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A System of Interaction and Structure IV: The Exponentials

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Abstract *We study some normalisation properties of the deep-inference proof system* NEL*, which can be seen both as 1) an extension of multiplicative exponential linear logic* (MELL) *by a certain non-commutative self-dual logical operator; and 2) an extension of system* BV *by the exponentials of linear logic. The interest of* NEL *resides in: 1) its being Turing complete, while the same for* MELL *is not known, and is widely conjectured not to be the case; 2) its inclusion of a self-dual, non-commutative logical operator that, despite its simplicity, cannot be axiomatised in any analytic sequent calculus system; 3) its ability to model the sequential composition of processes. We present several* decomposition *results for* NEL *and, as a consequence of those and via a* splitting *theorem, cut elimination. We use, for the first time, an induction measure based on flow graphs associated to the exponentials, which captures their rather complex behaviour in the normalisation process. The results are presented in the calculus of structures, which is the first, developed formalism in deep inference.*

Keywords: *Proof theory, deep inference, calculus of structures, linear logic, non-commutativity, cut elimination.*

1 Introduction

Non-commutative logical operators have a long tradition [Lam58, Yet90, Abr91, LMSS92, Ret97, AR00], and their proof theoretical properties have been studied in the sequent calculus [Gen34] and in proof nets [Gir87]. Recent research has shown that the sequent calculus is not adequate to deal with very simple forms of non-commutativity [Gug07, GS01, Tiu06b]. On the other hand, proof nets are not ideal for dealing with exponentials and additives, which are desirable for getting good computational power.

In this paper (that is the fourth in a series, of which two already appeared [Gug07, Tiu06b]) we show a logical system that joins a simple form of non-commutativity with commutative multiplicatives and with exponentials. This is done in the deep-inference formalism of the calculus of structures [GS01, Gug07], which overcomes the difficulties encountered in the sequent calculus and in proof nets. This paper contributes the following results:

1. We define a propositional logical system, called NEL (non-commutative exponential linear logic), which extends MELL (multiplicative exponential linear logic [Gir87]) by a non-commutative, self-dual logical operator called *seq*. This system, which was first presented in [GS01], is conservative over MELL augmented by the mix and nullary mix rules [AJ94, FR94, Ret93]. System NEL can equivalently be considered an extension of system BV [Gug07, Tiu06b] by the exponentials of linear logic. System NEL can be immediately understood by anybody acquainted with the sequent calculus, and is aimed at the same range of applications as MELL. In nearly all computer science languages, sequential composition plays a fundamental role, and it is therefore important to address it in a direct way, in logical representations of those languages. Perhaps surprisingly, parallel composition has been much easier to deal with, due to its commutative nature, which is more similar to the typical nature of traditional logics. The addition of seq opens new syntactic possibilities, for example in dealing with process algebras. It has been used already, in the purely multiplicative setting of system BV, to model prefixing in CCS [Bru02]. Languages and implementations have been realised, based on these deep-inference notions [Kah05a, Kah05b, Kah07a, Kah07b, Rei07, Kah06a].

2. We prove for NEL a property called *decomposition* (first pioneered in [GS01, Str03b]): we can transform every derivation into an equivalent one, composed of up to *eleven* derivations carried into up to *eleven disjoint* subsystems of NEL. We can study small subsystems of NEL in isolation and then compose them together with considerable more freedom than in the sequent calculus, where, for example, contraction can not be isolated in a derivation. Decomposition is made available in the calculus of structures by exploiting a new top-down symmetry of derivations. Since it is a basic compositional result, we expect applications to be very broad in range; we are especially excited about the possibilities in the semantics of derivations.

3. We prove cut elimination for NEL by use of decomposition and a technique that we call *splitting* (first pioneered in [Gug07]). In the calculus of structures, the traditional methods for proving cut elimination fail, due to the more general applicability of inference rules. The deep reason for this is in how the calculus deals with associativity. Splitting theorems are a uniform means of recovering control over the way logical operators associate; they allow us to manage the complex inductions required. The cut elimination argument becomes modular, because we can reduce the cut rule to several more primitive inference rules, each of which is separately shown admissible by way of splitting. Only one of these rules (an atomic form of cut) is infinitary, all the others enjoy the subformula property and can be used to extend the system without affecting provability. It is worth noting that this result about splitting holds also in the restriction of MELL (without mix and nullary mix), and is thus an alternative proof of cut elimination for that fragment of linear logic.

The points above correspond, respectively, to Sections 2, 3 and 4. Readers who are not interested in the proof theory of system NEL can just read Section 2.

Other systems extending linear logic with non-commutative operators are studied in [AR00, Rue00]. These are more traditional systems in the sequent calculus, for which a more limited proof theory can be developed. However, the calculus of structures allows us to design much simpler logics, while retaining analyticity, as witnessed by the fact that we have just one self-dual non-commutative operator instead of two dual ones.

While the techniques of decomposition and splitting have been presented in isolation previously, for different logics, they have not been used together, like in this paper. It appears that decomposition, in isolation, would be insufficient for proving cut elimination for NEL, while it seems to be possible to achieve cut elimination via splitting only (we leave this for future work). Anyway, decomposition provides for a much more refined analysis of proofs than otherwise possible. The main results of this paper have already been presented, without proof, in [GS02]. For several years, the proofs of the statements have been available in a manuscript on the web. The proofs in this paper are now much clearer and use a more efficient induction measure, the result of the familiarity that we acquired in a few years with normalisation in the calculus of structures.

We conclude this Introduction by a quick overview of deep inference and its main results, so to put this work in the context of this relatively new area of proof theory. The calculus of structures is the simplest formalism conceivable in deep inference, and, to a very large extent, its proof theory and its relations with computer science are now understood. Briefly, its achievements beyond what we consider here are:

- Classical [Brü03a, Brü03b, Brü06a, Brü06b, Brü06d, BG04, BT01, GG07, Str07a], intuitionistic [Tiu06a, Hor06], linear [Str03a, Str03b, Str02] and several modal logics [Brü06c, GT07, Hei05, SS05, Sto04, Sto07, Str07b] are expressed in *analytic* systems. Contrary to deep inference, Gentzen's methodology has difficulties dealing with modal logics, to the point that for many of them no analytic proof systems outside of deep inference are known. In particular, modal logics B and K5, which do not enjoy analytic presentations in Gentzen's formalisms, are expressed by *simple* analytic systems [Brü06c]. Proof search systems have been implemented for several logics [Kah04, Kah06a, Kah06b].

- Most deep-inference deductive systems consist entirely of local inference rules [Brü06d, DG04, Gug07, Sto07, Str02, Tiu06a]; a *local* inference rule is one whose computational complexity is a constant. Locality is a difficult property to achieve, and it is almost never achievable with Gentzen's methods [Brü03c], because of the necessity of duplicating formulae of unbounded size (lack of linearity). Thanks to locality, we could discover a new class of *proof nets* for classical logic [LS05b, LS05a, Str05a, Str05b] and linear logic with units [SL04, LS06]. Proof nets play a crucial role in understanding *semantics of proofs* (see [Gui06, Lam07, LS05a, LS06, McK05, McK06, Str07c] for deep-inference contributions).

- Some recent results show that deep inference allows for analytic formalisms that are exponentially more efficient than Gentzen's ones [BG07, Jap07], so contributing to the research on complexity lower bounds.

There are gentler introductions to deep inference and the calculus of structures than this paper. In general, we recommend [Brü03b]. The web page

`http://alessio.guglielmi.name/res/cos`

provides several pieces of introductory material, from different points of view.

2 The System

We call *calculus* a formalism, like natural deduction or the sequent calculus, for specifying logical systems. We say (*deductive*) *system* to indicate a collection of inference rules in a given calculus.

When defining a system in the sequent calculus one first has to define a language of formulae and sequents. Similarly, a system in the calculus of structures requires a language of *structures*, which are intermediate expressions between formulae and sequents.

We now define the language for system NEL and its variants. Intuitively, $[S_1 \parr \cdots \parr S_h]$ corresponds to a sequent $\vdash S_1, \ldots, S_h$ in linear logic, whose formulae are essentially connected by pars, subject to commutativity (and associativity). The structure $(S_1 \otimes \cdots \otimes S_h)$ corresponds to the associative and commutative tensor connection of $S_1, \ldots, S_h$. The structure $\langle S_1 \lhd \cdots \lhd S_h \rangle$ is associative and *non-commutative*: this corresponds to the new logical operator, called *seq*, that we add to those of MELL.

For reasons explained in [Gug07, GS01], dealing with seq involves adding the rules mix and its nullary version $\mathsf{mix0}$ (see [FR94, Ret93, AJ94]):

$$\mathsf{mix}\,\frac{\vdash \Phi \quad \vdash \Psi}{\vdash \Phi, \Psi} \qquad \text{and} \qquad \mathsf{mix0}\,\frac{}{\vdash} \quad .$$

This has the effect of collapsing the multiplicative units 1 and $\bot$: we will only have one unit $\circ$ common to par, times and seq.

Please notice that mix and $\mathsf{mix0}$ are not an artefact of the calculus of structures. See [Str03b, Str02, Str03a] for systems that do not involve mix.

2.1 Definition There are countably many *positive* and *negative atoms*. They, positive or negative, are denoted by a, b, ... *Structures* are denoted by S, P, Q, R, T, U, V, W, X and Z. The structures of the *language* NEL are generated by

$$S ::= a \mid \circ \mid [\underbrace{S \parr \cdots \parr S}_{>0}] \mid (\underbrace{S \otimes \cdots \otimes S}_{>0}) \mid \langle \underbrace{S \lhd \cdots \lhd S}_{>0} \rangle \mid {?S} \mid {!S} \mid \bar{S} \quad ,$$

where $\circ$, the *unit*, is not an atom and $\bar{S}$ is the *negation* of the structure S. Structures with a hole that does not appear in the scope of a negation are denoted by $S\{\ \}$. The structure R is a *substructure* of $S\{R\}$, and $S\{\ \}$ is its *context*. We simplify the

Associativity

$$
\begin{aligned}
[\vec{R} \parr [\vec{T}] \parr \vec{U}] &= [\vec{R} \parr \vec{T} \parr \vec{U}] \\
(\vec{R} \otimes (\vec{T}) \otimes \vec{U}) &= (\vec{R} \otimes \vec{T} \otimes \vec{U}) \\
\langle \vec{R} \lhd \langle \vec{T} \rangle \lhd \vec{U} \rangle &= \langle \vec{R} \lhd \vec{T} \lhd \vec{U} \rangle
\end{aligned}
$$

Commutativity

$$
\begin{aligned}
[\vec{R} \parr \vec{T}] &= [\vec{T} \parr \vec{R}] \\
(\vec{R} \otimes \vec{T}) &= (\vec{T} \otimes \vec{R})
\end{aligned}
$$

Unit

$$
\begin{aligned}
[\circ \parr \vec{R}] &= [\vec{R}] \\
(\circ \otimes \vec{R}) &= (\vec{R}) \\
\langle \circ \lhd \vec{R} \rangle &= \langle \vec{R} \rangle \\
\langle \vec{R} \lhd \circ \rangle &= \langle \vec{R} \rangle
\end{aligned}
$$

Singleton

$$[R] = (R) = \langle R \rangle = R$$

Negation

$$
\begin{aligned}
\bar{\circ} &= \circ \\
\overline{[R_1 \parr \cdots \parr R_h]} &= (\bar{R}_1 \otimes \cdots \otimes \bar{R}_h) \\
\overline{(R_1 \otimes \cdots \otimes R_h)} &= [\bar{R}_1 \parr \cdots \parr \bar{R}_h] \\
\overline{\langle R_1 \lhd \cdots \lhd R_h \rangle} &= \langle \bar{R}_1 \lhd \cdots \lhd \bar{R}_h \rangle \\
\overline{?R} &= !\bar{R} \\
\overline{!R} &= ?\bar{R} \\
\bar{\bar{R}} &= R
\end{aligned}
$$

Contextual Closure

$$\text{if } R = T \text{ then } S\{R\} = S\{T\}$$

Figure 1: Basic equations for the syntactic equivalence =

indication of context in cases where structural parentheses fill the hole exactly: for example, $S[R \parr T]$ stands for $S\{[R \parr T]\}$.

Structures come with equational theories establishing some basic, decidable algebraic laws by which structures are indistinguishable. These are analogous to the laws of associativity, commutativity, idempotency, and so on, usually imposed on sequents. The difference is that we merge the notions of formula and sequent, and we extend the equations to formulae. The structures of the language NEL are equivalent modulo the relation =, defined in Figure 1. There, $\vec{R}$, $\vec{T}$ and $\vec{U}$ stand for finite, non-empty sequences of structures (elements of the sequences are separated by $\parr$, $\lhd$, or $\otimes$, as appropriate in the context).

2.2 Definition An (*inference*) *rule* is any scheme

$$\rho \frac{T}{R} \quad ,$$

where ρ is the *name* of the rule, T is its *premise* and R is its *conclusion*; R or T, but not both, may be missing. Rule names are denoted by ρ. A (*deductive*) *system*, denoted by $\mathscr{S}$, is a set of rules. A *derivation* in a system $\mathscr{S}$ is a finite chain of instances of rules of $\mathscr{S}$, and is denoted by Δ; a derivation can consist of just one structure. The topmost structure in a derivation is called its *premise*; the bottommost structure is called *conclusion*. A derivation Δ whose premise is T, conclusion is R, and whose rules are in $\mathscr{S}$ is denoted by

$$\begin{array}{c} T \\ \mathscr{S} \Big\Vert \Delta \\ R \end{array} \quad .$$

The typical inference rules are of the kind

$$\rho \frac{S\{T\}}{S\{R\}} \quad .$$

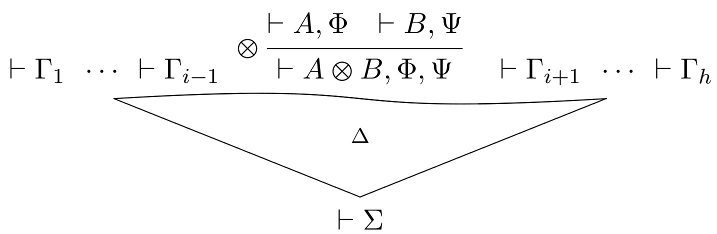
This rule scheme ρ specifies that if a structure matches R, in a context $S\{\ \}$, it can be rewritten as specified by T, in the same context $S\{\ \}$ (or vice versa if one reasons top-down). A rule corresponds to implementing in the deductive system *any axiom* $T \Rightarrow R$, where $\Rightarrow$ stands for the implication we model in the system, in our case linear implication. The case where the context is empty corresponds to the sequent calculus. For example, the linear logic sequent calculus rule

$$\otimes \frac{\vdash A, \Phi \quad \vdash B, \Psi}{\vdash A \otimes B, \Phi, \Psi}$$

could be simulated easily in the calculus of structures by the rule

$$\otimes' \frac{(\Gamma \otimes [A \parr \Phi] \otimes [B \parr \Psi])}{(\Gamma \otimes [(A \otimes B) \parr \Phi \parr \Psi])} \quad ,$$

where Φ and Ψ stand for multisets of formulae or their corresponding par structures. The structure Γ stands for the times structure of the other hypotheses in the derivation tree. More precisely, any sequent calculus derivation



containing the $\otimes$ rule can by simulated by

$$\begin{array}{c} \otimes' \dfrac{(\Gamma'_1 \otimes \ldots \otimes \Gamma'_{i-1} \otimes [A' \parr \Phi'] \otimes [B' \parr \Psi'] \otimes \Gamma'_{i+1} \otimes \ldots \otimes \Gamma'_h)}{(\Gamma'_1 \otimes \ldots \otimes \Gamma'_{i-1} \otimes [(A' \otimes B') \parr \Phi' \parr \Psi'] \otimes \Gamma'_{i+1} \otimes \ldots \otimes \Gamma'_h)} \\ \Big\Vert \Delta' \\ \Sigma' \end{array} \quad ,$$

in the calculus of structures, where Γ'_j, A', B', Φ', Ψ', Δ' and Σ' are obtained from their counterparts in the sequent calculus by the obvious translation. This means that by this method every system in the one-sided sequent calculus can be ported trivially to the calculus of structures.

Of course, in the calculus of structures, rules could be used as axioms of a generic Hilbert system, where there is no special, structural relation between T and R: then all the good proof theoretical properties of sequent systems would be lost. We will be careful to design rules in a way that is conservative enough to allow us to prove cut elimination, and such that they possess the subformula property.

In our systems, rules come in pairs,

$$\rho{\downarrow}\,\frac{S\{T\}}{S\{R\}}\ \text{(down version)} \qquad \text{and} \qquad \rho{\uparrow}\,\frac{S\{\bar{R}\}}{S\{\bar{T}\}}\ \text{(up version)} \quad .$$

Sometimes rules are self-dual, i.e., the up and down versions are identical, in which case we omit the arrows. This duality derives from the duality between $T \Rightarrow R$ and $\bar{R} \Rightarrow \bar{T}$, where $\Rightarrow$ is the implication and $\bar{(\cdot)}$ the negation of the logic. In the case of NEL these are linear implication and linear negation. We will be able to get rid of the up rules without affecting provability—after all, $T \Rightarrow R$ and $\bar{R} \Rightarrow \bar{T}$ are equivalent statements in many logics. Remarkably, the cut rule reduces into several up rules, and this makes for a modular decomposition of the cut elimination argument because we can eliminate up rules one independently from the other.

Let us now define system NEL by starting from an up-down symmetric variation, that we call SNEL. It is made by two sub-systems that we will call conventionally *interaction* and *structure*. The interaction fragment deals with negation, i.e., duality. It corresponds to identity and cut in the sequent calculus. In our calculus these rules become mutually top-down symmetric and both can be reduced to their atomic counterparts.

The structure fragment corresponds to logical and structural rules in the sequent calculus; it defines the logical operators. Differently from the sequent calculus, the operators need not be defined in isolation, rather complex contexts can be taken into consideration. In the following system we consider *pairs* of logical relations, one inside the other.

2.3 Definition In Figure 2, *system* SNEL is shown (*symmetric non-commutative exponential linear logic*). The rules $\mathsf{ai}{\downarrow}$, $\mathsf{ai}{\uparrow}$, s, $\mathsf{q}{\downarrow}$, $\mathsf{q}{\uparrow}$, $\mathsf{p}{\downarrow}$, $\mathsf{p}{\uparrow}$, $\mathsf{e}{\downarrow}$, $\mathsf{e}{\downarrow}$, $\mathsf{w}{\downarrow}$, $\mathsf{w}{\uparrow}$, $\mathsf{b}{\downarrow}$, $\mathsf{b}{\uparrow}$, $\mathsf{g}{\downarrow}$, and $\mathsf{g}{\uparrow}$ are called respectively *atomic interaction*, *atomic cut*, *switch*, *seq*, *coseq*, *promotion*, *copromotion*, *empty*, *coempty*, *weakening*, *coweakening*, *absorption*, *coabsorption*, *digging*, and *codigging*. The *down fragment* of SNEL is $\{\mathsf{ai}{\downarrow}, \mathsf{s}, \mathsf{q}{\downarrow}, \mathsf{p}{\downarrow}, \mathsf{e}{\downarrow}, \mathsf{w}{\downarrow}, \mathsf{b}{\downarrow}, \mathsf{g}{\downarrow}\}$, the *up fragment* is $\{\mathsf{ai}{\uparrow}, \mathsf{s}, \mathsf{q}{\uparrow}, \mathsf{p}{\uparrow}, \mathsf{e}{\uparrow}, \mathsf{w}{\uparrow}, \mathsf{b}{\uparrow}, \mathsf{g}{\uparrow}\}$.

There is a straightforward two-way correspondence between structures not involving seq and formulae of MELL: for example

$$![(?a \otimes b) \parr \bar{c} \parr !\bar{d}] \quad \text{corresponds to} \quad !((?a \otimes b) \parr c^{\perp} \parr !d^{\perp}) \quad ,$$

and vice versa. Units are mapped into $\circ$ (since $1 \equiv \perp$, when mix and $\mathsf{mix0}$ are added to MELL). System SNEL is just the merging of systems SBV and SELS shown

$$\mathsf{ai}\!\downarrow \frac{S\{\circ\}}{S[a \parr \bar{a}]} \qquad\qquad \mathsf{ai}\!\uparrow \frac{S(a \otimes \bar{a})}{S\{\circ\}}$$

$$\mathsf{s}\, \frac{S([R \parr U] \otimes T)}{S[(R \otimes T) \parr U]}$$

$$\mathsf{q}\!\downarrow \frac{S\langle [R \parr U] \lhd [T \parr V]\rangle}{S[\langle R \lhd T\rangle \parr \langle U \lhd V\rangle]} \qquad\qquad \mathsf{q}\!\uparrow \frac{S(\langle R \lhd U\rangle \otimes \langle T \lhd V\rangle)}{S\langle (R \otimes T) \lhd (U \otimes V)\rangle}$$

$$\mathsf{p}\!\downarrow \frac{S\{![R \parr T]\}}{S[!R \parr ?T]} \qquad\qquad \mathsf{p}\!\uparrow \frac{S(?R \otimes !T)}{S\{?(R \otimes T)\}}$$

$$\mathsf{e}\!\downarrow \frac{S\{\circ\}}{S\{!\circ\}} \qquad\qquad \mathsf{e}\!\uparrow \frac{S\{?\circ\}}{S\{\circ\}}$$

$$\mathsf{w}\!\downarrow \frac{S\{\circ\}}{S\{?R\}} \qquad\qquad \mathsf{w}\!\uparrow \frac{S\{!R\}}{S\{\circ\}}$$

$$\mathsf{b}\!\downarrow \frac{S[?R \parr R]}{S\{?R\}} \qquad\qquad \mathsf{b}\!\uparrow \frac{S\{!R\}}{S(!R \otimes R)}$$

$$\mathsf{g}\!\downarrow \frac{S\{??R\}}{S\{?R\}} \qquad\qquad \mathsf{g}\!\uparrow \frac{S\{!R\}}{S\{!!R\}}$$

Figure 2: System SNEL

in [Gug07, GS01, Str03b, Str03a]; there one can find details on the correspondence between our systems and linear logic.[1] The rules s, q↓ and q↑ are the same as in pomset logic viewed as a calculus of cographs [Ret99].

All equations are typical of a sequent calculus presentation, save those for units and contextual closure. Contextual closure just corresponds to equivalence being a congruence, it is a necessary ingredient of the calculus of structures. All other equations can be removed and replaced by rules (see, e.g., [Str05a]), as in the sequent calculus. This might prove necessary for certain applications. For our purposes, this setting makes for a much more compact presentation, at a more effective abstraction

[1]Note that there is a change of our system with respect to the system SELS in [Str03b] and the version of SNEL presented in the conference version [GS02] of this paper: Here we have added the rules e↓, e↑, g↓, and g↑, whereas previously we used the equations $??R = ?R$ and $!!R = !R$, as well as $!\circ = \circ = ?\circ$ in [GS02] and $!1 = 1$ and $?\bot = \bot$ in [Str03b] (see also [Str03a]). From the viewpoint of provability, there is no difference between the two approaches, but certain properties of the system (e.g., decomposition, see Section 3) can be demonstrated in a cleaner way. Also from the viewpoint of denotational semantics, our system is now easier accessible. For example in coherence spaces [Gir87] we do not have an isomorphism between $!R$ and $!!R$.

level.

Negation is involutive and can be pushed to atoms; it is convenient always to imagine it directly over atoms. Please note that negation does not swap arguments of seq, as happens in the systems of Lambek [Lam58] and Abrusci-Ruet [AR00]. The unit $\circ$ is self-dual and common to par, times and seq. One may think of it as a convenient way of expressing the empty sequence. Rules become very flexible in the presence of the unit. For example, the following notable derivation is valid:

$$\mathsf{q}{\downarrow}\frac{\mathsf{q}{\uparrow}\dfrac{(a \otimes b)}{\langle a \lhd b \rangle}}{[a \mathbin{⅋} b]} \qquad \equiv \qquad = \frac{= \dfrac{\mathsf{q}{\downarrow}\dfrac{= \dfrac{= \dfrac{\mathsf{q}{\uparrow}\dfrac{= \dfrac{(a \otimes b)}{(\langle a \lhd \circ \rangle \otimes \langle \circ \lhd b \rangle)}}{\langle (a \otimes \circ) \lhd (\circ \otimes b) \rangle}}{\langle a \lhd b \rangle}}{\langle [a \mathbin{⅋} \circ] \lhd [\circ \mathbin{⅋} b] \rangle}}{[\langle a \lhd \circ \rangle \mathbin{⅋} \langle \circ \lhd b \rangle]}}{[a \mathbin{⅋} b]}}{} \quad .$$

The right-hand side above is just a complicated way of writing the left-hand side. Using the "fake inference rule =" sometimes eases the reading of a derivation.

Each inference rule in Figure 2 corresponds to a linear implication that is sound in MELL plus mix and $\mathsf{mix0}$. For example, promotion corresponds to the implication $!(R \mathbin{⅋} T) \multimap (!R \mathbin{⅋} ?T)$. Notice that interaction and cut are atomic in SNEL; we can define their general versions as follows.

2.4 Definition The following rules are called *interaction* and *cut*:

$$\mathsf{i}{\downarrow}\frac{S\{\circ\}}{S[R \mathbin{⅋} \bar{R}]} \qquad \text{and} \qquad \mathsf{i}{\uparrow}\frac{S(R \otimes \bar{R})}{S\{\circ\}} \quad ,$$

where R and $\bar{R}$ are called *principal structures*.

The sequent calculus rule

$$\mathsf{cut}\frac{\vdash A, \Phi \quad \vdash A^{\perp}, \Psi}{\vdash \Phi, \Psi}$$

is realised as

$$\mathsf{i}{\uparrow}\frac{\mathsf{s}\dfrac{\mathsf{s}\dfrac{([A \mathbin{⅋} \Phi] \otimes [\bar{A} \mathbin{⅋} \Psi])}{[([A \mathbin{⅋} \Phi] \otimes \bar{A}) \mathbin{⅋} \Psi]}}{[(A \otimes \bar{A}) \mathbin{⅋} \Phi \mathbin{⅋} \Psi]}}{[\Phi \mathbin{⅋} \Psi]} \quad ,$$

where Φ and Ψ stand for multisets of formulae or their corresponding par structures. Notice how the tree shape of derivations in the sequent calculus is realised by making use of tensor structures: in the derivation above, the premise corresponds to the two branches of the cut rule. For this reason, in the calculus of structures rules are allowed to access structures deeply nested into contexts.

The cut rule in the calculus of structures can mimic the classical cut rule in the sequent calculus in its realisation of transitivity, but it is much more general. We believe a good way of understanding it is in thinking of the rule as being about lemmas *in context*. The sequent calculus cut rule generates a lemma valid in the most general context; the new cut rule does the same, but the lemma only affects the limited portion of structure that can interact with it.

We easily get the next two propositions, which say: 1) The interaction and cut rules can be reduced into their atomic forms—note that in the sequent calculus it is possible to reduce interaction to atomic form, but not cut. 2) The cut rule is as powerful as the whole up fragment of the system, and vice versa.

2.5 Definition A rule ρ is *derivable* in the system $\mathscr{S}$ if $\rho \notin \mathscr{S}$ and

$$\text{for every instance} \quad \rho\,\frac{T}{R} \quad \text{there exists a derivation} \quad \begin{array}{c} T \\ \mathscr{S}\,\Big\Vert\,\Delta \\ R \end{array} \quad .$$

The systems $\mathscr{S}$ and $\mathscr{S}'$ are *strongly equivalent* if

$$\text{for every derivation} \quad \begin{array}{c} T \\ \mathscr{S}\,\Big\Vert\,\Delta \\ R \end{array} \quad \text{there exists a derivation} \quad \begin{array}{c} T \\ \mathscr{S}'\,\Big\Vert\,\Delta' \\ R \end{array} \quad ,$$

and vice versa.

