphs and proof nets.

The translation from our proof nets to Girard's boxed proof nets of [9] is immediate: If $P \overset{\nu}{\triangleright} \Gamma \blacktriangleleft \sigma$ is a given proof net, then (1) for each $\exists$ in P draw a box around the sub-proof net which has as doors this $\exists$ and its partners in Γ; (2) replace in Γ every node A that is not a $\mathbb{\exists}$ by its floor $\lfloor A \blacktriangleleft \sigma \rfloor$, and remove all stretching edges and all $\mathbb{\exists}$-nodes, and finally (3) remove all $\exists$- and all $\parr$-nodes in P, and replace the $\otimes$-nodes in P by axiom links. For the converse translation we proceed in the opposite order. It is clear that in both directions correctness is preserved, i.e., the two criteria are equivalent. Both data structures contain the same information. However, Girard's boxed proof nets depend on the deductive structure of the sequent calculus. A box stands for the global view that the $\forall$-rule has in the sequent calculus, and the $\exists$-link is attached to it full premise and conclusion that are subject to the same side conditions as in the sequent calculus. The new proof nets presented in this paper make these side conditions explicit in the data structure, which is the reason why our definitions are a bit longer than Girard's.

The proof nets of [10] are obtained from the box proof nets by simply removing the boxes. In our setting this is equivalent to removing all $\exists$-nodes in P and all $\mathbb{\exists}$-nodes in Γ. Hence, this new data structure contains less information. This raises the question whether the other two representations contain reduntant data or whether Girard's box-free proof nets make more identifications, and whether the missing data can be recovered. The answer is that the proof nets of [10] make indeed more proof identifications. For example the following proofs of $\vdash \forall a.a, (\exists b.b \otimes [c \parr c^\bot])$ would be identified:


$\exists a.[(a^\bot \otimes a) \parr (c^\bot \otimes c)]$
$\forall a.a, \mathbb{\exists} a.(\boldsymbol{\exists} b.a^\bot \otimes [c \parr c^\bot])$

and


$[\exists a.(a^\bot \otimes a) \parr (c^\bot \otimes c)]$
$\forall a.a, (\mathbb{\exists} a.\boldsymbol{\exists} b.a^\bot \otimes [c \parr c^\bot])$

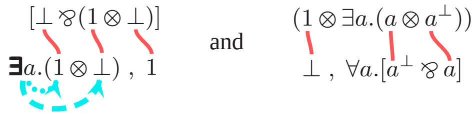
(6)

When translating back to box-nets, we must for each $\forall$-link introduce a box around its whole empire. This can be done because a proof net does not lose its correctness if a $\forall$-box is extended to a larger (correct) subnet, provided the bound variable does not occur freely in the new scope. In [10], Girard avoids this by variable renaming. The reason why this gives unique representants is the stability and uniqueness of empires in MLL^- proof nets. However, as already noted in [17], under the presence of the units, empires are no longer stable, i.e., due to the mobility of the $\bot$ the empire of an $\forall$-node might be different in different proof graphs, representing the same proof net.

Another reason for not using the solution of [10] is the desire to find a treatment for the quantifiers that is independent from the underlying propositional structure, i.e., that is also applicable to classical logic. While Girard's nets are tightly connected to the structure of MLL^--proof nets, our presentation is closely related to Miller's expansion trees [19] and the recent development by McKinley [18]. Thus, we can hope for a unified treatment of quantifiers in classical and linear logic.

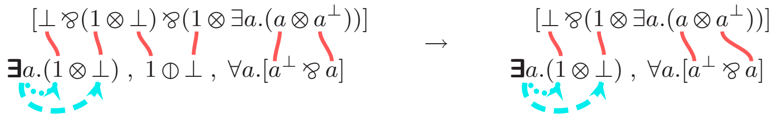
7 Concluding Remarks

We have investigated the relation between deep inference and proof nets and the sequent calculus for MLL2, and we have shown that this relation is much closer than one might expect. We did not go into the details of cut elimination because from the previous sections it should be clear that everything works as laid out in [9, 10] and [17, 23]. There are no technical surprises, and we have a confluent and terminating cut elimination procedure for our proof nets. An important consequence is that we have a category of proof nets: the objects are (simple) formulas and a map $A \to B$ is a proof net with conclusion $\vdash A^{\perp}, B$, where the composition of maps is defined by cut elimination. A detailed investigation of this category (which is *-autonomous [17]) has to be postponed to future research. The proof identifications made in this paper are motivated by the interplay between proof nets, calculus of structures, and sequent calculus. They should not be considered to be the final word. For example the proof nets by Girard [10] make more identifications, and the ones by Hughes [15] make less identifications. However, there are some observations about the units to be made here. The units can be expressed with the second-order quantifiers via $1 \equiv \forall a.[a^{\perp} \parr a]$ and $\perp \equiv \exists a.(a \otimes a^{\perp})$. An interesting question to ask is whether these logical equivalences should be isomorphisms in the categorification of the logic. In the category of coherent spaces [9] they are, but in our category of proof nets they are not: The two canonical maps $\forall a.[a^{\perp} \parr a] \to 1$ and $1 \to \forall a.[a^{\perp} \parr a]$ are given by:


$[\perp \parr (1 \otimes \perp)]$
$\boldsymbol{\exists} a.(1 \otimes \perp)$, 1
and
$(1 \otimes \exists a.(a \otimes a^{\perp}))$
$\perp$, $\forall a.[a^{\perp} \parr a]$

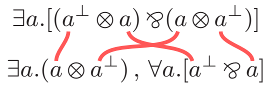
(7)

respectively. Composing them means performing this cut eliminating:


$[\perp \parr (1 \otimes \perp) \parr (1 \otimes \exists a.(a \otimes a^{\perp}))]$
$\boldsymbol{\exists} a.(1 \otimes \perp)$, $1 \oplus \perp$, $\forall a.[a^{\perp} \parr a]$
$\to$
$[\perp \parr (1 \otimes \exists a.(a \otimes a^{\perp}))]$
$\boldsymbol{\exists} a.(1 \otimes \perp)$, $\forall a.[a^{\perp} \parr a]$

(8)

If the two maps in (7) where isos, the result of (8) must be the same as the identity map $\forall a.[a^{\perp} \parr a] \to \forall a.[a^{\perp} \parr a]$ which is represented by the proof net


$\exists a.[(a^{\perp} \otimes a) \parr (a \otimes a^{\perp})]$
$\exists a.(a \otimes a^{\perp})$, $\forall a.[a^{\perp} \parr a]$

(9)

This is obviously not the case (even if we replaced $\exists a$ by $\exists\!\!\exists a.\boldsymbol{\exists} a$ as for Theorem 5.2). A similar situation occurs with the additive units, for which we have $0 \equiv \forall a.a$ and $\top \equiv \exists a.a$. Since we do not have 0 and $\top$ in the language, we cannot check whether we have these isos in our category. However, since 0 and $\top$ are commonly understood as initial and terminal objects of the category of proofs, we could ask whether $\forall a.a$ and $\exists a.a$ have this property: We clearly have a canonical proof for $\forall a.a \to A$ for every formula A, but it is *not* necessarily unique. The correct treatment of additive units in proof nets is still an open problem for future research.

References

1. Michael Barr. **-Autonomous Categories*, volume 752 of *Lecture Notes in Mathematics*. Springer-Verlag, 1979.
2. Gianluigi Bellin and Jacques van de Wiele. Subnets of proof-nets in MLL$^-$. In J.-Y. Girard, Y. Lafont, and L. Regnier, editors, *Advances in Linear Logic*, volume 222 of *London Mathematical Society Lecture Notes*, pages 249–270. Cambridge University Press, 1995.
3. Richard Blute, Robin Cockett, Robert Seely, and Todd Trimble. Natural deduction and coherence for weakly distributive categories. *Journal of Pure and Applied Algebra*, 113:229–296, 1996.
4. Kai Brünnler. *Deep Inference and Symmetry for Classical Proofs*. PhD thesis, Technische Universität Dresden, 2003.
5. Kai Brünnler and Alwen Fernanto Tiu. A local system for classical logic. In R. Nieuwenhuis and A. Voronkov, editors, *LPAR 2001*, volume 2250 of *LNAI*, pages 347–361. Springer-Verlag, 2001.
6. Samuel R. Buss. The undecidability of k-provability. *Annals of Pure and Applied Logic*, 53:72–102, 1991.
7. Vincent Danos and Laurent Regnier. The structure of multiplicatives. *Annals of Mathematical Logic*, 28:181–203, 1989.
8. H. Devarajan, Dominic Hughes, Gordon Plotkin, and Vaughan R. Pratt. Full completeness of the multiplicative linear logic of Chu spaces. In *14th IEEE Symposium on Logic in Computer Science (LICS 1999)*, 1999.
9. Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.
10. Jean-Yves Girard. Quantifiers in linear logic II. Preépublication de l'Equipe de Logique, Université Paris VII, Nr. 19, 1990.
11. Alessio Guglielmi. A system of interaction and structure. *ACM Transactions on Computational Logic*, 8(1), 2007.
12. Alessio Guglielmi and Tom Gundersen. Normalisation control in deep inference via atomic flows. Accepted for publication in *LMCS*, 2007.
13. Alessio Guglielmi and Lutz Straßburger. Non-commutativity and MELL in the calculus of structures. In Laurent Fribourg, editor, *Computer Science Logic, CSL 2001*, volume 2142 of *LNCS*, pages 54–68. Springer-Verlag, 2001.
14. Dominic Hughes. Simple free star-autonomous categories and full coherence. Preprint, 2005.
15. Dominic Hughes. Simple multiplicative proof nets with units. Preprint, 2005.
16. Yves Lafont. *Logique, Catégories et Machines*. PhD thesis, Université Paris 7, 1988.
17. François Lamarche and Lutz Straßburger. From proof nets to the free *-autonomous category. *Logical Methods in Computer Science*, 2(4:3):1–44, 2006.
18. Richard McKinley. On Herbrand's theorem and cut elimination (extended abstract). preprint, 2008.
19. Dale Miller. A compact representation of proofs. *Studia Logica*, 46(4):347–370, 1987.
20. Christian Retoré. *Réseaux et Séquents Ordonnés*. Thèse de Doctorat, spécialité mathématiques, Université Paris VII, February 1993.
21. Lutz Straßburger. *Linear Logic and Noncommutativity in the Calculus of Structures*. PhD thesis, Technische Universität Dresden, 2003.
22. Lutz Straßburger. From deep inference to proof nets. In *Structures and Deduction — The Quest for the Essence of Proofs (Satellite Workshop of ICALP 2005)*, 2005.
23. Lutz Straßburger and François Lamarche. On proof nets for multiplicative linear logic with units. In Jerzy Marcinkowski and Andrzej Tarlecki, editors, *Computer Science Logic, CSL 2004*, volume 3210 of *LNCS*, pages 145–159. Springer-Verlag, 2004.

