# A Local System for Linear Logic

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**Abstract.** In this paper I will present a deductive system for linear logic, in which all rules are *local*. In particular, the contraction rule is reduced to an atomic version, and there is no global promotion rule. In order to achieve this, it is necessary to depart from the sequent calculus and use the *calculus of structures*, which is a generalization of the one-sided sequent calculus. In a rule, premise and conclusion are not sequents, but *structures*, which are expressions that share properties of formulae and sequents.

## 1 Introduction

Since distributed computation moves more and more into the focus of research in theoretical computer science, it is also of interest to implement proof search in deductive systems in a distributed way. For this, it is desirable that the application of each inference rule consumes only a bounded amount of computational resources, i.e. time and space. But most deductive systems contain inference rules that do not have this property. Let me use as an example the well-known sequent calculus system for linear logic [5] (see Fig. 4). In particular, consider the following three rules,

$$?\mathsf{c}\,\frac{\vdash ?A, ?A, \Phi}{\vdash ?A, \Phi} \quad , \quad \&\,\frac{\vdash A, \Phi \quad \vdash B, \Phi}{\vdash A \mathbin{\&} B, \Phi} \quad \text{and} \quad !\,\frac{\vdash A, ?B_1, \ldots, ?B_n}{\vdash !A, ?B_1, \ldots, ?B_n} \quad ,$$

which are called *contraction, with* and *promotion*, respectively. If the contraction rule is applied in a proof search, going from bottom to top, the formula $?A$ has to be duplicated. Whatever mechanism is used for this duplication, it needs to inspect all of $?A$. In other words, the contraction rule needs a global view on $?A$. Further, the computational resources needed for applying contraction are not bounded, but depend on the size of $?A$. A similar situation occurs when the with rule is applied because the whole context $\Phi$ of the formula $A \mathbin{\&} B$ has to be copied. Another rule which involves a global knowledge of the context is the promotion rule, where for each formula in the context of $!A$ it has to be checked whether it has the form $?B$. In the sequel, inference rules, like contraction, with and promotion, that require such a global view on formulae or sequents of unbounded size, will be called *non-local.* All other rules are called *local* [2]. For example, the two rules

$$\otimes\,\frac{\vdash A, \Phi \quad \vdash B, \Psi}{\vdash A \otimes B, \Phi, \Psi} \qquad \text{and} \qquad \parr\,\frac{\vdash A, B, \Phi}{\vdash A \parr B, \Phi} \quad ,$$

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which are called *times* and *par*, respectively, are local because they do not need to look into the formulae $A$ and $B$ or their contexts. They require only a bounded amount of computational resources because it would suffice to operate on pointers to $A$ and $B$, which depend not on the size of the formulae.

Observe that sharing cannot be used for implementing a non-local rule because after copying a formula $A$ in an aplication of the contraction or with rule, both copies of $A$ might be used and modified in a very different way.

In [2] it has been shown that it is possible to design a local system (i.e. a deductive system in which all rules are local) for classical logic. Since linear logic plays an increasing role in computer science, it is natural to ask whether there is also a local system for linear logic. In this paper I will give a positive answer to this question.

The basic idea for making a system local is replacing each non-local inference rule by a local version, for instance, by restricting the application to atoms, which are formulae of bounded size. This idea is not new: the general (non-local) identity rule in the sequent calculus can be replaced by its atomic counterpart (which is local) without affecting provability.

To make the general contraction rule admissible for its atomic version, it is necessary to add new inference rules to the system in order to maintain completeness. As already observed in [2], these new rules cannot be given in the sequent calculus [4]. This makes it necessary to use another formalism for specifying deductive systems, namely, the calculus of structures [6, 7], which benefits from the following two features: First, the representation of sequents and formulae is merged into a single kind of expression, called *structure*. Second, inferences can be applied anywhere deep inside structures.

Locality is achieved by copying formulae stepwise, i.e. atom by atom, and by using the new rules to restore the original formula to be copied. Operationally this itself is not very interesting. The surprising fact is that this can be done *inside* a logical system without losing important properties like cut elimination, soundness and completeness. Further, the new top-down symmetry, which is unveiled by the calculus of structures, is kept in the local system.

In the next section, I will introduce the basic notions of the calculus of structures and present system $\mathsf{LS}$ for linear logic in the calculus of structures. Although this system is not local, it is a crucial step towards locality because the $\&$-rule is split into two parts: a purely multiplicative rule and an additive contraction rule. Further, the promotion rule is local, as already conceived in [7, 10]. In Section 3, I will show that system $\mathsf{LS}$ is equivalent to the well-known system $\mathsf{LL}$ for linear logic in the sequent calculus. I will show cut elimination for system $\mathsf{LS}$ in Section 4. Then, in Section 5, system $\mathsf{LS}$ will be made local: first the new additive contraction rule of system $\mathsf{LS}$ is reduced to an atomic version, and then contraction for the exponentials is reduced to the additives. The result will be system $\mathsf{LLS}$, a local system for propositional linear logic. The only drawback is that in system $\mathsf{LLS}$ the exponentials are not independent from the additives. But it is possible to consider the multiplicative additive fragment of system $\mathsf{LLS}$ separated from the exponentials.

## 2 Linear Logic in the Calculus of Structures

A system in the calculus of structures requires a language of *structures*, which are intermediate expressions between formulae and sequents. I will now define the language for the systems presented in this paper. Intuitively, the structure $[R_1, \ldots, R_h]$ corresponds to the sequent $\vdash R_1, \ldots, R_h$ in linear logic, whose formulae are essentially connected by pars, subject to commutativity and associativity. The structure $(R_1, \ldots, R_h)$ corresponds to the associative and commutative times connection of $R_1, \ldots, R_h$. The structures $\bullet[R_1, \ldots, R_h]\bullet$ and $\bullet(R_1, \ldots, R_h)\bullet$, which are also associative and commutative, correspond to the additive disjunction and conjunction, respectively, of $R_1, \ldots, R_h$.

**2.1 Definition** There are countably many *positive* and *negative propositional variables*, denoted with $a$, $b$, $c$, .... There are four *constants*, called *bottom, one, zero* and *top*, denoted with $\bot$, $\mathtt{1}$, $\mathtt{0}$ and $\top$, respectively. An *atom* is either a propositional variable or a constant. *Structures* are denoted with $P$, $Q$, $R$, ..., and are generated by

$$R ::= a \mid [\underbrace{R, \ldots, R}_{>0}] \mid (\underbrace{R, \ldots, R}_{>0}) \mid \bullet[\underbrace{R, \ldots, R}_{>0}]\bullet \mid \bullet(\underbrace{R, \ldots, R}_{>0})\bullet \mid\ !R \mid\ ?R \mid \bar{R} ,$$

where $a$ stands for any atom (positive or negative propositional variable or constant). A structure $[R_1, \ldots, R_h]$ is called a *par structure*, $(R_1, \ldots, R_h)$ is called a *times structure*, $\bullet[R_1, \ldots, R_h]\bullet$ is called a *plus structure*, $\bullet(R_1, \ldots, R_h)\bullet$ is called a *with structure*, $!R$ is called an *of-course structure*, and $?R$ is called a *why-not structure*; $\bar{R}$ is the *negation* of the structure $R$. Structures are considered to be equivalent modulo the relation $=$, which is the smallest congruence relation induced by the equations shown in Fig. 1, where $\boldsymbol{R}$ and $\boldsymbol{T}$ stand for finite, non-empty sequences of structures.

**2.2 Definition** In the same setting, we can define *structure contexts*, which are structures with a hole. Formally, they are generated by

$$S ::= \{\ \} \mid [R, S] \mid (R, S) \mid \bullet[R, S]\bullet \mid \bullet(R, S)\bullet \mid\ !S \mid\ ?S \quad .$$

Because of the De Morgan laws there is no need to include the negation into the definition of the context, which means that the structure that is plugged into the hole of a context will always be positive. Structure contexts will be denoted with $R\{\ \}$, $S\{\ \}$, $T\{\ \}$, .... Then, $S\{R\}$ denotes the structure that is obtained by replacing the hole $\{\ \}$ in the context $S\{\ \}$ by the structure $R$. The structure $R$ is a *substructure* of $S\{R\}$ and $S\{\ \}$ is its *context*. For a better readability, I will omit the context braces if no ambiguity is possible, e.g. I will write $S[R, T]$ instead of $S\{[R, T]\}$.