2.6 Proposition *The rule* $\mathsf{i}\!\downarrow$ *is derivable in* $\{\mathsf{ai}\!\downarrow, \mathsf{s}, \mathsf{q}\!\downarrow, \mathsf{p}\!\downarrow, \mathsf{e}\!\downarrow\}$, *and, dually, the rule* $\mathsf{i}\!\uparrow$ *is derivable in the system* $\{\mathsf{ai}\!\uparrow, \mathsf{s}, \mathsf{q}\!\uparrow, \mathsf{p}\!\uparrow, \mathsf{e}\!\uparrow\}$.

Proof: Induction on principal structures. We show the inductive cases for $\mathsf{i}\!\uparrow$:

$$\mathsf{i}\!\uparrow,\mathsf{i}\!\uparrow\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{S(P \otimes Q \otimes [\bar{P} \mathbin{\bindnasrepma} \bar{Q}])}{S(Q \otimes [(P \otimes \bar{P}) \mathbin{\bindnasrepma} \bar{Q}])}}{S[(P \otimes \bar{P}) \mathbin{\bindnasrepma} (Q \otimes \bar{Q})]}}{=\,\dfrac{S[\circ \mathbin{\bindnasrepma} \circ]}{S\{\circ\}}} \qquad \mathsf{i}\!\uparrow,\mathsf{i}\!\uparrow\,\dfrac{\mathsf{q}\!\uparrow\,\dfrac{S(\langle P \lhd Q\rangle \otimes \langle \bar{P} \lhd \bar{Q}\rangle)}{S\langle (P \otimes \bar{P}) \lhd (Q \otimes \bar{Q})\rangle}}{=\,\dfrac{S\langle \circ \lhd \circ\rangle}{S\{\circ\}}} \qquad \mathsf{e}\!\uparrow\,\dfrac{\mathsf{i}\!\uparrow\,\dfrac{\mathsf{p}\!\uparrow\,\dfrac{S(?P \otimes !\bar{P})}{S\{?(P \otimes \bar{P})\}}}{S\{?\circ\}}}{S\{\circ\}} \quad .$$

The cases for $\mathsf{i}\!\downarrow$ are dual. □

Note that in the proof above we tacitly used (for the sake of saving paper) another helpful notation: writing $\mathsf{i}\!\uparrow,\mathsf{i}\!\uparrow$ just means that two instances of $\mathsf{i}\!\uparrow$ applied one after the other, where the order does not matter.

2.7 Proposition *Each rule* $\rho\!\uparrow$ *in* SNEL *is derivable in* $\{\mathsf{i}\!\downarrow, \mathsf{i}\!\uparrow, \mathsf{s}, \rho\!\downarrow\}$, *and, dually, each rule* $\rho\!\downarrow$ *in* SNEL *is derivable in the system* $\{\mathsf{i}\!\downarrow, \mathsf{i}\!\uparrow, \mathsf{s}, \rho\!\uparrow\}$.

$$
\circ\!\downarrow \frac{}{\circ} \qquad \mathsf{ai}\!\downarrow \frac{S\{\circ\}}{S[a \parr \bar{a}]} \qquad \mathsf{e}\!\downarrow \frac{S\{\circ\}}{S\{!\circ\}}
$$

$$
\mathsf{s}\,\frac{S([R \parr U] \otimes T)}{S[(R \otimes T) \parr U]} \qquad \mathsf{q}\!\downarrow \frac{S\langle [R \parr U] \lhd [T \parr V]\rangle}{S[\langle R \lhd T\rangle \parr \langle U \lhd V\rangle]} \qquad \mathsf{p}\!\downarrow \frac{S\{![R \parr T]\}}{S[!R \parr ?T]}
$$

$$
\mathsf{w}\!\downarrow \frac{S\{\circ\}}{S\{?R\}} \qquad \mathsf{b}\!\downarrow \frac{S[?R \parr R]}{S\{?R\}} \qquad \mathsf{g}\!\downarrow \frac{S\{??R\}}{S\{?R\}}
$$

Figure 3: System NEL

Proof: Each instance

$$
\rho\!\uparrow \frac{S\{T\}}{S\{R\}}
$$

can be replaced by

$$
\mathsf{i}\!\downarrow \cfrac{S\{T\}}{\mathsf{s}\,\cfrac{S(T \otimes [R \parr \bar{R}])}{\rho\!\downarrow \cfrac{S[R \parr (T \otimes \bar{R})]}{\mathsf{i}\!\uparrow \cfrac{S[R \parr (T \otimes \bar{T})]}{S\{R\}}}}}
$$

and dually. □

In the calculus of structures, we call *core* the set of rules that is used to reduce interaction and cut to atomic form. We use the term *hard core* to denote the set of rules in the core other than atomic interaction/cut and empty/coempty. Rules that are not in the core are called *non-core*.

2.8 Definition The *core* of SNEL is $\{\mathsf{ai}\!\downarrow, \mathsf{ai}\!\uparrow, \mathsf{s}, \mathsf{q}\!\downarrow, \mathsf{q}\!\uparrow, \mathsf{p}\!\downarrow, \mathsf{p}\!\uparrow, \mathsf{e}\!\downarrow, \mathsf{e}\!\uparrow\}$, denoted by SNELc. The *hard core*, denoted by SNELh, is $\{\mathsf{s}, \mathsf{q}\!\downarrow, \mathsf{q}\!\uparrow, \mathsf{p}\!\downarrow, \mathsf{p}\!\uparrow\}$, and the *non-core* is $\{\mathsf{w}\!\downarrow, \mathsf{w}\!\uparrow, \mathsf{b}\!\downarrow, \mathsf{b}\!\uparrow, \mathsf{g}\!\downarrow, \mathsf{g}\!\uparrow\}$.

System SNEL is up-down symmetric, and the properties we saw are also symmetric. Provability is an asymmetric notion: we want to observe the possible conclusions that we can obtain from a unit premise. We now break the up-down symmetry by adding an inference rule with no premise, and we join this logical axiom to the down fragment of SNEL.

2.9 Definition The following rule is called *unit*:

$$
\circ\!\downarrow \frac{}{\circ} \quad .
$$

System NEL is shown in Figure 3.

As an immediate consequence of Propositions 2.6 and 2.7 we get:

2.10 Proposition *The systems* $\mathsf{NEL} \cup \{\mathsf{i}{\uparrow}\}$ *and* $\mathsf{SNEL} \cup \{\circ{\downarrow}\}$ *are strongly equivalent.*

2.11 Definition A derivation with no premise is called a *proof*, denoted by Π. A system $\mathscr{S}$ *proves* R if there is in the system $\mathscr{S}$ a proof Π whose conclusion is R, written

$$\begin{array}{c} \\ \mathscr{S}\,\big\Vert\,\Pi \\ R \end{array} \quad .$$

We say that a rule ρ is *admissible* for the system $\mathscr{S}$ if $\rho \notin \mathscr{S}$ and

$$\text{for every proof} \quad \begin{array}{c} \\ \mathscr{S} \cup \{\rho\}\,\big\Vert\,\Pi \\ R \end{array} \quad \text{there is a proof} \quad \begin{array}{c} \\ \mathscr{S}\,\big\Vert\,\Pi' \\ R \end{array} \quad .$$

Two systems are *equivalent* if they prove the same structures.

Except for cut and coweakening, all rules in the systems SNEL and NEL enjoy a subformula property (which we treat as an asymmetric property, by going from conclusion to premise): premises are made of substructures of the conclusions.

To get cut elimination, so as to have a system whose rules all enjoy the subformula property, we could just get rid of $\mathsf{ai}{\uparrow}$ and $\mathsf{w}{\uparrow}$, by proving their admissibility for the other rules. But we can do more than that: the whole up fragment of SNEL (except for s which also belongs to the down fragment) is admissible. This entails a *modular* scheme for proving cut elimination. In Sections 3 and 4 we will give the proof of the cut elimination theorem:

2.12 Theorem *System* NEL *is equivalent to* $\mathsf{SNEL} \cup \{\circ{\downarrow}\}$.

2.13 Corollary *The rule* $\mathsf{i}{\uparrow}$ *is admissible for system* NEL.

Any linear implication $T \multimap R$, i.e., $[\bar{T} \parr R]$, is connected to derivability by:

2.14 Corollary *For any two structures* T *and* R, *we have*

$$\begin{array}{c} T \\ \mathsf{SNEL}\,\big\Vert \\ R \end{array} \quad \textit{if and only if} \quad \begin{array}{c} \\ \mathsf{NEL}\,\big\Vert \\ [\bar{T} \parr R] \end{array} \quad .$$

Proof: For the first direction, perform the following transformations:

$$\begin{array}{c} T \\ \mathsf{SNEL}\,\big\Vert\,\Delta \\ R \end{array} \quad \overset{1}{\leadsto} \quad \begin{array}{c} [\bar{T} \parr T] \\ \mathsf{SNEL}\,\big\Vert\,\Delta' \\ [\bar{T} \parr R] \end{array} \quad \overset{2}{\leadsto} \quad \begin{array}{c} \mathsf{i}{\downarrow}\dfrac{\circ{\downarrow}\,\dfrac{}{\circ}}{[\bar{T} \parr T]} \\ \mathsf{SNEL}\,\big\Vert\,\Delta' \\ [\bar{T} \parr R] \end{array} \quad \overset{3}{\leadsto} \quad \begin{array}{c} \\ \mathsf{NEL}\,\big\Vert\,\Pi \\ [\bar{T} \parr R] \end{array} \quad .$$

In the first step we replace each structure S occurring inside Δ by $[\bar{T} \parr S]$, or, in other words, the derivation Δ' is obtained by putting Δ into the context $[\bar{T} \parr \{\ \}]$. This is then transformed into a proof by adding an instance of $\mathsf{i}{\downarrow}$ and $\circ{\downarrow}$. Then we apply Proposition 2.6 and cut elimination (Theorem 2.12) to obtain a proof in system NEL. For the other direction, we proceed as follows:

$$
\mathsf{NEL}\ \overline{\underline{\Big\Vert}}\ \Pi \atop [\bar{T} \parr R]
\quad\leadsto\quad
\begin{array}{c} \circ \\ \mathsf{NEL}\setminus\{\circ{\downarrow}\}\ \Big\Vert\ \Delta \\ [\bar{T} \parr R] \end{array}
\quad\leadsto\quad
\begin{array}{c} T \\ \mathsf{NEL}\setminus\{\circ{\downarrow}\}\ \Big\Vert\ \Delta' \\ \mathsf{s}\,\dfrac{(T \otimes [\bar{T} \parr R])}{[(T \otimes \bar{T}) \parr R]} \\ \mathsf{i}{\uparrow}\,\overline{\phantom{[(T \otimes \bar{T}) \parr R]}} \\ R \end{array}
\quad\leadsto\quad
\begin{array}{c} T \\ \mathsf{SNEL}\ \Big\Vert \\ R \end{array}
\quad ,
$$

where the first two steps are trivial, and the last one is an application of Proposition 2.6. □

It is easy to prove that system NEL is a conservative extension of MELL plus mix and $\mathsf{mix0}$ (see [Gug07, Str03a]). The locality properties shown in [GS01, Str03b] still hold in this system, of course. In particular, the promotion rule is local, as opposed to the same rule in the sequent calculus.

3 Decomposition

The new top-down symmetry of derivations in the calculus of structures allows to study properties that are not observable in the sequent calculus. The most remarkable results so far are decomposition theorems. In general, a decomposition theorem says that a given system $\mathscr{S}$ can be divided into n pairwise disjoint subsystems $\mathscr{S}_1, \ldots, \mathscr{S}_n$ such that every derivation Δ in system $\mathscr{S}$ can be rearranged as composition of n derivations $\Delta_1, \ldots, \Delta_n$, where Δ_i uses only rules of $\mathscr{S}_i$, for every $1 \leq i \leq n$.

System SNEL can be decomposed into eleven subsystems, and there are many different possibilities to transform a derivation into eleven subderivations. We state here only four of them, but, due to the modular proof, the others are evident.

3.1 Theorem (Decomposition) *For every derivation* $\Delta\;\begin{array}{c} T \\ \Vert \\ R \end{array}\mathsf{SNEL}$ *there are derivations*

$$
\begin{array}{rl}
& T \\
\{\mathsf{e}\downarrow\} & \Vert \\
& P_1 \\
\{\mathsf{g}\uparrow\} & \Vert \\
& P_2 \\
\{\mathsf{b}\uparrow\} & \Vert \\
& P_3 \\
\{\mathsf{ai}\downarrow\} & \Vert \\
& P_4 \\
\{\mathsf{w}\downarrow\} & \Vert \\
& P_5 \\
\mathsf{SNELh} & \Vert \\
& Q_5 \\
\{\mathsf{w}\uparrow\} & \Vert \\
& Q_4 \\
\{\mathsf{ai}\uparrow\} & \Vert \\
& Q_3 \\
\{\mathsf{b}\downarrow\} & \Vert \\
& Q_2 \\
\{\mathsf{g}\downarrow\} & \Vert \\
& Q_1 \\
\{\mathsf{e}\uparrow\} & \Vert \\
& R
\end{array}
\qquad
\begin{array}{rl}
& T \\
\{\mathsf{g}\uparrow\} & \Vert \\
& U_1 \\
\{\mathsf{b}\uparrow\} & \Vert \\
& U_2 \\
\{\mathsf{e}\downarrow\} & \Vert \\
& U_3 \\
\{\mathsf{w}\downarrow\} & \Vert \\
& U_4 \\
\{\mathsf{ai}\downarrow\} & \Vert \\
& U_5 \\
\mathsf{SNELh} & \Vert \\
& V_5 \\
\{\mathsf{ai}\uparrow\} & \Vert \\
& V_4 \\
\{\mathsf{w}\uparrow\} & \Vert \\
& V_3 \\
\{\mathsf{e}\uparrow\} & \Vert \\
& V_2 \\
\{\mathsf{b}\downarrow\} & \Vert \\
& V_1 \\
\{\mathsf{g}\downarrow\} & \Vert \\
& R
\end{array}
\qquad
\begin{array}{rl}
& T \\
\{\mathsf{e}\downarrow\} & \Vert \\
& W_1 \\
\{\mathsf{g}\uparrow\} & \Vert \\
& W_2 \\
\{\mathsf{b}\uparrow\} & \Vert \\
& W_3 \\
\{\mathsf{w}\uparrow\} & \Vert \\
& W_4 \\
\{\mathsf{ai}\downarrow\} & \Vert \\
& W_5 \\
\mathsf{SNELh} & \Vert \\
& Z_5 \\
\{\mathsf{ai}\uparrow\} & \Vert \\
& Z_4 \\
\{\mathsf{w}\downarrow\} & \Vert \\
& Z_3 \\
\{\mathsf{b}\downarrow\} & \Vert \\
& Z_2 \\
\{\mathsf{g}\downarrow\} & \Vert \\
& Z_1 \\
\{\mathsf{e}\uparrow\} & \Vert \\
& R
\end{array}
\qquad
\begin{array}{rl}
& T \\
\{\mathsf{g}\uparrow\} & \Vert \\
& T_1 \\
\{\mathsf{b}\uparrow\} & \Vert \\
& T_2 \\
\{\mathsf{w}\uparrow\} & \Vert \\
& T_3 \\
\{\mathsf{e}\downarrow\} & \Vert \\
& T_4 \\
\{\mathsf{ai}\downarrow\} & \Vert \\
& T_5 \\
\mathsf{SNELh} & \Vert \\
& R_5 \\
\{\mathsf{ai}\uparrow\} & \Vert \\
& R_4 \\
\{\mathsf{e}\uparrow\} & \Vert \\
& R_3 \\
\{\mathsf{w}\downarrow\} & \Vert \\
& R_2 \\
\{\mathsf{b}\downarrow\} & \Vert \\
& R_1 \\
\{\mathsf{g}\downarrow\} & \Vert \\
& R
\end{array}
$$

For simplicitity we will in the following call the four statements first, second, third, and fourth decomposition (from left to right).

Apart from a decomposition into eleven subsystems, the first and the second decomposition can also be read as a decomposition into three subsystems that could be called *creation*, *merging* and *destruction*. In the creation subsystem, each rule increases the size of the structure; in the merging system, each rule does some rearranging of substructures, without changing the size of the structures; and in the destruction system, each rule decreases the size of the structure. Here, the size of the structure incorporates not only the number of atoms in it, but also the modality-depth for each atom. In a decomposed derivation, the merging part is in the middle of the derivation, and (depending on your preferred reading of a derivation) the creation and destruction are at the top and at the bottom, as shown in the left of Figure 4. In system SNEL the merging part contains the rules s, $\mathsf{q}\downarrow$, $\mathsf{q}\uparrow$, $\mathsf{p}\downarrow$ and $\mathsf{p}\uparrow$, which coin-

$$
\begin{array}{rcl}
 & T & \\
\text{destruction} & \Uparrow\ \Downarrow & \text{creation} \\
 & T' & \\
\text{merging} & \Updownarrow & \\
 & R' & \\
\text{creation} & \Uparrow\ \Downarrow & \text{destruction} \\
 & R &
\end{array}
\qquad
\begin{array}{rc}
 & T \\
\text{empty modality (down)} & \Vert \\
 & T' \\
\text{noncore (up)} & \Vert \\
 & T'' \\
\text{interaction (down)} & \Vert \\
 & T''' \\
\text{hard core (up and down)} & \Vert \\
 & R''' \\
\text{interaction (up)} & \Vert \\
 & R'' \\
\text{noncore (down)} & \Vert \\
 & R' \\
\text{empty modality (up)} & \Vert \\
 & R
\end{array}
\qquad
\begin{array}{rc}
 & T \\
\text{noncore (up)} & \Vert \\
 & T' \\
\text{core (up and down)} & \Vert \\
 & R' \\
\text{noncore (down)} & \Vert \\
 & R
\end{array}
$$

Figure 4: Readings of the decompositions

cides with the hard core. In the top-down reading of a derivation, the creation part contains the rules $\mathsf{e}\downarrow$, $\mathsf{g}\uparrow$, $\mathsf{b}\uparrow$, $\mathsf{w}\downarrow$ and $\mathsf{ai}\downarrow$, and the destruction part consists of $\mathsf{e}\uparrow$, $\mathsf{g}\downarrow$, $\mathsf{b}\downarrow$, $\mathsf{w}\uparrow$ and $\mathsf{ai}\uparrow$. In the bottom-up reading, creation and destruction are exchanged.

Note that this kind of decomposition (creation, merging, destruction) is quite typical for logical systems presented in the calculus of structures, and is not restricted to system SNEL. It holds, for example, also for systems SBV and SELS [GS01, Str03b], for classical logic [BT01], and for full propositional linear logic [Str02].

The third decomposition allows a separation between hard core and noncore of the system, such that the up fragment and the down fragment of the noncore are not merged, as it is the case in the first and second decomposition. More precisely, we can separate the seven subsystems shown in the middle of Figure 4. The fourth decompostion is even stronger in this respect: it allows a complete separation between core and noncore, as shown on the right of Figure 4. This decomposition also plays a crucial rule for the cut elimination argument. Recall that cut elimination means to get rid of the entire up-fragment. Because of the decomposition, the elimination of the non-core up-fragment is now trivial (see Section 4 for details). Furthermore, recall that for cut elimination in the sequent calculus, the most problematic cases are usually the ones where cut interacts with rules like contraction and weakening, and that in our system these rules appear as the non-core down rules. In the third decomposition these are *below* the actual cut rules (i.e., the core up rules, cf. Propositions 2.6, 2.7, and 2.10) and can therefore no longer interfere with the cut elimination. This considerably simplifies our cut elimination argument in Section 4.

$$
\begin{array}{c} T \\ \mathsf{SNEL}\ \Vert \\ R \end{array}
\xrightarrow{1}
\begin{array}{c} T \\ \{\mathsf{e}\downarrow\}\ \Vert \\ W_1 \\ \mathscr{S}_1\ \Vert \\ Z_1 \\ \{\mathsf{e}\uparrow\}\ \Vert \\ R \end{array}
\xrightarrow{2}
\begin{array}{c} T \\ \{\mathsf{e}\downarrow\}\ \Vert \\ W_1 \\ \{\mathsf{g}\uparrow,\mathsf{b}\uparrow,\mathsf{w}\uparrow\}\ \Vert \\ W_4 \\ \mathscr{S}_2\ \Vert \\ Z_4 \\ \{\mathsf{g}\downarrow,\mathsf{b}\downarrow,\mathsf{w}\downarrow\}\ \Vert \\ Z_1 \\ \{\mathsf{e}\uparrow\}\ \Vert \\ R \end{array}
\xrightarrow{3}
\begin{array}{c} T \\ \{\mathsf{e}\downarrow\}\ \Vert \\ W_1 \\ \{\mathsf{g}\uparrow\}\ \Vert \\ W_2 \\ \{\mathsf{b}\uparrow\}\ \Vert \\ W_3 \\ \{\mathsf{w}\uparrow\}\ \Vert \\ W_4 \\ \mathscr{S}_2\ \Vert \\ Z_4 \\ \{\mathsf{w}\downarrow\}\ \Vert \\ Z_3 \\ \{\mathsf{b}\downarrow\}\ \Vert \\ Z_2 \\ \{\mathsf{g}\downarrow\}\ \Vert \\ Z_1 \\ \{\mathsf{e}\uparrow\}\ \Vert \\ R \end{array}
\xrightarrow{4}
\begin{array}{c} T \\ \{\mathsf{e}\downarrow\}\ \Vert \\ W_1 \\ \{\mathsf{g}\uparrow\}\ \Vert \\ W_2 \\ \{\mathsf{b}\uparrow\}\ \Vert \\ W_3 \\ \{\mathsf{w}\uparrow\}\ \Vert \\ W_4 \\ \{\mathsf{ai}\downarrow\}\ \Vert \\ W_5 \\ \mathsf{SNELh}\ \Vert \\ Z_5 \\ \{\mathsf{ai}\uparrow\}\ \Vert \\ Z_4 \\ \{\mathsf{w}\downarrow\}\ \Vert \\ Z_3 \\ \{\mathsf{b}\downarrow\}\ \Vert \\ Z_2 \\ \{\mathsf{g}\downarrow\}\ \Vert \\ Z_1 \\ \{\mathsf{e}\uparrow\}\ \Vert \\ R \end{array}
\xrightarrow{5}
\begin{array}{c} T \\ \{\mathsf{g}\uparrow\}\ \Vert \\ T_1 \\ \{\mathsf{b}\uparrow\}\ \Vert \\ T_2 \\ \{\mathsf{w}\uparrow\}\ \Vert \\ T_3 \\ \{\mathsf{e}\downarrow\}\ \Vert \\ T_4 \\ \{\mathsf{ai}\downarrow\}\ \Vert \\ T_5 \\ \mathsf{SNELh}\ \Vert \\ R_5 \\ \{\mathsf{ai}\uparrow\}\ \Vert \\ R_4 \\ \{\mathsf{e}\uparrow\}\ \Vert \\ R_3 \\ \{\mathsf{w}\downarrow\}\ \Vert \\ R_2 \\ \{\mathsf{b}\downarrow\}\ \Vert \\ R_1 \\ \{\mathsf{g}\downarrow\}\ \Vert \\ R \end{array}
$$

Figure 5: Obtaining the third and fourth decomposition

However, it is well-known that there is no free lunch. We cannot expect that the proof of the decomposition theorem is trivial. At least, we have to expect problems when the non-core rules (which in case of SNEL do all deal with the modalities ! and ?) do interact with the rules $\mathsf{p}\downarrow$ and $\mathsf{p}\uparrow$ (which are the only core rules that properly deal with ! and ?). The good news is that these are the only cases where the proof of the decomposition theorem becomes problematic.

We will now continue with a very brief sketch of the proof and in the remainder of the section we will fill in the details.

Proof of Theorem 3.1 (Sketch): The third and fourth decomposition are obtained via the five steps shown in Figure 5, where $\mathscr{S}_1 = \mathsf{SNEL} \setminus \{\mathsf{e}\downarrow, \mathsf{e}\uparrow\}$ and $\mathscr{S}_2 = \{\mathsf{ai}\downarrow, \mathsf{ai}\uparrow\} \cup \mathsf{SNELh}$. The first and second decomposition are reached as shown in Figure 6, where $\mathscr{S}_3 = \{\mathsf{ai}\downarrow, \mathsf{ai}\uparrow, \mathsf{w}\downarrow, \mathsf{w}\uparrow\} \cup \mathsf{SNELh}$. Some explanation: Step 1 is performed via a rather simple rule permutation. The rule $\mathsf{e}\downarrow$ is permuted up in the derivation, and the rule $\mathsf{e}\uparrow$ is permuted down via the dual procedure. The concept of permuting rules in the calculus of structures is explained in more detail in Section 3.1. Step 2 is the most critical one. In some sense it can also be considered as a simple rule permutation. However, contrary to Step 1, it is not obvious at all that Step 2 does

$$
\begin{array}{c}
T \\
\{\mathsf{e}\downarrow\}\,\Vert \\
P_1 \\
\{\mathsf{g}\uparrow\}\,\Vert \\
P_2 \\
\{\mathsf{b}\uparrow\}\,\Vert \\
P_3 \\
\{\mathsf{ai}\downarrow\}\,\Vert \\
P_4 \\
\{\mathsf{w}\downarrow\}\,\Vert \\
P_5 \\
\mathsf{SNELh}\,\Vert \\
Q_5 \\
\{\mathsf{w}\uparrow\}\,\Vert \\
Q_4 \\
\{\mathsf{ai}\uparrow\}\,\Vert \\
Q_3 \\
\{\mathsf{b}\downarrow\}\,\Vert \\
Q_2 \\
\{\mathsf{g}\downarrow\}\,\Vert \\
Q_1 \\
\{\mathsf{e}\uparrow\}\,\Vert \\
R
\end{array}
\quad\xleftarrow{7}\quad
\begin{array}{c}
T \\
\{\mathsf{e}\downarrow\}\,\Vert \\
P_1 \\
\{\mathsf{g}\uparrow\}\,\Vert \\
P_2 \\
\{\mathsf{b}\uparrow\}\,\Vert \\
P_3 \\
\mathscr{S}_3\,\Vert \\
Q_3 \\
\{\mathsf{b}\downarrow\}\,\Vert \\
Q_2 \\
\{\mathsf{g}\downarrow\}\,\Vert \\
Q_1 \\
\{\mathsf{e}\uparrow\}\,\Vert \\
R
\end{array}
\quad\xleftarrow{6}\quad
\begin{array}{c}
T \\
\mathsf{SNEL}\,\Vert \\
R
\end{array}
\quad\xrightarrow{8}\quad
\begin{array}{c}
T \\
\{\mathsf{g}\uparrow\}\,\Vert \\
U_1 \\
\{\mathsf{b}\uparrow\}\,\Vert \\
U_2 \\
\{\mathsf{e}\downarrow\}\,\Vert \\
U_3 \\
\mathscr{S}_3\,\Vert \\
V_3 \\
\{\mathsf{e}\uparrow\}\,\Vert \\
V_2 \\
\{\mathsf{b}\downarrow\}\,\Vert \\
V_1 \\
\{\mathsf{g}\downarrow\}\,\Vert \\
R
\end{array}
\quad\xrightarrow{9}\quad
\begin{array}{c}
T \\
\{\mathsf{g}\uparrow\}\,\Vert \\
U_1 \\
\{\mathsf{b}\uparrow\}\,\Vert \\
U_2 \\
\{\mathsf{e}\downarrow\}\,\Vert \\
U_3 \\
\{\mathsf{w}\downarrow\}\,\Vert \\
U_4 \\
\{\mathsf{ai}\downarrow\}\,\Vert \\
U_5 \\
\mathsf{SNELh}\,\Vert \\
V_5 \\
\{\mathsf{ai}\uparrow\}\,\Vert \\
V_4 \\
\{\mathsf{w}\uparrow\}\,\Vert \\
V_3 \\
\{\mathsf{e}\uparrow\}\,\Vert \\
V_2 \\
\{\mathsf{b}\downarrow\}\,\Vert \\
V_1 \\
\{\mathsf{g}\downarrow\}\,\Vert \\
R
\end{array}
$$

Figure 6: Obtaining the first and second decomposition

terminate: while permuting $\mathsf{g}\uparrow$, $\mathsf{b}\uparrow$, and $\mathsf{w}\uparrow$ up, new instances of $\mathsf{g}\downarrow$, $\mathsf{b}\downarrow$, and $\mathsf{w}\downarrow$ are introduced, and vice versa. For showing termination, we introduce in Section 3.3 the concept of !-?-flow-graph. Steps 3, 4 and 5 are again rather simple rule permutations and are detailed out in Section 3.1 as well. Steps 6 and 8 are essentially the same as Steps 1–3 and 5 with the only difference that the rules $\mathsf{w}\uparrow$ and $\mathsf{w}\downarrow$ do not need attention. Steps 7 and 9 are only slight variations of each other and are not more complicated than Step 4. They are also done in Section 3.1. One last remark: Treating the rules $\mathsf{g}\uparrow$, $\mathsf{b}\uparrow$, $\mathsf{w}\uparrow$ together in Step 2 and separating them afterwards in Step 3 has been done on purpose. Treating them separately from the very beginning would not give termination in the general case. □

3.1 Permutation of Rules

The basic idea of permuting rules is to change the order of two consecutive rule instances in a derivation without changing the essence of the derivation.