A Proofs

Proof of Theorem 3.1: We proceed by structural induction on the sequent proof to construct the deep inference proof. The only non-trivial cases are the rules for $\otimes$ and $\forall$. If the last rule application in the sequent proof is a $\otimes$, then we have by induction hypothesis two proofs

$$\begin{array}{c}1\\ \mathsf{MLL2}_{\mathsf{DI}\downarrow}\,\Big\Vert\,\mathscr{D}_1\\ [\Gamma \parr A]\end{array} \qquad \text{and} \qquad \begin{array}{c}1\\ \mathsf{MLL2}_{\mathsf{DI}\downarrow}\,\Big\Vert\,\mathscr{D}_2\\ [B \parr \Delta]\end{array}$$

From these we can built

$$\begin{array}{c}1\\ \mathsf{MLL2}_{\mathsf{DI}\downarrow}\,\Big\Vert\,\mathscr{D}_2\\ 1\!\downarrow\dfrac{[B \parr \Delta]}{[(1\otimes B)\parr \Delta]}\\ \mathsf{MLL2}_{\mathsf{DI}\downarrow}\,\Big\Vert\,\mathscr{D}_1\\ \mathsf{ls}\,\dfrac{[([\Gamma\parr A]\otimes B)\parr\Delta]}{[\Gamma\parr(A\otimes B)\parr\Delta]}\end{array}$$

In case of the $\forall$-rule, we have by induction hypothesis a proof

$$\begin{array}{c}1\\ \mathsf{MLL2}_{\mathsf{DI}\downarrow}\,\Big\Vert\,\mathscr{D}\\ [A \parr \Gamma]\end{array}$$

from which we get

$$\begin{array}{c}\mathsf{e}\!\downarrow\dfrac{1}{\forall a.1}\\ \mathsf{MLL2}_{\mathsf{DI}\downarrow}\,\Big\Vert\,\mathscr{D}\\ \mathsf{u}\!\downarrow\dfrac{\forall a.[A\parr\Gamma]}{\mathsf{f}\!\downarrow\dfrac{[\forall a.A\parr\exists a.\Gamma]}{[\forall a.A\parr\Gamma]}}\end{array}$$

Conversely, for translating a $\mathsf{MLL2}_{\mathsf{DI}\downarrow}$ proof $\mathscr{D}$ into the sequent calculus, we proceed by induction on the length of $\mathscr{D}$. We then translate

$$\begin{array}{c}1\\ \mathsf{MLL2}_{\mathsf{DI}\downarrow}\,\Big\Vert\,\mathscr{D}'\\ \rho\,\dfrac{A}{B}\end{array}$$

into

$$\mathsf{cut}\,\dfrac{\overset{\displaystyle\bigtriangledown_{\mathscr{D}_1}}{\vdash A} \qquad\qquad \overset{\displaystyle\bigtriangledown_{\mathscr{D}_2}}{\vdash A^{\perp}, B}}{\vdash B} \tag{10}$$

where $\mathscr{D}_1$ exists by induction hypothesis and $\mathscr{D}_2$ exists because every rule ρ of $\mathsf{MLL2_{DI\downarrow}}$ is a valid implication of MLL2. Finally, we apply cut elimination (Theorem 2.1). Remark: By using the proof nets introduced in this paper, this translation can be done without using cut elimination. □

Proof of Theorem 3.2: Given a proof in $\mathsf{MLL2_{DI\downarrow}} \cup \{\mathsf{i}{\uparrow}\}$, we translate it into $\mathsf{MLL2_{Seq}}$ as done in the proof of Theorem 3.1, eliminate the cut (Theorem 2.1), and translate the result back into $\mathsf{MLL2_{DI\downarrow}}$. When translating a sequent calculus proof with cuts into the calculus of structures as described in the proof of Theorem 3.1, then the sequent cut rule $\mathsf{cut}\,\dfrac{\vdash \Gamma, A \quad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}$ is simulated exactly by the rule $\mathsf{i}{\uparrow}\,\dfrac{S(A \otimes A^{\perp})}{S\{\bot\}}$. □

It is also possible to give a direct proof of Theorem 3.2 using only the calculus of structures (see, e.g., [21, 4, 11]), without referring to the sequent calculus.

Proof of Theorem 3.3: The construction is done in two phases. First, we permute all instances of $\mathsf{ai}{\downarrow}, \bot{\downarrow}, 1{\downarrow}, \mathsf{e}{\downarrow}$ to the top of the derivation. For $\mathsf{ai}{\downarrow}$ and $\mathsf{e}{\downarrow}$ this is trivial, because all steps are similar to the following:

$$\mathsf{e}{\downarrow}\,\dfrac{\sigma{\downarrow}\,\dfrac{S[A \parr B\{1\}]}{S[B\{1\} \parr A]}}{S[B\{\forall a.1\} \parr A]} \quad\to\quad \sigma{\downarrow}\,\dfrac{\mathsf{e}{\downarrow}\,\dfrac{S[A \parr B\{1\}]}{S[A \parr B\{\forall a.1\}]}}{S[B\{\forall a.1\} \parr A]}$$

For $\bot{\downarrow}$ and $1{\downarrow}$ there are some more cases to inspect. We show here only one because all others are similar:

$$1{\downarrow}\,\dfrac{\mathsf{u}{\downarrow}\,\dfrac{S\{\forall a.[A \parr B]\}}{S[\forall a.A \parr \exists a.B]}}{S[(1 \otimes \forall a.A) \parr \exists a.B]} \quad\to\quad \mathsf{rs}\,\dfrac{\mathsf{u}{\downarrow}\,\dfrac{1{\downarrow}\,\dfrac{S\{\forall a.[A \parr B]\}}{S(1 \otimes \forall a.[A \parr B])}}{S(1 \otimes [\forall a.A \parr \exists a.B])}}{S[(1 \otimes \forall a.A) \parr \exists a.B]}$$

Here, in order to permute the $1{\downarrow}$ above the $\mathsf{u}{\downarrow}$, we need an additional instance of rs (and possibly two instances of $\sigma{\downarrow}$). The situation is analogous if we permute the $1{\downarrow}$ over ls, rs, or $\alpha{\downarrow}$ (or $\mathsf{ai}{\downarrow}$ or $\bot{\downarrow}$, but this is not needed for this theorem). When permuting $\bot{\downarrow}$ up (instead of $1{\downarrow}$), then we need $\alpha{\downarrow}$ (and $\sigma{\downarrow}$) instead of rs. For a detailed analysis of this kind of permuation arguments, the reader is referred to [21].

In the second phase of the decomposition, all instances of $\mathsf{n}{\downarrow}$ and $\mathsf{f}{\downarrow}$ are permuted down to the bottom of the derivation. For the rule $\mathsf{n}{\downarrow}$ this is trivial since no rule can interfere (except for $\mathsf{f}{\downarrow}$, which is also permuted down). For permuting down the rule $\mathsf{f}{\downarrow}$, the problematic cases are as before caused by the rules $\mathsf{u}{\downarrow}$, ls, rs, and $\alpha{\downarrow}$. To get our result, we need an additional inference rule:

$$\mathsf{v}{\downarrow}\,\dfrac{S\{\exists a.[A \parr B]\}}{S[\exists a.A \parr \exists a.B]} \tag{11}$$

Now we can do the following replacement

$$\mathsf{f}{\downarrow}\dfrac{S(\exists a.[A \parr B] \otimes C)}{\mathsf{ls}\dfrac{S([A \parr B] \otimes C)}{S[A \parr (B \otimes C)]}} \quad\rightarrow\quad \mathsf{v}{\downarrow}\dfrac{S(\exists a.[A \parr B] \otimes C)}{\mathsf{ls}\dfrac{S([\exists a.A \parr \exists a.B] \otimes C)}{\mathsf{f}{\downarrow}\dfrac{S[\exists a.A \parr (\exists a.B \otimes C)]}{\mathsf{f}{\downarrow}\dfrac{S[A \parr (\exists a.B \otimes C)]}{S[A \parr (B \otimes C)]}}}}$$

and continue permuting the two new $\mathsf{f}{\downarrow}$ further down. Finally, we eliminate all instances of $\mathsf{v}{\downarrow}$ by permuting them up. This is trivial since no rule has an $\exists$ in its conclusion, except for $\mathsf{u}{\downarrow}$ and $\mathsf{n}{\downarrow}$. In the case of $\mathsf{u}{\downarrow}$ we can replace

$$\mathsf{u}{\downarrow}\dfrac{S\{\forall a.[A \parr [B \parr C]]\}}{\mathsf{v}{\downarrow}\dfrac{S[\forall a.A \parr \exists a.[B \parr C]]}{S[\forall a.A \parr [\exists a.B \parr \exists a.C]]}} \qquad \text{by} \qquad \alpha{\downarrow},\sigma{\downarrow}\dfrac{S\{\forall a.[A \parr [B \parr C]]\}}{\mathsf{u}{\downarrow}\dfrac{S\{\forall a.[[A \parr B] \parr C]\}}{\mathsf{u}{\downarrow}\dfrac{S[\forall a.[A \parr B] \parr \exists a.C]}{\alpha{\downarrow}\dfrac{S[[\forall a.A \parr \exists a.B] \parr \exists a.C]}{S[\forall a.A \parr [\exists a.B \parr \exists a.C]]}}}}$$

and in the case of $\mathsf{n}{\downarrow}$, we can replace

$$\mathsf{n}{\downarrow}\dfrac{S\{[A_1 \parr A_2]\langle a \backslash B\rangle\}}{\mathsf{v}{\downarrow}\dfrac{S\{\exists a.[A_1 \parr A_2]\}}{S[\exists a.A_1 \parr \exists a.A_2]}} \qquad \text{by} \qquad =\dfrac{S\{[A_1 \parr A_2]\langle a \backslash B\rangle\}}{\mathsf{n}{\downarrow}\dfrac{S[A_1\langle a \backslash B\rangle \parr A_2\langle a \backslash B\rangle]}{\mathsf{n}{\downarrow}\dfrac{S[A_1\langle a \backslash B\rangle \parr \exists a.A_2]}{S[\exists a.A_1 \parr \exists a.A_2]}}}$$

Because we start from a proof, i.e., the premise of the derivation is 1, all $\mathsf{v}{\downarrow}$ must eventually disappear. □

Proof of Theorem 3.4: For transforming $\mathscr{D}$ into $\mathscr{D}'$, we replace every instance of $\mathsf{f}{\downarrow}$ by a derivation using only the rules in Figure 3. For this, we proceed by structural induction on the formula A in the $\mathsf{f}{\downarrow}$. We show here only one case, the others are similar: If $A = (A' \otimes A'')$ then replace

$$\mathsf{f}{\downarrow}\dfrac{S\{\exists a.(A \otimes A'')\}}{S(A \otimes A'')} \qquad \text{by} \qquad \mathsf{w}{\downarrow}\dfrac{S\{\exists a.(A \otimes A'')\}}{\mathsf{f}{\downarrow}\dfrac{S(\exists a.A \otimes \exists a.A'')}{\mathsf{f}{\downarrow}\dfrac{S(A \otimes \exists a.A'')}{S(A \otimes A'')}}}$$

Conversely, for transforming $\mathscr{D}'$ into a derivation using only $\mathsf{n}{\downarrow}$ and $\mathsf{f}{\downarrow}$, note that $\mathsf{1f}{\downarrow}$, $\bot\mathsf{f}{\downarrow}$, $\mathsf{af}{\downarrow}$, and $\hat{\mathsf{a}}\mathsf{f}{\downarrow}$ are already instances of $\mathsf{f}{\downarrow}$. The rules x, $\mathsf{y}{\downarrow}$, $\mathsf{v}{\downarrow}$, and $\mathsf{w}{\downarrow}$ can be replaced as follows:

$$\mathsf{v}{\downarrow}\dfrac{S\{\exists a.[A \parr B]\}}{S[\exists a.A \parr \exists a.B]} \quad\rightarrow\quad \mathsf{n}{\downarrow}\dfrac{S\{\exists a.[A \parr B]\}}{\mathsf{n}{\downarrow}\dfrac{S\{\exists a.[\exists a.A \parr B]\}}{\mathsf{f}{\downarrow}\dfrac{S\{\exists a.[\exists a.A \parr \exists a.B]\}}{S[\exists a.A \parr \exists a.B]}}}$$

where in the two $\mathsf{n}{\downarrow}$, the variable a is substituted by itself. The other rules are handled similarly. □

Proof of Theorem 5.1: The pre-proof graphs obtained from the rules id and 1 are obviously correct. Then it is an easy exercise to check that all other rules preserve correctness. □

Proof of Theorem 5.2: We proceed by induction on the size of $P \overset{\nu}{\triangleright} \Gamma \blacktriangleleft \sigma$, i.e., the number of nodes in the graph. In the base case where our proof graph is just $\bot \triangleright 1$ we have an instance of the 1-rule and we are done.