**2.3 Example** Let $S\{\ \} = [(a, ![\{\ \}, ?a], \bar{b}), b]$ and $R = c$ and $T = (\bar{b}, \bar{c})$ then $S[R, T] = [(a, ![c, (\bar{b}, \bar{c}), ?a], \bar{b}), b]$.

**2.4 Definition** In the calculus of structures, an *inference rule* is a scheme of the kind $\rho \dfrac{T}{R}$, where $\rho$ is the *name* of the rule, $T$ is its *premise* and $R$ is its

| **Associativity** | **Commutativity** | **Negation** |
|---|---|---|
| $[\boldsymbol{R},[\boldsymbol{T}]]=[\boldsymbol{R},\boldsymbol{T}]$ | $[\boldsymbol{R},\boldsymbol{T}]=[\boldsymbol{T},\boldsymbol{R}]$ | $\bar{\bot}=1$ |
| $(\boldsymbol{R},(\boldsymbol{T}))=(\boldsymbol{R},\boldsymbol{T})$ | $(\boldsymbol{R},\boldsymbol{T})=(\boldsymbol{T},\boldsymbol{R})$ | $\bar{1}=\bot$ |
| $\{\!\!\{\boldsymbol{R},\{\!\!\{\boldsymbol{T}\}\!\!\}\}\!\!\}=\{\!\!\{\boldsymbol{R},\boldsymbol{T}\}\!\!\}$ | $\{\!\!\{\boldsymbol{R},\boldsymbol{T}\}\!\!\}=\{\!\!\{\boldsymbol{T},\boldsymbol{R}\}\!\!\}$ | $\bar{0}=\top$ |
| $\langle\!\langle\boldsymbol{R},\langle\!\langle\boldsymbol{T}\rangle\!\rangle\rangle\!\rangle=\langle\!\langle\boldsymbol{R},\boldsymbol{T}\rangle\!\rangle$ | $\langle\!\langle\boldsymbol{R},\boldsymbol{T}\rangle\!\rangle=\langle\!\langle\boldsymbol{T},\boldsymbol{R}\rangle\!\rangle$ | $\bar{\top}=0$ |
| **Units** | **Singleton** | $\overline{[R,T]}=(\bar{R},\bar{T})$ |
| $[\bot,\boldsymbol{R}]=[\boldsymbol{R}]$ | $[R]=R=(R)$ | $\overline{(R,T)}=[\bar{R},\bar{T}]$ |
| $(1,\boldsymbol{R})=(\boldsymbol{R})$ | $\{\!\!\{R\}\!\!\}=R=\langle\!\langle R\rangle\!\rangle$ | $\overline{\{\!\!\{R,T\}\!\!\}}=\langle\!\langle\bar{R},\bar{T}\rangle\!\rangle$ |
| $\{\!\!\{0,\boldsymbol{R}\}\!\!\}=\{\!\!\{\boldsymbol{R}\}\!\!\}$ | **Exponentials** | $\overline{\langle\!\langle R,T\rangle\!\rangle}=\{\!\!\{\bar{R},\bar{T}\}\!\!\}$ |
| $\langle\!\langle\top,\boldsymbol{R}\rangle\!\rangle=\langle\!\langle\boldsymbol{R}\rangle\!\rangle$ | $??R=?R$ | $\overline{?R}=!\bar{R}$ |
| $\{\!\!\{\bot,\bot\}\!\!\}=\bot=?\bot$ | $!!R=!R$ | $\overline{!R}=?\bar{R}$ |
| $\langle\!\langle 1,1\rangle\!\rangle=1=!1$ | | $\bar{\bar{R}}=R$ |

**Fig. 1.** Basic equations for the syntactic congruence =

*conclusion*. An inference rule is called an *axiom* if its premise is empty, i.e. the rule is of the shape $\rho\,\frac{}{R}$.

A typical rule has shape $\rho\,\frac{S\{T\}}{S\{R\}}$ and specifies a step of rewriting, by the implication $T \Rightarrow R$, inside a generic context $S\{\ \}$. Rules with empty contexts correspond to the case of the sequent calculus.

**2.5 Definition** A (*formal*) *system* $\mathscr{S}$ is a set of inference rules. A *derivation* $\Delta$ in a certain formal system is a finite chain of instances of inference rules in the system:

$$\rho\,\frac{R}{R'}\quad \rho'\,\frac{R'}{\vdots}\quad \rho''\,\frac{\vdots}{R''}\quad .$$

A derivation can consist of just one structure. The topmost structure in a derivation, if present, is called the *premise* of the derivation, and the bottommost structure is called its *conclusion*. A derivation $\Delta$ whose premise is $T$, whose conclusion is $R$, and whose inference rules are in $\mathscr{S}$ will be written as $\begin{array}{c}T\\ \Delta\Vert\mathscr{S}\\ R\end{array}$ . A *proof* $\Pi$ in the calculus of structures is a finite derivation whose topmost inference rule is an axiom. It will be denoted by $\begin{array}{c}\Pi\,\overline{\Vert}\,\mathscr{S}\\ R\end{array}$ .

$$\mathsf{ai}\!\downarrow \frac{S\{1\}}{S[a,\bar{a}]} \qquad \mathsf{ai}\!\uparrow \frac{S(a,\bar{a})}{S\{\bot\}} \qquad \text{Interaction}$$

$$\mathsf{s}\, \frac{S([R,U],T)}{S[(R,T),U]} \qquad \text{Multiplicatives}$$

$$\mathsf{d}\!\downarrow \frac{S(\!\bullet [R,U],[T,V] \bullet\!)}{S[(\!\bullet R,T \bullet\!),[\!\bullet U,V \bullet\!]]} \qquad \mathsf{d}\!\uparrow \frac{S([\!\bullet R,U \bullet\!],(\!\bullet T,V \bullet\!))}{S[\!\bullet (R,T),(U,V) \bullet\!]}$$

$$\mathsf{t}\!\downarrow \frac{S\{0\}}{S\{R\}} \qquad \mathsf{t}\!\uparrow \frac{S\{R\}}{S\{\top\}} \qquad \text{Additives}$$

$$\mathsf{c}\!\downarrow \frac{S[\!\bullet R,R \bullet\!]}{S\{R\}} \qquad \mathsf{c}\!\uparrow \frac{S\{R\}}{S(\!\bullet R,R \bullet\!)}$$

$$\mathsf{p}\!\downarrow \frac{S\{![R,T]\}}{S[!R,?T]} \qquad \mathsf{p}\!\uparrow \frac{S(?R,!T)}{S\{?(R,T)\}}$$

$$\mathsf{w}\!\downarrow \frac{S\{\bot\}}{S\{?R\}} \qquad \mathsf{w}\!\uparrow \frac{S\{!R\}}{S\{1\}} \qquad \text{Exponentials}$$

$$\mathsf{b}\!\downarrow \frac{S[?R,R]}{S\{?R\}} \qquad \mathsf{b}\!\uparrow \frac{S\{!R\}}{S(!R,R)}$$

**Fig. 2.** System $\mathsf{SLS}$

In the calculus of structures, rules come in pairs, a down-version $\rho\!\downarrow \dfrac{S\{T\}}{S\{R\}}$ and an up-version $\rho\!\uparrow \dfrac{S\{\bar{R}\}}{S\{\bar{T}\}}$. This duality derives from the duality between $T \Rightarrow R$ and $\bar{R} \Rightarrow \bar{T}$, where $\Rightarrow$ is the implication modelled in the system. In our case it is linear implication.

**2.6 Definition** *System* $\mathsf{SLS}$ is shown in Fig. 2.

The first $\mathsf{S}$ stands for "symmetric" or "self-dual", meaning that for each rule, its dual rule is also in the system. The $\mathsf{L}$ stands for linear logic and the last $\mathsf{S}$ stands for the calculus of structures.