3.1.1 Definition A rule π *permutes over* a rule ρ (or ρ *permutes under* π) if

$$\text{for every derivation}\quad \begin{array}{rc} & Q \\ \cline{2-2} \rho & U \\ \cline{2-2} \pi & P \end{array} \quad\text{there is a derivation}\quad \begin{array}{rc} & Q \\ \cline{2-2} \pi & V \\ \cline{2-2} \rho & P \end{array}$$

for some structure V.

For obtaining our decompositions, this definition is too strict. We would need, for example, that the rule $\mathsf{e}{\downarrow}$ permutes over all other rules in the system, which is not the case. We give a weaker concept:

3.1.2 Definition A rule π *permutes over* a rule ρ *by a system* $\mathscr{S}$, if

$$\text{for every derivation}\quad \begin{array}{rc} & Q \\ \cline{2-2} \rho & U \\ \cline{2-2} \pi & P \end{array} \quad\text{there is a derivation}\quad \begin{array}{rc} & Q \\ \cline{2-2} \pi & V \\ \cline{2-2} \rho & W \\ \mathscr{S} & \Big\Vert \\ & P \end{array}$$

for some structures V and W. Dually, ρ *permutes under* π *by* $\mathscr{S}$, if

$$\text{for every derivation}\quad \begin{array}{rc} & Q \\ \cline{2-2} \rho & U \\ \cline{2-2} \pi & P \end{array} \quad\text{there is a derivation}\quad \begin{array}{rc} & Q \\ \mathscr{S} & \Big\Vert \\ & W \\ \cline{2-2} \pi & V \\ \cline{2-2} \rho & P \end{array}$$

for some structures V and W.

Additionally, we will use the following terminology borrowed from term rewriting. In a rule instance

$$\rho\,\frac{S\{W\}}{S\{Z\}}$$

we call Z the *redex* and W the *contractum* of the rule's instance. If we have $Z = W$, then the rule instance is called *trivial*. (This can happen because of the equational theory and the involvement of the unit $\circ$.) In the following we will assume, without loss of generality, that the trivial rule instances are removed from all derivations.

When reading this section, the reader might notice some similarity to the analysis of critical pairs for local confluency in term rewriting. In fact, the basic idea is the same but the conceptual goal is different, as it is shown in Figure 7.

Here is the first lemma about rule permutations:

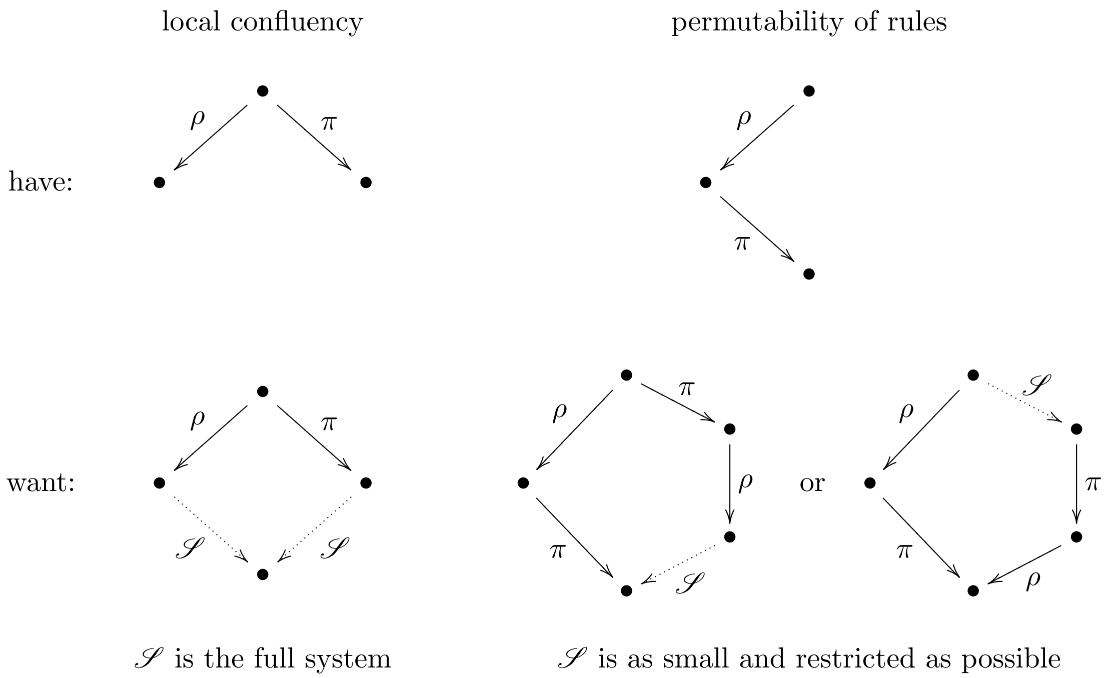


Figure 7: The analysis of critical pairs for local confluency and the permutability of rules

3.1.3 Lemma *The rule* $\mathsf{e}{\downarrow}$ *permutes over the rules* $\mathsf{e}{\uparrow}$, $\mathsf{ai}{\downarrow}$, $\mathsf{ai}{\uparrow}$, s, $\mathsf{q}{\downarrow}$, $\mathsf{q}{\uparrow}$, $\mathsf{p}{\downarrow}$, $\mathsf{p}{\uparrow}$, $\mathsf{w}{\uparrow}$, *and* $\mathsf{g}{\downarrow}$ *by the system* $\{\mathsf{s}, \mathsf{q}{\downarrow}, \mathsf{q}{\uparrow}\}$.

The proof of this lemma is a rather tedious case analysis in which most cases are trivial, and some cases are nontrivial. For the sake of completeness, this time we explain the case analysis in detail, and for similar lemmas that come later, we show only the nontrivial cases.

Proof of Lemma 3.1.3: Consider

$$\mathsf{e}{\downarrow}\,\dfrac{\rho\,\dfrac{S\{W\}}{S\{Z\}}}{S'\{Z'\}} \quad ,$$

where $\rho \in \{\mathsf{e}{\uparrow}, \mathsf{ai}{\downarrow}, \mathsf{ai}{\uparrow}, \mathsf{s}, \mathsf{q}{\downarrow}, \mathsf{q}{\uparrow}, \mathsf{p}{\downarrow}, \mathsf{p}{\uparrow}, \mathsf{w}{\uparrow}, \mathsf{g}{\downarrow}\} = \mathsf{SNEL} \setminus \{\mathsf{e}{\downarrow}, \mathsf{w}{\downarrow}, \mathsf{b}{\downarrow}, \mathsf{b}{\uparrow}, \mathsf{g}{\uparrow}\}$. We have to check all possibilities where the contractum $\circ$ of $\mathsf{e}{\downarrow}$ can appear inside $S\{Z\}$. We start with the two trivial cases:

(i) The contractum $\circ$ of $\mathsf{e}{\downarrow}$ is inside the context $S\{\ \}$. That means that $Z' = Z$, and we can replace

$$\mathsf{e}{\downarrow}\,\dfrac{\rho\,\dfrac{S\{W\}}{S\{Z\}}}{S'\{Z\}} \quad \leadsto \quad \rho\,\dfrac{\mathsf{e}{\downarrow}\,\dfrac{S\{W\}}{S'\{W\}}}{S'\{Z\}}$$

(ii) The contractum $\circ$ of $\mathsf{e}{\downarrow}$ is appears inside Z, but only inside a substructure of Z that is not affected by the rule ρ. Instead of getting too formal, we show an example:

$$\mathsf{e}{\downarrow}\,\frac{\mathsf{s}\,\dfrac{S([R\{\circ\} \parr U] \otimes T)}{S[(R\{\circ\} \otimes T) \parr U]}}{S[(R\{!\circ\} \otimes T) \parr U]} \quad\rightarrow\quad \mathsf{s}\,\frac{\mathsf{e}{\downarrow}\,\dfrac{S([R\{\circ\} \parr U] \otimes T)}{S([R\{!\circ\} \parr U] \otimes T)}}{S[(R\{!\circ\} \otimes T) \parr U]}$$

The cases where the $\circ$ appears inside U or T are similar. The same situation can occur with the rules $\mathsf{q}{\downarrow}$, $\mathsf{q}{\uparrow}$, $\mathsf{p}{\downarrow}$, $\mathsf{p}{\uparrow}$, and $\mathsf{g}{\downarrow}$.

The next case is in fact a subcase of (i), but for didactic resons we list it separately.

(iii) The contractum $\circ$ of $\mathsf{e}{\downarrow}$ is the redex of ρ (which is one of $\mathsf{e}{\uparrow}$, $\mathsf{ai}{\uparrow}$, $\mathsf{w}{\uparrow}$). Then we have

$$\mathsf{e}{\downarrow}\,\frac{\mathsf{ai}{\uparrow}\,\dfrac{S(a \otimes \bar{a})}{S\{\circ\}}}{S\{!\circ\}} \quad\rightarrow\quad = \frac{\mathsf{ai}{\uparrow}\,\dfrac{\mathsf{e}{\downarrow}\,\dfrac{=\dfrac{S(a \otimes \bar{a})}{S[(a \otimes \bar{a}) \parr \circ]}}{S[(a \otimes \bar{a}) \parr !\circ]}}{S[\circ \parr !\circ]}}{S\{!\circ\}}$$

Finally, we come to the case which is nontrivial. It is the one where we need the system $\{\mathsf{s}, \mathsf{q}{\downarrow}, \mathsf{q}{\uparrow}\}$.

(iv) The contractum $\circ$ of $\mathsf{e}{\downarrow}$ actively interferes with the rule ρ. This can happen because of the equational theory for $\circ$.

(a) Let $\rho = \mathsf{ai}{\downarrow}$ and consider the two derivations:

$$\mathsf{e}{\downarrow}\,\frac{=\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{S\{\circ\}}{S[a \parr \bar{a}]}}{S[(a \otimes \circ) \parr \bar{a}]}}{S[(a \otimes !\circ) \parr \bar{a}]} \quad\text{and}\quad \mathsf{e}{\downarrow}\,\frac{=\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{S\{\circ\}}{S[a \parr \bar{a}]}}{S[\langle a \lhd \circ\rangle \parr \bar{a}]}}{S[\langle a \lhd !\circ\rangle \parr \bar{a}]}$$

They can be replaced by

$$\mathsf{s}\,\frac{\mathsf{ai}{\downarrow}\,\dfrac{\mathsf{e}{\downarrow}\,\dfrac{=\dfrac{S\{\circ\}}{S(\circ \otimes \circ)}}{S(\circ \otimes !\circ)}}{S([a \parr \bar{a}] \otimes !\circ)}}{S[(a \otimes !\circ) \parr \bar{a}]} \quad\text{and}\quad \mathsf{q}{\downarrow}\,\frac{\mathsf{ai}{\downarrow}\,\dfrac{\mathsf{e}{\downarrow}\,\dfrac{=\dfrac{S\{\circ\}}{S\langle \circ \lhd \circ\rangle}}{S\langle \circ \lhd !\circ\rangle}}{S\langle [a \parr \bar{a}] \lhd !\circ\rangle}}{S[\langle a \lhd !\circ\rangle \parr \bar{a}]}$$

respectively. Here we used the rules s and $\mathsf{q}{\downarrow}$ to move the redex $!\circ$ of $\mathsf{e}{\downarrow}$ out of the way of the rule $\mathsf{ai}{\downarrow}$ such that the situation could be handled similarly to case (i). A similar situation can occur with the rules s, $\mathsf{p}{\downarrow}$, and $\mathsf{q}{\downarrow}$. We will not show all possibilities here, but it should be clear that they all work because of the same principle. We content ourselves of presenting only the most complicated case (where $\rho = \mathsf{q}{\downarrow}$):

$$
\mathsf{e}{\downarrow}\dfrac{=\dfrac{\mathsf{q}{\downarrow}\dfrac{S\langle[\langle R \lhd R'\rangle \parr U] \lhd [\langle T \lhd T'\rangle \parr V]\rangle}{S[\langle R \lhd R' \lhd T \lhd T'\rangle \parr \langle U \lhd V\rangle]}}{S[\langle R \lhd (\langle R' \lhd T\rangle \otimes \circ) \lhd T'\rangle \parr \langle U \lhd V\rangle]}}{S[\langle R \lhd (\langle R' \lhd T\rangle \otimes !\circ) \lhd T'\rangle \parr \langle U \lhd V\rangle]}
\quad\rightarrow\quad
\mathsf{q}{\uparrow}\dfrac{\mathsf{q}{\uparrow}\dfrac{\mathsf{s}\dfrac{\mathsf{q}{\downarrow}\dfrac{\mathsf{e}{\downarrow}\dfrac{=\dfrac{S\langle[\langle R \lhd R'\rangle \parr U] \lhd [\langle T \lhd T'\rangle \parr V]\rangle}{S(\langle[\langle R \lhd R'\rangle \parr U] \lhd [\langle T \lhd T'\rangle \parr V]\rangle \otimes \circ)}}{S(\langle[\langle R \lhd R'\rangle \parr U] \lhd [\langle T \lhd T'\rangle \parr V]\rangle \otimes !\circ)}}{S([\langle R \lhd R' \lhd T \lhd T'\rangle \parr \langle U \lhd V\rangle] \otimes !\circ)}}{S[(\langle R \lhd R' \lhd T \lhd T'\rangle \otimes !\circ) \parr \langle U \lhd V\rangle]}}{S[\langle R \lhd (\langle R' \lhd T \lhd T'\rangle \otimes !\circ)\rangle \parr \langle U \lhd V\rangle]}}{S[\langle R \lhd (\langle R' \lhd T\rangle \otimes !\circ) \lhd T'\rangle \parr \langle U \lhd V\rangle]}
$$

Here, two instances of $\mathsf{q}{\uparrow}$ and one instance of s are needed to move the $!\circ$ out of the way of $\mathsf{q}{\downarrow}$.

(b) Let $\rho = \mathsf{p}{\uparrow}$ and consider the two derivations

$$
\mathsf{e}{\downarrow}\dfrac{=\dfrac{\mathsf{p}{\uparrow}\dfrac{S(?(R \otimes R') \otimes !(T \otimes T'))}{S\{?(R \otimes R' \otimes T \otimes T')\}}}{S\{?(R \otimes [(R' \otimes T) \parr \circ] \otimes T')\}}}{S\{?(R \otimes [(R' \otimes T) \parr !\circ] \otimes T')\}}
\qquad\text{and}\qquad
\mathsf{e}{\downarrow}\dfrac{=\dfrac{\mathsf{p}{\uparrow}\dfrac{S(?(R \otimes R') \otimes !(T \otimes T'))}{S\{?(R \otimes R' \otimes T \otimes T')\}}}{S\{?(R \otimes \langle (R' \otimes T) \lhd \circ\rangle \otimes T')\}}}{S\{?(R \otimes \langle (R' \otimes T) \lhd !\circ\rangle \otimes T')\}}
$$

which can be replaced by:

$$
\mathsf{s}\dfrac{\mathsf{p}{\uparrow}\dfrac{\mathsf{e}{\downarrow}\dfrac{=\dfrac{S(?(R \otimes R') \otimes !(T \otimes T'))}{S(?(R \otimes [R' \parr \circ]) \otimes !(T \otimes T'))}}{S(?(R \otimes [R' \parr !\circ]) \otimes !(T \otimes T'))}}{S\{?(R \otimes [R' \parr !\circ] \otimes T \otimes T')\}}}{S\{?(R \otimes [(R' \otimes T) \parr !\circ] \otimes T')\}}
\qquad\text{and}\qquad
\mathsf{q}{\uparrow}\dfrac{\mathsf{p}{\uparrow}\dfrac{\mathsf{e}{\downarrow}\dfrac{=\dfrac{S(?(R \otimes R') \otimes !(T \otimes T'))}{S(?(R \otimes \langle R' \lhd \circ\rangle) \otimes !(T \otimes T'))}}{S(?(R \otimes \langle R' \lhd !\circ\rangle) \otimes !(T \otimes T'))}}{S\{?(R \otimes \langle R' \lhd !\circ\rangle \otimes T \otimes T')\}}}{S\{?(R \otimes \langle (R' \otimes T) \lhd !\circ\rangle \otimes T')\}}
$$

Here the $!\circ$ has not been moved to the outside but to the inside, such that the permutation could be handled as in case (ii) above. A similar situation can occur with the rules $\rho = \mathsf{s}, \mathsf{q}{\downarrow}, \mathsf{q}{\uparrow}$. Again, we do not show all possibilities. But the reader should be able to convince himself that it is always possible to move the $!\circ$ out of the way of ρ.[2] ☐

For completing Step 1 of the proof of Theorem 3.1, it is also necessary to permute $\mathsf{e}{\downarrow}$ over the rules $\mathsf{w}{\downarrow}$, $\mathsf{b}{\downarrow}$, $\mathsf{b}{\uparrow}$, and $\mathsf{g}{\uparrow}$, which have been left out in Lemma 3.1.3. The nontrivial cases are as follows:

- for $\mathsf{w}{\downarrow}$:

$$
\mathsf{e}{\downarrow}\dfrac{\mathsf{w}{\downarrow}\dfrac{S\{\circ\}}{S\{?R\{\circ\}\}}}{S\{?R\{!\circ\}\}}
\quad\rightarrow\quad
\mathsf{w}{\downarrow}\dfrac{S\{\circ\}}{S\{?R\{!\circ\}\}}
\tag{1}
$$

[2] A complete list of all possible cases can be found in [Str03a].

- for $\mathsf{b}{\downarrow}$:

$$\dfrac{\dfrac{S[?R\{\circ\} \parr R\{\circ\}]}{S\{?R\{\circ\}\}}\,\mathsf{b}{\downarrow}}{S\{?R\{!\circ\}\}}\,\mathsf{e}{\downarrow} \quad \rightarrow \quad \dfrac{\dfrac{S[?R\{\circ\} \parr R\{\circ\}]}{S[?R\{!\circ\} \parr R\{!\circ\}]}\,\mathsf{e}{\downarrow},\mathsf{e}{\downarrow}}{S\{?R\{!\circ\}\}}\,\mathsf{b}{\downarrow} \tag{2}$$

- for $\mathsf{b}{\uparrow}$:

$$\dfrac{\dfrac{S\{!R\{\circ\}\}}{S(!R\{\circ\} \otimes R\{\circ\})}\,\mathsf{b}{\uparrow}}{S(!R\{!\circ\} \otimes R\{\circ\})}\,\mathsf{e}{\downarrow} \quad \rightarrow \quad \dfrac{\dfrac{\dfrac{S\{!R\{\circ\}\}}{S\{!R\{!\circ\}\}}\,\mathsf{e}{\downarrow}}{S(!R\{!\circ\} \otimes R\{!\circ\})}\,\mathsf{b}{\uparrow}}{S(!R\{!\circ\} \otimes R\{\circ\})}\,\mathsf{w}{\uparrow} \tag{3}$$

- for $\mathsf{g}{\uparrow}$:

$$\dfrac{\dfrac{\dfrac{S\{!R\}}{S\{!!R\}}\,\mathsf{g}{\uparrow}}{S\{!(\circ \otimes !R)\}}\,=}{S\{!(!\circ \otimes !R)\}}\,\mathsf{e}{\downarrow} \quad \rightarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{S\{!R\}}{S\{!!R\}}\,\mathsf{g}{\uparrow}}{S\{!!!R\}}\,\mathsf{g}{\uparrow}}{S\{!(!!R \otimes !R)\}}\,\mathsf{b}{\uparrow}}{S\{!(!\circ \otimes !R)\}}\,\mathsf{w}{\uparrow} \tag{4}$$

Note that these cases do not follow the statement of Definitions 3.1.1 or 3.1.2, which is the reason why they have been left out in Lemma 3.1.3. But together with that lemma, they are sufficient to show by an easy inductive argument that in any SNEL derivation all instances of $\mathsf{e}{\downarrow}$ can be permuted to the top, and dually, all instances of $\mathsf{e}{\uparrow}$ can be permuted to the bottom. This completes Step 1 in the proof of Theorem 3.1.

The attentive reader might complain that the permutation of rules is a tedious business. However, the important point here is not the way it is done, but the fact that it can be done. No other deductive formalism allows such a freedom in moving around inference rules in a derivation. That this freedom has its price should not be surprising.

3.1.4 Lemma *The rules* $\mathsf{w}{\downarrow}$ *and* $\mathsf{ai}{\downarrow}$ *permute over the rules* $\mathsf{e}{\uparrow}$, $\mathsf{ai}{\downarrow}$, $\mathsf{ai}{\uparrow}$, s, $\mathsf{q}{\downarrow}$, $\mathsf{q}{\uparrow}$, $\mathsf{p}{\downarrow}$, $\mathsf{p}{\uparrow}$, $\mathsf{w}{\uparrow}$, *and* $\mathsf{g}{\downarrow}$ *by the system* $\{\mathsf{s},\mathsf{q}{\downarrow},\mathsf{q}{\uparrow}\}$.

Proof: The contractum of $\mathsf{w}{\downarrow}$ and $\mathsf{ai}{\downarrow}$ is the same as of $\mathsf{e}{\downarrow}$, namely $\circ$. Hence, this proof is the same as the one for Lemma 3.1.3. □

Clearly, this lemma is more than enough to show that in a derivation in the system $\{\mathsf{ai}{\downarrow},\mathsf{ai}{\uparrow}\} \cup \mathsf{SNELh} = \{\mathsf{ai}{\downarrow},\mathsf{ai}{\uparrow},\mathsf{s},\mathsf{q}{\downarrow},\mathsf{q}{\uparrow},\mathsf{p}{\downarrow},\mathsf{p}{\uparrow}\}$ all instances of $\mathsf{ai}{\downarrow}$ can be permuted to the top, and dually, all $\mathsf{ai}{\uparrow}$ can be permuted to the bottom. This completes Step 4 in the proof of Theorem 3.1. Similarly, Lemma 3.1.4 is used to complete Steps 7 and 9. Note that for Step 7, we additionally need to permute $\mathsf{ai}{\downarrow}$ over $\mathsf{w}{\downarrow}$, for which the only nontrivial case is similar to (1).

We will now continue with Step 3, for which the following lemma (and its dual) is sufficient.

3.1.5 Lemma *For every derivation* $\begin{array}{c} T \\ \{\mathsf{g}\uparrow,\mathsf{b}\uparrow,\mathsf{w}\uparrow\}\Big\| \\ R \end{array}$ *there is a derivation* $\begin{array}{c} T \\ \{\mathsf{g}\uparrow\}\Big\| \\ V \\ \{\mathsf{b}\uparrow\}\Big\| \\ U \\ \{\mathsf{w}\uparrow\}\Big\| \\ R \end{array}$.

Proof: This is again a simple rule permutation. First, all instances of $\mathsf{g}\uparrow$ are permuted up to the top. The trivial cases are as in Lemma 3.1.3. The only nontrivial cases are the following:

$$\mathsf{g}\uparrow\frac{\mathsf{b}\uparrow\dfrac{S\{!R\}}{S(!R\otimes R)}}{S(!!R\otimes R)} \quad\to\quad =\frac{\mathsf{w}\uparrow\dfrac{\mathsf{b}\uparrow\dfrac{\mathsf{b}\uparrow\dfrac{\mathsf{g}\uparrow\dfrac{S\{!R\}}{S\{!!R\}}}{S(!!R\otimes !R)}}{S(!!R\otimes !R\otimes R)}}{S(!!R\otimes\circ\otimes R)}}{S(!!R\otimes R)} \tag{5}$$

$$\mathsf{g}\uparrow\frac{\mathsf{b}\uparrow\dfrac{S\{!R\{!T\}\}}{S(!R\{!T\}\otimes R\{!T\})}}{S(!R\{!!T\}\otimes R\{!T\})} \quad\to\quad =\frac{\mathsf{w}\uparrow\dfrac{\mathsf{b}\uparrow\dfrac{\mathsf{b}\uparrow\dfrac{\mathsf{g}\uparrow\dfrac{S\{!R\{!T\}\}}{S\{!R\{!!T\}\}}}{S(!R\{!!T\}\otimes R(!!T))}}{S(!R\{!!T\}\otimes R(!!T\otimes !T))}}{S(!R\{!!T\}\otimes R(\circ\otimes !T))}}{S(!R\{!!T\}\otimes R\{!T\})} \tag{6}$$

Finally, all $\mathsf{w}\uparrow$ are permuted under the $\mathsf{b}\uparrow$, where

$$\mathsf{b}\uparrow\frac{\mathsf{w}\uparrow\dfrac{S\{!R\{!T\}\}}{S\{!R\{\circ\}\}}}{S(!R\{\circ\}\otimes R\{\circ\})} \quad\to\quad \mathsf{w}\uparrow,\mathsf{w}\uparrow\frac{\mathsf{b}\uparrow\dfrac{S\{!R\{!T\}\}}{S(!R\{!T\}\otimes R\{!T\})}}{S(!R\{\circ\}\otimes R\{\circ\})} \tag{7}$$

is the only nontrivial case. □

3.1.6 Remark Note that the decomposition of Lemma 3.1.5 does not allow much variation. We can neither permute $\mathsf{b}\uparrow$ over $\mathsf{g}\uparrow$, nor can we permute $\mathsf{w}\uparrow$ over $\mathsf{b}\uparrow$, as the following examples show:

$$\mathsf{b}\uparrow\frac{\mathsf{g}\uparrow\dfrac{!a}{!!a}}{(!!a\otimes !a)} \qquad\text{and}\qquad \mathsf{w}\uparrow\frac{\mathsf{b}\uparrow\dfrac{!a}{(!a\otimes a)}}{a}$$

3.1.7 Lemma *The rules* $\mathsf{g}\!\uparrow$*,* $\mathsf{b}\!\uparrow$*, and* $\mathsf{w}\!\uparrow$ *can be permuted over* $\mathsf{e}\!\downarrow$*.*

Proof: The only nontrival cases are the following.