If there are any $\parr$-roots in P or Γ, we simply remove them. If we remove a $\parr$-root in Γ, we have to apply the $\parr$-rule and can proceed by induction hypothesis. Note that by removing a $\parr$-root from P, the linking formula becomes a "linking sequent". This is the reason for the "modulo associativity and commutativity" in the statement of the theorem. If there is a $\boldsymbol{\exists}$-root in Γ then we can simply remove this node, which corresponds to applying the $\exists$-rule because its conclusion is $\lfloor\widehat{\Gamma \blacktriangleleft \sigma}\rfloor$, and we proceed by induction hypothesis.

We are now in a situation where all roots of our proof graph are either $\forall$-, $\exists$-, $\mathbb{\exists}$-, or $\otimes$-nodes. (By our transformation above, all $\exists$-nodes are inside the linking.) Let us first consider the case in which there are no $\otimes$-roots. By well-nestedness and connectedness, all of them quantify the same variable, the linking P consists of exactly one formula rooted by an $\exists$-node, and Γ contains exactly one $\forall$-root, all other roots being $\mathbb{\exists}$-nodes. Therefore, we can apply the $\forall$-rule, remove all root-nodes, and proceed by induction hypothesis.

Let us now consider the case where $\otimes$-roots are present (but no $\parr$- nor $\boldsymbol{\exists}$-roots). By the splitting tensor lemma (Lemma B.8, proved in Appendix B), we know that one of them must be splitting. This splitting tensor can either be inside the sequent Γ or inside the linking P. If it is inside Γ, we can immediately apply the $\otimes$-rule and proceed by induction hypothesis. If the splitting tensor is inside P, then there are two possibilities: either both children are dual atoms, or one child is a 1. Both cases handled exactly as in [17]. □

Proof of Proposition 5.3: We can proceed by structural induction on P to construct $\mathscr{D}$. The base case is trivial. Here are the four inductive cases:

$$
\mathsf{ai}\!\downarrow \frac{\{1\}}{[a^\perp \parr a]}
\qquad
\begin{array}{c} 1 \\ \Big\Vert \mathscr{D}' \\ \bot\!\downarrow \dfrac{A}{[\bot \parr A]} \end{array}
\qquad
\begin{array}{c} 1 \\ \Big\Vert \mathscr{D}' \\ 1\!\downarrow \dfrac{B}{(1 \otimes B)} \\ \Big\Vert \mathscr{D}'' \\ (A \otimes B) \end{array}
\qquad
\begin{array}{c} \mathsf{e}\!\downarrow \dfrac{1}{\forall a.1} \\ \Big\Vert \mathscr{D}' \\ \forall a.A \end{array}
$$

where $\mathscr{D}'$ and $\mathscr{D}''$ always exist by induction hypothesis. Conversely, we proceed by induction on the length of $\mathscr{D}$ to show that P is a linking formula. We show only the case where the bottommost rule in $\mathscr{D}$ is a $\mathsf{ai}\!\downarrow$, i.e., $\mathscr{D}$ is

$$
\begin{array}{c} 1 \\ \Big\Vert \mathscr{D}' \\ \bot\!\downarrow \dfrac{S\{1\}}{S[a^\perp \parr a]} \end{array} \quad .
$$

By induction hypothesis $S\{1\}^\perp = P\{\bot\}$ is a linking for some context $P\{\ \}$. Hence $S[a^\perp \parr a]^\perp = P(a \otimes a^\perp)$ is also a linking. The other cases are similar. □

In the following we also need the inference rules

$$\alpha\uparrow\,\frac{S(A\otimes(B\otimes C))}{S((A\otimes B)\otimes C)} \qquad \text{and} \qquad \sigma\uparrow\,\frac{S(A\otimes B)}{S(B\otimes A)} \tag{12}$$

which are the duals for $\alpha\downarrow$ and $\sigma\downarrow$, respectively.

We also use the following definition.

A.1 Definition If a linking has the shape $S_1(1 \otimes S_2(a \otimes a^\perp))$ for some contexts $S_1\{\ \}$ and $S_2\{\ \}$, then we say that the 1 *governs* the pair $(a \otimes a^\perp)$. Let P_1 and P_2 be two linkings. We say that P_1 *is weaker than* P_2, denoted by $P_1 \lesssim P_2$, if

- $\varnothing P_1 = \varnothing P_2$,
- P_1 and P_2 contain the same set of $\exists$-nodes, and for every $\exists$-node, its set of leaves is the same in P_1 and P_2, and
- whenever a 1 governs a pair $(a \otimes a^\perp)$ in P_2, then it also governs this pair in P_1.

Proof of Lemma 5.4: We prove that for any two linkings P_1 and P_2, the following are equivalent

1. $P_1 \lesssim P_2$.
2. There is a derivation

$$\begin{array}{c} P_1 \\ \{\alpha\downarrow,\sigma\downarrow,\mathsf{rs}\}\,\Big\Vert\,\mathscr{D} \\ P_2 \end{array}$$

3. There is a derivation

$$\begin{array}{c} P_2^\perp \\ \{\alpha\uparrow,\sigma\uparrow,\mathsf{ls}\}\,\Big\Vert\,\mathscr{D}' \\ P_1^\perp \end{array}$$

4. The simple pre-proof graph $P_2 \triangleright P_1^\perp$ is correct.

$1 \Rightarrow 2$: The only way in which P_1 and P_2 can differ from each other are the $\parr$-trees above the pairs $(a \otimes a^\perp)$ and where in these trees the 1-occurrences are attached. Therefore, the rules for associativity and commutativity of $\parr$ and the rule

$$\mathsf{rs}\,\frac{S(1\otimes[B\parr C])}{S[(1\otimes B)\parr C]}$$

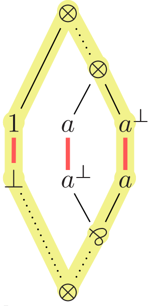
are sufficient to transform P_1 into P_2.

$2 \Rightarrow 3$: The derivation $\mathscr{D}'$ is the dual of $\mathscr{D}$.

$3 \Rightarrow 4$: We proceed by induction on the length of $\mathscr{D}'$. Clearly $P_2 \triangleright P_2^\perp$ is correct. Furthermore, all three inference rules $\alpha\uparrow$, $\sigma\uparrow$, and ls preserve correctness.

$4 \Rightarrow 1$: We have $\varnothing P_1 = \varnothing P_2$ because $P_2 \triangleright P_1^\perp$ is a simple proof graph. The second condition in Definition A.1 follows immediately from the well-nestedness of $P_2 \triangleright P_1^\perp$ and the fact that P_1 and P_2 are both linkings, i.e., do not contain $\forall$-nodes. Therefore, we only have to check the last condition. Assume, by way of contradiction, that there is a 1-occurrence which governs a pair $(a \otimes a^\perp)$ in P_2 but not in P_1, i.e., $P_2 = S_1(1 \otimes S_2(a \otimes a^\perp))$ for some contexts $S_1\{\ \}$ and $S_2\{\ \}$, and $P_1 = S_3[S_4\{1\} \parr S_5(a \otimes a^\perp)]$

for some contexts $S_3\{\ \}$, $S_4\{\ \}$, and $S_5\{\ \}$. This means we have the following situation in $P_2 \triangleright P_1^\perp$



which clearly fails the acyclicity condition. □

In the following proof we use a series of lemmas which are given in Appendix C. We also use the following notation: Let A and B be nodes in $\mathit{\Gamma}$ with $A \nleqslant B$ and $B \nleqslant A$. Then we write $A \overset{\mathit{\Gamma}}{\underset{\otimes}{\smile}} B$ if the first common ancestor of A and B is a $\otimes$, and we write $A \overset{\mathit{\Gamma}}{\underset{\parr}{\smile}} B$ if it is a $\parr$, or if A and B appear in different formulas of $\mathit{\Gamma}$.

Proof of Proposition 5.5: Let a simple pre-proof graph $P \overset{\nu}{\triangleright} \mathit{\Gamma}$ be given, and assume we have a linking $P' \lesssim P$ and derivation $\mathscr{D}$ as in (3) whose flow-graph determines ν. By Lemma 5.4 we have a derivation $\mathscr{D}_1$ such that

$$\begin{array}{rcl} & P^\perp & \\ \{\alpha\!\uparrow, \sigma\!\uparrow, \mathsf{ls}\} & \Big\| & \mathscr{D}_1 \\ & P'^\perp & \\ \{\alpha\!\downarrow, \sigma\!\downarrow, \mathsf{ls}, \mathsf{rs}, \mathsf{u}\!\downarrow\} & \Big\| & \mathscr{D} \\ & \mathit{\Gamma} & \end{array} \quad . \tag{13}$$

Now we proceed by induction on the length of $\mathscr{D}_1$ and $\mathscr{D}$ to show that $P \overset{\nu}{\triangleright} \mathit{\Gamma}$ is multiplicatively correct and well-nested. In the base case it is easy to see that $P \triangleright P^\perp$ has the desired properties. Now it remains to show that all rules $\alpha\!\downarrow, \sigma\!\downarrow, \alpha\!\uparrow, \sigma\!\uparrow, \mathsf{ls}, \mathsf{rs}, \mathsf{u}\!\downarrow$ preserve multiplicative correctness and well-nestedness. For multiplicative correctness it is easy: for $\mathsf{u}\!\downarrow$ it is trivial because it does not change the $\parr$-$\otimes$-structure of the graph, and for the other rules it is well-known. That well-nestedness is preserved is also easy to see: rules $\alpha\!\downarrow, \sigma\!\downarrow, \alpha\!\uparrow, \sigma\!\uparrow, \mathsf{ls}, \mathsf{rs}$ do not modify the $\forall$-$\exists$-structure of the graph, and therefore trivially preserve Conditions 1–4 in Definition 4.4. For the no-down-path condition it suffices to observe that it cannot happen that a $\underset{\parr}{\smile}$ is changed into $\underset{\otimes}{\smile}$ while going down in a derivation. Finally, it is easy to see that $\mathsf{u}\!\downarrow$ preserves all five conditions in Definition 4.4.

Conversely, assume $P \overset{\nu}{\triangleright} \mathit{\Gamma}$ is well-nested and multiplicatively correct. For constructing $\mathscr{D}$, we will again need the rule $\mathsf{v}\!\downarrow$ that has already been used in the proof of Theorem 3.3.

We proceed by induction on the distance between $P^\perp$ and $\mathit{\Gamma}$. For defining this formally, let A be a simple formula and define $\#_{\parr} A$ to be the number of pairs $\langle a, b\rangle$ with $a, b \in \mathscr{O}A$ and $a \overset{A}{\underset{\parr}{\smile}} b$, and define $\#_{\exists} A$ to be the number of $\exists$-nodes in A. Now observe that $P^{\perp}$ and $\mathit{\Gamma}$ have the same set of leaves. We can therefore define

$$\delta_{\parr}\langle P^{\perp}, \mathit{\Gamma}\rangle = \#_{\parr}\mathit{\Gamma} - \#_{\parr}P^{\perp}$$
$$\delta_{\exists}\langle P^{\perp}, \mathit{\Gamma}\rangle = \#_{\exists}\mathit{\Gamma} - \#_{\exists}P^{\perp}$$

Note that because of acyclicity it can never happen that for some $a, b \in \text{🍃}\mathit{\Gamma}$ we have $a \overset{P^{\perp}}{\underset{\parr}{\smile}} b$ and $a \overset{\mathit{\Gamma}}{\underset{\otimes}{\smile}} b$. Therefore $\delta_{\parr}\langle P^{\perp}, \mathit{\Gamma}\rangle$ is the number of pairs $a, b \in \text{🍃}\mathit{\Gamma}$ with $a \overset{P^{\perp}}{\underset{\otimes}{\smile}} b$ and $a \overset{\mathit{\Gamma}}{\underset{\parr}{\smile}} b$. Furthermore, observe that by definition there cannot be any $\exists$-node in $P^{\perp}$. Hence $\delta_{\exists}\langle P^{\perp}, \mathit{\Gamma}\rangle = \#_{\exists}\mathit{\Gamma}$. Now define the *distance between $P^{\perp}$ and $\mathit{\Gamma}$* to be the pair

$$\delta\langle P^{\perp}, \mathit{\Gamma}\rangle \;=\; \langle \delta_{\parr}\langle P^{\perp}, \mathit{\Gamma}\rangle, \delta_{\exists}\langle P^{\perp}, \mathit{\Gamma}\rangle\rangle$$

where we assume the lexicographic ordering.