**2.7 Definition** The rules of system $\mathsf{SLS}$ are called *atomic interaction* ($\mathsf{ai}\!\downarrow$), *atomic cut* ($\mathsf{ai}\!\uparrow$), *switch* ($\mathsf{s}$), *additive* ($\mathsf{d}\!\downarrow$), *coadditive* ($\mathsf{d}\!\uparrow$), *thinning* ($\mathsf{t}\!\downarrow$), *cothinning* ($\mathsf{t}\!\uparrow$), *contraction* ($\mathsf{c}\!\downarrow$), *cocontraction* ($\mathsf{c}\!\uparrow$), *promotion* ($\mathsf{p}\!\downarrow$), *copromotion* ($\mathsf{p}\!\uparrow$), *weakening* ($\mathsf{w}\!\downarrow$), *coweakening* ($\mathsf{w}\!\uparrow$), *absorption* ($\mathsf{b}\!\downarrow$) and *coabsorption* ($\mathsf{b}\!\uparrow$). The set $\{\mathsf{ai}\!\downarrow,\mathsf{s},\mathsf{d}\!\downarrow,\mathsf{t}\!\downarrow,\mathsf{c}\!\downarrow,\mathsf{p}\!\downarrow,\mathsf{w}\!\downarrow,\mathsf{b}\!\downarrow\}$ is called the *down fragment* and $\{\mathsf{ai}\!\uparrow,\mathsf{s},\mathsf{d}\!\uparrow,\mathsf{t}\!\uparrow,\mathsf{c}\!\uparrow,\mathsf{p}\!\uparrow,\mathsf{w}\!\uparrow,\mathsf{b}\!\uparrow\}$ is called the *up fragment*.

**2.8 Definition** The rules

$$\mathsf{i}\!\downarrow \frac{S\{1\}}{S[R,\bar{R}]} \qquad \text{and} \qquad \mathsf{i}\!\uparrow \frac{S(R,\bar{R})}{S\{\bot\}}$$

are called *interaction* and *cut*, respectively.

The rules $\mathsf{ai}\!\downarrow$ and $\mathsf{ai}\!\uparrow$ are obviously instances of the rules $\mathsf{i}\!\downarrow$ and $\mathsf{i}\!\uparrow$ above. It is well known that in many systems in the sequent calculus, the identity rule can be reduced to its atomic version. In the calculus of structures we can do the same. But furthermore, by duality, we can do the same to the cut rule, which is not possible in the sequent calculus.

**2.9 Definition** A rule $\rho$ is *derivable* in a system $\mathscr{S}$ if $\rho \notin \mathscr{S}$ and for every application of $\rho \frac{T}{R}$ there is a derivation $\begin{array}{c} T \\ \Delta \| \mathscr{S} \\ R \end{array}$.

**2.10 Proposition** *The rule* $\mathsf{i}\!\downarrow$ *is derivable in* $\{\mathsf{ai}\!\downarrow, \mathsf{s}, \mathsf{d}\!\downarrow, \mathsf{p}\!\downarrow\}$. *Dually, the rule* $\mathsf{i}\!\uparrow$ *is derivable in the system* $\{\mathsf{ai}\!\uparrow, \mathsf{s}, \mathsf{d}\!\uparrow, \mathsf{p}\!\uparrow\}$.

**Proof:** For a given application of $\mathsf{i}\!\downarrow \frac{S\{1\}}{S[R,\bar{R}]}$, by structural induction on $R$, we will construct an equivalent derivation that contains only $\mathsf{ai}\!\downarrow$, $\mathsf{s}$, $\mathsf{d}\!\downarrow$ and $\mathsf{p}\!\downarrow$. If $R$ is an atom, then the given instance of $\mathsf{i}\!\downarrow$ is an instance of $\mathsf{ai}\!\downarrow$. Otherwise apply the induction hypothesis to one of the following derivations:

$$\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{i}\!\downarrow\dfrac{\mathsf{i}\!\downarrow\dfrac{S\{1\}}{S[Q,\bar{Q}]}}{S([P,\bar{P}],[Q,\bar{Q}])}}{S[Q,([P,\bar{P}],\bar{Q})]}}{S[P,Q,(\bar{P},\bar{Q})]} \quad , \quad \mathsf{d}\!\downarrow\dfrac{\mathsf{i}\!\downarrow\dfrac{\mathsf{i}\!\downarrow\dfrac{S{\bullet\!(}1,1{)\!\bullet}}{S{\bullet\!(}1,[Q,\bar{Q}]{)\!\bullet}}}{S{\bullet\!(}[P,\bar{P}],[Q,\bar{Q}]{)\!\bullet}}}{S[{\bullet\![}P,Q{]\!\bullet},{\bullet\!(}\bar{P},\bar{Q}{)\!\bullet}]} \quad \text{or} \quad \mathsf{p}\!\downarrow\dfrac{\mathsf{i}\!\downarrow\dfrac{S\{!1\}}{S\{![P,\bar{P}]\}}}{S[?P,!\bar{P}]} \quad .$$

The second statement is dual to the first. □

There is another such result involved here, that has already been observed in [6]. If the rules $\mathsf{i}\!\downarrow$, $\mathsf{i}\!\uparrow$ and $\mathsf{s}$ are in a system, then any other rule $\rho$ makes its corule $\rho'$, i.e. the rule obtained from $\rho$ by exchanging and negating premise and conclusion, derivable. Let $\rho \frac{S\{P\}}{S\{Q\}}$ be given. Then any instance of $\rho' \frac{S\{\bar{Q}\}}{S\{\bar{P}\}}$ can be replaced by the following derivation:

$$\mathsf{i}\!\uparrow\dfrac{\rho\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{i}\!\downarrow\dfrac{S\{\bar{Q}\}}{S(\bar{Q},[P,\bar{P}])}}{S[(\bar{Q},P),\bar{P}]}}{S[(\bar{Q},Q),\bar{P}]}}{S\{\bar{P}\}} \quad .$$

**2.11 Proposition** *Every rule* $\rho\!\uparrow$ *is derivable in* $\{\mathsf{i}\!\downarrow, \mathsf{i}\!\uparrow, \mathsf{s}, \rho\!\downarrow\}$.

$$\mathsf{1}{\downarrow}\,\frac{}{\mathsf{1}} \qquad \mathsf{ai}{\downarrow}\,\frac{S\{\mathsf{1}\}}{S[a,\bar{a}]} \qquad \mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{d}{\downarrow}\,\frac{S\llparenthesis[R,U],[T,V]\rrparenthesis}{S[\llparenthesis R,T\rrparenthesis,\llparenthesis U,V\rrparenthesis]} \qquad \mathsf{t}{\downarrow}\,\frac{S\{\mathsf{0}\}}{S\{R\}} \qquad \mathsf{c}{\downarrow}\,\frac{S\llparenthesis R,R\rrparenthesis}{S\{R\}}$$

$$\mathsf{p}{\downarrow}\,\frac{S\{![R,T]\}}{S[!R,?T]} \qquad \mathsf{w}{\downarrow}\,\frac{S\{\bot\}}{S\{?R\}} \qquad \mathsf{b}{\downarrow}\,\frac{S[?R,R]}{S\{?R\}}$$

**Fig. 3.** System $\mathsf{LS}$

Propositions 2.10 and 2.11 together say, that the general cut rule $\mathsf{i}{\uparrow}$ is as powerful as the whole up fragment of the system and vice versa.

So far we are only able to describe derivations. In order to describe proofs, we need an axiom.

**2.12 Definition** The following rule is called *one*: $\mathsf{1}{\downarrow}\,\frac{}{\mathsf{1}}$ .

In the language of linear logic it simply says that $\vdash \mathsf{1}$ is provable. I will put this rule in the down fragment of system $\mathsf{SLS}$ and by this break the top-down symmetry of derivations and observe proofs.