- for $\mathsf{g}\!\uparrow$:

$$\mathsf{g}\!\uparrow \frac{\mathsf{e}\!\downarrow \dfrac{S\{\circ\}}{S\{!\circ\}}}{S\{!!\circ\}} \qquad \rightarrow \qquad \mathsf{e}\!\downarrow \frac{\mathsf{e}\!\downarrow \dfrac{S\{\circ\}}{S\{!\circ\}}}{S\{!!\circ\}}$$

- for $\mathsf{b}\!\uparrow$:

$$\mathsf{b}\!\uparrow \frac{\mathsf{e}\!\downarrow \dfrac{S\{\circ\}}{S\{!\circ\}}}{S(!\circ \otimes \circ)} \qquad \rightarrow \qquad = \frac{\mathsf{e}\!\downarrow \dfrac{S\{\circ\}}{S\{!\circ\}}}{S(!\circ \otimes \circ)}$$

- for $\mathsf{w}\!\uparrow$:

$$\mathsf{w}\!\uparrow \frac{\mathsf{e}\!\downarrow \dfrac{S\{\circ\}}{S\{!\circ\}}}{S\{\circ\}} \qquad \rightarrow \qquad S\{\circ\}$$

In all of them the instance of $\mathsf{g}\!\uparrow$, $\mathsf{b}\!\uparrow$, and $\mathsf{w}\!\uparrow$, which is permuted up disappears. The trivial cases are as in case (i) of Lemma 3.1.3. Case (ii) in the proof of that lemma cannot occur here. ☐

This completes Step 5. For completing Steps 6 and 8, note that they are almost identical to Steps 1 to 3 and 5, with the only difference that the rules $\mathsf{w}\!\uparrow$ and $\mathsf{w}\!\downarrow$ are ommitted.

After this tour de force of simple rule permutations, the proof of Theorem 3.1 is completed, except for Step 2. At first sight one might expect that this can also be done by simple rule permutations. So, let us attempt to permute all $\mathsf{g}\!\uparrow$, $\mathsf{b}\!\uparrow$, and $\mathsf{w}\!\uparrow$ up to the top of a derivation.

3.1.8 Permuting $\mathsf{g}\!\uparrow$, $\mathsf{b}\!\uparrow$, $\mathsf{w}\!\uparrow$ up: Consider a derivation

$$\pi \frac{\rho \dfrac{S\{W\}}{S\{Z\}}}{P} \quad ,$$

where $\rho \in \mathsf{SNEL} \setminus \{\mathsf{g}\!\uparrow, \mathsf{b}\!\uparrow, \mathsf{w}\!\uparrow, \mathsf{e}\!\downarrow, \mathsf{e}\!\uparrow\}$ and $\pi \in \{\mathsf{g}\!\uparrow, \mathsf{b}\!\uparrow, \mathsf{w}\!\uparrow\}$. The trivial cases (i) and (ii) are as in the proof of Lemma 3.1.3. Then there is another (almost) trivial case which does not correspond to a case in the proof of Lemma 3.1.3.

(iii) The redex Z of ρ is inside the contractum of π, i.e., we have one of the following three situations

$$\mathsf{g}\!\uparrow \frac{\rho \dfrac{S\{!R\{W\}\}}{S\{!R\{Z\}\}}}{S\{!!R\{Z\}\}} \qquad \mathsf{b}\!\uparrow \frac{\rho \dfrac{S\{!R\{W\}\}}{S\{!R\{Z\}\}}}{S(!R\{Z\} \otimes R\{Z\})} \qquad \mathsf{w}\!\uparrow \frac{\rho \dfrac{S\{!R\{W\}\}}{S\{!R\{Z\}\}}}{S\{\circ\}}$$

which can be replaced by

$$
\mathsf{g}{\uparrow}\,\frac{\displaystyle \rho\,\frac{S\{!R\{W\}\}}{S\{!!R\{W\}\}}}{S\{!!R\{Z\}\}}
\qquad
\mathsf{b}{\uparrow}\,\frac{\displaystyle \rho,\rho\,\frac{S\{!R\{W\}\}}{S(!R\{W\}\otimes R\{W\})}}{S(!R\{Z\}\otimes R\{Z\})}
\qquad
\mathsf{w}{\uparrow}\,\frac{S\{!R\{W\}\}}{S\{\circ\}}
\tag{8}
$$

respectively.

The next case corresponds to case (iv) in the proof of Lemma 3.1.3.

(iv) The contractum $!R$ of π actively interferes with the redex Z of ρ. This can only happen with $\rho \in \{\mathsf{w}{\downarrow}, \mathsf{b}{\downarrow}, \mathsf{p}{\downarrow}\}$. If ρ is $\mathsf{w}{\downarrow}$ or $\mathsf{b}{\downarrow}$, then the situation is similar to (1) and (2) above. If $\rho = \mathsf{p}{\downarrow}$, then we have one of

$$
\mathsf{g}{\uparrow}\,\frac{\displaystyle \mathsf{p}{\downarrow}\,\frac{S\{![R \parr T]\}}{S[!R \parr ?T]}}{S[!!R \parr ?T]}
\qquad
\mathsf{b}{\uparrow}\,\frac{\displaystyle \mathsf{p}{\downarrow}\,\frac{S\{![R \parr T]\}}{S[!R \parr ?T]}}{S[(!R \otimes R) \parr ?T]}
\qquad
\mathsf{w}{\uparrow}\,\frac{\displaystyle \mathsf{p}{\downarrow}\,\frac{S\{![R \parr T]\}}{S[!R \parr ?T]}}{S[\circ \parr ?T]}
$$

which can be replaced by (respectively):

$$
\begin{array}{l}
\mathsf{g}{\uparrow}\,\dfrac{S\{![R \parr T]\}}{S\{!![R \parr T]\}} \\
\mathsf{p}{\downarrow}\,\dfrac{}{S\{![!R \parr ?T]\}} \\
\mathsf{p}{\downarrow}\,\dfrac{}{S[!!R \parr ??T]} \\
\mathsf{g}{\downarrow}\,\dfrac{}{S[!!R \parr ?T]}
\end{array}
\qquad
\begin{array}{l}
\mathsf{b}{\uparrow}\,\dfrac{S\{![R \parr T]\}}{S(![R \parr T] \otimes [R \parr T])} \\
\mathsf{p}{\downarrow}\,\dfrac{}{S([!R \parr ?T] \otimes [R \parr T])} \\
\mathsf{s}\,\dfrac{}{S[([!R \parr ?T] \otimes R) \parr T]} \\
\mathsf{s}\,\dfrac{}{S[(!R \otimes R) \parr ?T \parr T]} \\
\mathsf{b}{\downarrow}\,\dfrac{}{S[(!R \otimes R) \parr ?T]}
\end{array}
\qquad
\begin{array}{l}
\mathsf{w}{\uparrow}\,\dfrac{S\{![R \parr T]\}}{S\{\circ\}} \\
=\,\dfrac{}{S[\circ \parr \circ]} \\
\mathsf{w}{\downarrow}\,\dfrac{}{S[\circ \parr ?T]}
\end{array}
$$

This means that there is indeed no objection against permuting all instances of $\mathsf{g}{\uparrow}$, $\mathsf{b}{\uparrow}$, and $\mathsf{w}{\uparrow}$ up to the top of a derivation, and then (by duality) permute all $\mathsf{g}{\downarrow}$, $\mathsf{b}{\downarrow}$, and $\mathsf{w}{\downarrow}$ down to the bottom. However, the problem is that while permuting $\mathsf{g}{\uparrow}$, $\mathsf{b}{\uparrow}$, $\mathsf{w}{\uparrow}$ up, we introduce, in case (iv), new instances of $\mathsf{g}{\downarrow}$, $\mathsf{b}{\downarrow}$, $\mathsf{w}{\downarrow}$, and dually, while permuting $\mathsf{g}{\downarrow}$, $\mathsf{b}{\downarrow}$, $\mathsf{w}{\downarrow}$ down, we introduce new instances of $\mathsf{g}{\uparrow}$, $\mathsf{b}{\uparrow}$, $\mathsf{w}{\uparrow}$. This means that this permuting up and down could run forever. At least, it is not obvious that it terminates eventually, as it is the case with Steps 1, 4, 7 and 9 in the proof of Theorem 3.1.

Please note that there is no obvious induction measure related to the size of the derivation that could be used for showing termination. The up and down permutation of $\mathsf{w}{\uparrow}$ and $\mathsf{w}{\downarrow}$ alone is unproblematic because at each critical case the disturbing instance of $\mathsf{p}{\downarrow}$ or $\mathsf{p}{\uparrow}$ is destroyed (but for convenience we will deal with all six rules $\mathsf{g}{\uparrow}$, $\mathsf{b}{\uparrow}$, $\mathsf{w}{\uparrow}$ and $\mathsf{g}{\downarrow}$, $\mathsf{b}{\downarrow}$, $\mathsf{w}{\downarrow}$ together). The up and down permutation of $\mathsf{g}{\uparrow}$, $\mathsf{b}{\uparrow}$ and $\mathsf{g}{\downarrow}$, $\mathsf{b}{\downarrow}$ is very problematic, however. The rules $\mathsf{g}{\uparrow}$ and $\mathsf{g}{\downarrow}$ cause a duplication of the disturbing instance of promotion, and the permutation of $\mathsf{b}{\uparrow}$ and $\mathsf{b}{\downarrow}$ causes an even worse increase in the size of the derivation. In fact, the ρ in the middle derivation in (8) could be an instance of a promotion that is disturbing for another $\mathsf{g}{\uparrow}$ or $\mathsf{b}{\uparrow}$.

Clearly, a different technology is needed here, and will be introduced in the next sections.

3.2 Order theoretic preliminaries

Let $\langle A, \leq \rangle$ be a partial order. If $a, b \in A$ and $a \leq b$ and $a \neq b$, then we write $a < b$. Recall that the order $\langle A, \leq \rangle$ is *well-founded* iff there is no infinite strictly descending chain $a_0 > a_1 > a_2 > \cdots$. Let now $A^{\#}$ denote the free commutative monoid generated by A, i.e., the set $\mathbb{N}^A$ of all functions from A to the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ of natural numbers, that have value 0 almost everywhere. Equivalently, we can define $A^{\#}$ by taking the set A^* of all finite words over A, and disregarding the order of the letters inside a word $u \in A^*$. We can write an element $u \in A^{\#}$ as a formal sum

$$u = \sum_{a \in A} u_a a \tag{9}$$

where $a \in A$ and $u_a \in \mathbb{N}$ such that $u_a > 0$ for only finitely many a. If $u_a > 0$, then we say that a *occurs* in u. When we think of u as a finite word over A, then u_a is the number of occurrences of the letter a in u, which is the only information that matters when we live in the free commutative monoid. The monoid operation of two elements $u, v \in A^{\#}$ is defined as their sum in the obvious way:

$$u + v \;=\; \sum_{a \in A} u_a a \;+\; \sum_{a \in A} v_a a \;=\; \sum_{a \in A} (u_a + v_a) a$$

For simplicity, we can see A as a subset of $A^{\#}$ by identifying $a \in A$ with $1a \in A^{\#}$. For $v, u \in A^{\#}$ we write $v \prec u$ if there are $w, z \in A^{\#}$ and $a \in A$, such that $u = w + a$, and $v = w + z$ and $b < a$ for all b occurring in z. We define $\leq$ to be the reflexive, transitive closure of $\prec$.

3.2.1 Theorem *If $\langle A, \leq \rangle$ is a well-founded order, then $\langle A^{\#}, \leq \rangle$ is a well-founded order.*

This theorem is well-known, see, e.g., [Reu89, Théorème 2.6] for a variation of it. The proof is a direct application of König's lemma [Kön50, Satz 6.6].

In this paper we will use Theorem 3.2.1 for the set $A = \omega \times (\omega + 1) \times \omega$, equipped with the lexicographic ordering, where $\omega = \{0, 1, 2, \ldots\}$ and $\omega + 1 = \omega \cup \{\omega\}$ are both equipped with the natural ordering.

3.3 !-?-Flow-Graphs

3.3.1 Definition For instances of the rules $\mathsf{g}\downarrow$, $\mathsf{g}\uparrow$, $\mathsf{b}\downarrow$, $\mathsf{b}\uparrow$, $\mathsf{w}\downarrow$, $\mathsf{w}\uparrow$, and $\mathsf{p}\downarrow$, $\mathsf{p}\uparrow$ we define their *principal structure* as indicated below with a grey background:

$$\mathsf{g}\!\downarrow \frac{S\{??T\}}{S\{?T\}} \quad , \qquad \mathsf{b}\!\downarrow \frac{S[?T \parr T]}{S\{?T\}} \quad , \qquad \mathsf{w}\!\downarrow \frac{S\{\circ\}}{S\{?T\}} \quad ,$$

$$\mathsf{g}\!\uparrow \frac{S\{!R\}}{S\{!!R\}} \quad , \qquad \mathsf{b}\!\uparrow \frac{S\{!R\}}{S(!R \otimes R)} \quad , \qquad \mathsf{w}\!\uparrow \frac{S\{!R\}}{S\{\circ\}} \quad ,$$

$$
\begin{array}{cll}
\text{(i)} & \rho\,\dfrac{S\{!R\}}{S'\{!R\}} & \rho\,\dfrac{S\{?T\}}{S'\{?T\}} \\[2ex]
\text{(ii)} & \rho\,\dfrac{S\{!R\{W\}\}}{S\{!R\{Z\}\}} & \rho\,\dfrac{S\{?T\{W\}\}}{S\{?T\{Z\}\}} \\[2ex]
\text{(iii)} & \mathsf{b}\!\uparrow\dfrac{S\{!R\}}{S(!R \otimes R)} & \mathsf{b}\!\downarrow\dfrac{S[?T \mathbin{\bindnasrepma} T]}{S\{?T\}} \\[2ex]
\text{(iv)} & \mathsf{g}\!\uparrow\dfrac{S\{!R\}}{S\{!!R\}} & \mathsf{g}\!\downarrow\dfrac{S\{??T\}}{S\{?T\}} \\[2ex]
\text{(v)} & \mathsf{b}\!\uparrow\dfrac{S\{!V\{!R\}\}}{S(!V\{!R\} \otimes V\{!R\})} & \mathsf{b}\!\downarrow\dfrac{S[?U\{?T\} \mathbin{\bindnasrepma} U\{?T\}]}{S\{?U\{?T\}\}} \\[2ex]
\text{(vi)} & \mathsf{b}\!\downarrow\dfrac{S[?U\{!R\} \mathbin{\bindnasrepma} U\{!R\}]}{S\{?U\{!R\}\}} & \mathsf{b}\!\uparrow\dfrac{S\{!V\{?T\}\}}{S(!V\{?T\} \otimes V\{?T\})} \\[2ex]
\text{(vii)} & \mathsf{p}\!\downarrow\dfrac{S\{![R \mathbin{\bindnasrepma} T]\}}{S[!R \mathbin{\bindnasrepma} ?T]} & \mathsf{p}\!\uparrow\dfrac{S(?T \otimes !R)}{S\{?(T \otimes R)\}}
\end{array}
$$

Figure 8: Edges in the !-?-flow-graph

$$
\mathsf{p}\!\downarrow\dfrac{S\{![R \mathbin{\bindnasrepma} T]\}}{S[!R \mathbin{\bindnasrepma} ?T]} \quad , \qquad \mathsf{p}\!\uparrow\dfrac{S(?T \otimes !R)}{S\{?(T \otimes R)\}} \qquad .
$$

I.e., if $\rho \in \{\mathsf{g}\!\downarrow, \mathsf{b}\!\downarrow, \mathsf{w}\!\downarrow\}$, then its principal structure is the redex $?T$ of ρ. If $\rho \in \{\mathsf{g}\!\uparrow, \mathsf{b}\!\uparrow, \mathsf{w}\!\uparrow\}$, then its principal structure is the contractum $!R$ of ρ. If $\rho = \mathsf{p}\!\downarrow$, then its principal structure is the !-substructure of its redex, and if $\rho = \mathsf{p}\!\uparrow$, then its principal structure is the ?-substructure of its contractum.

The basic idea of the !-?-flow-graph of a derivation is to mark the "path" that is taken by the principal structures of instances of $\mathsf{g}\!\uparrow$, $\mathsf{g}\!\downarrow$, $\mathsf{b}\!\uparrow$, $\mathsf{b}\!\downarrow$, $\mathsf{w}\!\uparrow$, $\mathsf{w}\!\downarrow$ while they are travelling up and down in the derivation. Formally, the !-?-flow-graph is defined as follows:

3.3.2 Definition Let Δ be a derivation in SNEL. The *!-?-flow-graph* of Δ is a directed graph, denoted by $G_{!?}(\Delta)$, whose vertices are the occurrences of !- and ?-substructures appearing in Δ. Two such substructures are connected via an edge in $G_{!?}(\Delta)$ if they appear inside the premise and the conclusion of an inference rule according to the prescriptions in Figure 8.

When visualizing the !-?-flow-graph of a derivation, we draw it inside the derivation, as indicated in Figure 8, and as shown in the example below:

$$
\begin{array}{rc}
 & !![!a \parr a] \\ \cline{2-2}
\mathsf{p}\!\downarrow & ![?!a \parr !a] \\ \cline{2-2}
\mathsf{b}\!\downarrow & !?!a \\ \cline{2-2}
\mathsf{b}\!\uparrow & (!?!a \otimes ?!a) \\ \cline{2-2}
\mathsf{p}\!\uparrow & ?(?!a \otimes !a) \\ \cline{2-2}
\mathsf{p}\!\uparrow & ??(!a \otimes a)
\end{array}
\tag{10}
$$

The first two cases in Figure 8 are straightforward: The rule ρ either modifies the context of $!R$ or $?T$, or ρ works inside $!R$ or $?T$, without touching the modality. Cases (iii) and (iv) take care of the modalities that are actively involved in the redex and contractum of the absorption and digging rules. Cases (v) and (vi) involve a duplication of a modality structure due to absorption, which causes a branching in the !-?-flow-graph. The most interesting case is (vii). It takes care of the situation in case (iv) in 3.1.8. Note that in Figure 8 the cases (vi) and (vii) are the only ones where we have a "forking" in the graph. In cases (iv) and (v) the situation is better described as "merging", and in all other cases the situation is purely "linear".

For two vertices U and V of $G_{!?}(\Delta)$, we write $U \underset{\Delta}{\curvearrowright} V$ if there is an edge from U to V in $G_{!?}(\Delta)$. We use $\overset{+}{\underset{\Delta}{\curvearrowright}}$ to denote the transitive closure of $\underset{\Delta}{\curvearrowright}$, and $\overset{*}{\underset{\Delta}{\curvearrowright}}$ to denote the reflexive transitive closure of $\underset{\Delta}{\curvearrowright}$. We use the standard notions of paths and cycles in directed graphs:

3.3.3 Definition A *path* in the !-?-flow-graph of a derivation Δ is a sequence of vertices $V_0, V_1, \ldots, V_n$, such that $V_{i-1} \underset{\Delta}{\curvearrowright} V_i$ for each $i \in \{1, \ldots, n\}$. A *cycle* is a path such that the first vertex and the last vertex are identical. The !-?-flow-graph of a derivation is *acyclic*, if it does not contain any cycle, i.e., there is no vertex V with $V \overset{+}{\underset{\Delta}{\curvearrowright}} V$. A path p is called *cyclic*, if there is a vertex which occurs more than once in p.

Clearly, every cycle is a cyclic path, and the !-?-flow-graph of a derivation is acyclic, if and only if it contains no cyclic path. To come back to our example in (10), consider the following four excerpts from its !-?-flow-graph:

$$
\begin{array}{rc}
 & !![!a \parr a] \\ \cline{2-2}
\mathsf{p}\!\downarrow & ![?!a \parr !a] \\ \cline{2-2}
\mathsf{b}\!\downarrow & !?!a \\ \cline{2-2}
\mathsf{b}\!\uparrow & (!?!a \otimes ?!a) \\ \cline{2-2}
\mathsf{p}\!\uparrow & ?(?!a \otimes !a) \\ \cline{2-2}
\mathsf{p}\!\uparrow & ??(!a \otimes a)
\end{array}
\qquad
\begin{array}{rc}
 & !![!a \parr a] \\ \cline{2-2}
\mathsf{p}\!\downarrow & ![?!a \parr !a] \\ \cline{2-2}
\mathsf{b}\!\downarrow & !?!a \\ \cline{2-2}
\mathsf{b}\!\uparrow & (!?!a \otimes ?!a) \\ \cline{2-2}
\mathsf{p}\!\uparrow & ?(?!a \otimes !a) \\ \cline{2-2}
\mathsf{p}\!\uparrow & ??(!a \otimes a)
\end{array}
\qquad
\begin{array}{rc}
 & !![!a \parr a] \\ \cline{2-2}
\mathsf{p}\!\downarrow & ![?!a \parr !a] \\ \cline{2-2}
\mathsf{b}\!\downarrow & !?!a \\ \cline{2-2}
\mathsf{b}\!\uparrow & (!?!a \otimes ?!a) \\ \cline{2-2}
\mathsf{p}\!\uparrow & ?(?!a \otimes !a) \\ \cline{2-2}
\mathsf{p}\!\uparrow & ??(!a \otimes a)
\end{array}
\qquad
\begin{array}{rc}
 & !![!a \parr a] \\ \cline{2-2}
\mathsf{p}\!\downarrow & ![?!a \parr !a] \\ \cline{2-2}
\mathsf{b}\!\downarrow & !?!a \\ \cline{2-2}
\mathsf{b}\!\uparrow & (!?!a \otimes ?!a) \\ \cline{2-2}
\mathsf{p}\!\uparrow & ?(?!a \otimes !a) \\ \cline{2-2}
\mathsf{p}\!\uparrow & ??(!a \otimes a)
\end{array}
\tag{11}
$$

The first example shows a path, where the first and the last vertex in the path are marked with a light grey background. The subgraph indicated in the second example is not a path (direction matters). The third example shows a cycle, and the last example a cyclic path (again, first and last vertex are marked). In particular, the !-?-flow-graph in (10) is not acyclic.

3.3.4 Definition A vertex V in $G_{!?}(\Delta)$ is called a *!-vertex* if it is a !-structure, and *?-vertex* if it is a ?-structure. Note that an edge from a !-vertex to a !-vertex always goes upwards in a derivation. Hence, we call a path that contains only !-vertices an *up-path*. Similarly, a path with only ?-vertices is called a *down-path*. Edges from !-vertices to ?-vertices or from ?-vertices to !-vertices are called *flipping edges*. The number of flipping edges in a path p is called the *flipping number* of p, denoted by $\mathsf{fl}(p)$.

For example, the path indicated in the leftmost derivation in (11) has flipping number 2, and the two paths in the second derivation in (11) have both flipping number 0.

3.3.5 Definition Let Δ be a derivation. A vertex V in $G_{!?}(\Delta)$ is called a p*-vertex* if it is the principal structure of a $\mathsf{p}\downarrow$ or $\mathsf{p}\uparrow$. The vertex V is called a b*-vertex* if it is the principal structure of a $\mathsf{b}\downarrow$ or $\mathsf{b}\uparrow$.

3.4 The Induction Measure

3.4.1 Definition The p*-number* of a path p in $G_{!?}(\Delta)$, denoted by $\mathsf{p}(p)$, is the number of p-vertices occurring in p. If p is cyclic, the vertices with multiple occurrences in p are counted as many times as they occur in p.

For example, the path p indicated in the leftmost example in (11), we have $\mathsf{p}(p) = 2$. The rightmost one has $\mathsf{p}(p) = 3$ if the path passes through the cycle once, and $\mathsf{p}(p) = 5$ if the path passes through the cycle twice, and so on. Note that we do not have $\mathsf{p}(p) = \mathsf{fl}(p)$ in general. But we have always $\mathsf{p}(p) \geq \mathsf{fl}(p)$.

3.4.2 Definition Let Δ be a derivation and let V be a vertex in $G_{!?}(\Delta)$. Then the p*-number* of V in Δ, denoted by $\mathsf{p}(V)$, is defined as follows:

$$\mathsf{p}(V) = \sup\{\, \mathsf{p}(p) \mid p \text{ is a path starting in } V \,\} \quad . \tag{12}$$

For a rule instance ρ in Δ of the kind $\mathsf{g}\downarrow$, $\mathsf{b}\downarrow$, $\mathsf{w}\downarrow$, or $\mathsf{g}\uparrow$, $\mathsf{b}\uparrow$, $\mathsf{w}\uparrow$, we define its p*-number*, denoted by $\mathsf{p}_\Delta(\rho)$ to be the p-number of its principal structure.

In other words, for determining $\mathsf{p}(V)$, we take the maximum of all $\mathsf{p}(p)$, where p ranges over all paths that have V as starting vertex. If one of these paths is cyclic, then $\mathsf{p}(V) = \omega$.

For example, consider again the derivation in (10). Below we show it again twice where in each derivation one vertex of the !-?-flow-graph is marked. Let us denote them by V_1 and V_2, respectively.

$$
\begin{array}{rc}
 & !![!a \parr a] \\ \cline{2-2}
\mathsf{p}\!\downarrow & ![?!a \parr !a] \\ \cline{2-2}
\mathsf{b}\!\downarrow & !?!a \\ \cline{2-2}
\mathsf{b}\!\uparrow & (!?!a \otimes ?!a) \\ \cline{2-2}
\mathsf{p}\!\uparrow & ?(?!a \otimes !a) \\ \cline{2-2}
\mathsf{p}\!\uparrow & ??(!a \otimes a)
\end{array}
\qquad\qquad
\begin{array}{rc}
 & !![!a \parr a] \\ \cline{2-2}
\mathsf{p}\!\downarrow & ![?!a \parr !a] \\ \cline{2-2}
\mathsf{b}\!\downarrow & !?!a \\ \cline{2-2}
\mathsf{b}\!\uparrow & (!?!a \otimes ?!a) \\ \cline{2-2}
\mathsf{p}\!\uparrow & ?(?!a \otimes !a) \\ \cline{2-2}
\mathsf{p}\!\uparrow & ??(!a \otimes a)
\end{array}
\tag{13}
$$

On the left, we have shown all paths starting in V_1. There are only two of them, one has p-number 1 and the other has p-number 0. Hence $\mathsf{p}(V_1) = 1$. On the right we have shown all paths starting in V_2. Because of the cycle, we have $\mathsf{p}(V_2) = \omega$.

3.4.3 Definition Let Δ be a derivation. A *look-back tree* t in $G_{!?}(\Delta)$ is a subgraph which is a directed tree such that the edges are directed towards the root, and such that every path from a leaf to the root in t contains at most one flipping edge, and such that every branching vertex of t, i.e., every vertex with two incoming edges is the principle structure of an instance of $\mathsf{g}\!\downarrow$ or $\mathsf{g}\!\uparrow$. The b-*number* of a look-back tree t, denoted by $\mathsf{b}(t)$, is the number of b-vertices occurring in t.

Note that because of the restriction of the flipping number of paths in t to 1, a look-back tree cannot be cyclic.

Consider for example the following derivations in which we exhibited subgraphs of the !-?-flow-graph.

$$
\begin{array}{rc}
 & ![a \parr ?b \parr b] \\ \cline{2-2}
\mathsf{p}\!\downarrow & [!a \parr ?[?b \parr b]] \\ \cline{2-2}
\mathsf{b}\!\downarrow & [!a \parr ??b] \\ \cline{2-2}
\mathsf{g}\!\downarrow & [!a \parr ?b] \\ \cline{2-2}
\mathsf{b}\!\uparrow & [(!a \otimes a) \parr ?b]
\end{array}
\qquad
\begin{array}{rc}
 & (?a \otimes ![b \parr c]) \\ \cline{2-2}
\mathsf{p}\!\downarrow & (?a \otimes [!b \parr ?c]) \\ \cline{2-2}
\mathsf{s} & [(?a \otimes !b) \parr ?c] \\ \cline{2-2}
\mathsf{p}\!\uparrow & [?(a \otimes b) \parr ?c]
\end{array}
\qquad
\begin{array}{rc}
 & [???a \parr ??a] \\ \cline{2-2}
\mathsf{g}\!\downarrow & [??a \parr ??a] \\ \cline{2-2}
\mathsf{g}\!\downarrow & [??a \parr ?a] \\ \cline{2-2}
\mathsf{b}\!\downarrow & ??a \\ \cline{2-2}
\mathsf{g}\!\downarrow & ?a
\end{array}
\tag{14}
$$

On the left we have a look-back tree, and its b-number is two. Its root and the two b-vertices are marked with a grey background. The second example in (14) is not a look-back tree because of the two flippings in the path. The third example is not a look-back because there is a branching vertex (marked with grey background) that is not the principle structure of an instance of $\mathsf{g}\!\downarrow$ or $\mathsf{g}\!\uparrow$.