Let us now pick in $\mathit{\Gamma}$ a pair of dual atoms, say $a^{\perp}$ and a, which appear in the same "axiom link" in P, i.e., P is $P(a \otimes a^{\perp})$. We now make a case analysis on the relative position of $a^{\perp}$ and a to each other in $\mathit{\Gamma}$. Because of acyclicity we must have $a^{\perp} \overset{\mathit{\Gamma}}{\underset{\parr}{\smile}} a$. This means $\mathit{\Gamma} = S[A\{a^{\perp}\} \parr B\{a\}]$ for some contexts $S\{\;\}$, $A\{\;\}$, and $B\{\;\}$. Without loss of generality, we assume that neither A nor B has a $\parr$ as root (otherwise apply $\alpha{\downarrow}$ and $\sigma{\downarrow}$). There are the following cases:

1. $A\{\;\}$ and $B\{\;\}$ have both a quantifier as root. Then both must quantify the same variable (because of the same-depth-condition and the same-variable-condition), and at least one of them must be an $\exists$ (because of the one-$\exists$-condition and the one-$\forall$-condition). Assume, without loss of generality, that $A\{a^{\perp}\} = \forall b.A'\{a^{\perp}\}$ and $B\{a\} = \exists b.B'\{a\}$. Then by Lemma C.3 we have that $P \overset{\nu}{\triangleright} \mathit{\Gamma}'$ with $\mathit{\Gamma}' = S\{\forall b.[A'\{a^{\perp}\} \parr B'\{a\}]\}$ is also correct. We can therefore apply the $\mathsf{u}{\downarrow}$-rule and proceed by induction hypothesis because $\delta\langle P^{\perp}, \mathit{\Gamma}'\rangle$ is strictly smaller than $\delta\langle P^{\perp}, \mathit{\Gamma}\rangle$. If A and B have both an $\exists$ as root, the situation is the same, except that we apply $\mathsf{v}{\downarrow}$-rule instead of $\mathsf{u}{\downarrow}$.
2. One of $A\{\;\}$ and $B\{\;\}$ has a quantifier as root and the other has a $\otimes$ as root. Without loss of generality, let $A\{\;\} = \forall b.A'\{\;\}$ and $B\{\;\} = (B'\{\;\} \otimes B'')$, i.e., $\mathit{\Gamma} = S[\forall b.A'\{a^{\perp}\} \parr (B'\{a\} \otimes B'')]$. Then by Lemma C.1 we have that $P \overset{\nu}{\triangleright} \mathit{\Gamma}'$ with $\mathit{\Gamma}' = S([\forall b.A'\{a^{\perp}\} \parr B'\{a\}] \otimes B'')$ is also correct. We can therefore apply the ls-rule and proceed by induction hypothesis because $\delta\langle P^{\perp}, \mathit{\Gamma}'\rangle$ is strictly smaller than $\delta\langle P^{\perp}, \mathit{\Gamma}\rangle$.
3. One of $A\{\;\}$ and $B\{\;\}$ has a quantifier as root and the other is just $\{\;\}$. This is impossible because it is a violation of the same-depth-condition.
4. $A\{\;\}$ and $B\{\;\}$ have both a $\otimes$ as root. Without loss of generality, assume that $\mathit{\Gamma} = S[(A'' \otimes A'\{a^{\perp}\}) \parr (B'\{a\} \otimes B'')]$. Then, we have by Lemma C.2 that $P \overset{\nu}{\triangleright} \mathit{\Gamma}'$ is correct, with either $\mathit{\Gamma}' = S([(A'' \otimes A'\{a^{\perp}\}) \parr B'\{a\}] \otimes B'')$ or $\mathit{\Gamma}' = S(A'' \otimes [A'\{a^{\perp}\} \parr (B'\{a\} \otimes B'')])$. In one case we apply the rs-rule, and in the other the ls-rule. In both cases we have that $\delta\langle P^{\perp}, \mathit{\Gamma}'\rangle$ is strictly smaller than $\delta\langle P^{\perp}, \mathit{\Gamma}\rangle$. Therefore we can proceed by induction hypothesis.
5. One of $A\{\;\}$ and $B\{\;\}$ has a $\otimes$ as root and the other is just $\{\;\}$. Without loss of generality, $\mathit{\Gamma} = S[a^{\perp} \parr (B'\{a\} \otimes B'')]$. Then, by Lemma C.4, we have that $P \overset{\nu}{\triangleright} \mathit{\Gamma}'$ with $\mathit{\Gamma}' = S([a^{\perp} \parr B'\{a\}] \otimes B'')$, is also correct. We can therefore apply the ls-rule and proceed by induction hypothesis (as before $\delta\langle P^{\perp}, \mathit{\Gamma}'\rangle$ is strictly smaller than $\delta\langle P^{\perp}, \mathit{\Gamma}\rangle$).

6. If $A\{\ \}$ and $B\{\ \}$ are both just $\{\ \}$, i.e., $\Gamma = S[a^\perp \parr a]$, then do nothing and pick another pair of dual atoms.

We continue until we cannot proceed any further by applying these cases. This means, all pairs of dual atoms in $\oslash\Gamma$ are in a situation as in case 6 above. Now observe that a formula is the negation of a linking formula iff it is generated by the grammar

$$\mathscr{N} ::= 1 \mid [\mathscr{A}^\perp \parr \mathscr{A}] \mid [\perp \parr \mathscr{N}] \mid (\mathscr{N} \otimes \mathscr{N}) \mid \forall \mathscr{A}.\mathscr{N}$$

Consequently, the only thing that remains to do is to bring the all $\perp$ to the left-hand side of a $\parr$. This can be done in a similar fashion as we brought pairs $[a^\perp \parr a]$ together, by applying $\alpha\!\downarrow, \sigma\!\downarrow, \mathsf{ls}, \mathsf{rs}, \mathsf{u}\!\downarrow$. This makes Γ the negation of a linking. (Because of well-nestedness, there can be no $\exists$-nodes left.) Let us call this linking formula P'. Now we have a proof graph $P \triangleright P'^\perp$. By Lemma 5.4 we have $P' \lesssim P$.

It remains to remove all instances of $\mathsf{v}\!\downarrow$, which is done exactly as in the proof of Theorem 3.3. □

Proof of Proposition 5.6: We begin by extracting $\Gamma \blacktriangleleft \sigma$ from $\mathscr{D}$. For this, we proceed by induction on the length of $\mathscr{D}$. In the base case, let $\Gamma = D = C$ and σ be empty. In the inductive case let $\mathscr{D}$ be

$$\begin{array}{c} D \\ \{\mathsf{n}\!\downarrow, \mathsf{f}\!\downarrow\} \Big\| \mathscr{D}' \\ \rho\,\dfrac{C'}{C} \end{array}$$

where ρ is either

$$\mathsf{f}\!\downarrow \frac{S\{\exists a.A\}}{S\{A\}} \qquad \text{or} \qquad \mathsf{n}\!\downarrow \frac{S\{A\langle a \backslash B\rangle\}}{S\{\exists a.A\}}$$

and let $\Gamma' \blacktriangleleft \sigma'$ be obtained by induction hypothesis from $\mathscr{D}'$. In particular, $C' = \lfloor \Gamma' \blacktriangleleft \sigma' \rfloor$.

- If ρ is $\mathsf{f}\!\downarrow$, then we construct Γ from Γ' as follows: If the $\exists$ to which $\mathsf{f}\!\downarrow$ is applied appears in Γ' as ordinary $\exists$, then replace it by a $\exists$-node, and let $\sigma = \sigma'$. If the $\exists$ is in fact a $\boldsymbol{\exists}$, then completely remove it, and let σ be obtained from σ' by removing all edges adjacent to that $\boldsymbol{\exists}$. In both cases the same-formula-condition and the no-capture-condition (4.7-1 and 4.7-2) are satisfied for $\Gamma \blacktriangleleft \sigma$ by induction hypothesis (because $\Gamma' \blacktriangleleft \sigma'$ is appropriate). The not-free-condition (4.7-3) holds because otherwise the $\mathsf{f}\!\downarrow$ would not be a valid rule application.
- If ρ is $\mathsf{n}\!\downarrow$, we insert an $\boldsymbol{\exists}$-node at the position where the $\mathsf{n}\!\downarrow$-rule is applied and let σ be obtained from σ' by adding a positive (resp. negative) edge from this new $\boldsymbol{\exists}$ to every occurrence of B in C' which is replaced by a (resp. $a^\perp$) in C. Then clearly the same-formula-condition is satisfied since it is everywhere the same B which is substituted. Let us now assume by way of contradiction, that the no-capture-condition is violated. This means we have A_1, A_2, B_1, B_2 such that $A_1 \curvearrowright B_1$ and $A_2 \curvearrowright B_2$ and $A_1 \preccurlyeq A_2$ and $B_1 \preccurlyeq B_2$ and $B_1 \not\preccurlyeq A_2$. Note that by the definition of stretching we have that A_1, A_2, B_1, B_2 all sit on the same branch in Γ. Therefore we must have that $A'_2 \preccurlyeq B_1$, where A'_2 is child of A_2. Since the no-capture-condition is satisfied for $\Gamma' \blacktriangleleft \sigma'$ we have that either A_1 or A_2 is the newly introduced $\boldsymbol{\exists}$. Note that it cannot be A_2 because then B_1 would not be visible in $\lfloor \Gamma' \blacktriangleleft \sigma' \rfloor$ because it has been replaced by the variable a bound in A_1. Since B_2 is inside B_1 it would also

be invisible in $\lfloor \Gamma' \blacktriangleleft \sigma' \rfloor$. Hence the new $\boldsymbol{\exists}$ must be A_1. Without loss of generality, let $A_1 = \boldsymbol{\exists} a.A'_1$. Then our $\mathsf{n}{\downarrow}$-instance must look like

$$\mathsf{n}{\downarrow}\,\frac{S\{A'_1\{\mathsf{Q}b.A'_2\{B_1\{b\}\}\}\}}{S\{\exists a.\tilde{A}'_1\{\mathsf{Q}b.\tilde{A}'_2\{a\}\}\}} \tag{14}$$

where a is substituted by $B_1\{b\}$ everywhere inside $\tilde{A}'_1\{\mathsf{Q}b.\tilde{A}'_2\{a\}\}$ and Q is either $\forall$ or $\exists$. Clearly, the variable b is captured. Therefore (14) is not a valid rule application. Hence, the no-capture-condition must be satisfied. Finally, the not-free-condition could only be violated in a situation as above where A_2 is a $\reflectbox{$\mathbb{E}$}$-node. But since (14) is not valid, the not-free-condition does also hold.

Conversely, for constructing $\mathscr{D}$ from $\Gamma \blacktriangleleft \sigma$, we proceed by induction on the number of $\reflectbox{$\mathbb{E}$}$ and $\boldsymbol{\exists}$ in Γ. The base case is trivial. Now pick in Γ an $\reflectbox{$\mathbb{E}$}$ or $\boldsymbol{\exists}$ which is minimal wrt. $\preccurlyeq$, i.e., has no other $\reflectbox{$\mathbb{E}$}$ or $\boldsymbol{\exists}$ as ancestor.