**2.13 Definition** The system $\{\mathsf{1}{\downarrow}, \mathsf{ai}{\downarrow}, \mathsf{s}, \mathsf{d}{\downarrow}, \mathsf{t}{\downarrow}, \mathsf{c}{\downarrow}, \mathsf{p}{\downarrow}, \mathsf{w}{\downarrow}, \mathsf{b}{\downarrow}\}$, shown in Fig. 3, that is obtained from the down fragment of system $\mathsf{SLS}$ together with the axiom, is called *linear logic in the calculus of structures*, or *system* $\mathsf{LS}$.

Observe that in every proof in system $\mathsf{LS}$, the rule $\mathsf{1}{\downarrow}$ occurs exactly once, namely as the topmost rule of the proof.

**2.14 Definition** Two systems $\mathscr{S}$ and $\mathscr{S}'$ are *strongly equivalent* if for every derivation $\begin{array}{c}T\\ \Delta\Vert\mathscr{S}\\ R\end{array}$ there is a derivation $\begin{array}{c}T\\ \Delta'\Vert\mathscr{S}'\\ R\end{array}$ , and vice versa.

**2.15 Theorem** *The systems* $\mathsf{SLS}\cup\{\mathsf{1}{\downarrow}\}$ *and* $\mathsf{LS}\cup\{\mathsf{i}{\uparrow}\}$ *are strongly equivalent.*

**Proof:** Immediate consequence of Propositions 2.10 and 2.11. □

## 3 Equivalence to Linear Logic in the Sequent Calculus

In this section I will first recall the well-known sequent calculus system $\mathsf{LL}$ for linear logic [5], and then show that it is equivalent to the systems defined in the previous section, and by this, justify their names. More precisely, system $\mathsf{SLS}$ corresponds to $\mathsf{LL}$ with cut, and system $\mathsf{LS}$ corresponds to $\mathsf{LL}$ without cut.

**3.1 Definition** *Formulae* (denoted with $A, B, C, \ldots$) are built from propositional variables (denoted with $a, b, c, \ldots$), their duals ($a^\bot, b^\bot, c^\bot, \ldots$) and the constants $\bot, \mathsf{1}, \mathsf{0}, \top$ by means of the (binary) connectives $⅋, \otimes, \oplus, \&$ and the modalities $!, ?$. *Linear negation* $(\cdot)^\perp$ is defined on formulae by De Morgan equations:

$$
\begin{array}{rclcrcl}
1^\perp & := & \perp & \qquad & \perp^\perp & := & 1 \\
\top^\perp & := & 0 & & 0^\perp & := & \top \\
(a)^\perp & := & a^\perp & & (a^\perp)^\perp & := & a \\
(A \otimes B)^\perp & := & A^\perp \parr B^\perp & & (A \parr B)^\perp & := & A^\perp \otimes B^\perp \\
(A \mathbin{\&} B)^\perp & := & A^\perp \oplus B^\perp & & (A \oplus B)^\perp & := & A^\perp \mathbin{\&} B^\perp \\
(!A)^\perp & := & ?A^\perp & & (?A)^\perp & := & !A^\perp
\end{array}
$$

*Linear implication* $\multimap$ is defined by $A \multimap B = A^\perp \parr B$.

It follows immediately that $A = A^{\perp\perp}$ for each formula $A$.

**3.2 Definition** A *sequent* is an expression of the kind $\vdash A_1, \ldots, A_h$, where $h \geqslant 0$ and the comma between the formulae $A_1, \ldots, A_h$ stands for multiset union. Multisets of formulae are denoted with $\Phi$ and $\Psi$.

**3.3 Definition** *Derivations*, are trees where the nodes are sequents to which a finite number (possibly zero) of instances of the inference rules shown in Fig. 4 are applied. The sequents in the leaves are called *premises*, and the sequent in the root is the *conclusion*. A derivation with no premises is a *proof*, denoted with $\Pi$. A sequent $\vdash \Phi$ is *provable* if there is a proof $\Pi$ with conclusion $\vdash \Phi$.

**3.4 Definition** The functions $\underline{\cdot}_{\mathsf{S}}$ and $\underline{\cdot}_{\mathsf{L}}$ define the obvious translations between $\mathsf{LL}$ formulae and $\mathsf{LS}$ structures:

$$
\begin{array}{rcll}
\underline{a}_{\mathsf{S}} & = & a & , \\
\underline{\perp}_{\mathsf{S}} & = & \perp & , \\
\underline{1}_{\mathsf{S}} & = & 1 & , \\
\underline{0}_{\mathsf{S}} & = & 0 & , \\
\underline{\top}_{\mathsf{S}} & = & \top & , \\
\underline{A \parr B}_{\mathsf{S}} & = & [\underline{A}_{\mathsf{S}}, \underline{B}_{\mathsf{S}}] & , \\
\underline{A \otimes B}_{\mathsf{S}} & = & (\underline{A}_{\mathsf{S}}, \underline{B}_{\mathsf{S}}) & , \\
\underline{A \oplus B}_{\mathsf{S}} & = & \{\!\bullet\, \underline{A}_{\mathsf{S}}, \underline{B}_{\mathsf{S}} \,\bullet\!\} & , \\
\underline{A \mathbin{\&} B}_{\mathsf{S}} & = & \langle\!\bullet\, \underline{A}_{\mathsf{S}}, \underline{B}_{\mathsf{S}} \,\bullet\!\rangle & , \\
\underline{?A}_{\mathsf{S}} & = & ?\underline{A}_{\mathsf{S}} & , \\
\underline{!A}_{\mathsf{S}} & = & !\underline{A}_{\mathsf{S}} & , \\
\underline{A^\perp}_{\mathsf{S}} & = & \overline{\underline{A}_{\mathsf{S}}} & ,
\end{array}
\qquad
\begin{array}{rcll}
\underline{a}_{\mathsf{L}} & = & a & , \\
\underline{\perp}_{\mathsf{L}} & = & \perp & , \\
\underline{1}_{\mathsf{L}} & = & 1 & , \\
\underline{0}_{\mathsf{L}} & = & 0 & , \\
\underline{\top}_{\mathsf{L}} & = & \top & , \\
\underline{[R_1, \ldots, R_h]}_{\mathsf{L}} & = & \underline{R_1}_{\mathsf{L}} \parr \cdots \parr \underline{R_h}_{\mathsf{L}} & , \\
\underline{(R_1, \ldots, R_h)}_{\mathsf{L}} & = & \underline{R_1}_{\mathsf{L}} \otimes \cdots \otimes \underline{R_h}_{\mathsf{L}} & , \\
\underline{\{\!\bullet\, R_1, \ldots, R_h \,\bullet\!\}}_{\mathsf{L}} & = & \underline{R_1}_{\mathsf{L}} \oplus \cdots \oplus \underline{R_h}_{\mathsf{L}} & , \\
\underline{\langle\!\bullet\, R_1, \ldots, R_h \,\bullet\!\rangle}_{\mathsf{L}} & = & \underline{R_1}_{\mathsf{L}} \mathbin{\&} \cdots \mathbin{\&} \underline{R_h}_{\mathsf{L}} & , \\
\underline{?R}_{\mathsf{L}} & = & ?\underline{R}_{\mathsf{L}} & , \\
\underline{!R}_{\mathsf{L}} & = & !\underline{R}_{\mathsf{L}} & , \\
\underline{\bar{R}}_{\mathsf{L}} & = & (\underline{R}_{\mathsf{L}})^\perp & .
\end{array}
$$

The domain of $\underline{\cdot}_{\mathsf{S}}$ is extended to sequents by

$$
\underline{\vdash}_{\mathsf{S}} = \perp \quad \text{and}
$$
$$
\underline{\vdash A_1, \ldots, A_h}_{\mathsf{S}} = [\underline{A_1}_{\mathsf{S}}, \ldots, \underline{A_h}_{\mathsf{S}}] \quad , \text{ for } h \geqslant 0 \quad .
$$

