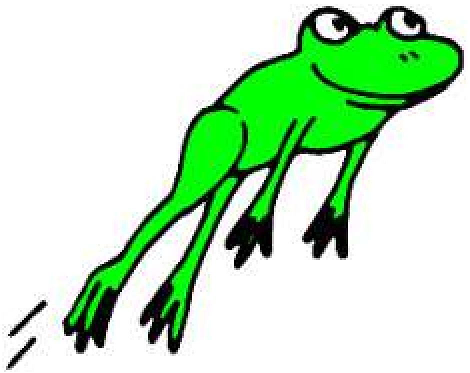
A gentle introduction to deep inference

11. Subatomic proof theory



ESSLLI 2025

Victoria Barrett and Lutz Straßburger

Another deep inference proof system : from SMLSg to SBV

$$i{\downarrow}\,\frac{1}{A \parr \bar{A}} \qquad i{\uparrow}\,\frac{A \otimes \bar{A}}{\perp}$$

$$s\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C}$$

Another deep inference proof system : from SMLSg to SBV

$$\mathsf{i}\!\downarrow \frac{1}{A \parr \bar{A}} \qquad \mathsf{i}\!\uparrow \frac{A \otimes \bar{A}}{\bot}$$

$$\mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C}$$

Add a connective $\lhd$ which is

- associative $(A \lhd B) \lhd C = A \lhd (B \lhd C)$
- self-dual $\overline{A \lhd B} = \bar{A} \lhd \bar{B}$
- non-commutative $A \lhd B \neq B \lhd A$

Another deep inference proof system : from SMLSg to SBV

$$i{\downarrow}\frac{1}{A \parr \bar{A}} \qquad i{\uparrow}\frac{A \otimes \bar{A}}{\bot}$$

$$s\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C}$$

Add a connective $\triangleleft$ which is

- associative $(A \triangleleft B) \triangleleft C = A \triangleleft (B \triangleleft C)$
- self-dual $\overline{A \triangleleft B} = \bar{A} \triangleleft \bar{B}$
- non-commutative $A \triangleleft B \neq B \triangleleft A$

2 cases for decomposing identity to atomic form :

Another deep inference proof system : from SMLSg to SBV

$$\text{i}\downarrow\frac{1}{A \parr \bar{A}} \qquad \text{i}\uparrow\frac{A \otimes \bar{A}}{\bot}$$

$$\text{s}\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C}$$

Add a connective $\triangleleft$ which is

- associative $(A \triangleleft B) \triangleleft C = A \triangleleft (B \triangleleft C)$
- self-dual $\overline{A \triangleleft B} = \bar{A} \triangleleft \bar{B}$
- non-commutative $A \triangleleft B \neq B \triangleleft A$

2 cases for decomposing identity to atomic form :

$$\frac{1}{(B \otimes C) \parr (\bar{B} \parr \bar{C})}$$

Another deep inference proof system : from SMLSg to SBV

$$\text{i}\!\downarrow\frac{1}{A \parr \bar{A}} \qquad \text{i}\!\uparrow\frac{A \otimes \bar{A}}{\bot}$$

$$\mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C}$$

Add a connective $\lhd$ which is

- associative $(A \lhd B) \lhd C = A \lhd (B \lhd C)$
- self-dual $\overline{A \lhd B} = \bar{A} \lhd \bar{B}$
- non-commutative $A \lhd B \neq B \lhd A$

2 cases for decomposing identity to atomic form :

$$= \frac{1}{2\mathsf{s}\,\dfrac{\text{i}\!\downarrow\dfrac{1}{B \parr \bar{B}} \otimes \text{i}\!\downarrow\dfrac{1}{C \parr \bar{C}}}{(B \otimes C) \parr (\bar{B} \parr \bar{C})}}$$