- If we pick an $\reflectbox{$\mathbb{E}$}$, say $\Gamma = S\{\reflectbox{$\mathbb{E}$} a.A\}$, then let $\Gamma' = S\{\exists a.A\}$. By the not-free-condition, a does not appear free in $\lfloor A \blacktriangleleft \sigma \rfloor$. Hence
$$\mathsf{f}{\downarrow}\,\frac{\lfloor \Gamma' \blacktriangleleft \sigma \rfloor}{\lfloor \Gamma \blacktriangleleft \sigma \rfloor}$$
is a proper application of $\mathsf{f}{\downarrow}$.
- If we pick an $\boldsymbol{\exists}$, say $\Gamma = S\{\boldsymbol{\exists} a.A\}$, then let $\Gamma' = S\{A\}$ and let σ' be obtained from σ by removing all edges coming out of the selected $\boldsymbol{\exists} a$. We now have to check that
$$\mathsf{n}{\downarrow}\,\frac{\lfloor \Gamma' \blacktriangleleft \sigma' \rfloor}{\lfloor \Gamma \blacktriangleleft \sigma \rfloor}$$
is a proper application of $\mathsf{n}{\downarrow}$. Indeed, by the same-formula-condition, every occurrence of a bound by $\boldsymbol{\exists} a$ in $\lfloor \Gamma \blacktriangleleft \sigma \rfloor$ is substituted by the same formula in $\lfloor \Gamma' \blacktriangleleft \sigma' \rfloor$. The no-capture-condition ensures that no other variable is captured by this.

In both cases we have that $\lceil \Gamma' \blacktriangleleft \sigma' \rceil = \lceil \Gamma \blacktriangleleft \sigma \rceil$. Therefore we can proceed by induction hypothesis. □

Proof of Theorem 5.7: Apply Theorem 3.3 and Propositions 5.3, 5.5, and 5.6. We get a pre-proof graph $P \overset{\nu}{\triangleright} \Gamma \blacktriangleleft \sigma$ with $P^{\perp} = A$ and $\lceil \Gamma \blacktriangleleft \sigma \rceil = B$ and $\lfloor \Gamma \blacktriangleleft \sigma \rfloor = C$. □

Proof of Theorem 5.8: Propositions 5.3, 5.5, and 5.6 give us for a $P \overset{\nu}{\triangleright} \Gamma \blacktriangleleft \sigma$ the derivations

$$\begin{array}{rcl} & 1 & \\ \{\mathsf{ai}{\downarrow},\bot{\downarrow},1{\downarrow},\mathsf{e}{\downarrow}\} & \Big\| & \mathscr{D}'_1 \\ & P'^{\perp} & \\ \{\alpha{\downarrow},\sigma{\downarrow},\mathsf{ls},\mathsf{rs},\mathsf{u}{\downarrow}\} & \Big\| & \mathscr{D}'_2 \\ & \lceil \Gamma \blacktriangleleft s \rceil & \\ \{\mathsf{n}{\downarrow},\mathsf{f}{\downarrow}\} & \Big\| & \mathscr{D}_3 \\ & \lfloor \Gamma \blacktriangleleft s \rfloor & \end{array}$$

where $P' \lesssim P$. Note that together with Lemma 5.4, we also have

$$\begin{array}{r@{\;}c@{\;}l} & 1 & \\ \{\mathsf{ai}\!\downarrow, \bot\!\downarrow, 1\!\downarrow, \mathsf{e}\!\downarrow\} & \Big\| & \mathscr{D}_1 \\ & P^{\bot} & \\ \{\alpha\!\uparrow, \sigma\!\uparrow, \alpha\!\downarrow, \sigma\!\downarrow, \mathsf{ls}, \mathsf{rs}, \mathsf{u}\!\downarrow\} & \Big\| & \mathscr{D}_2 \\ & \lceil \Gamma \blacktriangleleft s \rceil & \\ \{\mathsf{n}\!\downarrow, \mathsf{f}\!\downarrow\} & \Big\| & \mathscr{D}_3 \\ & \lfloor \Gamma \blacktriangleleft s \rfloor & \end{array}$$

□

B The splitting tensor lemma

For proving our sequentialization (into the sequent calculus) we need the so-called "splitting tensor lemma", which is a well-known fact for the purely multiplicative case [9]. Unfortunately, due to the presence of the quantifiers and the units, the proof of the splitting tensor lemma is slightly more complicated than in the purely multiplicative case. This means, for the sake of completeness, we have to prove it here again. We follow closely the presentation in [2]. We need the concept of a *weak* (pre-)proof graph $P \overset{\nu}{\triangleright} \Gamma \blacktriangleleft \sigma$ which is a (pre-)proof graph in which the linking P does not have to be a formula but can be a sequent, i.e., some of the root-$\parr$s are removed.

B.1 Definition Let π_1 and π_2 be weak pre-proof graphs. We say π_1 is a *subpregraph* of π_2, written as $\pi_1 \subseteq \pi_2$ if all nodes appearing in π_1 are also present in π_2. We say π_1 is a *subgraph* of π_2 if $\pi_1 \subseteq \pi_2$, and π_1 and π_2 are both multiplicatively correct (i.e, for the time being we ignore well-nestedness and appropriateness). A *door* of π_1 is any root node (in P or in Γ) of π_1.

B.2 Lemma *Let π' and π'' be subgraphs of some weak proof graph π.*

(i) *The subpregraph $\pi' \cup \pi''$ is a subgraph of π if and only if $\pi' \cap \pi'' \neq \emptyset$.*

(ii) *If $\pi' \cap \pi'' \neq \emptyset$ then $\pi' \cap \pi''$ is a subgraph of π.*

Proof: Intersection and union in the statement of that lemma have to be understood in the canonical sense: An edge/node/link appears in in $\pi' \cap \pi''$ (resp. $\pi' \cup \pi''$) if it appears in both, π' and π'' (resp. in at least one of π' or π''). For proving the lemma, let us first note that because in π every switching is acyclic, also in every subpregraph of π every switching is acyclic, in particular also in $\pi' \cup \pi''$ and $\pi' \cap \pi''$. Therefore, we need only to consider the connectedness condition.

(i) If $\pi' \cap \pi'' = \emptyset$ then every switching of $\pi' \cup \pi''$ must be disconnected. Conversely, if $\pi' \cap \pi'' \neq \emptyset$, then every switching of $\pi' \cup \pi''$ must be connected (in every switching of $\pi' \cup \pi''$ every node in $\pi' \cap \pi''$ must be connected to every node in π' and to every node in π'', because π' and π'' are both multiplicatively correct).

(ii) Let $\pi' \cap \pi'' \neq \emptyset$ and let s be a switching for $\pi' \cup \pi''$. Then s is connected and acyclic by (i). Let s'_π, s''_π, and $s_{\pi' \cap \pi''}$, be the restrictions of s to π', π'', and $\pi' \cap \pi''$, respectively. Now let A and B be two nodes in $\pi' \cap \pi''$. Then A and B are connected by a path in s'_π because π' is correct, and by a path in s''_π because π'' is correct. Since s is acyclic, the two paths must be the same and therefore be contained in $s_{\pi' \cap \pi''}$. □

B.3 Lemma *Let π be a weak proof graph, and let A be a node appearing in π. Then there is a subgraph π' of π, that has A as a door.*

Proof: For proving this lemma, we need the following notation. Let π be a proof graph, let A be some node in π, and let s be a switching for π. Then we write $s(\pi, A)$ for the graph obtained as follows:

– If A is a child of a binary node B in π, and there is an edge from B to A in s, then remove that edge and let $s(\pi, A)$ be the connected component of (the remainder of) s that contains A.

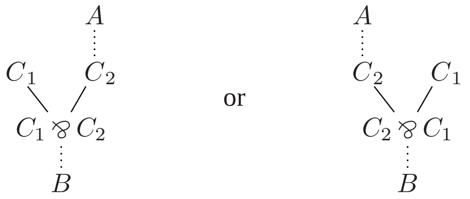
– Otherwise let $s(\pi, A)$ be just s.

Now let

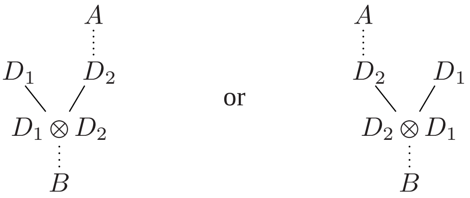
$$\pi' = \bigcap_s s(\pi, A)$$

where s ranges over all possible switchings of π. (Note that it could happen that formally π' is not a subpregraph because some edges in the formula trees might be missing. We graciously add these missing edges to π' such that it becomes a subpregraph.) Clearly, A is in π'.

We are now going to show that A is a door of π'. By way of contradiction, assume it is not. This means there is ancestor B of A that is in $\bigcap_s s(\pi, A)$. Now choose a switching $\hat{s}$ such that whenever there is a ⅋ node between A and B, i.e.,



then $\hat{s}$ chooses C_2 (i.e., removes the edge between C_1 and its parent).[7] Then there must be a $\otimes$ between A and B:



Otherwise B would not be in π' (because we remove every edge from A to its parent). Now suppose we have chosen the uppermost such $\otimes$. Then the path connecting A and D_1 in $\hat{s}(\pi, A)$ cannot pass through D_2 (by the construction of $\hat{s}(\pi, A)$). But this means that in $\hat{s}$ (where the edge between A and its parent is not removed) there are two distinct paths connecting A and D_1, which contradicts the acyclicity of $\hat{s}$.

Now we have to show that π' is a subgraph. Let s be a switching for π'. Since π' is a subpregraph of π, we have that s is acyclic. Now let $\tilde{s}$ be an extension of s to π. Then s is the restriction of $\tilde{s}(\pi, A)$ to π', and hence connected. □

[7] Note that there is a mistake in [2].

B.4 Definition Let π be a weak proof graph, and let A be a node in π. The *kingdom of A in π*, denoted by kA, is the smallest subgraph of π, that has A as a door. Similarly, the *empire of A in π*, denoted by eA, is the largest subgraph of π, that has A as a door. We define $A \ll B$ iff $A \in kB$, where A and B can be any nodes in π.

An immediate consequence of Lemmas B.2 and B.3 is that kingdom and empire always exist.

B.5 Remark The subgraph π' constructed in the proof of Lemma B.3 is in fact the empire of A. But we will not need this fact later and will not prove it here.

B.6 Lemma *Let π be a weak proof graph, and let A, A', B, and B' be nodes in π, such that A and B are distinct, A' is a child of A, and B' is a child of B. Now suppose that $B' \in eA'$. Then we have that $B \notin eA'$ if and only if $A \in kB$.*

Proof: We have $B' \in eA' \cap kB$. Hence, $\pi' = eA' \cap kB$ and $\pi'' = eA' \cup kB$ are subnets of π. By way of contradiction, let $B \notin eA'$ and $A \notin kB$. Then π'' has A' as door and is larger than eA' because it contains B. This contradicts the definition of eA'. On the other hand, if $B \in eA'$ and $A \in kB$ then π' has B as door and is smaller than kB because it does not contain A. This contradicts the definition of kB. □

B.7 Lemma *Let A and B be nodes in a weak proof graph $P \overset{\nu}{\triangleright} \Gamma \blacktriangleleft \sigma$. If $A \ll B$ and $B \ll A$, then either A and B are the same node or they are dual leaf-nodes connected via an edge in ν.*

Proof: If a and $a^{\perp}$ are two dual leaf-nodes connected via ν, then clearly $ka = ka^{\perp}$. Now let A and B be two distinct non-leaf nodes with $A \in kB$ and $B \in kA$. Then $kA \cap kB$ is a subgraph and hence $kA = kA \cap kB = kB$. We have three cases:

– If A is a quantifier node, then the result of removing A from kB is still a subgraph, contradicting the minimality of kB.

– If $A = A' \parr A''$ then the result of removing A from kB is still a subgraph, contradicting the minimality of kB.