**3.5 Theorem** *If a given structure $R$ is provable in system $\mathsf{SLS} \cup \{1\!\downarrow\}$, then its translation $\vdash \underline{R}_{\mathsf{L}}$ is provable in $\mathsf{LL}$ (with cut).*

$$
\mathsf{id}\,\frac{}{\vdash A, A^{\perp}} \qquad \mathsf{cut}\,\frac{\vdash A, \Phi \quad \vdash A^{\perp}, \Psi}{\vdash \Phi, \Psi}
$$

$$
\otimes\,\frac{\vdash A, \Phi \quad \vdash B, \Psi}{\vdash A \otimes B, \Phi, \Psi} \qquad \invamp\,\frac{\vdash A, B, \Phi}{\vdash A \invamp B, \Phi} \qquad \perp\,\frac{\vdash \Phi}{\vdash \perp, \Phi} \qquad \mathsf{1}\,\frac{}{\vdash \mathsf{1}}
$$

$$
\&\,\frac{\vdash A, \Phi \quad \vdash B, \Phi}{\vdash A \mathbin{\&} B, \Phi} \qquad \oplus_1\,\frac{\vdash A, \Phi}{\vdash A \oplus B, \Phi} \qquad \oplus_2\,\frac{\vdash B, \Phi}{\vdash A \oplus B, \Phi} \qquad \top\,\frac{}{\vdash \top, \Phi}
$$

$$
?\mathsf{d}\,\frac{\vdash A, \Phi}{\vdash ?A, \Phi} \qquad ?\mathsf{c}\,\frac{\vdash ?A, ?A, \Phi}{\vdash ?A, \Phi} \qquad ?\mathsf{w}\,\frac{\vdash \Phi}{\vdash ?A, \Phi} \qquad !\,\frac{\vdash A, ?B_1, \ldots, ?B_n}{\vdash !A, ?B_1, \ldots, ?B_n}\ (n \geqslant 0)
$$

**Fig. 4.** System $\mathsf{LL}$ in the sequent calculus

**Proof:** All equations shown in Fig. 1 correspond to logical equivalences in linear logic. Further, for every rule $\rho\,\dfrac{S\{R\}}{S\{T\}}$ in $\mathsf{SLS}$, the sequent $\vdash (\underline{R}_{\mathsf{L}})^{\perp}, \underline{T}_{\mathsf{L}}$ is provable in $\mathsf{LL}$. Hence, $\vdash (\underline{S\{R\}}_{\mathsf{L}})^{\perp}, \underline{S\{T\}}_{\mathsf{L}}$ is also provable in $\mathsf{LL}$. Use this and

$$
\mathsf{cut}\,\frac{\vdash \underline{S\{R\}}_{\mathsf{L}} \quad \vdash (\underline{S\{R\}}_{\mathsf{L}})^{\perp}, \underline{S\{T\}}_{\mathsf{L}}}{\vdash \underline{S\{T\}}_{\mathsf{L}}}
$$

to proceed inductively over the length of a given proof $\begin{array}{c} \Vert\!\!\!\!\top \mathsf{SLS}\cup\{1\!\downarrow\} \\ \rho\,\dfrac{S\{R\}}{S\{T\}} \end{array}$ . □

**3.6 Theorem** (a) *If a given sequent* $\vdash \Phi$ *is provable in* $\mathsf{LL}$ *(with cut), then the structure* $\underline{\vdash \Phi}_{\mathsf{S}}$ *is provable in system* $\mathsf{SLS} \cup \{1\!\downarrow\}$. (b) *If a given sequent* $\vdash \Phi$ *is cut-free provable in* $\mathsf{LL}$*, then the structure* $\underline{\vdash \Phi}_{\mathsf{S}}$ *is provable in system* $\mathsf{LS}$.

**Proof:** Let $\Pi$ be the proof of $\vdash \Phi$ in $\mathsf{LL}$. By structural induction on $\Pi$, construct a proof $\underline{\Pi}_{\mathsf{S}}$ of $\underline{\vdash \Phi}_{\mathsf{S}}$ in system $\mathsf{SLS} \cup \{1\!\downarrow\}$ (or system $\mathsf{LS}$ if $\Pi$ is cut-free). Here, I will show only the case where $\&\,\dfrac{\vdash A, \Phi \quad \vdash B, \Phi}{\vdash A \mathbin{\&} B, \Phi}$ is the last rule applied in $\Pi$. Then let $\underline{\Pi}_{\mathsf{S}}$ be the proof

$$
\begin{array}{c}
1\!\downarrow\,\dfrac{}{\mathsf{1}} \\
\Delta_1 \Vert \mathsf{SLS} \\
\llparenthesis [\underline{A}_{\mathsf{S}}, \underline{\Phi}_{\mathsf{S}}], \mathsf{1} \rrparenthesis \\
\Delta_2 \Vert \mathsf{SLS} \\
\mathsf{d}\!\downarrow\,\dfrac{\llparenthesis [\underline{A}_{\mathsf{S}}, \underline{\Phi}_{\mathsf{S}}], [\underline{B}_{\mathsf{S}}, \underline{\Phi}_{\mathsf{S}}] \rrparenthesis}{\mathsf{c}\!\downarrow\,\dfrac{[\llparenthesis \underline{A}_{\mathsf{S}}, \underline{B}_{\mathsf{S}} \rrparenthesis, \llparenthesis \underline{\Phi}_{\mathsf{S}}, \underline{\Phi}_{\mathsf{S}} \rrparenthesis]}{[\llparenthesis \underline{A}_{\mathsf{S}}, \underline{B}_{\mathsf{S}} \rrparenthesis, \underline{\Phi}_{\mathsf{S}}]}}
\end{array} ,
$$

where $\Delta_1$ and $\Delta_2$ exist by induction hypothesis. □

## 4 Cut Elimination

By inspecting the rules of system $\mathsf{SLS}$, one can observe that the only infinitary rules are atomic cut, cothinning and coweakening. This means, that in order to obtain a finitary system, one could get rid only of the rules $\mathsf{ai}\!\uparrow$, $\mathsf{t}\!\uparrow$ and $\mathsf{w}\!\uparrow$. But we can get more: the whole up fragment is admissible (except for the switch rule, which also belongs to the down fragment).

**4.1 Definition** A rule $\rho$ is *admissible* for a system $\mathscr{S}$ if $\rho \notin \mathscr{S}$ and for every proof $\underset{R}{\overset{\Pi}{\|}}\scriptstyle{\mathscr{S}\cup\{\rho\}}$ there is a proof $\underset{R}{\overset{\Pi'}{\|}}\scriptstyle{\mathscr{S}}$. Two systems $\mathscr{S}$ and $\mathscr{S}'$ are *equivalent* if for every proof $\underset{R}{\overset{\Pi}{\|}}\scriptstyle{\mathscr{S}}$ there is a proof $\underset{R}{\overset{\Pi'}{\|}}\scriptstyle{\mathscr{S}'}$, and vice versa.

**4.2 Theorem (Cut elimination)** *System* $\mathsf{LS}$ *is equivalent to every subsystem of* $\mathsf{SLS} \cup \{1\!\downarrow\}$ *containing* $\mathsf{LS}$.

**Proof:** A proof in $\mathsf{SLS} \cup \{1\!\downarrow\}$ can be transformed into a proof in $\mathsf{LL}$ (by Theorem 3.5), to which we can apply the cut elimination procedure in the sequent calculus. The cut-free proof in $\mathsf{LL}$ can then be transformed into a proof in system $\mathsf{LS}$ by Theorem 3.6. □

**4.3 Corollary** *The rule* $\mathsf{i}\!\uparrow$ *is admissible for system* $\mathsf{LS}$.

**Proof:** Immediate consequence of Theorems 2.15 and 4.2. □

The proof of Theorem 4.2 relies on the results of the previous section together with the well-known cut elimination proof for linear logic in the sequent calculus. But it should be mentioned here that it is also possible to prove Theorem 4.2 directly inside the calculus of structures, without using the sequent calculus, by using the technique of *splitting* [8].

## 5 Local Linear Logic

In this section, I will start from system $\mathsf{SLS}$ and produce a strongly equivalent system, in which all rules are local.