3.4.4 Definition Let Δ be a derivation and let V be a vertex in $G_{!?}(\Delta)$. We define the b-*number* of V, denoted by $\mathsf{b}(V)$, as follows:

$$\mathsf{b}(V) = \sup\{\, \mathsf{b}(t) \mid t \text{ is a look-back tree with root } V \,\} \quad . \tag{15}$$

Note that for the p-number of a vertex, we look forward in the graph, and for the b-number we look backwards. Furthermore, for the b-number we consider only paths with flipping number 0 or 1, and we allow branchings as in case (iv) of Figure 8, but never as in cases (v), (vi), and (vii) of that Figure.

To see some example, consider again the rightmost derivation in (14). Let us denote the $?a$-occurrence in the conclusion by V_3. The first two derivations below in (16) show two look-back tree with V_3 as root. The third derivation shows a look-back tree of the $!![!a \parr a]$-vertex in the premisse of the derivation in (10). Let us denote that vertex by V_4.

$$
\begin{array}{rc}
 & [???a \parr ??a] \\ \cline{2-2}
\mathsf{g}\!\downarrow & [??a \parr ??a] \\ \cline{2-2}
\mathsf{g}\!\downarrow & [??a \parr ?a] \\ \cline{2-2}
\mathsf{b}\!\downarrow & ??a \\ \cline{2-2}
\mathsf{g}\!\downarrow & ?a
\end{array}
\qquad
\begin{array}{rc}
 & [???a \parr ??a] \\ \cline{2-2}
\mathsf{g}\!\downarrow & [??a \parr ??a] \\ \cline{2-2}
\mathsf{g}\!\downarrow & [??a \parr ?a] \\ \cline{2-2}
\mathsf{b}\!\downarrow & ??a \\ \cline{2-2}
\mathsf{g}\!\downarrow & ?a
\end{array}
\qquad
\begin{array}{rc}
 & !![!a \parr a] \\ \cline{2-2}
\mathsf{p}\!\downarrow & ![?!a \parr !a] \\ \cline{2-2}
\mathsf{b}\!\downarrow & !?!a \\ \cline{2-2}
\mathsf{b}\!\uparrow & (!?!a \otimes ?!a) \\ \cline{2-2}
\mathsf{p}\!\uparrow & ?(?!a \otimes !a) \\ \cline{2-2}
\mathsf{p}\!\uparrow & ??(!a \otimes a)
\end{array}
\tag{16}
$$

We have $\mathsf{b}(V_3) = 1$. The first look-back tree has b-number 1 and the second one has b-number 0. We have $\mathsf{b}(V_4) = 2$ because both instances, $\mathsf{b}\!\uparrow$ and $\mathsf{b}\!\downarrow$ have their principal structure as vertex in the indicated look-back tree.

3.4.5 Definition Let Δ be a derivation and $?Z\{!R\}$ a structure occurring in Δ. Then we say that the ?-vertex $?Z\{!R\}$ is in the *onion* $\circledcirc(!R)$ of the !-vertex $!R$. Dually, we define the *onion* of a ?-vertex $?T$, denoted by $\circledcirc(?T)$, to be the set of all !-vertices that have this occurrence of $?T$ as substructure. For every rule instance ρ in Δ of the kind $\mathsf{g}\!\downarrow$, $\mathsf{b}\!\downarrow$, $\mathsf{w}\!\downarrow$, or $\mathsf{g}\!\uparrow$, $\mathsf{b}\!\uparrow$, $\mathsf{w}\!\uparrow$, we define its *onion* $\circledcirc_\Delta(\rho)$ in Δ to be the onion of its principal structure. The *onion* b-*number* of ρ in Δ, denoted by $\mathsf{b}\circledcirc_\Delta(\rho)$, is the sum of the b-numbers of the vertices in its onion, i.e.,

$$
\mathsf{b}\circledcirc_\Delta(\rho) = \sum_{V \in \circledcirc_\Delta(\rho)} \mathsf{b}(V) \quad .
$$

For example, consider the bottommost occurrence of $!a$ in the derivation in (10). It is marked in the leftmost derivation in (11). Its onion consists of the two ?-structures in the conclusion of the derivation. Both have b-number 1. Hence, the onion b-number of that $!a$ is 2.

Finally, we define the *status* of a rule instance to be either 0 or 1, such that it is 1 if the rule is of the kind $\mathsf{g}\!\downarrow$, $\mathsf{b}\!\downarrow$, $\mathsf{w}\!\downarrow$, or $\mathsf{g}\!\uparrow$, $\mathsf{b}\!\uparrow$, $\mathsf{w}\!\uparrow$, and not yet at its final destination at the top or the bottom of the derivation. The status is 0 if the rule does not play any further role in the up-down-permutation. The motivation of this is that Step 2 of our decomposition process (see Figure 5) is completed if and only if all rules instances in the derivation have status 0. Formally, the status is defined as follows.

3.4.6 Definition Let $\mathsf{SNEL}' = \mathsf{SNEL} \setminus \{\mathsf{e}\!\downarrow, \mathsf{e}\!\uparrow\}$, let Δ be a derivation in SNEL', and let ρ be a rule instance inside Δ. Then ρ splits Δ into two parts:

$$\begin{array}{c} Q \\ \mathsf{SNEL}' \big\Vert \Delta_1 \\ \rho \dfrac{S\{W\}}{S\{Z\}} \\ \mathsf{SNEL}' \big\Vert \Delta_2 \\ P \end{array}$$

The *status* of ρ in Δ, denoted by $\mathsf{st}_\Delta(\rho)$ is 1 if we have one of the following two cases:

- $\rho \in \{\mathsf{g}\!\uparrow, \mathsf{b}\!\uparrow, \mathsf{w}\!\uparrow\}$ and Δ_1 contains an instance of a rule in $\mathsf{SNEL}' \setminus \{\mathsf{g}\!\uparrow, \mathsf{b}\!\uparrow, \mathsf{w}\!\uparrow\}$, or
- $\rho \in \{\mathsf{g}\!\downarrow, \mathsf{b}\!\downarrow, \mathsf{w}\!\downarrow\}$ and Δ_2 contains an instance of a rule in $\mathsf{SNEL}' \setminus \{\mathsf{g}\!\downarrow, \mathsf{b}\!\downarrow, \mathsf{w}\!\downarrow\}$.

Otherwise $\mathsf{st}_\Delta(\rho) = 0$.

The reason for using SNEL' is that the rules $\mathsf{e}\!\downarrow$ and $\mathsf{e}\!\uparrow$ are not considered in Step 2 of Figure 5. However, all statements in this section about SNEL' are also valid for SNEL.

Now we are using the status, the p-number, and the onion b-number of a rule instance to define its rank.

3.4.7 Definition For a rule instance ρ of the kind $\mathsf{g}\!\downarrow$, $\mathsf{b}\!\downarrow$, $\mathsf{w}\!\downarrow$ or $\mathsf{g}\!\uparrow$, $\mathsf{b}\!\uparrow$, $\mathsf{w}\!\uparrow$ inside a derivation Δ, we define its *rank* $\mathsf{rk}_\Delta(\rho) \in \omega \times (\omega + 1) \times \omega$ to be the triple

$$\mathsf{rk}_\Delta(\rho) = \langle \mathsf{st}_\Delta(\rho), \mathsf{p}_\Delta(\rho), \mathsf{b}{\circledcirc}_\Delta(\rho) \rangle \quad .$$

For the whole of Δ, we define the *rank* $\mathsf{rk}(\Delta) \in (\omega \times (\omega + 1) \times \omega)^{\#}$ to be the formal sum of the ranks of its occurrences of $\mathsf{g}\!\downarrow, \mathsf{b}\!\downarrow, \mathsf{w}\!\downarrow, \mathsf{g}\!\uparrow, \mathsf{b}\!\uparrow, \mathsf{w}\!\uparrow$, i.e.,

$$\mathsf{rk}(\Delta) = \sum_{\substack{\rho \text{ in } \Delta \text{ and } \rho \text{ is one of} \\ \mathsf{g}\downarrow,\mathsf{b}\downarrow,\mathsf{w}\downarrow,\mathsf{g}\uparrow,\mathsf{b}\uparrow,\mathsf{w}\uparrow}} \mathsf{rk}_\Delta(\rho) \quad .$$

We define the *down-rank* of Δ, denoted by $\mathsf{rk}^{\downarrow}(\Delta)$ by considering only the down-rules $\mathsf{g}\!\downarrow, \mathsf{b}\!\downarrow, \mathsf{w}\!\downarrow$ in the formal sum:

$$\mathsf{rk}^{\downarrow}(\Delta) = \sum_{\rho \text{ in } \Delta \text{ and } \rho \text{ is one of } \mathsf{g}\downarrow,\mathsf{b}\downarrow,\mathsf{w}\downarrow} \mathsf{rk}_\Delta(\rho) \quad .$$

Similarly, the *up-rank* $\mathsf{rk}^{\uparrow}(\Delta)$ takes only the up-rules $\mathsf{g}\!\uparrow, \mathsf{b}\!\uparrow, \mathsf{w}\!\uparrow$ into account:

$$\mathsf{rk}^{\uparrow}(\Delta) = \sum_{\rho \text{ in } \Delta \text{ and } \rho \text{ is one of } \mathsf{g}\uparrow,\mathsf{b}\uparrow,\mathsf{w}\uparrow} \mathsf{rk}_\Delta(\rho) \quad .$$

It follows immediately from the definition that $\mathsf{rk}(\Delta) = \mathsf{rk}^{\downarrow}(\Delta) + \mathsf{rk}^{\uparrow}(\Delta)$. For example, in (10), we have that the rank of the $\mathsf{b}{\downarrow}$ instance is $\langle 1, \omega, 1\rangle$ and the rank of the $\mathsf{b}{\uparrow}$ instance is $\langle 1, 0, 0\rangle$. Hence, the rank of the whole derivation is the formal sum $\langle 1, \omega, 1\rangle + \langle 1, 0, 0\rangle$.

3.5 Permutations again

After what has been said in Section 3.2, it should be clear what is coming now. Namely, we will use the rank of the derivation as induction measure to show that the permutation process for achieving Step 2, as indicated at the end of Section 3.1, does indeed terminate. For this, let us inspect what happens to the rank of a derivation during the permutation process. Consider again the cases in 3.1.8. We begin with the trivial cases (cf. the proof of Lemma 3.1.3).

3.5.1 Permuting $\mathsf{g}{\uparrow}$, $\mathsf{b}{\uparrow}$, $\mathsf{w}{\uparrow}$ up: Let a derivation Δ be given. As in 3.1.8, Consider a subderivation

$$\begin{array}{rc} & S\{W\} \\ \cline{2-2} \rho & S\{Z\} \\ \cline{2-2} \pi & P \end{array} \quad , \tag{17}$$

where $\rho \in \mathsf{SNEL}' \setminus \{\mathsf{g}{\uparrow}, \mathsf{b}{\uparrow}, \mathsf{w}{\uparrow}\}$ and $\pi \in \{\mathsf{g}{\uparrow}, \mathsf{b}{\uparrow}, \mathsf{w}{\uparrow}\}$. In the following case analysis we replace (as done in Section 3.1) in Δ the subderivation in (17) by a new subderivation with the same premise and conclusion. We use Δ' to denote the result of this replacement.

(i) The contractum $!R$ of π is inside the context $S\{\ \}$. Here is an example with $\pi = \mathsf{g}{\uparrow}$ and $\rho = \mathsf{s}$:

$$\begin{array}{rc} & S'\{!R\}\{([P \parr U] \otimes T)\} \\ \cline{2-2} \mathsf{s} & S'\{!R\}\{[(P \otimes T) \parr U]\} \\ \cline{2-2} \mathsf{g}{\uparrow} & S'\{!!R\}\{[(P \otimes T) \parr U]\} \end{array} \quad \to \quad \begin{array}{rc} & S'\{!R\}\{([P \parr U] \otimes T)\} \\ \cline{2-2} \mathsf{g}{\uparrow} & S'\{!!R\}\{([P \parr U] \otimes T)\} \\ \cline{2-2} \mathsf{s} & S'\{!!R\}\{[(P \otimes T) \parr U]\} \end{array}$$

Here, we used $S'\{\ \}\{\ \}$ to denote a context with two independent holes, and we used bold light lines to indicate bunches of parallel paths going through the derivation. Clearly, in this case, neither $\mathsf{p}_{\Delta}(\pi)$ nor $\mathsf{b}{\circledcirc}_{\Delta}(\pi)$ change their value (but $\mathsf{st}_{\Delta}(\pi)$ could go down). The important fact to observe is that the rank of all other rules in Δ remains unchanged in Δ'. Hence, $\mathsf{rk}^{\uparrow}(\Delta') \leq \mathsf{rk}^{\uparrow}(\Delta)$ and $\mathsf{rk}^{\downarrow}(\Delta') = \mathsf{rk}^{\downarrow}(\Delta)$.

(ii) The contactum $!R$ of π appears inside the redex Z of ρ, but only inside a substructure of Z that is not affected by ρ. Again, we exhibit an example with $\pi = \mathsf{g}{\uparrow}$ and $\rho = \mathsf{s}$:

$$\begin{array}{rc} & S([P\{!R\} \parr U] \otimes T) \\ \cline{2-2} \mathsf{s} & S[(P\{!R\} \otimes T) \parr U] \\ \cline{2-2} \mathsf{g}{\uparrow} & S[(P\{!!R\} \otimes T) \parr U] \end{array} \quad \to \quad \begin{array}{rc} & S([P\{!R\} \parr U] \otimes T) \\ \cline{2-2} \mathsf{g}{\uparrow} & S([P\{!!R\} \parr U] \otimes T) \\ \cline{2-2} \mathsf{s} & S[(P\{!!R\} \otimes T) \parr U] \end{array}$$

As in the previous case, the values of $\mathsf{p}_\Delta(\pi)$ and $\mathsf{b}\circledcirc_\Delta(\pi)$ are not affected. This is trivial for $\rho \in \{\mathsf{s}, \mathsf{q}\!\downarrow, \mathsf{q}\!\uparrow\}$, and we leave it as an instructive exercise to the reader to verify it also for $\rho = \mathsf{p}\!\downarrow$. For $\rho = \mathsf{p}\!\uparrow$, the value of $\mathsf{p}_\Delta(\pi)$ remains unchanged, but $\mathsf{b}\circledcirc_\Delta(\pi)$ could go down. As in the previous case, $\mathsf{st}_\Delta(\pi)$ could go down, and the rank of all other rules in Δ remains unchanged in Δ'. Hence, $\mathsf{rk}^\uparrow(\Delta') \leq \mathsf{rk}^\uparrow(\Delta)$ and $\mathsf{rk}^\downarrow(\Delta') = \mathsf{rk}^\downarrow(\Delta)$.

(iii) The redex Z of ρ is inside the contractum $!R$ of π.

(a) If $\pi = \mathsf{w}\!\uparrow$, then

$$\mathsf{w}\!\uparrow\dfrac{\rho\,\dfrac{S'\{!R\{W\}\}}{S'\{!R\{Z\}\}}}{S'\{\circ\}} \qquad \rightarrow \qquad \mathsf{w}\!\uparrow\dfrac{S'\{!R\{W\}\}}{S'\{\circ\}}$$

We have $\mathsf{rk}(\Delta') \leq \mathsf{rk}(\Delta)$ because ρ is removed.

(b) If $\pi = \mathsf{g}\!\uparrow$, then

$$\mathsf{g}\!\uparrow\dfrac{\rho\,\dfrac{S'\{!R\{W\}\}}{S'\{!R\{Z\}\}}}{S'\{!!R\{Z\}\}} \qquad \rightarrow \qquad \rho\,\dfrac{\mathsf{g}\!\uparrow\dfrac{S'\{!R\{W\}\}}{S'\{!!R\{W\}\}}}{S'\{!!R\{Z\}\}}$$

Note that the onion of ρ is changed (if ρ is an instance of $\mathsf{w}\!\downarrow$, $\mathsf{b}\!\downarrow$, or $\mathsf{g}\!\downarrow$). But the b-number of the !-vertex in the premise of the derivations above is the same as the sum of the b-numbers of the two !-vertices in the conlusion. Hence, the onion b-number of ρ does not change. Therefore $\mathsf{rk}(\Delta') \leq \mathsf{rk}(\Delta)$.

(c) If $\pi = \mathsf{b}\!\uparrow$, then the situation is not entirely trivial, because ρ gets duplicated:

$$\mathsf{b}\!\uparrow\dfrac{\rho\,\dfrac{S'\{!R\{W\}\}}{S'\{!R\{Z\}\}}}{S'(!R\{Z\} \otimes R\{Z\})} \qquad \rightarrow \qquad \rho\,\dfrac{\rho\,\dfrac{\mathsf{b}\!\uparrow\dfrac{S'\{!R\{W\}\}}{S'(!R\{W\} \otimes R\{W\})}}{S'(!R\{W\} \otimes R\{Z\})}}{S'(!R\{Z\} \otimes R\{Z\})}$$

We distiguish the following cases:

(1) If ρ does not involve any modalities, i.e., $\rho \in \{\mathsf{ai}\!\downarrow, \mathsf{ai}\!\uparrow, \mathsf{s}, \mathsf{q}\!\downarrow, \mathsf{q}\!\uparrow\}$, then situation is similar to cases (i) and (ii) above. No rule changes its rank, except that we could have that the status of the $\mathsf{b}\!\uparrow$ goes down. Hence, we have $\mathsf{rk}^\uparrow(\Delta') \leq \mathsf{rk}^\uparrow(\Delta)$ and $\mathsf{rk}^\downarrow(\Delta') = \mathsf{rk}^\downarrow(\Delta)$.

(2) If $\rho = \mathsf{p}\!\downarrow$, then

$$\mathsf{b}\!\uparrow\dfrac{\mathsf{p}\!\downarrow\dfrac{S'\{!R\{![P \parr T]\}\}}{S'\{!R[!P \parr ?T]\}}}{S'(!R[!P \parr ?T] \otimes R[!P \parr ?T])} \qquad \rightarrow \qquad \mathsf{p}\!\downarrow\dfrac{\mathsf{p}\!\downarrow\dfrac{\mathsf{b}\!\uparrow\dfrac{S'\{!R\{![P \parr T]\}\}}{S'(!R\{![P \parr T]\} \otimes R\{![P \parr T]\})}}{S'(!R\{![P \parr T]\} \otimes R[!P \parr ?T])}}{S'(!R[!P \parr ?T] \otimes R[!P \parr ?T])}$$

As before, $\mathsf{p}_\Delta(\pi)$ and $\mathsf{b}\circledcirc_\Delta(\pi)$ do not change. However, the p-number, as well as the onion b-number of other rules might go down because some paths disappear. Hence, $\mathsf{rk}^\uparrow(\Delta') \leq \mathsf{rk}^\uparrow(\Delta)$ and $\mathsf{rk}^\downarrow(\Delta') \leq \mathsf{rk}^\downarrow(\Delta)$.

(3) If $\rho = \mathsf{p}{\uparrow}$, then

$$\mathsf{b}{\uparrow}\dfrac{\mathsf{p}{\uparrow}\dfrac{S'\{!R(?T \otimes !P)\}}{S'\{!R\{?(T \otimes P)\}\}}}{S'(!R\{?(T \otimes P)\} \otimes R\{?(T \otimes P)\})} \quad \to \quad \mathsf{p}{\uparrow}\dfrac{\mathsf{p}{\uparrow}\dfrac{\mathsf{b}{\uparrow}\dfrac{S'\{!R(?T \otimes !P)\}}{S'(!R(?T \otimes !P) \otimes R(?T \otimes !P))}}{S'(!R(?T \otimes !P) \otimes R\{?(T \otimes P)\})}}{S'(!R\{?(T \otimes P)\} \otimes R\{?(T \otimes P)\})}$$

Again, neither $\mathsf{p}_\Delta(\pi)$ nor $\mathsf{b}{\circledcirc}_\Delta(\pi)$ can change (but $\mathsf{st}_\Delta(\pi)$ could go down). No other rule in Δ changes its rank. Althought the $\mathsf{p}{\uparrow}$-instance is duplicated, no path changes its p-number or its b-number. Hence, $\mathsf{rk}^{\uparrow}(\Delta') \leq \mathsf{rk}^{\uparrow}(\Delta)$ and $\mathsf{rk}^{\downarrow}(\Delta') = \mathsf{rk}^{\downarrow}(\Delta)$.

(4) Finally, we have to consider the case where $\rho \in \{\mathsf{g}{\downarrow}, \mathsf{b}{\downarrow}, \mathsf{w}{\downarrow}\}$. We show only the case for $\rho = \mathsf{g}{\downarrow}$:

$$\mathsf{b}{\uparrow}\dfrac{\mathsf{g}{\downarrow}\dfrac{S'\{!R(??T)\}}{S'\{!R\{?T\}\}}}{S'(!R\{?T\} \otimes R\{?T\})} \quad \to \quad \mathsf{g}{\downarrow}\dfrac{\mathsf{g}{\downarrow}\dfrac{\mathsf{b}{\uparrow}\dfrac{S'\{!R(??T)\}}{S'(!R\{??T\} \otimes R\{??T\})}}{S'(!R\{??T\} \otimes R\{?T\})}}{S'(!R\{?T\} \otimes R\{?T\})}$$

Again, neither $\mathsf{p}_\Delta(\pi)$ nor $\mathsf{b}{\circledcirc}_\Delta(\pi)$ can change (but $\mathsf{st}_\Delta(\pi)$ could go down). Hence, $\mathsf{rk}^{\uparrow}(\Delta') \leq \mathsf{rk}^{\uparrow}(\Delta)$. However, the number of $\mathsf{g}{\downarrow}$ instances in the derivation is increased. But both new instances of $\mathsf{g}{\downarrow}$ have strictly smaller rank in Δ' than the original $\mathsf{g}{\downarrow}$ in Δ, because their onion b-number is reduced by 1. Hence, $\mathsf{rk}^{\downarrow}(\Delta') < \mathsf{rk}^{\downarrow}(\Delta)$. The same holds for $\rho = \mathsf{b}{\downarrow}$ and $\rho = \mathsf{w}{\downarrow}$. Note that for this, it is crucial that the look-back tree of a vertex in the onion (that is used for computing the onion b-number) is acyclic.

(iv) The crucial case is where the contractum $!R$ of π actively interferes with the redex Z of ρ. There are four subcases:

(a) For $\rho = \mathsf{w}{\downarrow}$, the situation is dual to case (iii.a). We show only the case $\pi = \mathsf{g}{\uparrow}$:

$$\mathsf{g}{\uparrow}\dfrac{\mathsf{w}{\downarrow}\dfrac{S\{\circ\}}{S\{?Z\{!R\}\}}}{S\{?Z\{!!R\}\}} \quad \to \quad \mathsf{w}{\downarrow}\dfrac{S\{\circ\}}{S\{?Z\{!!R\}\}}$$

We have $\mathsf{rk}^{\uparrow}(\Delta') < \mathsf{rk}^{\uparrow}(\Delta)$ because π disappears, and $\mathsf{rk}^{\downarrow}(\Delta') \leq \mathsf{rk}^{\downarrow}(\Delta)$ because the status of the $\mathsf{w}{\downarrow}$ could go down.

(b) For $\rho = \mathsf{g}{\downarrow}$, the situation is dual to case (iii.b). We again show only the case $\pi = \mathsf{g}{\uparrow}$:

$$\mathsf{g}{\uparrow}\dfrac{\mathsf{g}{\downarrow}\dfrac{S\{??Z\{!R\}\}}{S\{?Z\{!R\}\}}}{S\{?Z\{!!R\}\}} \quad \to \quad \mathsf{g}{\downarrow}\dfrac{\mathsf{g}{\uparrow}\dfrac{S\{??Z\{!R\}\}}{S\{??Z\{!!R\}\}}}{S\{?Z\{!!R\}\}}$$

We have $\mathsf{rk}^{\uparrow}(\Delta') \leq \mathsf{rk}^{\uparrow}(\Delta)$ and $\mathsf{rk}^{\downarrow}(\Delta') \leq \mathsf{rk}^{\downarrow}(\Delta)$ because the status of both rules could go down, and nothing else changes, for the same reason as explained in (iii.b).

(c) For $\rho = \mathsf{b}{\downarrow}$ the permutations are dual to the ones in case (iii.c.4) above. For $\pi = \mathsf{w}{\uparrow}$, we have

$$\mathsf{w}{\uparrow}\,\dfrac{\mathsf{b}{\downarrow}\,\dfrac{S[?Z\{!R\} \parr Z\{!R\}]}{S\{?Z\{!R\}\}}}{S\{?Z\{\circ\}\}} \quad\to\quad \mathsf{b}{\downarrow}\,\dfrac{\mathsf{w}{\uparrow}\,\dfrac{\mathsf{w}{\uparrow}\,\dfrac{S[?Z\{!R\} \parr Z\{!R\}]}{S[?Z\{!R\} \parr Z\{\circ\}]}}{S[?Z\{\circ\} \parr Z\{\circ\}]}}{S\{?Z\{\circ\}\}}$$

For $\pi = \mathsf{g}{\uparrow}$, we have

$$\mathsf{g}{\uparrow}\,\dfrac{\mathsf{b}{\downarrow}\,\dfrac{S[?Z\{!R\} \parr Z\{!R\}]}{S\{?Z\{!R\}\}}}{S\{?Z\{!!R\}\}} \quad\to\quad \mathsf{b}{\downarrow}\,\dfrac{\mathsf{g}{\uparrow}\,\dfrac{\mathsf{g}{\uparrow}\,\dfrac{S[?Z\{!R\} \parr Z\{!R\}]}{S[?Z\{!R\} \parr Z\{!!R\}]}}{S[?Z\{!!R\} \parr Z\{!!R\}]}}{S\{?Z\{!!R\}\}}$$

And for $\pi = \mathsf{b}{\uparrow}$, we have

$$\mathsf{b}{\uparrow}\,\dfrac{\mathsf{b}{\downarrow}\,\dfrac{S[?Z\{!R\} \parr Z\{!R\}]}{S\{?Z\{!R\}\}}}{S\{?Z(!R \otimes R)\}} \quad\to\quad \mathsf{b}{\downarrow}\,\dfrac{\mathsf{b}{\uparrow}\,\dfrac{\mathsf{b}{\uparrow}\,\dfrac{S[?Z\{!R\} \parr Z\{!R\}]}{S[?Z\{!R\} \parr Z(!R \otimes R)]}}{S[?Z(!R \otimes R) \parr Z(!R \otimes R)]}}{S\{?Z(!R \otimes R)\}}$$

In all three cases, the rule π is duplicated. But both copies have a smaller onion b-number in Δ'. Hence $\mathsf{rk}^{\uparrow}(\Delta') < \mathsf{rk}^{\uparrow}(\Delta)$. As in case (iii.c.4) above, this crucially relies on the fact that the look-back tree of a vertex is acyclic. We also have $\mathsf{rk}^{\downarrow}(\Delta') \leq \mathsf{rk}^{\downarrow}(\Delta)$ because the status of the $\mathsf{b}{\downarrow}$ could go down, and nothing else changes.