– If $A = A' \otimes A''$ then $kA = kA' \cup kA'' \cup \{A' \otimes A''\}$. Hence $B \in kA'$ or $B \in kA''$. This contradicts Lemma B.6, which says that $B \notin eA'$ and $B \notin eA''$. □

From Lemma B.7 it immediately follows that $\ll$ is a partial order on the nodes of a weak proof graph π. We make crucial use of this fact in in the proof of the splitting tensor lemma:

B.8 Lemma *Let $P \overset{\nu}{\triangleright} \Gamma \blacktriangleleft \sigma$ be a weak proof graph in which no root (in P or Γ) is an $\parr$- or $\exists$-node. If there are $\otimes$-roots in P or Γ, then at least one of them is splitting, i.e., by removing that $\otimes$, the graph becomes disconnected.*

Proof: Choose among the $\otimes$-roots of $P \overset{\nu}{\triangleright} \Gamma \blacktriangleleft \sigma$ one which is maximal w.r.t. $\ll$. Without loss of generality, assume it is $A_i = A'_i \otimes A''_i$. We will now show that it is splitting, i.e., $\pi = \{A'_i \otimes A''_i\} \cup eA'_i \cup eA''_i$. By way of contradiction, assume $A'_i \otimes A''_i$ is not splitting. This means we have somewhere in π a node B with two children B' and B'' such that $B' \in eA'_i$ and $B'' \in eA''_i$, and therefore $B \notin eA'_i$ and $B \notin eA''_i$. We also know that $A_j \preccurlyeq B$ for some other root node A_j. We have now two cases to consider

– If A_j is a $\otimes$-node, say $A_j = A'_j \otimes A''_j$, then $B \in kA_j$ and therefore $kB \subseteq kA_j$. But by Lemma B.6 we have $A_i \in kB$ and therefore $A_i \in kA_j$, which contradicts the maximality of A_i w.r.t. $\ll$.

– Otherwise A_j is a $\forall$-, $\exists$-, or $\reflectbox{$\mathbb{E}$}$-node. Then B is inside a box which has A_j as a door. Since eA'_i and eA''_i are both multiplicatively correct, we have a switching s with two paths, $A'_i \frown\!\circledS\!\frown B'$ and $A''_i \frown\!\circledS\!\frown B''$. Both paths must enter the box at some point. This can happen only through a door. And because of the acyclicity condition the two paths must come in through two different doors. At most one of them can be in the linking P, because otherwise the one-$\exists$-condition (4.4-3) would be violated. But if one of the doors is in P and the other in $\mathit{\Gamma}$, we have immediately a violation of the acyclicity condition. (For every box we can always construct a switching with a direct path from the $\exists$-door in P to any chosen door in $\mathit{\Gamma}$. Hence both doors must be inside $\mathit{\Gamma}$. But this violates the no-down-path condition (4.4-5), because there is a down path between the two doors going through $A'_i \otimes A''_i$. Contradiction. □

C Properties of simple proof graphs

In this appendix we present a series of lemmas which are needed for the proof of Proposition 5.5, and whose role in the big picture is similar to the role of the "splitting tensor lemma" for the sequent calculus.

In the following we will sometimes identify a sequent $\vdash A_1, \ldots, A_n$ with the formula $[A_1 \parr \cdots \parr A_n]$.

C.1 Lemma *Let*

$$P(a \otimes a^{\perp}) \overset{\nu}{\triangleright} S[\forall b.A'\{a^{\perp}\} \parr (B'\{a\} \otimes B'')] \tag{15}$$

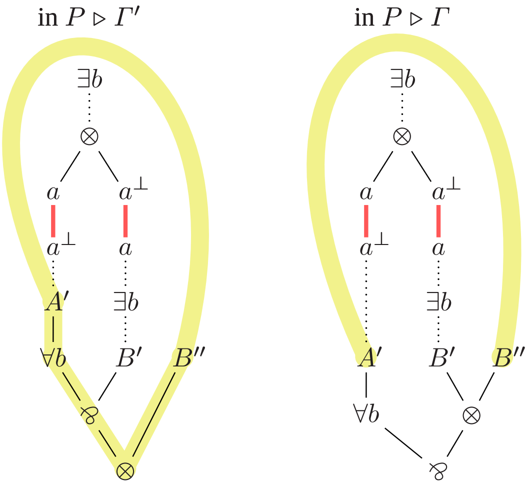
be a simple proof graph, where $S\{\ \}$, $A'\{\ \}$, and $B'\{\ \}$ are arbitrary contexts, $P\{\ \}$ is a linking formula context, and ν pairs up the shown occurrences of a and $a^{\perp}$. Then

$$P(a \otimes a^{\perp}) \overset{\nu}{\triangleright} S([\forall b.A'\{a^{\perp}\} \parr B'\{a\}] \otimes B'') \tag{16}$$

is also correct.

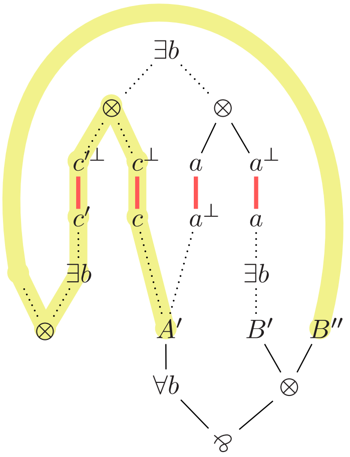
Proof: Let us abbreviate (15) by $P \overset{\nu}{\triangleright} \mathit{\Gamma}$ and (16) by $P \overset{\nu}{\triangleright} \mathit{\Gamma}'$. By way of contradiction, assume that $P \overset{\nu}{\triangleright} \mathit{\Gamma}'$ is not correct.

If it is not multiplicatively correct then there is a switching s which is either disconnected or cyclic. If it is disconnected, then we get from s immediately a disconnected switching for $P \overset{\nu}{\triangleright} \mathit{\Gamma}$. So, let us assume s is cyclic. The only modification from $\mathit{\Gamma}$ to $\mathit{\Gamma}'$ that could produce such a cycle is the change from $A'\{a^{\perp}\} \overset{\mathit{\Gamma}}{\parr} B''$ to $A'\{a^{\perp}\} \overset{\mathit{\Gamma}'}{\otimes} B''$. Hence, we must have a path $A'\{a^{\perp}\} \frown\!\circledS\!\frown B''$, which is also present in $P \overset{\nu}{\triangleright} \mathit{\Gamma}$. Note that this path cannot pass through $a^{\perp}$ and a because otherwise we could use $(B'\{a\} \otimes B'')$ to get a cyclic switching for $P \overset{\nu}{\triangleright} \mathit{\Gamma}$. Furthermore, because $P \overset{\nu}{\triangleright} \mathit{\Gamma}$ is well-nested, there is an $\exists b$-node inside $B'\{a\}$ below a. We can draw the following pictures to visualize the situation:


in $P \triangleright \Gamma'$

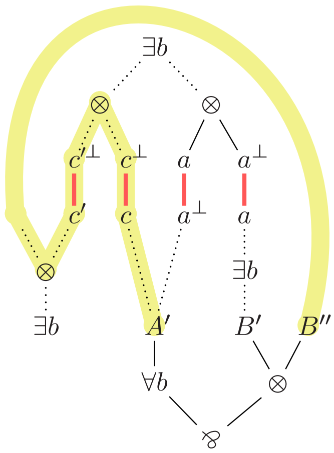
in $P \triangleright \Gamma$

Now, let c be the leaf at which our path leaves $A'\{a^\perp\}$ and goes into P, and let c' be the leaf at which it leaves P and comes back into Γ. by well-nestedness of $P \overset{\nu}{\triangleright} \Gamma$, there must be some $\exists b$-node somewhere in Γ below c'. We also know that our path, coming into Γ at c', goes first down, and at some point goes up again. This turning point must be some $\otimes$-node below c'. Since the $\exists b$-node and the $\otimes$-node are both on the path from c' to the root of the formula, one must be an ancestor of the other. Let us first assume the $\otimes$ is below the $\exists b$. Then our path is of the shape

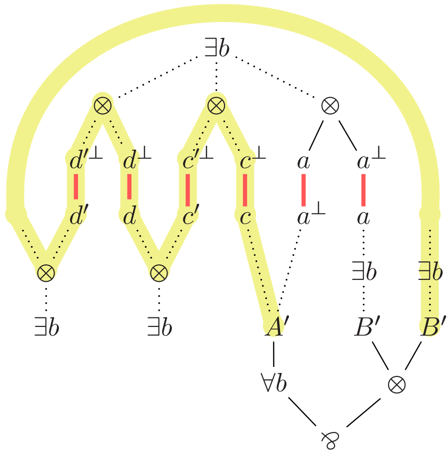


(17)

This, however, is a contradiction to the well-nestedness of $P \overset{\nu}{\triangleright} \Gamma$ because it violates the no-down-path-condition (4.4-5) because there is a path between the $\exists b$ below the c' and the $\exists b$ below the a. Therefore the $\otimes$ must be above the $\exists b$. The situation is now as follows:

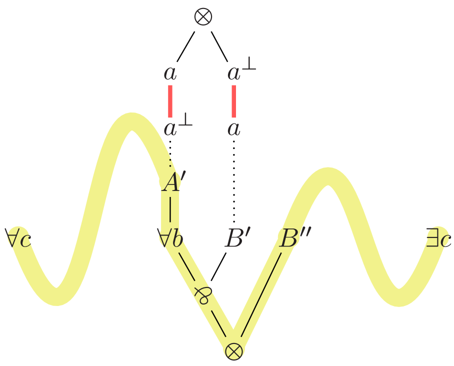


From the $\otimes$, the path must go up again. Without loss of generality, assume it leaves Γ at d and reenters Γ at d'. For the same reasons as above, there must be an $\exists b$ and a $\otimes$ below d'. And so on. There are two possibilities: either at some point the $\otimes$ is below the $\exists b$, which gives us a violation of the no-down-path-condition as in (17), or we reach eventually B'':



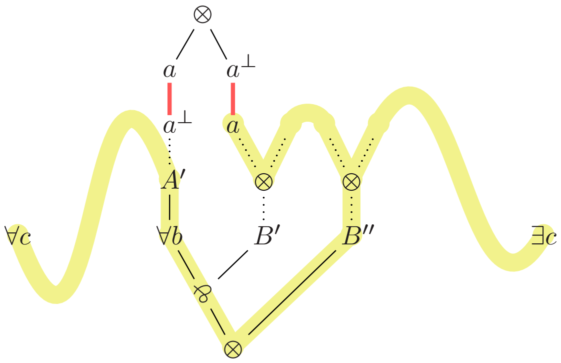
For the same reasons as above, there must be an $\exists b$ inside B'', and we get immediately a violation of the no-down-path-condition because of the short path between the two $\exists b$ above B' and B''. Consequently, $P \overset{\nu}{\triangleright} \Gamma'$ must be multiplicatively correct.