Before discussing the new system, let us first detect the non-local rules of system $\mathsf{SLS}$ (Fig. 2). Obviously the rules $\mathsf{c}\!\downarrow$ and $\mathsf{b}\!\downarrow$ are non-local because they involve the duplication of a structure of unbounded size. Also their corules $\mathsf{c}\!\uparrow$ and $\mathsf{b}\!\uparrow$ are non-local because they involve the comparison and the deletion of structures of unbounded size (or, again a duplication if one reasons top-down). Similarly, I consider the rules $\mathsf{t}\!\downarrow$ and $\mathsf{w}\!\downarrow$, as well as their corules $\mathsf{t}\!\uparrow$ and $\mathsf{w}\!\uparrow$ to be non-local because they involve the deletion or introduction of structures of unbounded size. Of course, one could argue that the deletion of a structure of unbounded size can be considered local because it might suffice to delete just the pointer to the structure. But then garbage collection becomes a problem. Further, the symmetry exhibited in system $\mathsf{SLS}$ should be carried through to the local system, and therefore, the locality of a rule should be invariant under forming the contrapositive, i.e. the corule.

Observe that all other rules of system $\mathsf{SLS}$ are already local. In particular, the two rules $\mathsf{ai}\!\downarrow$ and $\mathsf{ai}\!\uparrow$ only involve atoms. The switch rule can be implemented by changing the marking of two nodes and exchanging two pointers, as already observed in [2]:

$$
\begin{array}{ccccc}
 & S[\;] & \\
 \swarrow & & \searrow \\
(\;) & & T \\
\swarrow \quad \searrow & & \\
R \qquad U & &
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{ccccc}
 & S(\;) & \\
 \swarrow & & \searrow \\
[\;] & & U \\
\swarrow \quad \searrow & & \\
R \qquad T & &
\end{array}
\quad .
$$

For the rules $\mathsf{d}\!\downarrow$, $\mathsf{d}\!\uparrow$, $\mathsf{p}\!\downarrow$ and $\mathsf{p}\!\uparrow$, the situation is similar.

Let us now have a first glimpse at the new system, which is called *system* SLLS (the new L stands for "local") and which is shown in Fig. 5. The reader should not be frightened by the complexity of the system. I will discuss the purpose of the rules later on. At this point I will draw the attention to the fact that all rules in system SLLS are local. Either they handle only atoms, or their implementation can be realized by exchanging pointers in a similar way as for the switch rule. At this point the reader might observe that the equations shown in Fig. 1 are not local. However, they can be made local by implementing them in the in the same way as the inference rules. This means that system SLLS is indeed a local system. It remains to show that it is linear logic.

In order to do so, I will show that it is strongly equivalent to system SLS. Further, I will define system LLS by adding the axiom $1\!\downarrow$ to the down fragment of system SLLS. System LLS will be strongly equivalent to system LS. As a corollary we get a cut elimination result for the local system.

Consider now the rules $\mathsf{ac}\!\downarrow$ and $\mathsf{ac}\!\uparrow$ of system SLLS. They are called *atomic contraction* and *atomic cocontraction*, respectively. They replace their general non-local counterparts $\mathsf{c}\!\downarrow$ and $\mathsf{c}\!\uparrow$. But they are not powerful enough to ensure completeness. For this reason, the *medial* rule $\mathsf{m}$ together with its variations $\mathsf{m}_1\!\downarrow$, $\mathsf{m}_1\!\uparrow$, $\mathsf{m}_2\!\downarrow$, $\mathsf{m}_2\!\uparrow$, $\mathsf{l}_1\!\downarrow$, $\mathsf{l}_1\!\uparrow$, $\mathsf{l}_2\!\downarrow$, $\mathsf{l}_2\!\uparrow$ is introduced. These rules have the same purpose as the medial rule of [2]. There are more medial rules in linear logic because there are more connectives.

**5.1 Proposition** *The rule* $\mathsf{c}\!\downarrow$ *is derivable in* $\{\mathsf{ac}\!\downarrow, \mathsf{m}, \mathsf{m}_1\!\downarrow, \mathsf{m}_2\!\downarrow, \mathsf{l}_1\!\downarrow, \mathsf{l}_2\!\downarrow\}$. *Dually, the rule* $\mathsf{c}\!\uparrow$ *is derivable in* $\{\mathsf{ac}\!\uparrow, \mathsf{m}, \mathsf{m}_1\!\uparrow, \mathsf{m}_2\!\uparrow, \mathsf{l}_1\!\uparrow, \mathsf{l}_2\!\uparrow\}$.

**Proof:** For a given instance $\mathsf{c}\!\downarrow \dfrac{S\Lbag R, R\Rbag}{S\{R\}}$, proceed by structural induction on $R$. If $R = (P, Q)$ (where $P \neq 1 \neq Q$), then apply the induction hypothesis to

$$
\mathsf{m}_2\!\downarrow \dfrac{S\Lbag (P,Q),(P,Q)\Rbag}{\mathsf{c}\!\downarrow \dfrac{S(\Lbag P,P\Rbag,\Lbag Q,Q\Rbag)}{\mathsf{c}\!\downarrow \dfrac{S(\Lbag P,P\Rbag,Q)}{S(P,Q)}}} \quad .
$$

The other cases are similar. □

Let us now consider the rules rules $\mathsf{at}\!\downarrow$, $\mathsf{at}\!\uparrow$, called *atomic thinning* and *atomic cothinning*, respectively. Again, they are the replacement for the general thinning and cothinning rules, and in order to keep completeness, we need to add the rules $\mathsf{nm}\!\downarrow$, $\mathsf{nm}\!\uparrow$, $\mathsf{nm}_1\!\downarrow$, $\mathsf{nm}_1\!\uparrow$, $\mathsf{nm}_2\!\downarrow$, $\mathsf{nm}_2\!\uparrow$, $\mathsf{nl}_1\!\downarrow$, $\mathsf{nl}_1\!\uparrow$, $\mathsf{nl}_2\!\downarrow$, $\mathsf{nl}_2\!\uparrow$, which are the nullary

$$\mathsf{ai}\!\downarrow \frac{S\{1\}}{S[a,\bar{a}]} \qquad \mathsf{ai}\!\uparrow \frac{S(a,\bar{a})}{S\{\bot\}}$$

$$\mathsf{s}\, \frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{d}\!\downarrow \frac{S(\!\bullet [R,U],[T,V] \bullet\!)}{S[(\!\bullet R,T \bullet\!),\{\!\bullet U,V \bullet\!\}]} \qquad \mathsf{d}\!\uparrow \frac{S(\{\!\bullet R,U \bullet\!\},(\!\bullet T,V \bullet\!))}{S\{\!\bullet (R,T),(U,V) \bullet\!\}}$$

$$\mathsf{p}\!\downarrow \frac{S\{![R,T]\}}{S[!R,?T]} \qquad \mathsf{p}\!\uparrow \frac{S(?R,!T)}{S\{?(R,T)\}}$$

$$\mathsf{at}\!\downarrow \frac{S\{0\}}{S\{a\}} \qquad \mathsf{ac}\!\downarrow \frac{S\{\!\bullet a,a \bullet\!\}}{S\{a\}} \qquad \mathsf{ac}\!\uparrow \frac{S\{a\}}{S(\!\bullet a,a \bullet\!)} \qquad \mathsf{at}\!\uparrow \frac{S\{a\}}{S\{\top\}}$$

$$\mathsf{nm}\!\downarrow \frac{S\{0\}}{S(\!\bullet 0,0 \bullet\!)} \qquad \mathsf{m}\, \frac{S\{\!\bullet (\!\bullet R,U \bullet\!),(\!\bullet T,V \bullet\!) \bullet\!\}}{S(\!\bullet \{\!\bullet R,T \bullet\!\},\{\!\bullet U,V \bullet\!\} \bullet\!)} \qquad \mathsf{nm}\!\uparrow \frac{S\{\!\bullet \top,\top \bullet\!\}}{S\{\top\}}$$

$$\mathsf{nm}_1\!\downarrow \frac{S\{0\}}{S[0,0]} \qquad \mathsf{m}_1\!\downarrow \frac{S\{\!\bullet [R,U],[T,V] \bullet\!\}}{S[\{\!\bullet R,T \bullet\!\},\{\!\bullet U,V \bullet\!\}]} \qquad \mathsf{m}_1\!\uparrow \frac{S((\!\bullet R,U \bullet\!),(\!\bullet T,V \bullet\!))}{S(\!\bullet (R,T),(U,V) \bullet\!)} \qquad \mathsf{nm}_1\!\uparrow \frac{S(\top,\top)}{S\{\top\}}$$