(d) The most interesting case is when $\rho = \mathsf{p}{\downarrow}$. We have the following situations:

(1) For $\pi = \mathsf{g}{\uparrow}$:

$$\mathsf{g}{\uparrow}\,\dfrac{\mathsf{p}{\downarrow}\,\dfrac{S\{![R \parr T]\}}{S[!R \parr ?T]}}{S[!!R, ?T]} \quad\to\quad \mathsf{g}{\downarrow}\,\dfrac{\mathsf{p}{\downarrow}\,\dfrac{\mathsf{p}{\downarrow}\,\dfrac{\mathsf{g}{\uparrow}\,\dfrac{S\{![R \parr T]\}}{S\{!![R \parr T]\}}}{S\{![!R \parr ?T]\}}}{S[!!R \parr ??T]}}{S[!!R \parr ?T]}$$

A single $\mathsf{g}{\uparrow}$ is replaced by a $\mathsf{g}{\uparrow}$ and a $\mathsf{g}{\downarrow}$. We clearly have $\mathsf{rk}^{\uparrow}(\Delta') \leq \mathsf{rk}^{\uparrow}(\Delta)$ because the status of the $\mathsf{g}{\uparrow}$-instance could go down. If $G_{!?}(\Delta)$ is acyclic, then also its p-number goes down. Note that no other up-rule changes its rank. We cannot make any statements about $\mathsf{rk}^{\downarrow}(\Delta)$. But, observe that if $G_{!?}(\Delta)$ is acyclic, then the p-number of the new $\mathsf{g}{\downarrow}$ is strictly smaller than the p-number of the original $\mathsf{g}{\uparrow}$. Hence, if $G_{!?}(\Delta)$ is acyclic, then $\mathsf{rk}(\Delta') < \mathsf{rk}(\Delta)$. Note that even in the case of acyclicity of $G_{!?}(\Delta)$, we *do not* have $\mathsf{rk}^{\downarrow}(\Delta') \leq \mathsf{rk}^{\downarrow}(\Delta)$.

(2) For $\pi = \mathsf{b}{\uparrow}$ we have:

$$
\mathsf{b}{\uparrow}\dfrac{\mathsf{p}{\downarrow}\dfrac{S\{![R \parr T]\}}{S[!R \parr ?T]}}{S[(!R \otimes R) \parr ?T]}
\quad\rightarrow\quad
\mathsf{b}{\downarrow}\dfrac{\mathsf{s}\dfrac{\mathsf{s}\dfrac{\mathsf{p}{\downarrow}\dfrac{\mathsf{b}{\uparrow}\dfrac{S\{![R \parr T]\}}{S(![R \parr T] \otimes [R \parr T])}}{S([!R \parr ?T] \otimes [R \parr T])}}{S[([!R \parr ?T] \otimes R) \parr T]}}{S[(!R \otimes R) \parr ?T \parr T]}}{S[(!R \otimes R) \parr ?T]}
\tag{18}
$$

This case is similar to the one for $\mathsf{g}{\uparrow}$ above, but slightly more complicated. The $\mathsf{b}{\uparrow}$-instance is replaced by a $\mathsf{b}{\uparrow}$ and a $\mathsf{b}{\downarrow}$. By this, it can happen that the onion b-number of other down rules in Δ is increased. But note that no up-rule can change its onion b-number. (This is the reason for allowing one flipping edge in a path in the look-back tree in Definition 3.4.3, instead of forbidding any flipping edge. Note that cases (iii.c.4) and (iv.c) above would also work without the flipping edges in the look-back tree.) Since, as before, the status of the $\mathsf{b}{\uparrow}$ could go down, we have $\mathsf{rk}^{\uparrow}(\Delta') \leq \mathsf{rk}^{\uparrow}(\Delta)$. But since the rank of some down-rules can become bigger, we cannot compare $\mathsf{rk}(\Delta')$ with $\mathsf{rk}(\Delta)$. Nonetheless, it is important to mention that if $G_{!?}(\Delta)$ is acyclic, then the p-number of the new $\mathsf{b}{\downarrow}$ is strictly smaller than the p-number of the original $\mathsf{b}{\uparrow}$.

(3) For $\pi = \mathsf{w}{\uparrow}$ we have:

$$
\mathsf{w}{\uparrow}\dfrac{\mathsf{p}{\downarrow}\dfrac{S\{![R \parr T]\}}{S[!R \parr ?T]}}{S[\circ \parr ?T]}
\quad\rightarrow\quad
\mathsf{w}{\downarrow}\dfrac{=\dfrac{\mathsf{w}{\uparrow}\dfrac{S\{![R \parr T]\}}{S\{\circ\}}}{S[\circ \parr \circ]}}{S[\circ \parr ?T]}
$$

This case is simpler than the other two because the instance of $\mathsf{p}{\downarrow}$ disappears. Hence, we have $\mathsf{rk}^{\uparrow}(\Delta') \leq \mathsf{rk}^{\uparrow}(\Delta)$ and if $G_{!?}(\Delta)$ is acyclic also $\mathsf{rk}(\Delta') < \mathsf{rk}(\Delta)$. But we do *not* have $\mathsf{rk}^{\downarrow}(\Delta') \leq \mathsf{rk}^{\downarrow}(\Delta)$.

This case analysis is enough to show the following three lemmas.

3.5.2 Lemma

Every derivation $\mathsf{SNEL}' \begin{array}{c} P \\ \Big\| \\ Q \end{array} \Delta$ *can be transformed into* $\begin{array}{r} P \\ \{\mathsf{g}{\uparrow}, \mathsf{b}{\uparrow}, \mathsf{w}{\uparrow}\} \Big\| \\ P' \\ \mathsf{SNEL}' \setminus \{\mathsf{g}{\uparrow}, \mathsf{b}{\uparrow}, \mathsf{w}{\uparrow}\} \Big\| \\ Q \end{array}$.

Proof: This transformation is obtained by permuting all instances of $\mathsf{g}\uparrow$, $\mathsf{b}\uparrow$, $\mathsf{w}\uparrow$ to the top of a derivation as described in 3.5.1. Termination is ensured by using as measure the pair $\langle \mathsf{rk}^{\uparrow}(\Delta), \delta\rangle$ in a lexicographic ordering, where δ is the number of rule instances in the derivation above the topmost instance of $\mathsf{g}\uparrow$, $\mathsf{b}\uparrow$, or $\mathsf{w}\uparrow$ with status 1. If we always choose this topmost instance of $\mathsf{g}\uparrow$, $\mathsf{b}\uparrow$, or $\mathsf{w}\uparrow$ with status 1 for performing the next permutation step, then this measure always goes down, and by of Theorem 3.2.1, this is well-founded. □

3.5.3 Lemma

Every derivation $\begin{array}{c} P \\ \mathsf{SNEL}' \Big\| \Delta \\ Q \end{array}$ *can be transformed into* $\begin{array}{r} P \\ \mathsf{SNEL}' \setminus \{\mathsf{g}\downarrow, \mathsf{b}\downarrow, \mathsf{w}\downarrow\} \Big\| \\ Q' \\ \{\mathsf{g}\downarrow, \mathsf{b}\downarrow, \mathsf{w}\downarrow\} \Big\| \\ Q \end{array}$.

Proof: Dual to the previous lemma. □

3.5.4 Lemma *If the !-?-flow-graph of a derivation*

$$\begin{array}{c} P \\ \mathsf{SNEL}' \Big\| \Delta \\ Q \end{array}$$

is acyclic, then Δ *can be transformed into a derivation* Δ'

$$\begin{array}{r} Q \\ \{\mathsf{g}\uparrow, \mathsf{b}\uparrow, \mathsf{w}\uparrow\} \Big\| \\ Q' \\ \{\mathsf{ai}\downarrow, \mathsf{ai}\uparrow, \mathsf{s}, \mathsf{q}\downarrow, \mathsf{q}\uparrow, \mathsf{p}\downarrow, \mathsf{p}\uparrow\} \Big\| \\ P' \\ \{\mathsf{g}\downarrow, \mathsf{b}\downarrow, \mathsf{w}\downarrow\} \Big\| \\ P \end{array} \quad . \tag{19}$$

Proof: The derivation Δ' is obtained from Δ by a sequence of transformations:

$$\Delta = \Delta_0 \rightsquigarrow \Delta_1 \rightsquigarrow \Delta_2 \rightsquigarrow \Delta_3 \rightsquigarrow \ldots \rightsquigarrow \Delta' \quad , \tag{20}$$

where Δ_{i+1} is obtained from Δ_i by permuting all instances of $\mathsf{g}\uparrow$, $\mathsf{b}\uparrow$, $\mathsf{w}\uparrow$ up to the top of the derivation if i is even, and by permuting all instances of $\mathsf{g}\downarrow$, $\mathsf{b}\downarrow$, $\mathsf{w}\downarrow$ down to the bottom of the derivation if i is odd. Each of these single steps is well-defined because of Lemma 3.5.2 and Lemma 3.5.3. Now assume i is even and $i \geq 2$. Then there are no instances of $\mathsf{g}\uparrow$, $\mathsf{b}\uparrow$, or $\mathsf{w}\uparrow$ in Δ_{i+1}, and all instances of $\mathsf{g}\downarrow$, $\mathsf{b}\downarrow$, $\mathsf{w}\downarrow$ in Δ_{i+1} have been introduced by case (iv.d) in 3.5.1. Hence, for each ρ' of the kind $\mathsf{g}\!\downarrow$, $\mathsf{b}\!\downarrow$, $\mathsf{w}\!\downarrow$ in Δ_{i+1}, there is a ρ of the kind $\mathsf{g}\!\uparrow$, $\mathsf{b}\!\uparrow$, $\mathsf{w}\!\uparrow$ in Δ_i, with $\mathsf{st}_{\Delta_i}(\rho) = 1$ and $\mathsf{p}_{\Delta_i}(\rho) > \mathsf{p}_{\Delta_{i+1}}(\rho')$, and therefore $\mathsf{rk}_{\Delta_i}(\rho) > \mathsf{rk}_{\Delta_{i+1}}(\rho')$. Hence $\mathsf{rk}(\Delta_i) > \mathsf{rk}(\Delta_{i+1})$. By a similar argument we can conclude that $\mathsf{rk}(\Delta_i) > \mathsf{rk}(\Delta_{i+1})$ for all odd i with $i > 1$. By Theorem 3.2.1, we can conclude that the process indicated in (20) terminates eventually. The resulting derivation Δ' is of the desired shape (19). $\square$

Note that the argument in the previous proof is necessary because of case (iv.d.2) in 3.5.1. In all other permutation steps the rank does not increase.

As the derivation in (10) shows, we cannot hope for a lemma saying that $G_{!?}(\Delta)$ is always acyclic. Nonetheless, the decomposition terminates for (10), and the result is shown in Figure 9. Since in that figure, the !-?-flow-graph is acyclic, the cycle must have been broken eventually. For understanding how this is happening, we will now continue with an investigation in the structure of cycles in the flow-graph, and how they are broken. Before, we exhibit another example of a derivation with a cycle in its !-?-flow-graph:

$$
\begin{array}{rc}
\mathsf{p}\!\downarrow,\mathsf{p}\!\downarrow & \underline{!(![b \parr a] \otimes ![c \parr d])} \\
\mathsf{b}\!\uparrow & \underline{!([?b \parr !a] \otimes [!c \parr ?d])} \\
\mathsf{s},\mathsf{s} & \underline{(!([?b \parr !a] \otimes [!c \parr ?d]) \otimes [?b \parr !a] \otimes [!c \parr ?d])} \\
\mathsf{s},\mathsf{s} & \underline{(![!a \parr (?b \otimes !c) \parr ?d] \otimes [?b \parr !a] \otimes [!c \parr ?d])} \\
\mathsf{p}\!\uparrow,\mathsf{p}\!\uparrow & \underline{(![!a \parr (?b \otimes !c) \parr ?d] \otimes [?b \parr (!a \otimes ?d) \parr !c])} \\
\mathsf{g}\!\uparrow & \underline{(![!a \parr ?(b \otimes c) \parr ?d] \otimes [?b \parr ?(a \otimes d) \parr !c])} \\
 & (![!!a \parr ?(b \otimes c) \parr ?d] \otimes [?b \parr ?(a \otimes d) \parr !c])
\end{array}
\tag{21}
$$

This derivation can be used to explain why we have Steps 2 and 3 in the proof of Theorem 3.1 (see Figure 5), instead of doing something like

$$
\begin{array}{c} W_1 \\ \mathsf{SNEL}' \Big\| \\ Z_1 \end{array}
\quad \xrightarrow{2'} \quad
\begin{array}{c} W_1 \\ \{\mathsf{g}\!\uparrow\} \Big\| \\ W_2 \\ \mathsf{SNEL}' \setminus \{\mathsf{g}\!\downarrow, \mathsf{g}\!\uparrow\} \Big\| \\ Z_2 \\ \{\mathsf{g}\!\downarrow\} \Big\| \\ Z_1 \end{array}
\quad \xrightarrow{2''} \quad
\begin{array}{c} W_1 \\ \{\mathsf{g}\!\uparrow\} \Big\| \\ W_2 \\ \{\mathsf{b}\!\uparrow\} \Big\| \\ W_3 \\ \mathsf{SNEL}' \setminus \{\mathsf{g}\!\downarrow, \mathsf{g}\!\uparrow, \mathsf{b}\!\downarrow, \mathsf{b}\!\uparrow\} \Big\| \\ Z_3 \\ \{\mathsf{b}\!\downarrow\} \Big\| \\ Z_2 \\ \{\mathsf{g}\!\downarrow\} \Big\| \\ Z_1 \end{array}
\quad \cdots
$$

Running Step 2′ on the derivation in (21) does indeed fail because of non-termination. If we apply all the transformations of 3.5.1 together with the ones in (5) and (6) (and their duals), then the instances of $\mathsf{g}\uparrow$ (and $\mathsf{g}\downarrow$) get caught in the cycle in (21), and the process will run forever. Only if the $\mathsf{b}\uparrow$ is permuted up together with the $\mathsf{g}\uparrow$, the process does terminate. The reason is that when the instance of $\mathsf{b}\uparrow$ is permuted over the two $\mathsf{p}\downarrow$ on the top of the derivation, the cycle is broken, because some edges in the !-?-flow-graph disappear. This shows the importance of case (iii.c.2) in 3.5.1, and motivates the following definition.

3.5.5 Definition A cycle c in $G_{!?}(\Delta)$ is called *forked* if there is an instance of

$$\mathsf{b}\uparrow\frac{S\{!R\}}{S(!R\otimes R)}\qquad\text{or}\qquad\mathsf{b}\downarrow\frac{S[?T\parr T]}{S\{?T\}}$$

inside Δ such that both copies of R of the redex of the $\mathsf{b}\uparrow$, or both copies of T in the contractum of $\mathsf{b}\downarrow$ contain vertices of the cycle. We say that such an instance of $\mathsf{b}\uparrow$ or $\mathsf{b}\downarrow$ *forks* the cycle c. The number of $\mathsf{b}\uparrow$ and $\mathsf{b}\downarrow$ that fork a cycle c is called the *forking number* of c, denoted by $\mathsf{fk}(c)$. A cycle c with $\mathsf{fk}(c)=0$ is called *unforked.*

The cycles in (10) and (21) are both forked. The one in (10) has forking number 2 (since both, the $\mathsf{b}\downarrow$ and the $\mathsf{b}\uparrow$ fork the cycle), and the cycle in (21) has forking number 1. Let us now state the key property of !-?-flow-graphs, that in the end makes the decomposition possible.

3.5.6 Theorem *There is no derivation Δ in* SNEL*, such that $G_{!?}(\Delta)$ contains an unforked cycle.*

We will postpone the proof of this theorem to the next section. Let us now see how this theorem can be used to show that all forked cycles are eventually broken. Thus, Lemma 3.5.4 gives us our desired result.

3.5.7 Lemma *Let Δ be a derivation that contains no instances of $\mathsf{g}\downarrow$, $\mathsf{b}\downarrow$, $\mathsf{w}\downarrow$, and let Δ' be the outcome of applying Lemma 3.5.2 to Δ. If Δ' contains a cycle c with $\mathsf{fk}(c)=n$ for some $n>0$, then it also contains a cycle c' with $\mathsf{fk}(c')=n-1$.*

Proof: The cycle c is forked by n instances of $\mathsf{b}\downarrow$ that have all been introduced by the transformation shown in (18). Now consider the topmost $\mathsf{b}\downarrow$ that forks c. The introduction of this $\mathsf{b}\downarrow$ causes a duplication of all up-paths and down-paths through T (we are still referring to (18)). Furthermore, the continued up-permutation of the $\mathsf{b}\uparrow$ (that caused the introduction of the $\mathsf{b}\downarrow$) causes a duplication of all flipping edges connecting up-paths and down-paths through T (see cases (iii.c.2) and (iii.c.3) in 3.5.1). Therefore, for every path starting or ending inside the right-hand side copy of T in the contractum of the $\mathsf{b}\downarrow$, we have a path starting or ending at the same place inside the left-hand side copy of T. Hence, from c, we can construct another cycle c', which does not use the right-hand side copy of T, as it is visualized in Figure 10. Thus, the $\mathsf{b}\downarrow$ does not fork c'. Hence $\mathsf{fk}(c')=n-1$. □

$$
\begin{array}{rc}
 & !![!a \parr a] \\ \cline{2-2}
\mathsf{b}\!\uparrow & (!![!a \parr a] \otimes ![!a \parr a]) \\ \cline{2-2}
\mathsf{b}\!\uparrow & (!![!a \parr a] \otimes ![!a \parr a] \otimes ![!a \parr a]) \\ \cline{2-2}
\mathsf{b}\!\uparrow & (!![!a \parr a] \otimes ![!a \parr a] \otimes ![(!a \otimes a) \parr a]) \\ \cline{2-2}
\mathsf{b}\!\uparrow & (!![!a \parr a] \otimes ![!a \parr a] \otimes ![(!a \otimes a) \parr a] \otimes [(!a \otimes a) \parr a]) \\ \cline{2-2}
\mathsf{b}\!\uparrow & (!![!a \parr a] \otimes ![!a \parr a] \otimes ![!a \parr a] \otimes ![(!a \otimes a) \parr a] \otimes [(!a \otimes a) \parr a]) \\ \cline{2-2}
\mathsf{p}\!\downarrow & (![?!a \parr !a] \otimes ![!a \parr a] \otimes ![!a \parr a] \otimes ![(!a \otimes a) \parr a] \otimes [(!a \otimes a) \parr a]) \\ \cline{2-2}
\mathsf{p}\!\downarrow & (![?!a \parr !a] \otimes [?!a \parr !a] \otimes ![!a \parr a] \otimes ![(!a \otimes a) \parr a] \otimes [(!a \otimes a) \parr a]) \\ \cline{2-2}
\mathsf{p}\!\downarrow & (![?!a \parr !a] \otimes [?!a \parr !a] \otimes [?!a \parr !a] \otimes ![(!a \otimes a) \parr a] \otimes [(!a \otimes a) \parr a]) \\ \cline{2-2}
\mathsf{p}\!\downarrow & (![?!a \parr !a] \otimes [?!a \parr !a] \otimes [?!a \parr !a] \otimes [?(!a \otimes a) \parr !a] \otimes [(!a \otimes a) \parr a]) \\ \cline{2-2}
\mathsf{s} & (![?!a \parr !a] \otimes [?!a \parr !a] \otimes [?!a \parr !a] \otimes [?(!a \otimes a) \parr (!a \otimes [(!a \otimes a) \parr a])]) \\ \cline{2-2}
\mathsf{s} & (![?!a \parr !a] \otimes [?!a \parr !a] \otimes [?!a \parr !a] \otimes [?(!a \otimes a) \parr (!a \otimes a) \parr (!a \otimes a)]) \\ \cline{2-2}
\mathsf{s} & (![?!a \parr !a] \otimes [?!a \parr !a] \otimes [?(!a \otimes a) \parr (!a \otimes a) \parr ([?!a \parr !a] \otimes !a \otimes a)]) \\ \cline{2-2}
\mathsf{s} & [(![?!a \parr !a] \otimes [?!a \parr !a] \otimes [?(!a \otimes a) \parr (!a \otimes a)]) \parr ([?!a \parr !a] \otimes !a \otimes a)] \\ \cline{2-2}
\mathsf{s} & [(![?!a \parr !a] \otimes [?(!a \otimes a) \parr ([?!a \parr !a] \otimes !a \otimes a)]) \parr ([?!a \parr !a] \otimes !a \otimes a)] \\ \cline{2-2}
\mathsf{s} & [(![?!a \parr !a] \otimes ?(!a \otimes a)) \parr ([?!a \parr !a] \otimes !a \otimes a) \parr ([?!a \parr !a] \otimes !a \otimes a)] \\ \cline{2-2}
\mathsf{p}\!\uparrow & [?([?!a \parr !a] \otimes !a \otimes a) \parr ([?!a \parr !a] \otimes !a \otimes a) \parr ([?!a \parr !a] \otimes !a \otimes a)] \\ \cline{2-2}
\mathsf{s} & [?[(([(?!a \otimes !a) \parr !a] \otimes a)] \parr ([?!a \parr !a] \otimes !a \otimes a) \parr ([?!a \parr !a] \otimes !a \otimes a)] \\ \cline{2-2}
\mathsf{s} & [?[(?!a \otimes !a) \parr (!a \otimes a)] \parr ([?!a \parr !a] \otimes !a \otimes a) \parr ([?!a \parr !a] \otimes !a \otimes a)] \\ \cline{2-2}
\mathsf{s} & [?[(?!a \otimes !a) \parr (!a \otimes a)] \parr ([(?!a \otimes !a) \parr !a] \otimes a) \parr ([?!a \parr !a] \otimes !a \otimes a)] \\ \cline{2-2}
\mathsf{s} & [?[(?!a \otimes !a) \parr (!a \otimes a)] \parr (?!a \otimes !a) \parr (!a \otimes a) \parr ([?!a \parr !a] \otimes !a \otimes a)] \\ \cline{2-2}
\mathsf{s} & [?[(?!a \otimes !a) \parr (!a \otimes a)] \parr (?!a \otimes !a) \parr (!a \otimes a) \parr ([(?!a \otimes !a) \parr !a] \otimes a)] \\ \cline{2-2}
\mathsf{s} & [?[(?!a \otimes !a) \parr (!a \otimes a)] \parr (?!a \otimes !a) \parr (!a \otimes a) \parr (?!a \otimes !a) \parr (!a \otimes a)] \\ \cline{2-2}
\mathsf{p}\!\uparrow & [?[?(!a \otimes a) \parr (!a \otimes a)] \parr (?!a \otimes !a) \parr (!a \otimes a) \parr (?!a \otimes !a) \parr (!a \otimes a)] \\ \cline{2-2}
\mathsf{p}\!\uparrow & [?[?(!a \otimes a) \parr (!a \otimes a)] \parr ?(!a \otimes a) \parr (!a \otimes a) \parr (?!a \otimes !a) \parr (!a \otimes a)] \\ \cline{2-2}
\mathsf{p}\!\uparrow & [?[?(!a \otimes a) \parr (!a \otimes a)] \parr ?(!a \otimes a) \parr (!a \otimes a) \parr ?(!a \otimes a) \parr (!a \otimes a)] \\ \cline{2-2}
\mathsf{b}\!\downarrow & [??(!a \otimes a) \parr ?(!a \otimes a) \parr (!a \otimes a) \parr ?(!a \otimes a) \parr (!a \otimes a)] \\ \cline{2-2}
\mathsf{b}\!\downarrow & [??(!a \otimes a) \parr ?(!a \otimes a) \parr ?(!a \otimes a) \parr (!a \otimes a)] \\ \cline{2-2}
\mathsf{b}\!\downarrow & [??(!a \otimes a) \parr ?(!a \otimes a) \parr ?(!a \otimes a)] \\ \cline{2-2}
\mathsf{b}\!\downarrow & [??(!a \otimes a) \parr ?(!a \otimes a)] \\ \cline{2-2}
\mathsf{b}\!\downarrow & ??(!a \otimes a)
\end{array}
$$

Figure 9: Result of applying the decomposition to the derivation in (10)

3.5.8 Lemma *Let Δ be a derivation in* SNEL'*, and let Δ' be the result of applying two permutation steps to Δ (i.e., first permute all* $\mathsf{g}\!\downarrow$, $\mathsf{b}\!\downarrow$, $\mathsf{w}\!\downarrow$ *down, and second permute all* $\mathsf{g}\!\uparrow$, $\mathsf{b}\!\uparrow$, $\mathsf{w}\!\uparrow$ *up). Then $G_{!?}(\Delta')$ is acyclic.*

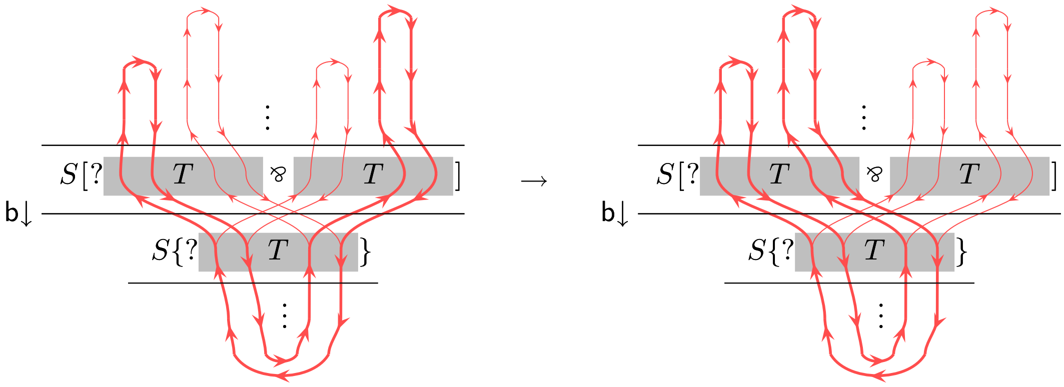


Figure 10: The basic idea of the proof of Lemma 3.5.7

Proof: This follows immediately by way of contradiction from Theorem 3.5.6 and Lemma 3.5.7 by an induction on the forking number of the cycle. $\square$

Now Step 2 of the decomposition theorems (see Figure 5) is obtained via Lemmas 3.5.8 and 3.5.4. It remains to show that unforked cycles cannot exist, which is the purpose of the next section.

3.6 Switch and Seq

The deep reason for the impossibility of unforked cycles in a !-?-flow-graph has nothing to do with the modalities ! and ?, but is caused by a fundamental property of derivations in the system $\{\mathsf{s}, \mathsf{q}\downarrow, \mathsf{q}\uparrow\}$. This property is stated in the following lemma (a similar result has already been shown by Retoré [Ret99]):

3.6.1 Lemma *Let $n > 0$ and let $a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}$ be $2n$ different atoms. Further, let $W_0, \ldots, W_{n-1}, Z_0, \ldots, Z_{n-1}$ be structures, such that*

- *$W_i = [a_i \parr b_i]$ or $W_i = \langle a_i \triangleleft b_i \rangle$, for every $i = 0, \ldots, n-1$,*
- *$Z_j = (b_j \otimes a_{j+1})$ or $Z_j = \langle b_j \triangleleft a_{j+1} \rangle$, for every $j = 0, \ldots, n-1$ (where the indices are counted modulo n).*

Then there is no derivation

$$\begin{array}{c} (W_0 \otimes W_1 \otimes \ldots \otimes W_{n-1}) \\ \{\mathsf{s}, \mathsf{q}\downarrow, \mathsf{q}\uparrow\} \Big\| \tilde{\Delta} \\ [Z_0 \parr Z_1 \parr \ldots \parr Z_{n-1}] \end{array} \qquad (22)$$

Before giving the proof of it, let us state and prove the second lemma of this section, which says that an unforked cycle in the !-?-flow-graph of a derivation Δ

can be transformed into a derivation $\tilde{\Delta}$ as shown in (22) above. The basic idea is to remove from Δ everything that does not belong to the cycle, and then construct $\tilde{\Delta}$ such that the !-?-flow-graph of Δ becomes the atomic flow-graph of $\tilde{\Delta}$.

To make this technically precise, note that in every cycle c in a !-?-flow-graph, the following numbers are all equal:

- the number of maximal !-up-paths in c,
- the number of maximal ?-down-paths in c,
- the number of flipping edges in c from a !-vertex to a ?-vertex, and
- the number of flipping edges in c from a ?-vertex to a !-vertex.