Let us therefore assume $P \overset{\nu}{\triangleright} \Gamma'$ is not well-nested. The same-depth-condition and the same-variable-condition (4.4-1 and 4.4-2) must hold in $P \overset{\nu}{\triangleright} \Gamma'$ because they hold in $P \overset{\nu}{\triangleright} \Gamma$ and the quantifier structure is identical in Γ and Γ'. For the same reasons also the one-$\exists$-condition and the one-$\forall$-condition (4.4-3 and 4.4-4) must hold in $P \overset{\nu}{\triangleright} \Gamma'$. Therefore, it must be the no-down-path-condition which is violated. This means we must have in Γ' two quantifier nodes, say $\forall c$ and $\exists c$, connected by a path $\forall c \leadsto\textcircled{s}\leadsto \exists c$ in some switching s. Because this path is not present in $P \overset{\nu}{\triangleright} \Gamma$ it must pass through the new $\otimes$ between $\forall b.A'\{a^{\perp}\}$ and B'', as follows:



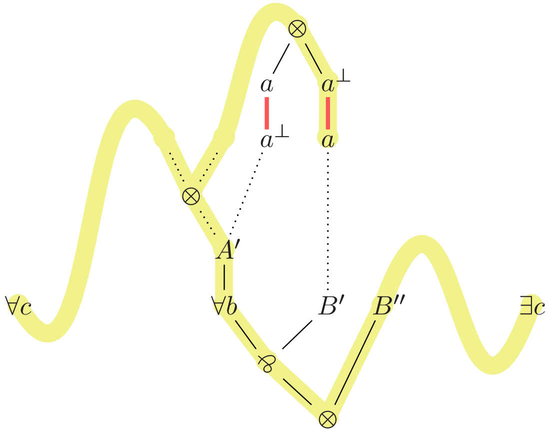
(18)

Since $P \overset{\nu}{\triangleright} \Gamma'$ is multiplicatively correct, the switching s must be connected. Therefore there is in s a path from the $\forall b$-node to the a inside B'. This new path must follow the path between $\forall c$ and $\exists c$ for some steps in one direction. Hence, we either have



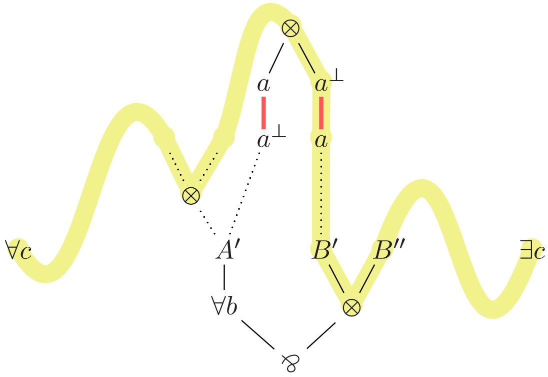
(19)

or



(20)

Clearly, (19) violates the acyclicity condition for $P \overset{\nu}{\triangleright} \Gamma'$ as well as for $P \overset{\nu}{\triangleright} \Gamma$. And from (20), we can obtain a switching for $P \overset{\nu}{\triangleright} \Gamma$ with a path $\forall c \overset{\textcircled{s}}{\frown} \exists c$ as follows:



(21)

Contradiction. (Note that although in (20) and (21) the path does not go through the $a^\perp$ inside A', this case is not excluded by the argument.) □

C.2 Lemma *Let*

$$P(a \otimes a^\perp) \overset{\nu}{\triangleright} S[(A'' \otimes A'\{a^\perp\}) \parr (B'\{a\} \otimes B'')] \tag{22}$$

be a simple proof graph. Then at least one of

$$P(a \otimes a^\perp) \overset{\nu}{\triangleright} S([(A'' \otimes A'\{a^\perp\}) \parr B'\{a\}] \otimes B'') \tag{23}$$

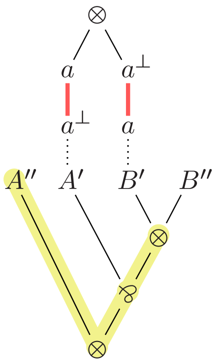
and

$$P(a \otimes a^\perp) \overset{\nu}{\triangleright} S(A'' \otimes [A'\{a^\perp\} \parr (B'\{a\} \otimes B'')]) \tag{24}$$

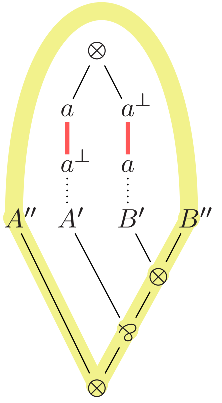
is also correct.

Proof: We will abbreviate (22) by $P \triangleright \Gamma$, (23) by $P \triangleright \Gamma'$, and (24) by $P \triangleright \Gamma''$.

We start by showing that both, $P \triangleright \Gamma'$ and $P \triangleright \Gamma''$ have to be multiplicatively correct. We consider here only the acyclicity condition and leave connectedness to the reader. Suppose by way of contradiction, that there is a switching s' for $P \triangleright \Gamma'$ that is cyclic. Then the cycle must pass through A'', the root $\otimes$ and the $\parr$ as follows:

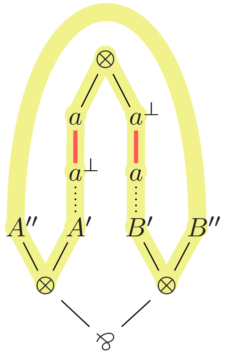


Otherwise we could construct a switching with the same cycle in π. If our cycle continues through B'', i.e.,

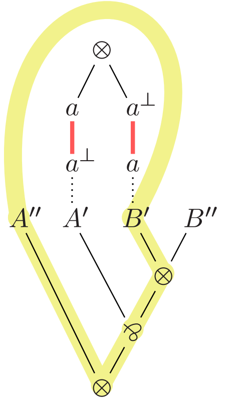


(25)

then we can use the path from A'' to B'' (which cannot go through A' or B') to construct a cyclic switching s in $P \triangleright \Gamma$ as follows:



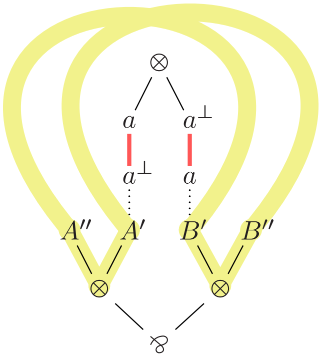
Hence, the cycle in s' goes through B', giving us a path from A'' to B' (not passing through A'):



(26)

By the same argumentation we get a switching s'' in $P \triangleright \Gamma''$ with a path from A' to B'', not going through B'. From s' and s'', we can now construct a switching s for $P \triangleright \Gamma$

with a cycle as follows:



which contradicts the correctness of $P \triangleright \Gamma$.

We now have to show that $P \triangleright \Gamma'$ and $P \triangleright \Gamma''$ are both well-nested. This can be done in almost literally the same way as in the proof of Lemma C.1. □

C.3 Lemma *Let*

$$P(a \otimes a^\perp) \overset{\nu}{\triangleright} S[\forall b.A'\{a^\perp\} \parr \exists b.B'\{a\}] \tag{27}$$

be a simple proof graph. Then

$$P(a \otimes a^\perp) \overset{\nu}{\triangleright} S\{\forall b.[A'\{a^\perp\} \parr B'\{a\}]\} \tag{28}$$

is also correct.

Proof: Multiplicative correctness of (28) follows immediately, because the $\otimes$-$\parr$-structure is the same as in (27). Furthermore, all five conditions in Definition 4.4 are obviously preserved by going from (27) to (28). Hence (28) is correct. □

C.4 Lemma *Let*

$$P(a \otimes a^\perp) \overset{\nu}{\triangleright} S[a^\perp \parr (B'\{a\} \otimes B'')] \tag{29}$$

be a simple proof graph. Then

$$P(a \otimes a^\perp) \overset{\nu}{\triangleright} S([a^\perp \parr B'\{a\}] \otimes B'') \tag{30}$$

is also correct.

Proof: As before, we abbreviate (29) by $P \triangleright \Gamma$ and (30) by $P \triangleright \Gamma'$. Well-nestedness of $P \triangleright \Gamma'$ follows trivially from the well-nestedness of $P \triangleright \Gamma$. By way of contradiction, assume $P \triangleright \Gamma'$ is not multiplicatively correct. Since connectedness is trivial, assume there is a cyclic switching s. If the cycle does not involve the $\otimes$ between $a^\perp$ and B'', then we immediately have a cyclic switching for $P \triangleright \Gamma$. Since the cycle involves $a^\perp$, it must also involve a. Therefore it must leave $B'\{a\}$ at some other leaf, and finally enter

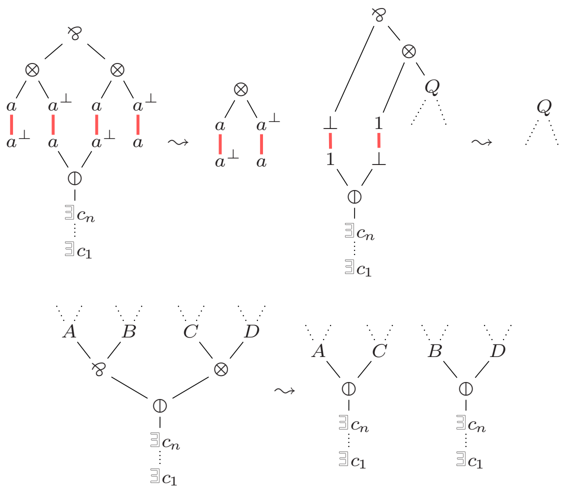
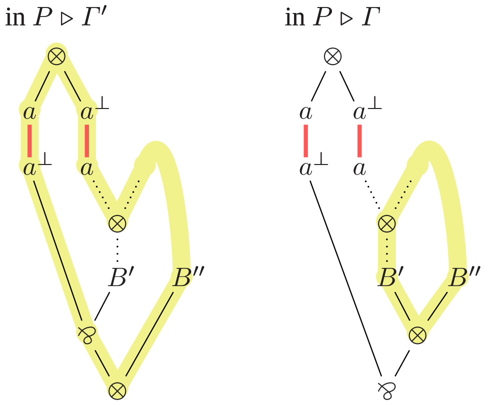


Fig. 9. Cut reduction for MLL2 proof nets (Part 1)

B'' from above, as shown below on the left.



This allows us to construct a cyclic switching for $P \triangleright \Gamma$, as shown on the right above. Contradiction. □

D Cut elimination

For the convenience of the referee we show how cut elimination works for the proof nets introduced in this paper. We will be brief because the essential ingredients have already been shown in [9] and [17]. graph $P \overset{\nu}{\triangleright} \Gamma \blacktriangleleft \sigma$ is a special binary connective $\obar$, such that whenever we have $A \obar B$ in Γ, then we must have $\lfloor A \blacktriangleleft \sigma \rfloor = \lfloor B \blacktriangleleft \sigma \rfloor^{\perp}$.[8] Morally, a $\obar$ may occur only at the root of a formula in Γ. However, due to well-nestedness we must allow cuts to have $\exists$-nodes as ancestors. Then the $\obar$ is treated in the correctness

[8] Note that it does not mean $A = B^{\perp}$, because Γ is expanded.

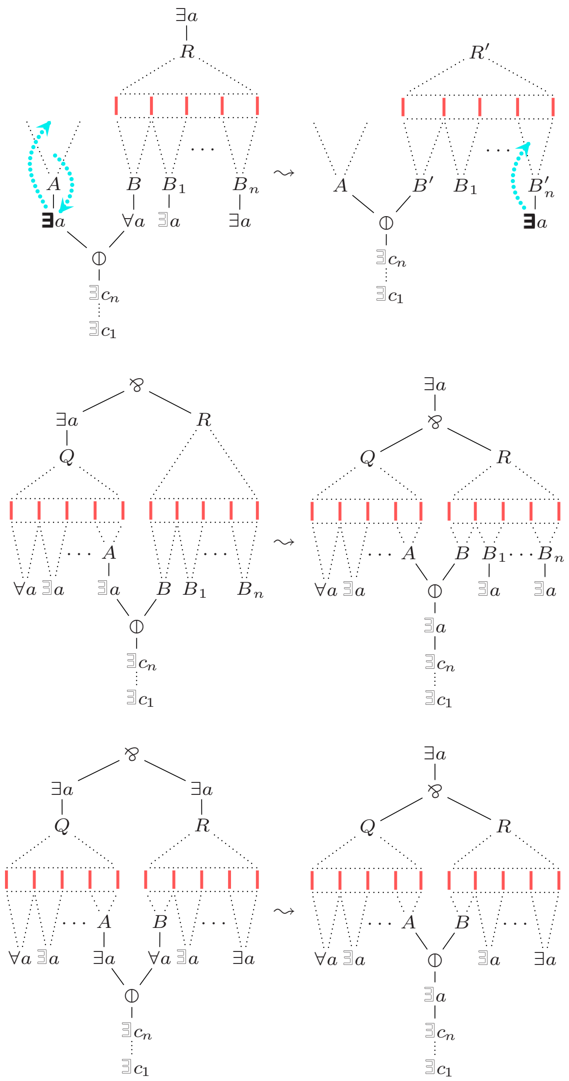


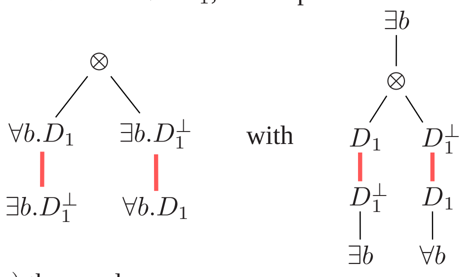
Fig. 10. Cut reduction for MLL2 proof nets (Part 2)

criterion in exactly the same way as the $\otimes$, and sequentialization does also hold for proof graphs with cut.