$$\mathsf{nm}_2\!\downarrow \frac{S\{0\}}{S(0,0)} \qquad \mathsf{m}_2\!\downarrow \frac{S\{\!\bullet (R,U),(T,V) \bullet\!\}}{S(\{\!\bullet R,T \bullet\!\},\{\!\bullet U,V \bullet\!\})} \qquad \mathsf{m}_2\!\uparrow \frac{S[(\!\bullet R,U \bullet\!),(\!\bullet T,V \bullet\!)]}{S(\!\bullet [R,T],[U,V] \bullet\!)} \qquad \mathsf{nm}_2\!\uparrow \frac{S[\top,\top]}{S\{\top\}}$$

$$\mathsf{nl}_1\!\downarrow \frac{S\{0\}}{S\{?0\}} \qquad \mathsf{l}_1\!\downarrow \frac{S\{\!\bullet ?R,?T \bullet\!\}}{S\{?\{\!\bullet R,T \bullet\!\}\}} \qquad \mathsf{l}_1\!\uparrow \frac{S\{!(\!\bullet R,T \bullet\!)\}}{S(\!\bullet !R,!T \bullet\!)} \qquad \mathsf{nl}_1\!\uparrow \frac{S\{!\top\}}{S\{\top\}}$$

$$\mathsf{nl}_2\!\downarrow \frac{S\{0\}}{S\{!0\}} \qquad \mathsf{l}_2\!\downarrow \frac{S\{\!\bullet !R,!T \bullet\!\}}{S\{!\{\!\bullet R,T \bullet\!\}\}} \qquad \mathsf{l}_2\!\uparrow \frac{S\{?(\!\bullet R,T \bullet\!)\}}{S(\!\bullet ?R,?T \bullet\!)} \qquad \mathsf{nl}_2\!\uparrow \frac{S\{?\top\}}{S\{\top\}}$$

$$\mathsf{nz}\!\downarrow \frac{S\{\bot\}}{S\{?0\}} \qquad \mathsf{z}\!\downarrow \frac{S[?R,T]}{S\{?\{\!\bullet R,T \bullet\!\}\}} \qquad \mathsf{z}\!\uparrow \frac{S\{!(\!\bullet R,T \bullet\!)\}}{S(!R,T)} \qquad \mathsf{nz}\!\uparrow \frac{S\{!\top\}}{S\{1\}}$$

**Fig. 5.** System SLLS

versions of the medial rules. In the local system for classical logic [2] these rules are hidden in the equational theory for structures. It might be argued about doing the same for linear logic. In this presentation I chose not to do so because of the following reasons: First, not all of them are equivalences in linear logic, e.g. we have $0 \multimap ?0$ but not $?0 \multimap 0$, and second, for obvious reasons I want to use the same equational theory for both systems, SLS and SLLS. But the new equations, e.g. $0 = !0$, would be redundant for system SLS.

**5.2 Proposition** *The rule* $\mathsf{t}{\downarrow}$ *is derivable in* $\{\mathsf{at}{\downarrow}, \mathsf{nm}{\downarrow}, \mathsf{nm}_1{\downarrow}, \mathsf{nm}_2{\downarrow}, \mathsf{nl}_1{\downarrow}, \mathsf{nl}_2{\downarrow}\}$. *Dually,* $\mathsf{t}{\uparrow}$ *is derivable in* $\{\mathsf{at}{\uparrow}, \mathsf{nm}{\uparrow}, \mathsf{nm}_1{\uparrow}, \mathsf{nm}_2{\uparrow}, \mathsf{nl}_1{\uparrow}, \mathsf{nl}_2{\uparrow}\}$.

**Proof:** Similar to Proposition 5.1. □

The problem that arises now is that the rules $\mathsf{w}{\downarrow}, \mathsf{w}{\uparrow}, \mathsf{b}{\downarrow}, \mathsf{b}{\uparrow}$ cannot be reduced to their atomic versions in the same way as this has been done for the rules $\mathsf{t}{\downarrow}, \mathsf{t}{\uparrow}, \mathsf{c}{\downarrow}, \mathsf{c}{\uparrow}$. This is not really a surprise since the basic idea of the exponentials is to guard whole subformulas such that no arbitrary weakening and contraction is possible. For this reason, I will reduce the rules $\mathsf{w}{\downarrow}, \mathsf{w}{\uparrow}, \mathsf{b}{\downarrow}, \mathsf{b}{\uparrow}$ to the rules $\mathsf{t}{\downarrow}, \mathsf{t}{\uparrow}, \mathsf{c}{\downarrow}, \mathsf{c}{\uparrow}$, respectively, by using the well-known equivalence

$$!(A \mathbin{\&} B) \equiv\ !A \otimes\, !B \quad ,$$

which is encoded in the rules $\mathsf{z}{\downarrow}, \mathsf{z}{\uparrow}$ and their nullary versions $\mathsf{nz}{\downarrow}, \mathsf{nz}{\uparrow}$.

**5.3 Proposition** *The rule* $\mathsf{w}{\downarrow}$ *is derivable in* $\{\mathsf{nz}{\downarrow}, \mathsf{t}{\downarrow}\}$, *and the rule* $\mathsf{b}{\downarrow}$ *is derivable in* $\{\mathsf{z}{\downarrow}, \mathsf{c}{\downarrow}\}$. *Dually, the rule* $\mathsf{w}{\uparrow}$ *is derivable in* $\{\mathsf{nz}{\uparrow}, \mathsf{t}{\uparrow}\}$, *and the rule* $\mathsf{b}{\uparrow}$ *is derivable in* $\{\mathsf{z}{\uparrow}, \mathsf{c}{\uparrow}\}$.

**Proof:** Use the derivations

$$\mathsf{t}{\downarrow}\frac{\mathsf{nz}{\downarrow}\dfrac{S\{\bot\}}{S\{?0\}}}{S\{?R\}} \qquad \text{and} \qquad \mathsf{c}{\downarrow}\frac{\mathsf{z}{\downarrow}\dfrac{S[?R,R]}{S\{?\{\!|R,R|\!\}\}}}{S\{?R\}} \quad ,$$

for $\mathsf{w}{\downarrow}$ and $\mathsf{b}{\downarrow}$, respectively. □

All new rules, introduced in this section, are sound, i.e. correspond to linear implications. More precisely, we have:

**5.4 Proposition** *The rules* $\mathsf{m}$, $\mathsf{m}_1{\downarrow}$, $\mathsf{m}_2{\downarrow}$, $\mathsf{l}_1{\downarrow}$, $\mathsf{l}_2{\downarrow}$ *are derivable in* $\{\mathsf{t}{\downarrow}, \mathsf{c}{\downarrow}\}$, *the rule* $\mathsf{z}{\downarrow}$ *is derivable in* $\{\mathsf{t}{\downarrow}, \mathsf{b}{\downarrow}\}$, *the rules* $\mathsf{nm}{\downarrow}$, $\mathsf{nm}_1{\downarrow}$, $\mathsf{nm}_2{\downarrow}$, $\mathsf{nl}_1{\downarrow}$, $\mathsf{nl}_2{\downarrow}$ *are derivable in* $\{\mathsf{t}{\downarrow}\}$, *and the rule* $\mathsf{nz}{\downarrow}$ *is derivable in* $\{\mathsf{w}{\downarrow}\}$. *Dually, the rules* $\mathsf{m}$, $\mathsf{m}_1{\uparrow}$, $\mathsf{m}_2{\uparrow}$, $\mathsf{l}_1{\uparrow}$, $\mathsf{l}_2{\uparrow}$ *are derivable in* $\{\mathsf{t}{\uparrow}, \mathsf{c}{\uparrow}\}$, *the rule* $\mathsf{z}{\uparrow}$ *is derivable in* $\{\mathsf{t}{\uparrow}, \mathsf{b}{\uparrow}\}$, *the rules* $\mathsf{nm}{\uparrow}$, $\mathsf{nm}_1{\uparrow}$, $\mathsf{nm}_2{\uparrow}$, $\mathsf{nl}_1{\uparrow}$, $\mathsf{nl}_2{\uparrow}$ *are derivable in* $\{\mathsf{t}{\uparrow}\}$, *and the rule* $\mathsf{nz}{\uparrow}$ *is derivable in* $\{\mathsf{w}{\uparrow}\}$.