We call this number the *characteristic number* of c. For example, the cycle in the derivation in (10) has characteristic number 1, and the one in (21) has characteristic number 2.

3.6.2 Lemma *Let Δ be a derivation in SNEL' such that $G_{!?}(\Delta)$ contains an unforked cycle c. Then there is a derivation*

$$\{\mathsf{s},\mathsf{q}\downarrow,\mathsf{q}\uparrow\}\,\begin{array}{c}([a_0 \parr b_0] \otimes [a_1 \parr b_1] \otimes \ldots \otimes [a_{n-2} \parr b_{n-2}] \otimes [a_{n-1} \parr b_{n-1}])\\ \Big\Vert\,\tilde{\Delta}\\ [(b_0 \otimes a_1) \parr (b_1 \otimes a_2) \parr \ldots \parr (b_{n-2} \otimes a_{n-1}) \parr (b_{n-1} \otimes a_0)]\end{array} \tag{23}$$

for some atoms $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}$, where $n > 0$ is the characteristic number of c.

Proof: First, we transform Δ into a derivation Δ' which contains only rules from $\mathsf{SNEL}' \setminus \{\mathsf{g}\downarrow, \mathsf{b}\downarrow, \mathsf{w}\downarrow, \mathsf{w}\uparrow\}$ and in which the cycle is preserved. This is done by moving the rules $\mathsf{g}\downarrow$, $\mathsf{b}\downarrow$, and $\mathsf{w}\downarrow$ down in the derivation by applying Lemma 3.5.3, and by moving all instances of $\mathsf{w}\uparrow$ also down in derivation (by applying the dual of Lemma 3.1.4, together with (7)):

$$\mathsf{SNEL}'\,\begin{array}{c}P\\ \Big\Vert\,\Delta\\ Q\end{array} \quad\leadsto\quad \begin{array}{r}\mathsf{SNEL}' \setminus \{\mathsf{g}\downarrow, \mathsf{b}\downarrow, \mathsf{w}\downarrow, \mathsf{w}\uparrow\}\\ \\ \{\mathsf{g}\downarrow, \mathsf{b}\downarrow, \mathsf{w}\downarrow, \mathsf{w}\uparrow\}\end{array}\begin{array}{c}P\\ \Big\Vert\,\Delta'\\ Q'\\ \Big\Vert\\ Q\end{array} \quad .$$

Since c is unforked, no transformation step destroys the cycle, which is therefore still present in $G_{!?}(\Delta')$.

We continue the proof by marking some structures occurring in Δ'. We start by marking all !- and ?-vertices of c by $!^{\bullet}$ and $?^{\bullet}$, respectively. Since c is unforked, it cannot happen that a $!^{\bullet}$- or $?^{\bullet}$-structure occurs inside another $!^{\bullet}$- or $?^{\bullet}$-structure (as it would be the case in the example in (10)). Now we replace every $!^{\bullet}$ by $!^{\bullet}_i$ and every $?^{\bullet}$ by $?^{\bullet}_j$ for some $i, j \in \{0, \ldots, n-1\}$, such that

- two $!^{\bullet}$-vertices in the same up-path get the same index, and two $?^{\bullet}$ in the same down-path get the same index, and
- every flipping edge in c goes from a $!^{\bullet}_i$ to a $?^{\bullet}_i$ vertex, or from a $?^{\bullet}_i$ to a $!^{\bullet}_{i+1}$ vertex, where the addition is modulo n.

Note that at every flipping edge from a $!^{\bullet}_i$ to a $?^{\bullet}_i$ vertex there is another edge in $G_{!?}(\Delta)$ also starting at $!^{\bullet}_i$, which continues the up-path marked by $!^{\bullet}_i$ up to the top of the derivation, We mark all !-vertices on this path by $!^{\blacktriangle}_i$. Since there are no instances of $\mathsf{b}{\downarrow}$ left in Δ', the $!^{\blacktriangle}_i$ up-path is never forked, and since there are no $\mathsf{e}{\downarrow}$ and no $\mathsf{w}{\downarrow}$ in Δ', this path does not end before the top of the derivation. Hence, the premise P of Δ' contains exactly n substructures, marked by $!^{\blacktriangle}_0$, $!^{\blacktriangle}_1$, $\ldots$, $!^{\blacktriangle}_{n-1}$. Let us call them $!^{\blacktriangle}_0 W_0$, $!^{\blacktriangle}_1 W_1$, $\ldots$, $!^{\blacktriangle}_{n-1} W_{n-1}$. We also have n instances of $\mathsf{p}{\downarrow}$ in Δ, marked as follows:

$$\mathsf{p}{\downarrow}\,\frac{S\{!^{\blacktriangle}_i[R \parr T]\}}{S[!^{\bullet}_i R \parr ?^{\bullet}_i T]} \tag{24}$$

Now we proceed similarly and mark the continuations of the $?^{\bullet}_i$-down-paths by $?^{\blacktriangledown}_i$, i.e., we obtain n instances of $\mathsf{p}{\uparrow}$ marked as

$$\mathsf{p}{\uparrow}\,\frac{S(?^{\bullet}_i T \otimes !^{\bullet}_{i+1} R)}{S\{?^{\blacktriangledown}_i (T \otimes R)\}} \tag{25}$$

However, note that now it can happen that we meet during the marking process a proper forking vertex, due to the presence of $\mathsf{b}{\uparrow}$:

$$\mathsf{b}{\uparrow}\,\frac{S\{!V\{?^{\blacktriangledown}_i T\}\}}{S(!V\{?T\} \otimes V\{?T\})} \quad .$$

then we continue the marking in only one side, namely, into that copy of $V\{?T\}$ in the redex of $\mathsf{b}{\uparrow}$, that contains already a $!^{\bullet}$-, $?^{\bullet}$-, $!^{\blacktriangle}$-, or $?^{\blacktriangledown}$-marking. Note that it cannot happen that both copies of $V\{?T\}$ contain such a marking because the cycle is unforked. If neither side contains a marking, we arbitrarily pick one side. Since there are no $\mathsf{e}{\uparrow}$ and $\mathsf{w}{\uparrow}$ in Δ', the $?^{\blacktriangledown}$-paths cannot end in the middle of the derivation. Hence, the conclusion Q' of Δ' contains exactly n different marked $?^{\blacktriangledown}$-structures, that we denote by $?^{\blacktriangledown}_0 Z_0$, $?^{\blacktriangledown}_1 Z_1$, $\ldots$, $?^{\blacktriangledown}_{n-1} Z_{n-1}$. Now we remove in Δ' every modality that is not marked, and we replace every atom that is not inside a marked structure by the unit $\circ$. The important point is that after this rather drastic change we still have a correct derivation. Every rule instance in Δ' remains valid, or becomes vacuous, i.e., premise and conclusion are identical. Note that here we make crucial use of the fact that the cycle is unforked: Doing this deletion to a $\mathsf{b}{\uparrow}$ which forks c would yield an incorrect inference step.

Let us call the new derivation Δ''. Its premise P'' is made from the structures $!^{\blacktriangle}_0 W_0$, $!^{\blacktriangle}_1 W_1$, $\ldots$, $!^{\blacktriangle}_{n-1} W_{n-1}$ by using only the binary connectives $\otimes$, $\lhd$, and $\parr$, and its conclusion Q'' s made from $?^{\blacktriangledown}_0 Z_0$, $?^{\blacktriangledown}_1 Z_1$, $\ldots$, $?^{\blacktriangledown}_{n-1} Z_{n-1}$ by using only $\otimes$, $\lhd$, and $\parr$. Now note that for arbitrary structures A and B, we have the following three derivations:

$$
\begin{array}{rc}
 & (A \otimes B) \\
= & \overline{(\langle A \lhd \circ\rangle \otimes \langle \circ \lhd B\rangle)} \\
\mathsf{q}\!\uparrow & \overline{\langle (A \otimes \circ) \lhd (\circ \otimes B)\rangle} \\
= & \overline{\langle A \lhd B\rangle}
\end{array}
\quad \text{and} \quad
\begin{array}{rc}
 & (A \otimes B) \\
= & \overline{(A \otimes [\circ \parr B])} \\
\mathsf{s} & \overline{[(A \otimes \circ) \parr B]} \\
= & \overline{[A \parr B]}
\end{array}
\quad \text{and} \quad
\begin{array}{rc}
 & \langle A \lhd B\rangle \\
= & \overline{\langle [A \parr \circ] \lhd [\circ \parr B]\rangle} \\
\mathsf{q}\!\downarrow & \overline{[\langle A \lhd \circ\rangle \parr \langle \circ \lhd B\rangle]} \\
= & \overline{[A \parr B]}
\end{array}
$$

Hence, we can extend Δ'' as follows:

$$
\begin{array}{c}
(!^{\blacktriangle}_0 W_0 \otimes !^{\blacktriangle}_1 W_1 \otimes \cdots \otimes !^{\blacktriangle}_{n-1} W_{n-1}) \\
\{\mathsf{q}\!\uparrow, \mathsf{s}\} \Big\Vert \\
P'' \\
\Big\Vert\, \Delta'' \\
Q'' \\
\{\mathsf{q}\!\downarrow, \mathsf{s}\} \Big\Vert \\
[?^{\blacktriangledown}_0 Z_0 \parr ?^{\blacktriangledown}_1 Z_1 \parr \cdots \parr ?^{\blacktriangledown}_{n-1} Z_{n-1}]
\end{array}
\tag{26}
$$

Let us use Δ''' to denote the derivation in (26). We finally obtain $\tilde{\Delta}$ from Δ''' by replacing every $!^{\bullet}_i$-structure by a_i, every $?^{\bullet}_i$-structure by b_i, every $!^{\blacktriangle}_i$-structure by $[a_i \parr b_i]$, and every $?^{\blacktriangledown}_i$-structure by (b_i, a_{i+1}). Clearly, every inference rule remains valid, or becomes vacuous, as for example the instances of $\mathsf{p}\!\downarrow$ in (24) and $\mathsf{p}\!\uparrow$ in (25):

$$
\mathsf{p}\!\downarrow \frac{S\{!^{\blacktriangle}_i [R \parr T]\}}{S[!^{\bullet}_i R \parr ?^{\bullet}_i T]} \qquad \to \qquad = \frac{S[a_i \parr b_i]}{S[a_i \parr b_i]}
$$

and

$$
\mathsf{p}\!\uparrow \frac{S(?^{\bullet}_i T \otimes !^{\bullet}_{i+1} R)}{S\{?^{\blacktriangledown}_i (T \otimes R)\}} \qquad \to \qquad = \frac{S(b_i \otimes a_{i+1})}{S(b_i \otimes a_{i+1})}
$$

If a rule does not become vacuous, it must be one of s, $\mathsf{q}\!\downarrow$, and $\mathsf{q}\!\uparrow$. ☐

Proof of Lemma 3.6.1: The proof is carried out by induction on the pair $\langle n, q\rangle$, where q is the number of seq-structures in the conclusion, and we endorse the lexicographic ordering on $\mathbb{N} \times \mathbb{N}$. The base case (i.e., $n = 1$) is trivial. For the inductive case we assume by way of contradiction the existence of the derivation $\tilde{\Delta}$ in (22) and consider the bottommost rule instance ρ. There are three cases.

(i) $\rho = \mathsf{q}\!\uparrow$. There is only one possibility to apply this rule:

$$
\begin{array}{c}
(W_0 \otimes W_1 \otimes \ldots \otimes W_{n-1}) \\
\{\mathsf{s}, \mathsf{q}\!\downarrow, \mathsf{q}\!\uparrow\} \Big\Vert\, \Delta' \\
\mathsf{q}\!\uparrow \dfrac{[Z_0 \parr \cdots \parr Z_{j-1} \parr (b_j \otimes a_{j+1}) \parr Z_{j+1} \parr \cdots \parr Z_{n-1}]}{[Z_0 \parr \cdots \parr Z_{j-1} \parr \langle b_j \lhd a_{j+1}\rangle \parr Z_{j+1} \parr \cdots \parr Z_{n-1}]}
\end{array}
$$

We can apply the induction hypothesis to Δ' because the number n did not change and the number q of seq-structures in the conlusion did decrease by 1. Hence we get a contradiction.

(ii) $\rho = \mathsf{q}{\downarrow}$. There are several possibilities to apply this rule. We show here only two representative cases and leave the others to the reader because they are very similar. The complete case analysis can be found in [Str03a].

(a) If we have

$$\mathsf{q}{\downarrow}\,\frac{\begin{array}{c}(W_0 \otimes W_1 \otimes \ldots \otimes W_{n-1})\\ \{\mathsf{s},\mathsf{q}{\downarrow},\mathsf{q}{\uparrow}\}\,\Big\Vert\,\Delta'\\ [\langle [b_0 \parr b_i] \lhd [a_1 \parr a_{i+1}]\rangle \parr Z_1 \parr \cdots \parr Z_{i-1} \parr Z_{i+1} \parr \cdots \parr Z_{n-1}]\end{array}}{[\langle b_0 \lhd a_1\rangle \parr Z_1 \parr \cdots \parr Z_{i-1} \parr \langle b_i \lhd a_{i+1}\rangle \parr Z_{i+1} \parr \cdots \parr Z_{n-1}]}$$

then Δ' remains valid if we replace a_m and b_m by $\circ$ for every $m > i$ and for $m = 0$. This gives us the derivation

$$\begin{array}{c}(W_1 \otimes \cdots \otimes W_i)\\ \{\mathsf{s},\mathsf{q}{\downarrow},\mathsf{q}{\uparrow}\}\,\Big\Vert\,\Delta''\\ [\langle b_i \lhd a_1\rangle \parr Z_1 \parr \cdots \parr Z_{i-1}]\end{array}$$

which is a contradiction to the induction hypothesis because $i < n$.

(b) Consider

$$\mathsf{q}{\downarrow}\,\frac{\begin{array}{c}(W_0 \otimes W_1 \otimes \ldots \otimes W_{n-1})\\ \{\mathsf{s},\mathsf{q}{\downarrow},\mathsf{q}{\uparrow}\}\,\Big\Vert\,\Delta'\\ [\langle b_0 \lhd [a_1 \parr Z_{k_1} \parr \cdots \parr Z_{k_v}]\rangle \parr Z_{h_1} \parr \cdots \parr Z_{h_s}]\end{array}}{[\langle b_0 \lhd a_1\rangle \parr Z_1 \parr \cdots \parr Z_{n-1}]}$$

where $\{1,\ldots,n-1\} \setminus \{k_1,\ldots,k_v\} = \{h_1,\ldots,h_s\}$ and $s = n - v - 1$ and (without loss of generality) $k_1 < k_2 < \ldots < k_v$. As before, the derivation Δ' remains valid if we replace a_m and b_m by $\circ$ for every m with $1 \le m \le k_v$. Then we get

$$\begin{array}{c}(W_0 \otimes W_{k_v+1} \otimes \cdots \otimes W_{n-1})\\ \{\mathsf{s},\mathsf{q}{\downarrow},\mathsf{q}{\uparrow}\}\,\Big\Vert\,\Delta''\\ [\langle b_0 \lhd a_{k_v+1}\rangle \parr Z_{k_v+1} \parr \cdots \parr Z_{n-1}]\end{array}$$

which is (as before) a contradiction to the induction hypothesis because $v \ge 1$.

(iii) $\rho = \mathsf{s}$. This is similar to the case for $\mathsf{q}{\downarrow}$. But note that a situation like in (ii.a) cannot happen for s. □

Proof of Theorem 3.5.6: The existence of an unforked cycle in $G_{!?}(\Delta)$ implies by Lemma 3.6.2 the existence of a derivation as in (23). By Lemma 3.6.1, this is impossible. □

4 Cut Elimination

The classical arguments for proving cut elimination in the sequent calculus rely on the following property: when the principal formulae in a cut are active in both branches, they determine which rules are applied immediately above the cut. This is a consequence of the fact that formulae have a root connective, and logical rules only hinge on that, and nowhere else in the formula.

This property does not necessarily hold in the calculus of structures. Further, since rules can be applied anywhere deep inside structures, everything can happen above a cut. This complicates considerably the task of proving cut elimination. On the other hand, a great simplification is made possible in the calculus of structures by the reduction of cut to its atomic form, which happens simply and independently of cut elimination. The remaining difficulty is actually understanding what happens, while going up in a proof, *around* the atoms produced by an atomic cut. The two atoms of an atomic cut can be produced inside any structure, and they do not belong to distinct branches, as in the sequent calculus: complex interactions with their context are possible. As a consequence, our techniques are largely different than the traditional ones.

Two approaches to cut elimination in the calculus of structures have been explored in previous papers: in [GS01, Str03b] we relied on permutations of rules, in [BT01] Brünnler and Tiu relied on semantics, and in [Brü03a] Brünnler presents a simple syntactic method that employs the atomicity of cut together with certain proof theoretical properties of classical logic. In this paper we use a third technique, called *splitting* [Gug07], which has the advantage of being more uniform than the one based on permutations and which yields a much simpler case analysis. It also establishes a deep connection to the sequent calculus, at least for the fragments of systems that allow for a sequent calculus presentation (in this case, the commutative fragment). Since many systems are expressed in the sequent calculus, our method appears to be entirely general; still it is independent of the sequent calculus and of a complete semantics.

Splitting can be best understood by considering a sequent system with no weakening and absorption (or contraction). Consider for example multiplicative linear logic: If we have a proof of the sequent $\vdash F\{A \otimes B\}, \Gamma$, where $F\{A \otimes B\}$ is a formula that contains the subformula $A \otimes B$, we know for sure that somewhere in the proof there is one and only one instance of the $\otimes$ rule, which splits A and B along with their context. This is indicated in Figure 11. We can consider, as shown at the left, the proof for the given sequent as composed of three pieces, Δ, Π_1 and Π_2. In the calculus of structures, many different proofs correspond to the sequent calculus one: they differ for the possible sequencing of rules, and because rules in the calculus of structures have smaller granularity and larger applicability. But, among all these proofs, there must also be one that fits the scheme at the right of Figure 11. This precisely illustrates the idea behind the splitting technique.

The derivation Δ in Figure 11 implements a *context reduction* and a proper splitting. We can state, in general, these principles as follows:

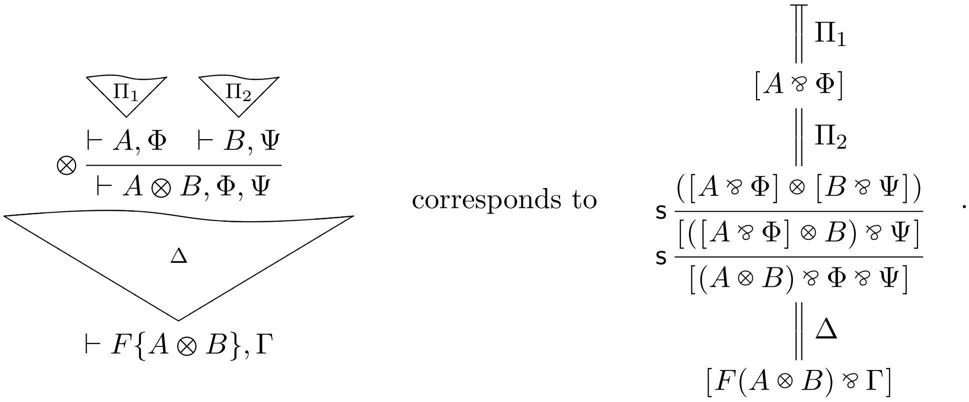


Figure 11: The basic idea behind splitting and context reduction

1. Context reduction: If $S\{R\}$ is provable, then $S\{\ \}$ can be reduced to the structure $[\{\ \} \parr U]$, such that $[R \parr U]$ is provable. In the example above, $[F\{\ \} \parr \Gamma]$ is reduced to $[\{\ \} \parr \Gamma']$, for some Γ'.

2. Splitting: If $[(R \otimes T) \parr P]$ is provable, then P can be reduced to $[P_1 \parr P_2]$, such that $[R \parr P_1]$ and $[T \parr P_2]$ are provable. In the example above Γ' is reduced to $[\Phi \parr \Psi]$. Splitting holds for all logical operators.

Context reduction is in turn proved by splitting, which is then at the core of the matter. The biggest difficulty resides in proving splitting, and this mainly requires finding the right induction measure.

4.1 Splitting

In this section we will state and prove splitting, as we will need it for cut elimination. For notational convenience, we define *system* NELc to be the system obtained from NEL by removing the non-core rules:

$$\mathsf{NELc} = \mathsf{NEL} \setminus \{\mathsf{w}\!\downarrow, \mathsf{b}\!\downarrow, \mathsf{g}\!\downarrow\} = \{\circ\!\downarrow, \mathsf{ai}\!\downarrow, \mathsf{s}, \mathsf{q}\!\downarrow, \mathsf{p}\!\downarrow, \mathsf{e}\!\downarrow\} = \mathsf{SNELc}\!\downarrow \cup \{\circ\!\downarrow\} \quad .$$

4.1.1 Lemma (Splitting) *Let R, T, P be any NEL structures.*

(i) *If $[(R \otimes T) \parr P]$ is provable in NELc, then there are structures P_R and P_T, such that*

$$\begin{array}{c} [P_R \parr P_T] \\ \mathsf{NELc}\,\Big\| \\ P \end{array} \qquad \textit{and} \qquad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\|} \\ [R \parr P_R] \end{array} \qquad \textit{and} \qquad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\|} \\ [T \parr P_T] \end{array} \quad .$$

(ii) *If $[\langle R \lhd T \rangle \parr P]$ is provable in NELc, then there are structures P_R and P_T, such* *that*

$$\begin{array}{c}\langle P_R \lhd P_T \rangle\\ \Big\Vert \mathsf{NELc}\\ P\end{array} \quad \textit{and} \quad \mathsf{NELc}\,\Big\Vert\!\!\!\overline{}\; [R \parr P_R] \quad \textit{and} \quad \mathsf{NELc}\,\Big\Vert\!\!\!\overline{}\; [T \parr P_T] \quad .$$

Proof: We prove both statements simultaneously by structural induction on the number of atoms in the conclusion and the length (number of rule instances) of the proof, ordered lexicographically. Without loss of generality, assume $R \neq \circ \neq T$ (otherwise both statements are trivially true).

(i) Consider the bottommost rule instance ρ in the proof of $[(R \otimes T) \parr P]$. We can distinguish between three different kinds of cases:

(a) The first kind appears when the redex of ρ is inside R, T or P. Then we have the following situation:

$$\begin{array}{c}\mathsf{NELc}\,\Big\Vert\!\!\!\overline{}\;\Pi\\ \rho\,\dfrac{[(R' \otimes T) \parr P]}{[(R \otimes T) \parr P]}\end{array}$$

where we can apply the induction hypothesis to Π because it is one rule shorter (if $\rho = \mathsf{ai}{\downarrow}$ also the conclusion is smaller). We get

$$\begin{array}{c}[P_{R'} \parr P_T]\\ \mathsf{NELc}\,\Big\Vert\,\Delta_P\\ P\end{array} \quad \text{and} \quad \mathsf{NELc}\,\Big\Vert\!\!\!\overline{}\;\Pi_R \; [R' \parr P_R] \quad \text{and} \quad \mathsf{NELc}\,\Big\Vert\!\!\!\overline{}\;\Pi_T \; [T \parr P_T]$$

From Π_R, we can get

$$\begin{array}{c}\mathsf{NELc}\,\Big\Vert\!\!\!\overline{}\;\Pi'_R\\ \rho\,\dfrac{[R' \parr P_R]}{[R \parr P_R]}\end{array}$$

and we are done. If the redex of ρ is inside T or P, the situation is similar.

(b) In the second kind of case the substructure $(R \otimes T)$ is inside the redex of ρ, but is not modified by ρ. These cases can be compared with the "commutative cases" in the usual sequent calculus cut elimination argument. We show only one representative example (a complete case analysis can be found in [Gug07] and [Str03a]): Suppose we have

$$\begin{array}{c}\mathsf{NELc}\,\Big\Vert\!\!\!\overline{}\;\Pi\\ \mathsf{q}{\downarrow}\,\dfrac{[\langle [(R \otimes T) \parr P_1 \parr P_3] \lhd P_2 \rangle \parr P_4]}{[(R \otimes T) \parr \langle P_1 \lhd P_2 \rangle \parr P_3 \parr P_4]}\end{array}$$

We can apply the induction hypothesis to Π because it is one rule shorter (the size of the conclusion does not change). This gives us

$$\begin{array}{c}\langle Q_1 \lhd Q_2 \rangle \\ \mathsf{NELc} \Big\Vert \Delta_1 \\ P_4\end{array} \quad \text{and} \quad \begin{array}{c}\mathsf{NELc} \Big\Vert \Pi_1 \\ [(R \otimes T) \mathbin{⅋} P_1 \mathbin{⅋} P_3 \mathbin{⅋} Q_1]\end{array} \quad \text{and} \quad \begin{array}{c}\mathsf{NELc} \Big\Vert \Pi_2 \\ [P_2 \mathbin{⅋} Q_2]\end{array}$$

We can apply the induction hypothesis again to Π_1, because now the the number of atoms in the conclusion is strictly smaller (because we can assume that the instance of $\mathsf{q}\downarrow$ is not trivial). We get

$$\begin{array}{c}[P_R \mathbin{⅋} P_T] \\ \mathsf{NELc} \Big\Vert \Delta_2 \\ [P_1 \mathbin{⅋} P_3 \mathbin{⅋} Q_1]\end{array} \quad \text{and} \quad \begin{array}{c}\mathsf{NELc} \Big\Vert \Pi_R \\ [R \mathbin{⅋} P_R]\end{array} \quad \text{and} \quad \begin{array}{c}\mathsf{NELc} \Big\Vert \Pi_T \\ [T \mathbin{⅋} P_T]\end{array}$$

From Δ_1, Δ_2 and Π_2 we can build the following derivation

$$\begin{array}{c}[P_R \mathbin{⅋} P_T] \\ \mathsf{NELc} \Big\Vert \Delta_2 \\ [P_1 \mathbin{⅋} P_3 \mathbin{⅋} Q_1] \\ \mathsf{NELc} \Big\Vert \Pi_2 \\ \mathsf{q}\downarrow \dfrac{[\langle [P_1 \mathbin{⅋} Q_1] \lhd [P_2 \mathbin{⅋} Q_2] \rangle \mathbin{⅋} P_3]}{[\langle P_1 \lhd P_2 \rangle \mathbin{⅋} P_3 \mathbin{⅋} \langle Q_1 \lhd Q_2 \rangle]} \\ \mathsf{NELc} \Big\Vert \Delta_1 \\ [\langle P_1 \lhd P_2 \rangle \mathbin{⅋} P_3 \mathbin{⅋} P_4]\end{array}$$

and we are done. All other cases in this group are similar.