The cut reduction relation $\rightsquigarrow$ is defined on pre-proof graphs as shown in Figures 9 and 10. The reductions not involving quantifiers are exactly as shown in [17]. If we encounter a cut between two binary connectives, then we replace $[A \parr B] \obar (C \otimes D)$ by two smaller cuts $A \obar C$ and $B \obar D$. Note that if $\lfloor [A \parr B] \blacktriangleleft \sigma \rfloor = \lfloor (C \otimes D) \blacktriangleleft \sigma \rfloor^{\perp}$ then $\lfloor A \blacktriangleleft \sigma \rfloor = \lfloor C \blacktriangleleft \sigma \rfloor^{\perp}$ and $\lfloor B \blacktriangleleft \sigma \rfloor = \lfloor D \blacktriangleleft \sigma \rfloor^{\perp}$. If we have an atomic cut $a^{\perp} \obar a$, then we must have in P two "axiom links" $(a^{\perp} \otimes a)$, which are by the leaf mapping ν attached to the two atoms in the cut. It was shown in [17] that the two pairs $(a^{\perp} \otimes a)$ can, under the equivalence relation in Definition 4.9, be brought next to each other such that P has $[(a \otimes a^{\perp}) \parr (a \otimes a^{\perp})]$ as subformula. We can replace this by a single $(a^{\perp} \otimes a)$ and remove the cut. If we encounter a cut $1 \obar \perp$ on the units, we must have in the linking a corresponding $\perp$ and a subformula $(1 \otimes Q)$, which can (for the same reasons as for the atomic cut) be brought together, such that we have in P a subformula $[\perp \parr (1 \otimes Q)]$. We replace this by Q and remove the cut.

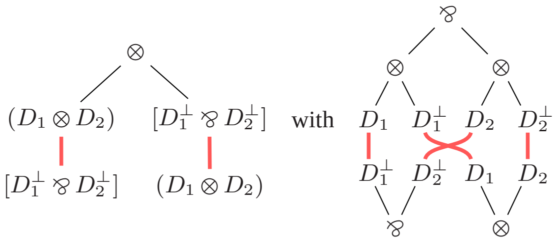
Let us now consider the cuts that involve the quantifiers. There are three cases, one for each of $\boldsymbol{\exists}$, $\mathbb{\exists}$, and $\exists$. The first two correspond to the ones in [9]. The third one does not appear in [9] because there is never a $\exists$-node created when a sequent calculus proof is translated into a proof net.

If one of the cut formulas is an $\boldsymbol{\exists}$-node, then the other must be an $\forall$, which quantifies the same variable, say we have $\boldsymbol{\exists} a.A \obar \forall a.B$. Then we pick a stretching edge starting from $\boldsymbol{\exists} a.A$. Let C be the node where it ends and let $D = \lfloor C \blacktriangleleft \sigma \rfloor$. Note that by Condition 4.7-1, D is independent from the choice of the edge in case there are many of them. (If there are only negative edges, then let $D = \lfloor C \blacktriangleleft \sigma \rfloor^{\perp}$. If there are no stretching edges at all, then let $D = a$. Now we can inside the box of $\forall a.B$ substitute a everywhere by D. Then we remove all the doors of the $\forall a.B$-box and replace the cut by $A \obar B$. There are two subtleties involved in this case. First, “removing a door” means for a $\mathbb{\exists}$ that the node is removed, but for and $\exists$, it means that the node is replaced by an $\boldsymbol{\exists}$ and a stretching edge is added for every a and $a^{\perp}$ bound by the $\exists$-node to be removed. Second, by substituting a with D we get “axiom links” which are not atomic anymore, but it is straightforward to make them atomic again: one proceeds by structural induction on D. If $D = \forall b.D_1$, then replace



(31)

and if $D = (D_1 \otimes D_2)$ then replace



(32)

The cases for $\exists a.D_1$ and $[D_1 \parr D_2]$ are similar.

If one of the two cut formulas is a $\mathbb{\exists}$-node, then the other one can be anything. Say, we have $\mathbb{\exists} a.A \obar B$. Let eB be the *empire* of B, i.e, largest sub-proof graph of $P \overset{\nu}{\triangleright} \Gamma \blacktriangleleft \sigma$ that has B as a conclusion. Let $B_1, \ldots, B_n$ be the other doors of eB inside

Γ, and let R be the door of eB in P. If eB has more than one root-node inside the linking P, then we can rearrange the $\parr$-nodes in P via the equivalence in 4.9 such that eB has a single $\parr$-root in P. Furthermore, as in the case of the atomic cut we can use the equivalence in 4.9 to get in P a subformula $[\exists a.Q \parr R]$ where $\exists a.Q$ is the partner of $\mathbb{\exists} a.A$. Now we replace on P the formula $[\exists a.Q \parr R]$ by $\exists a.[Q \parr R]$ and in Γ the formulas $B_1, \dots, B_n$ by $\mathbb{\exists} a.B_1, \dots, \mathbb{\exists} a.B_n$. Put in plain words, we have pulled the whole empire of B inside the box of $\mathbb{\exists} a.A$. But now we have a little problem: Morally, we should replace the cut $\mathbb{\exists} a.A \obar B$ by $A \obar B$; the cut is also pulled inside the box. But by this we would break our correctness criterion, namely, the same-depth-condition 4.4-1. To solve this problem, we allow cut-nodes to have $\mathbb{\exists}$-nodes as ancestors, and we replace the cut $\mathbb{\exists} a.A \obar B$ by $\mathbb{\exists} a.(A \obar B)$. Note that this does not cause problems for the other cut reduction steps because we can just keep all $\mathbb{\exists}$-ancestors when we replace a cut by a smaller one.

Finally, there is the cut between an ordinary $\exists$-node and a $\forall$-node, say $\exists a.A \obar \forall a.B$. Then we do not pull the whole empire of $\forall a.B$ inside the box of $\exists a.A$ but only the $\forall a.B$-box. This is the same as merging the two boxes into one. Formally, let $\exists a.Q$ and $\exists a.R$ be the partners of $\exists a.A$ and $\forall a.B$, respectively. Again, for the same reasons as in the case of the atomic cut, we can assume that we have the configuration $[\exists a.Q \parr \exists a.R]$ in P, which we replace by $\exists a.[Q \parr R]$. The cut is replaced by $\mathbb{\exists} a.(A \obar B)$.

This cut reduction relation is defined *a priori* only on pre-proof graphs. For a pre-proof graph $P \overset{\nu}{\triangleright} \Gamma \blacktriangleleft \sigma$ and a cut $A \obar B$ in Γ, we say the cut is *ready*, if the cut can immediately be reduced without further modification of P. We now can show the following:

D.1 Theorem *The cut reduction relation preserves correctness and is well-defined on proof nets.*

Proof: That correctness is preserved follows immediately from inspecting the six cases. To show that cut reduction is well-defined on proof nets we need to verify the following two facts:

1. Whenever the same cut is reduced in two different representations of the same proof net, then the two results also represent the same proof net.
2. Whenever there is a cut in a proof net, then this cut can be reduced, i.e., there is a representant to which the corresponding reduction step in Figures 9 and 10 can be applied.

For the first statement, it suffices to observe that whenever one of the basic equivalence steps in Definition 4.9 can be performed in the non-reduced net, then the same step can be performed in the reduced net or is vacuous in the reduced net. For the second statement we have to make a case analysis on the type of cut: If the cut is $[A \parr B] \obar (C \otimes D)$ or $\boldsymbol{\exists} a.A \obar \forall a.B$, then it is trivial because these cuts are always ready. Let us now consider a cut $\exists a.A \obar \forall a.B$. Clearly, the two boxes of which $\exists a$ and $\forall a$ are doors each have a single door $\exists a$ in P, and their first common ancestor is a $\parr$ (because of the acyclicity condition). Therefore, the linking is of the shape $P[S_1\{\exists a.Q\} \parr S_2\{\exists a.R\}]$ for some contexts $S_1\{\ \}$ and $S_2\{\ \}$. Now we proceed by induction on the size of $S_1\{\ \}$ and $S_2\{\ \}$ and make a case analysis on their root-nodes:[9]

[9] Note the similarity to the proof of Proposition 5.5.

- Both contexts are empty. In this case the linking has the desired shape, and we are done.
- One of them has a $\parr$-root. In this case we apply associativity of the $\parr$ and proceed by induction hypothesis.
- One of them has an $\exists$-node as root. This is impossible because it would violate the well-nested condition.
- One of them has a $\otimes$-root, and the other context is empty. Without loss of generality, the linking is of the shape $P[(1 \otimes S'_1\{\exists a.Q\}) \parr \exists a.R]$. We claim, that in this case the correctness is preserved if we replace the linking by $P(1 \otimes [S'_1\{\exists a.Q\} \parr \exists a.R])$. We leave the proof of this claim to the reader because it is very similar to the proof of Lemma C.4. Hence, we can proceed by induction hypothesis.
- Both contexts have a $\otimes$-root. Then the linking is of the shape
$$P[(1 \otimes S'_1\{\exists a.Q\}) \parr (1 \otimes S'_2\{\exists a.R\})] \quad .$$
Now we claim that we can replace this linking with one of
$$P(1 \otimes [S'_1\{\exists a.Q\} \parr (1 \otimes S'_2\{\exists a.R\})])$$
and
$$P(1 \otimes [(1 \otimes S'_1\{\exists a.Q\}) \parr S'_2\{\exists a.R\}])$$
without destroying correctness. Again, we leave the proof to the reader because it is almost the same as the proof of Lemma C.2. As before, we can proceed by induction hypothesis.

For a cut $\exists a.A \obar B$ we proceed similarly. The only difference is that we first have to apply associativity and commutativity of $\parr$ to bring the proof graph in a form where the empire eB has a single root R in the linking. For cuts $a \obar a^\perp$ and $1 \obar \perp$ we can also proceed similarly. □

D.2 Theorem *The cut reduction relation $\leadsto$ is terminating and confluent.*

Proof: Termination has already been shown in [9], and we will not repeat it here. For showing confluence it suffices to show local confluence. We will do this first for proof graphs. Suppose we have two cuts which are ready in a given proof graph. We claim that the result of reducing them is independent from the order of the reduction. There is only one critical pair, since the only possibility for overlapping redexes is when one cut is $\boldsymbol{\exists} a.A \obar \forall a.B$ and the other is $\exists a.C \obar \forall a.D$ and the formulas $\forall a.B$ and $\exists a.C$ are doors of the same box. If we reduce first the cut $\boldsymbol{\exists} a.A \obar \forall a.B$, then we do first the substitution in the $\forall a.B$-box, remove its border, change the second cut to $\boldsymbol{\exists} a.C' \obar \forall a.D$, and then do the same substitution in the $\forall a.D$-box and remove its border. If we reduce first the cut $\exists a.C \obar \forall a.D$, then we merge the two boxes into one, and then do the substitution and remove the border of the box. Clearly, the result is the same in both cases. Hence, we have local confluence for the cut reduction on proof graphs. In the case of proof nets, it can happen that the two cuts are ready in two different representants. With the method shown in the previous proof we can try to construct a representant in which both cuts are ready. There are only two cases in which this fails. The first is when we have two atomic cuts using the same “axiom link”. But then the result of reducing the two is a single axiom link, independent from the order. The second case is when we have two cuts $\exists a.A \obar \forall a.B$ and $\exists a.C \obar \forall a.D$ where $\forall a.B$ and $\exists a.C$ are doors of the same box. Here the result of reducing the two will be a big box which is the merge of all three boxes, independent of the order in which the two cuts are reduced. □