**Proof:** For the rule $\mathsf{m}_1{\downarrow}$, use the derivation

$$\begin{array}{ll}
 & S\{\!|[R,U],[T,V]|\!\} \\
\mathsf{t}{\downarrow} & \overline{S\{\!|[R,U],[T,\{\!|U,V|\!\}]|\!\}} \\
\mathsf{t}{\downarrow} & \overline{S\{\!|[R,U],[\{\!|R,T|\!\},\{\!|U,V|\!\}]|\!\}} \\
\mathsf{t}{\downarrow} & \overline{S\{\!|[R,\{\!|U,V|\!\}],[\{\!|R,T|\!\},\{\!|U,V|\!\}]|\!\}} \\
\mathsf{t}{\downarrow} & \overline{S\{\!|[\{\!|R,T|\!\},\{\!|U,V|\!\}],[\{\!|R,T|\!\},\{\!|U,V|\!\}]|\!\}} \\
\mathsf{c}{\downarrow} & \overline{S[\{\!|R,T|\!\},\{\!|U,V|\!\}]}
\end{array} \quad .$$

The other cases are similar. □

**5.5 Theorem** *Systems* SLLS *and* SLS *are strongly equivalent.*

**Proof:** Immediate consequence of Propositions 5.1–5.4. □

Observe that Propositions 5.1–5.4 show that there is a certain modularity involved in the equivalence of the two systems. For instance, the user can choose to have either the rules $\mathsf{at}{\downarrow}, \mathsf{nm}{\downarrow}, \mathsf{nm}_1{\downarrow}, \mathsf{nm}_2{\downarrow}, \mathsf{nl}_1{\downarrow}, \mathsf{nl}_2{\downarrow}$ or the rule $\mathsf{t}{\downarrow}$ in the system without affecting the other rules.

$$\mathsf{1}\!\downarrow \frac{}{\mathsf{1}} \qquad \mathsf{ai}\!\downarrow \frac{S\{\mathsf{1}\}}{S[a,\bar{a}]} \qquad \mathsf{nm_1}\!\downarrow \frac{S\{\mathsf{0}\}}{S[\mathsf{0},\mathsf{0}]} \qquad \mathsf{m_1}\!\downarrow \frac{S\llbracket [R,U],[T,V] \rrbracket}{S[\llbracket R,T \rrbracket, \llbracket U,V \rrbracket]}$$

$$\mathsf{s} \frac{S([R,U],T)}{S[(R,T),U]} \qquad \mathsf{nm_2}\!\downarrow \frac{S\{\mathsf{0}\}}{S(\mathsf{0},\mathsf{0})} \qquad \mathsf{m_2}\!\downarrow \frac{S\llbracket (R,U),(T,V) \rrbracket}{S(\llbracket R,T \rrbracket, \llbracket U,V \rrbracket)}$$

$$\mathsf{d}\!\downarrow \frac{S\llparenthesis [R,U],[T,V] \rrparenthesis}{S[\llparenthesis R,T \rrparenthesis, \llbracket U,V \rrbracket]} \qquad \mathsf{nm}\!\downarrow \frac{S\{\mathsf{0}\}}{S\llparenthesis \mathsf{0},\mathsf{0} \rrparenthesis} \qquad \mathsf{m} \frac{S\llbracket \llparenthesis R,U \rrparenthesis, \llparenthesis T,V \rrparenthesis \rrbracket}{S\llparenthesis \llbracket R,T \rrbracket, \llbracket U,V \rrbracket \rrparenthesis}$$

$$\mathsf{p}\!\downarrow \frac{S\{![R,T]\}}{S[!R,?T]} \qquad \mathsf{nl_1}\!\downarrow \frac{S\{\mathsf{0}\}}{S\{?\mathsf{0}\}} \qquad \mathsf{l_1}\!\downarrow \frac{S\llbracket ?R,?T \rrbracket}{S\{?\llbracket R,T \rrbracket\}}$$

$$\mathsf{at}\!\downarrow \frac{S\{\mathsf{0}\}}{S\{a\}} \qquad \mathsf{nl_2}\!\downarrow \frac{S\{\mathsf{0}\}}{S\{!\mathsf{0}\}} \qquad \mathsf{l_2}\!\downarrow \frac{S\llbracket !R,!T \rrbracket}{S\{!\llbracket R,T \rrbracket\}}$$

$$\mathsf{ac}\!\downarrow \frac{S\llbracket a,a \rrbracket}{S\{a\}} \qquad \mathsf{nz}\!\downarrow \frac{S\{\bot\}}{S\{?\mathsf{0}\}} \qquad \mathsf{z}\!\downarrow \frac{S[?R,T]}{S\{?\llbracket R,T \rrbracket\}}$$

**Fig. 6.** System $\mathsf{LLS}$

**5.6 Definition** *System* $\mathsf{LLS}$, shown in Fig. 6, is obtained from the down fragment of system $\mathsf{SLLS}$ by adding the axiom $\mathsf{1}\!\downarrow$.

**5.7 Theorem** *The systems* $\mathsf{LLS}$ *and* $\mathsf{LS}$ *are strongly equivalent.*

**Proof:** Again, use Propositions 5.1–5.4. □

**5.8 Corollary (Cut elimination)** *System* $\mathsf{LLS}$ *is equivalent to every subsystem of* $\mathsf{SLLS} \cup \{\mathsf{1}\!\downarrow\}$ *containing* $\mathsf{LLS}$.

**5.9 Corollary** *The rule* $\mathsf{i}\!\uparrow$ *is admissible for system* $\mathsf{LLS}$.

## 6 Conclusions and Future Work

There are two results presented in this paper. First, the work of [7, 10] is extended by showing that also full linear logic can benefit from its presentation in the calculus of structures. In particular, the rules for the additives are split into two parts, namely, a purely multiplicative part (the rules $\mathsf{d}\!\downarrow$ and $\mathsf{d}\!\uparrow$) and an explicit contraction (the rules $\mathsf{c}\!\downarrow$ and $\mathsf{c}\!\uparrow$), whereas in the sequent calculus, contraction is contained implicitly in the rules for the additives. The second achievement of this paper is to show that in the calculus of structures it is possible to reduce contraction in linear logic to an atomic version, which is not possible in the sequent calculus. Apart from their independent interest, those two insights might also help to extend the work in [3], which captures a process algebra in a purely logical way.

Although system SLLS is quite big, one should observe that all rules either deal directly with atoms or follow a certain common pattern in which two, three, or four substructures are rearranged. In fact, the discovery of the rules for system SLLS contributed to the development of a general recipe for designing systems in the calculus of structures. The deep reasons behind these regularities are not known and are the topic of current investigation.

A problem of the local system is that proof search becomes very non deterministic. However, since the system is still logical and bears a certain uniformity, it offers some possibilities for analyses on synchronicity in order to focus proofs in a similar way as it has been done in the sequent calculus [9, 1]. It should be of major interest of future investigation to control the non-determinism in system LLS, because then one is able to use it for proof search in a distributed way.

Another focus of future research lies in the so-called decomposition theorems. There are already preliminary results showing that similar decomposition theorems as shown in [2] can also be achieved for system SLLS. But in order to relate decomposition to cut elimination it is necessary to find a decomposition that separates core from non-core, where the core of the system contains the rules that are needed to reduce the interaction rules to atoms, i.e. the rules $\mathsf{s}, \mathsf{d}\downarrow, \mathsf{d}\uparrow, \mathsf{p}\downarrow, \mathsf{p}\uparrow$ in system SLLS. Although such a decomposition exists for multiplicative exponential linear logic [7, 10] and a non-commutative extension of it [8], it defied any proof for full linear logic so far.

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