(c) In the last type of case the substructure $(R \otimes T)$ is destroyed by ρ. These cases can be compared to the "key cases" in a standard sequent calculus cut elimination argument. We have only one possibility. The most general situation is as follows:

$$\begin{array}{c}\mathsf{NELc} \Big\Vert \Pi \\ \mathsf{s} \dfrac{[([(R_1 \otimes T_1) \mathbin{⅋} P_1] \otimes R_2 \otimes T_2) \mathbin{⅋} P_2]}{[(R_1 \otimes R_2 \otimes T_1 \otimes T_2) \mathbin{⅋} P_1 \mathbin{⅋} P_2]}\end{array}$$

where one of R_1 and R_2 might be $\circ$, but not both of them (similarly for T_1 and T_2). As before, we can apply the induction hypothesis to Π and get

$$\begin{array}{c}[Q_1 \mathbin{⅋} Q_2] \\ \mathsf{NELc} \Big\Vert \Delta_1 \\ P_2\end{array} \quad \text{and} \quad \begin{array}{c}\mathsf{NELc} \Big\Vert \Pi_1 \\ [(R_1 \otimes T_1) \mathbin{⅋} P_1 \mathbin{⅋} Q_1]\end{array} \quad \text{and} \quad \begin{array}{c}\mathsf{NELc} \Big\Vert \Pi_2 \\ [(R_2 \otimes T_2) \mathbin{⅋} Q_2]\end{array}$$

We can apply the induction hypothesis again to Π_1 and Π_2. (Because we assume that the instance of s is not trivial, the conclusions are strictly smaller than the one of the original proof.) We get:

$$\begin{array}{c}[P_{R_1} \mathbin{\bindnasrepma} P_{T_1}]\\ \mathsf{NELc} \Big\Vert \Delta_3\\ [P_1 \mathbin{\bindnasrepma} Q_1]\end{array} \quad\text{and}\quad \begin{array}{c}\mathsf{NELc} \Big\Vert \Pi_{R_1}\\ [R_1 \mathbin{\bindnasrepma} P_{R_1}]\end{array} \quad\text{and}\quad \begin{array}{c}\mathsf{NELc} \Big\Vert \Pi_{T_1}\\ [T_1 \mathbin{\bindnasrepma} P_{T_1}]\end{array}$$

and

$$\begin{array}{c}[P_{R_2} \mathbin{\bindnasrepma} P_{T_2}]\\ \mathsf{NELc} \Big\Vert \Delta_4\\ Q_2\end{array} \quad\text{and}\quad \begin{array}{c}\mathsf{NELc} \Big\Vert \Pi_{R_2}\\ [R_2 \mathbin{\bindnasrepma} P_{R_2}]\end{array} \quad\text{and}\quad \begin{array}{c}\mathsf{NELc} \Big\Vert \Pi_{T_2}\\ [T_2 \mathbin{\bindnasrepma} P_{T_2}]\end{array}$$

Now let $P_R = [P_{R_1} \mathbin{\bindnasrepma} P_{R_2}]$ and $P_T = [P_{T_1} \mathbin{\bindnasrepma} P_{T_2}]$. We can build

$$\begin{array}{c}[P_{R_1} \mathbin{\bindnasrepma} P_{R_2} \mathbin{\bindnasrepma} P_{T_1} \mathbin{\bindnasrepma} P_{T_2}]\\ \mathsf{NELc} \Big\Vert \Delta_4\\ [P_{R_1} \mathbin{\bindnasrepma} P_{T_1} \mathbin{\bindnasrepma} Q_2]\\ \mathsf{NELc} \Big\Vert \Delta_3\\ [P_1 \mathbin{\bindnasrepma} Q_1 \mathbin{\bindnasrepma} Q_2]\\ \mathsf{NELc} \Big\Vert \Delta_1\\ [P_1 \mathbin{\bindnasrepma} P_2]\end{array} \quad\text{and}\quad \begin{array}{c}\mathsf{NELc} \Big\Vert \Pi_{R_1}\\ [R_1 \mathbin{\bindnasrepma} P_{R_1}]\\ \mathsf{NELc} \Big\Vert \Pi_{R_2}\\ \mathsf{s}\,\dfrac{[(R_1 \otimes [R_2 \mathbin{\bindnasrepma} P_{R_2}]) \mathbin{\bindnasrepma} P_{R_1}]}{[(R_1 \otimes R_2) \mathbin{\bindnasrepma} P_{R_1} \mathbin{\bindnasrepma} P_{R_2}]}\end{array}$$

and a similar proof of $[(T_1 \otimes T_2) \mathbin{\bindnasrepma} P_{T_1} \mathbin{\bindnasrepma} P_{T_2}]$, and we are done.

(ii) The case for $[\langle R \lhd T\rangle \mathbin{\bindnasrepma} P]$ is similar to the one for $[(R \otimes T) \mathbin{\bindnasrepma} P]$, and we leave it to the reader. □

4.1.2 Lemma (Splitting for Modalities) *Let R and P be any* NEL *structures.*

(i) *If $[!R \mathbin{\bindnasrepma} P]$ is provable in* NELc*, then there are structures $P_1, \dots, P_h$ for some $h \geq 0$, such that*

$$\begin{array}{c}[?P_1 \mathbin{\bindnasrepma} \cdots \mathbin{\bindnasrepma} ?P_h]\\ \mathsf{NELc} \Big\Vert\\ P\end{array} \quad\textit{and}\quad \begin{array}{c}\mathsf{NELc} \Big\Vert\\ [R \mathbin{\bindnasrepma} P_1 \mathbin{\bindnasrepma} \cdots \mathbin{\bindnasrepma} P_h]\end{array} \quad .$$

(ii) *If $[?R \mathbin{\bindnasrepma} P]$ is provable in* NELc*, then there is a structure P_R, such that*

$$\begin{array}{c}!P_R\\ \mathsf{NELc} \Big\Vert\\ P\end{array} \quad\textit{and}\quad \begin{array}{c}\mathsf{NELc} \Big\Vert\\ [R \mathbin{\bindnasrepma} P_R]\end{array} \quad .$$

Proof: The proof is similar to the previous one. We use the same induction measure and the same pattern in the case analysis as before.

(i) We consider again the bottommost rule instance ρ in the proof of $[!R \parr P]$, and we have the same three classes of cases as in the proof of Lemma 4.1.1.

(a) The redex of ρ is inside R or P. This case is the similar as in the proof of Lemma 4.1.1.

(b) The substructure $!R$ is inside the redex of ρ, but is not changed by ρ. This case is almost literally the same as for Lemma 4.1.1. We only have to replace $(R \otimes T)$ by $!R$, and

$$\begin{array}{c}[P_R \parr P_T]\\ \mathsf{NELc}\,\Big\Vert\,\Delta_2\\ [P_1 \parr P_3 \parr Q_1]\end{array} \qquad \text{by} \qquad \begin{array}{c}[?P_1 \parr \cdots \parr ?P_h]\\ \mathsf{NELc}\,\Big\Vert\,\Delta_2\\ [P_1 \parr P_3 \parr Q_1]\end{array}$$

(As for the previous lemma, the full details can be found in [Str03a].)

(c) The substructure $!R$ is destroyed by ρ. There are two possibilities ($\rho = \mathsf{e}\!\downarrow$ and $\rho = \mathsf{p}\!\downarrow$):

$$\begin{array}{c}\mathsf{NELc}\,\overset{-}{\Big\Vert}\\ \mathsf{e}\!\downarrow \dfrac{[\circ \parr P]}{[!\circ \parr P]}\end{array} \qquad \text{and} \qquad \begin{array}{c}\mathsf{NELc}\,\overset{-}{\Big\Vert}\,\Pi\\ \mathsf{p}\!\downarrow \dfrac{[![R \parr P_1] \parr Q_2]}{[!R \parr ?P_1 \parr Q_2]}\end{array}$$

For $\rho = \mathsf{e}\!\downarrow$ we are done immediately by letting $h = 0$. For $\rho = \mathsf{p}\!\downarrow$ we can apply the induction hypothesis to Π and get structures $P_2, \ldots, P_h$ such that

$$\begin{array}{c}[?P_2 \parr \cdots \parr ?P_h]\\ \mathsf{NELc}\,\Big\Vert\\ Q_2\end{array} \qquad \text{and} \qquad \begin{array}{c}\mathsf{NELc}\,\overset{-}{\Big\Vert}\\ [R \parr P_1 \parr P_2 \parr \cdots \parr P_h]\end{array}$$

We immediately get

$$\begin{array}{c}[?P_1 \parr ?P_2 \parr \cdots \parr ?P_h]\\ \mathsf{NELc}\,\Big\Vert\\ [?P_1 \parr Q_2]\end{array} \quad .$$

(ii) As before, consider the bottommost rule instance ρ in the proof of $[?R \parr P]$.

(a) The redex of ρ is inside R or P. This case is the similar as before.

(b) The substructure $?R$ is inside the redex of ρ, but is not changed by ρ. As before, this case is almost literally the same as in the proof of Lemma 4.1.1. This time we have to replace $(R \otimes T)$ by $?R$, and

$$\begin{array}{c}[P_R \parr P_T]\\ \mathsf{NELc}\,\Big\Vert\,\Delta_2\\ [P_1 \parr P_3 \parr Q_1]\end{array} \qquad \text{by} \qquad \begin{array}{c}!P_R\\ \mathsf{NELc}\,\Big\Vert\,\Delta_2\\ [P_1 \parr P_3 \parr Q_1]\end{array}$$

(c) The substructure $?R$ is destroyed by ρ. For this case there is only one possibility:

$$\mathsf{p}{\downarrow}\,\frac{\overset{\mathsf{NELc}\,\big\Vert\,\Pi}{[![R \parr P_1] \parr P_2]}}{[?R \parr !P_1 \parr P_2]}$$

We can apply part (i) of the lemma and get

$$\begin{array}{c}[?Q_1 \parr \cdots \parr ?Q_h]\\ \mathsf{NELc}\,\big\Vert\,\Delta\\ P_2\end{array} \qquad \text{and} \qquad \begin{array}{c}\mathsf{NELc}\,\big\Vert\,\Pi_R\\ [R \parr P_1 \parr Q_1 \parr \cdots \parr Q_h]\end{array}$$

Now let $P_R = [P_1 \parr Q_1 \parr \ldots \parr Q_h]$. We can build

$$\begin{array}{c}![P_1 \parr Q_1 \parr \cdots \parr Q_h]\\ \{\mathsf{p}{\downarrow}\}\,\big\Vert\\ [!P_1 \parr ?Q_1 \parr \cdots \parr ?Q_h]\\ \mathsf{NELc}\,\big\Vert\,\Delta\\ [!P_1 \parr P_2]\end{array}$$

as desired. □

4.1.3 Lemma (Splitting for Atoms) *Let a be any atom and P be any* NEL *structure.*

If there is a proof $\begin{array}{c}\mathsf{NELc}\,\big\Vert\\ [a \parr P]\end{array}$ *then there is a derivation* $\begin{array}{c}\bar{a}\\ \mathsf{NELc}\,\big\Vert\\ P\end{array}$.

Proof: After the previous two proofs this is an almost trivial exercise: The case (a) is as before, and for (b), we have to replace $(R \otimes T)$ by a, and

$$\begin{array}{c}[P_R \parr P_T]\\ \mathsf{NELc}\,\big\Vert\,\Delta_2\\ [P_1 \parr P_3 \parr Q_1]\end{array} \qquad \text{by} \qquad \begin{array}{c}\bar{a}\\ \mathsf{NELc}\,\big\Vert\,\Delta_2\\ [P_1 \parr P_3 \parr Q_1]\end{array} .$$

For case (c), the only possibility is

$$\mathsf{ai}{\downarrow}\,\frac{\overset{\mathsf{NELc}\,\big\Vert\,\Pi'}{P_1}}{[a, \bar{a}, P_1]}$$

from which we immediately get

$$\mathsf{NELc}\,\left\Vert\begin{array}{c}\bar{a}\\ \\ [\bar{a},P_1]\end{array}\right. .$$

as desired. □

4.2 Context Reduction

The idea of context reduction is to reduce a problem that concerns an arbitrary (deep) context $S\{\ \}$ to a problem that concerns only a shallow context $[\{\ \} \mathbin{\bindnasrepma} P]$. In the case of cut elimination, for example, we will then be able to apply splitting.

Before giving the statement, we need to define the *modality depth* of a context $S\{\ \}$ to be the number of ! and ? in whose scope the $\{\ \}$ occurs. In the following lemma, the $\{\ \}$ is treated as ordinary atom.

4.2.1 Lemma (Context Reduction) *Let R be a* NEL *structure and $S\{\ \}$ be a context. If $S\{R\}$ is provable in* NELc*, then there is a structure P_R, such that*

$$\begin{array}{c}!\cdots![\{\ \}\mathbin{\bindnasrepma} P_R]\\ \mathsf{NELc}\,\Vert\,\Delta\\ S\{\ \}\end{array} \qquad \textit{and} \qquad \begin{array}{c}\mathsf{NELc}\,\overline{\Vert}\,\Pi\\ [R\mathbin{\bindnasrepma} P_R]\end{array}$$

where the number of ! *in front of $[\{\ \}\mathbin{\bindnasrepma} P_R]$ is the modality depth of $S\{\ \}$.*

Proof: We proceed by structural induction on the context $S\{\ \}$. The base case when $S\{\ \}=\{\ \}$ is trivial. Now we can distinguish four cases

(a) $S\{\ \} = [(S'\{\ \}\otimes T)\mathbin{\bindnasrepma} P]$ where,without loss of generality, $T\neq\circ$. Note that we do allow $P=\circ$. We can apply splitting (Lemma 4.1.1) to the proof of $[(S'\{R\}\otimes T)\mathbin{\bindnasrepma} P]$ and get:

$$\begin{array}{c}[P_S\mathbin{\bindnasrepma} P_T]\\ \mathsf{NELc}\,\Vert\,\Delta_P\\ P\end{array} \qquad \text{and} \qquad \begin{array}{c}\mathsf{NELc}\,\overline{\Vert}\,\Pi_S\\ [S'\{R\}\mathbin{\bindnasrepma} P_S]\end{array} \qquad \text{and} \qquad \begin{array}{c}\mathsf{NELc}\,\overline{\Vert}\,\Pi_T\\ [T\mathbin{\bindnasrepma} P_T]\end{array}$$

Because $T\neq\circ$ we can now apply the induction hypothesis to Π_S and get:

$$\begin{array}{c}!\cdots![\{\ \}\mathbin{\bindnasrepma} P_R]\\ \mathsf{NELc}\,\Vert\,\Delta'\\ [S'\{\ \}\mathbin{\bindnasrepma} P_S]\end{array} \qquad \text{and} \qquad \begin{array}{c}\mathsf{NELc}\,\overline{\Vert}\,\Pi\\ [R\mathbin{\bindnasrepma} P_R]\end{array}$$

From this we can build

$$\begin{array}{c} !\cdots![\{\ \}\parr P_R] \\ \mathsf{NELc}\,\Big\Vert\,\Delta' \\ [S'\{\ \}\parr P_S] \\ \mathsf{NELc}\,\Big\Vert\,\Pi_T \\ \mathsf{s}\,\dfrac{[(S'\{\ \}\otimes[T\parr P_T])\parr P_S]}{[(S'\{\ \}\otimes T)\parr P_S\parr P_T]} \\ \mathsf{NELc}\,\Big\Vert\,\Delta_P \\ [(S'\{\ \}\otimes T)\parr P] \end{array}$$

as desired.

(b) The cases $S\{\ \} = [\langle S'\{\ \}\lhd T\rangle\parr P]$ and $S\{\ \} = [\langle T\lhd S'\{\ \}\rangle\parr P]$ are handled similarly to (a).

(c) If $S\{\ \} = [!S'\{\ \}\parr P]$, then we can apply splitting (Lemma 4.1.2) to the proof of $[!S'\{R\}\parr P]$ and get:

$$\begin{array}{c} [?P_1\parr\cdots\parr ?P_h] \\ \mathsf{NELc}\,\Big\Vert\,\Delta_P \\ P \end{array} \qquad\text{and}\qquad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert}\,\Pi_S \\ [S'\{R\}\parr P_1\parr\cdots\parr P_h] \end{array} \quad .$$

By applying the induction hypothesis to Π_S we get P_R such that

$$\begin{array}{c} !\cdots![\{\ \}\parr P_R] \\ \mathsf{NELc}\,\Big\Vert\,\Delta' \\ [S'\{\ \}\parr P_1\parr\cdots\parr P_h] \end{array} \qquad\text{and}\qquad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert}\,\Pi \\ [R\parr P_R] \end{array}$$

From this we can build

$$\begin{array}{c} !!\cdots![\{\ \}\parr P_R] \\ \mathsf{NELc}\,\Big\Vert\,\Delta' \\ ![S'\{\ \}\parr P_1\parr\cdots\parr P_h] \\ \{\mathsf{p}{\downarrow}\}\,\Big\Vert \\ [!S'\{\ \}\parr ?P_1\parr\cdots\parr ?P_h] \\ \mathsf{NELc}\,\Big\Vert\,\Delta_P \\ [!S'\{\ \}\parr P] \end{array}$$

Note that in this case the number of ! in front of $[\{\ \}\parr P_R]$ increases.

(d) The case where $S\{\ \} = [?S'\{\ \}\parr P]$ is similar to (c). $\square$

4.3 Elimination of the Up Fragment

In this section, we will first show four lemmas, which are all easy consequences of splitting and which say that the core up rules of system SNEL are admissible if they are applied in a shallow context $[\{\ \} \parr P]$. Then we will show how context reduction is used to extend these lemmas to any context. As a result, we get a proof of cut elimination that can be considered modular, in the sense that the four core up rules $\mathsf{ai}\!\uparrow$, $\mathsf{q}\!\uparrow$, $\mathsf{p}\!\uparrow$, and $\mathsf{e}\!\uparrow$ are shown to be admissible independently from each other.

4.3.1 Lemma *Let P be a structure and let a be an atom. If $[(a \otimes \bar{a}) \parr P]$ is provable in NELc, then P is also provable in NELc.*

Proof: Apply splitting to the proof of $[(a \otimes \bar{a}) \parr P]$. This yields:

$$\begin{array}{c} [P_a \parr P_{\bar{a}}] \\ \mathsf{NELc}\,\Big\| \\ P \end{array} \quad \text{and} \quad \begin{array}{c} \mathsf{NELc}\,\Big\| \\ [a \parr P_a] \end{array} \quad \text{and} \quad \begin{array}{c} \mathsf{NELc}\,\Big\| \\ [\bar{a} \parr P_{\bar{a}}] \end{array} \quad .$$

By applying Lemma 4.1.3, we get a derivation from $\bar{a}$ to P_a and one from a to $P_{\bar{a}}$. From these we can build our proof

$$\begin{array}{c} \circ\!\downarrow \dfrac{}{\circ} \\ \mathsf{ai}\!\downarrow \dfrac{\phantom{[\bar{a} \parr a]}}{[\bar{a} \parr a]} \\ \mathsf{NELc}\,\Big\| \\ [P_a \parr P_{\bar{a}}] \\ \mathsf{NELc}\,\Big\| \\ P \end{array}$$

as desired. □

4.3.2 Lemma *Let R, T, U, V and P be any NEL structures. If $[(\langle R \lhd U \rangle \otimes \langle T \lhd V \rangle) \parr P]$ is provable in NELc, then $[\langle (R \otimes T) \lhd (U \otimes V) \rangle \parr P]$ is also provable in NELc.*

Proof: By applying splitting several times to the proof of $[(\langle R \lhd U \rangle \otimes \langle T \lhd V \rangle) \parr P]$, we get structures P_R, P_T, P_U, and P_V such that

$$\begin{array}{c} [\langle P_R \lhd P_U \rangle \parr \langle P_T \lhd P_V \rangle] \\ \mathsf{NELc}\,\Big\| \\ P \end{array} \qquad \begin{array}{c} \mathsf{NELc}\,\Big\| \\ [R \parr P_R] \end{array} \qquad \begin{array}{c} \mathsf{NELc}\,\Big\| \\ [U \parr P_U] \end{array} \qquad \begin{array}{c} \mathsf{NELc}\,\Big\| \\ [T \parr P_T] \end{array} \qquad \begin{array}{c} \mathsf{NELc}\,\Big\| \\ [V \parr P_V] \end{array}$$

By putting things together, we can build the proof

$$
\mathsf{s},\mathsf{s},\mathsf{s},\mathsf{s}\;\mathsf{q}\!\downarrow,\mathsf{q}\!\downarrow\;
\begin{array}{c}
\mathsf{NELc}\,\Big\Vert\\
\dfrac{\dfrac{\langle([R\parr P_R]\otimes[T\parr P_T])\lhd([U\parr P_U]\otimes[V\parr P_V])\rangle}{\langle[(R\otimes T)\parr P_R\parr P_T]\lhd[(U\otimes V)\parr P_U\parr P_V]\rangle}}{[\langle(R\otimes T)\lhd(U\otimes V)\rangle\parr\langle P_R\lhd P_U\rangle\parr\langle P_T\lhd P_V\rangle]}\\
\mathsf{NELc}\,\Big\Vert\\
[\langle(R\otimes T)\lhd(U\otimes V)\rangle\parr P]
\end{array}
$$

as desired. □

4.3.3 Lemma *Let R, T and P be any* NEL *structures. If $[(?R\otimes !T)\parr P]$ is provable in* NELc*, then $[?(R\otimes T)\parr P]$ is also provable in* NELc.

Proof: As above, we apply splitting several times to the proof of $[(?R\otimes !T)\parr P]$ and get structures $P_R, P_1, \ldots, P_h$ such that:

$$
\begin{array}{c}
[!P_R\parr ?P_1\parr\cdots\parr ?P_h]\\
\mathsf{NELc}\,\Big\Vert\\
P
\end{array}
\quad\text{and}\quad
\begin{array}{c}
\mathsf{NELc}\,\Big\Vert\\
[R\parr P_R]
\end{array}
\quad\text{and}\quad
\begin{array}{c}
\mathsf{NELc}\,\Big\Vert\\
[T\parr P_1\parr\cdots\parr P_h]
\end{array}
$$

By putting things together, we can build the proof

$$
\begin{array}{c}
\mathsf{NELc}\,\Big\Vert\\
\mathsf{s},\mathsf{s}\,\dfrac{!([R\parr P_R]\otimes[T\parr P_1\parr\cdots\parr P_h])}{![(R\otimes T)\parr P_R\parr P_1\parr\cdots\parr P_h]}\\
\{\mathsf{p}\!\downarrow\}\,\Big\Vert\\
[?(R\otimes T)\parr !P_R\parr ?P_1\parr\cdots\parr ?P_h]\\
\mathsf{NELc}\,\Big\Vert\\
[?(R\otimes T)\parr P]
\end{array}
$$

as desired. □

4.3.4 Lemma *Let P be any* NEL *structure. If $[?\circ\parr P]$ is provable in* NELc*, then $[\circ\parr P]$ is also provable in* NELc.

Proof: This is now a trivial exercise, that we leave to the reader. □

By the use of context reduction, we can extend the statements of Lemmas 4.3.1–4.3.4 from shallow contexts $[\{\ \}\parr P]$ to arbitrary contexts $S\{\ \}$.

4.3.5 Lemma *Let R, T, U and V be any structures, let a be an atom and let $S\{\ \}$ be any context. Then we have the following*

(i) *If $S(a \otimes \bar{a})$ is provable in* NELc*, then so is $S\{\circ\}$.*

(ii) *If $S(\langle R \lhd U\rangle \otimes \langle T \lhd V\rangle)$ is provable in* NELc*, then so is $S\langle(R \otimes T) \lhd (U \otimes V)\rangle$.*

(iii) *If $S(?R \otimes !T)$ is provable in* NELc*, then so is $S\{?(R \otimes T)\}$.*

(iv) *If $S\{?\circ\}$ is provable in* NELc*, then so is $S\{\circ\}$.*

Proof: All four statements are proved similarly. We will here show only the third: Let a proof of $S(?R \otimes !T)$ be given and apply context reduction, to get a structure P, such that

$$\begin{array}{c} !\cdots![\{\ \} \parr P] \\ \mathsf{NELc}\,\Big\Vert\,\Delta \\ S\{\ \} \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathsf{NELc}\,\Big\Vert\,\Pi \\ [(?R \otimes !T) \parr P] \end{array}$$

By Lemma 4.3.3 there is a proof Π' of $[?(R \otimes T) \parr P]$. By plugging $?(R \otimes T)$ into the hole of Δ, we can build

$$\begin{array}{c} \{\circ\downarrow, \mathsf{e}\downarrow\}\,\Big\Vert \\ !\cdots!\circ \\ \mathsf{NELc}\,\Big\Vert\,\Pi' \\ !\cdots![?(R \otimes T) \parr P] \\ \mathsf{NELc}\,\Big\Vert\,\Delta \\ S\{?(R \otimes T)\} \end{array}$$

It is obvious that the other statements are proved in the same way. □

4.3.6 Lemma *If a structure R is provable in* $\mathsf{NELc} \cup \{\mathsf{ai}\uparrow, \mathsf{q}\uparrow, \mathsf{p}\uparrow, \mathsf{e}\uparrow\}$ *then it is also provable in* NELc*.*

Proof: The instances of the rules $\mathsf{ai}\uparrow, \mathsf{q}\uparrow, \mathsf{p}\uparrow, \mathsf{e}\uparrow$ are removed one after the other (starting with the topmost one) via Lemma 4.3.5. □

Now we can very easily give a proof for the cut elimination theorem for the system NEL.

Proof of Theorem 2.12: Cut elimination is obtained in two steps:

$$\begin{array}{r} \circ\downarrow\,\dfrac{}{\circ} \\ \mathsf{SNEL}\,\Big\Vert \\ R \end{array} \quad \xrightarrow{1} \quad \begin{array}{r} \mathsf{NELc} \cup \{\mathsf{ai}\uparrow, \mathsf{q}\uparrow, \mathsf{p}\uparrow, \mathsf{e}\uparrow\}\,\Big\Vert \\ R' \\ \{\mathsf{w}\downarrow, \mathsf{b}\downarrow, \mathsf{g}\downarrow\}\,\Big\Vert \\ R \end{array} \quad \xrightarrow{2} \quad \begin{array}{r} \mathsf{NELc}\,\Big\Vert \\ R' \\ \{\mathsf{w}\downarrow, \mathsf{b}\downarrow, \mathsf{g}\downarrow\}\,\Big\Vert \\ R \end{array}$$

Step 1 is an application of the fourth decomposition (Theorem 3.1). The instances of $\mathsf{g}\uparrow, \mathsf{b}\uparrow, \mathsf{w}\uparrow$ disappear because their premise must be the unit $\circ$, which is impossible. Step 2 is just Lemma 4.3.6. $\square$

This technique shows how admissibility can be proved uniformly, both for cut rules (the atomic ones) and the other up rules, which are actually very different rules than cut. So, our technique is more general than cut elimination in the sequent calculus, for two reasons:

1. it applies to connectives that admit no sequent calculus definition, as seq;

2. it can be used to show admissibility of non-infinitary rules that involve no negation, like $\mathsf{q}\uparrow$ and $\mathsf{p}\uparrow$.

5 Conclusions and Future Work

We have defined the logical system NEL, which integrates multiplicative commutativity and non-commutativity, together with exponentials. This has been done in the formalism of the calculus of structures, which allows us to obtain very simple systems. In addition, we get properties of locality, atomicity and modularity that do not hold in other known calculi.

Proving cut elimination in deep inference is more difficult than in the sequent calculus. However, the methods we used are more general than the traditional ones, and, we believe, unveil some fundamental properties of logical systems that were previously hidden. We make an essential use of a top-down symmetric notion of derivation, which leads to a reduction of the cut rule into constituents which are dual to the common logical rules.

System NEL was originally inspired by Retoré's pomset logic [Ret97]. There is research in progress to show that the multiplicative fragments of his logic and ours coincide. In this case, our system and the work [Tiu06b] would explain why sequentialising pomset logic has been so hard and unfruitful. It should be possible to extend our system NEL to other logical operators, perhaps to full linear logic, and also to the self-dual modality associated to Retoré's non-commutative operator [Ret94]. In this paper we limited ourselves to the bare necessary to include MELL.

In [Str03c], it is shown that NEL is Turing-complete. This result establishes an interesting boundary to MELL, whose decidability is still an open problem. If it turns out, as many believe, that MELL is decidable, then the boundary with undecidability is crossed by our simple extension to seq. This would give a precise technical content to the perceived difficulty of getting Turing-completeness for MELL, namely the trouble in realising the tape of a Turing machine. In this sense, our sequentiality would be even more strongly motivated by a basic computational mechanism.

One of the biggest open problems we have is understanding when and why decomposition theorems work. They seem to have a strong relation to the notion of core system, but we fail to understand the deep reasons for this. For the time being we observe that decomposition theorems hold for all logics we studied so far (classical, linear and several commutative/non-commutative systems).

We believe that there is a close relation to the theory of structads that has recently been developed by Lamarche [Lam01]. The exploration of this promises to be an active area of research.

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