Another deep inference proof system : from SMLSg to SBV

$$\mathsf{i}{\downarrow}\frac{1}{A \parr \bar{A}} \qquad \mathsf{i}{\uparrow}\frac{A \otimes \bar{A}}{\bot}$$

$$\mathsf{s}\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C}$$

Add a connective $\lhd$ which is

- associative $(A \lhd B) \lhd C = A \lhd (B \lhd C)$
- self-dual $\overline{A \lhd B} = \bar{A} \lhd \bar{B}$
- non-commutative $A \lhd B \neq B \lhd A$

2 cases for decomposing identity to atomic form :

$$= \frac{1}{\mathsf{2s}\dfrac{\mathsf{i}{\downarrow}\dfrac{1}{B \parr \bar{B}} \otimes \mathsf{i}{\downarrow}\dfrac{1}{C \parr \bar{C}}}{(B \otimes C) \parr (\bar{B} \parr \bar{C})}}$$

$$\mathsf{i}{\downarrow}\frac{1}{(B \lhd C) \parr (\bar{B} \lhd \bar{C})}$$

Another deep inference proof system : from SMLSg to SBV

$$\text{i}{\downarrow}\,\frac{1}{A \parr \bar{A}} \qquad \text{i}{\uparrow}\,\frac{A \otimes \bar{A}}{\bot}$$

$$\text{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C}$$

Add a connective $\triangleleft$ which is

- associative $\quad (A \triangleleft B) \triangleleft C = A \triangleleft (B \triangleleft C)$
- self-dual $\quad \overline{A \triangleleft B} = \bar{A} \triangleleft \bar{B}$
- non-commutative $\quad A \triangleleft B \neq B \triangleleft A$

2 cases for decomposing identity to atomic form :

$$=\frac{1}{\text{2s}\,\dfrac{\text{i}{\downarrow}\,\dfrac{1}{B \parr \bar{B}} \otimes \text{i}{\downarrow}\,\dfrac{1}{C \parr \bar{C}}}{(B \otimes C) \parr (\bar{B} \parr \bar{C})}}$$

$$=\frac{1}{\dfrac{\text{i}{\downarrow}\,\dfrac{1}{B \parr \bar{B}} \qquad \text{i}{\downarrow}\,\dfrac{1}{C \parr \bar{C}}}{(B \triangleleft C) \parr (\bar{B} \triangleleft \bar{C})}}$$

Another deep inference proof system: from SMLSg to SBV

$$\mathsf{i}{\downarrow}\frac{1}{A \parr \bar{A}} \qquad \mathsf{i}{\uparrow}\frac{A \otimes \bar{A}}{\bot}$$

$$\mathsf{s}\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C}$$

Add a connective $\lhd$ which is

- associative $\quad (A \lhd B) \lhd C = A \lhd (B \lhd C)$
- self-dual $\quad \overline{A \lhd B} = \bar{A} \lhd \bar{B}$
- non-commutative $\quad A \lhd B \neq B \lhd A$

2 cases for decomposing identity to atomic form:

$$2\mathsf{s}\frac{=\dfrac{1}{\mathsf{i}{\downarrow}\dfrac{1}{B \parr \bar{B}} \otimes \mathsf{i}{\downarrow}\dfrac{1}{C \parr \bar{C}}}}{(B \otimes C) \parr (\bar{B} \parr \bar{C})} \qquad \mathsf{q}{\downarrow}\frac{=\dfrac{1}{\mathsf{i}{\downarrow}\dfrac{1}{B \parr \bar{B}} \lhd \mathsf{i}{\downarrow}\dfrac{1}{C \parr \bar{C}}}}{(B \lhd C) \parr (\bar{B} \lhd \bar{C})}$$

Another deep inference proof system : SBV

$$\mathsf{q}{\downarrow}\,\frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)} \qquad \mathsf{ai}{\downarrow}\,\frac{\circ}{a \parr \bar{a}}$$

$$\mathsf{q}{\uparrow}\,\frac{(A \lhd B) \otimes (C \lhd D)}{(A \otimes C) \lhd (B \otimes D)} \qquad \mathsf{ai}{\uparrow}\,\frac{a \otimes \bar{a}}{\circ}$$

$$\mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C}$$

$$A \otimes B = B \otimes A$$
$$(A \otimes B) \otimes C = A \otimes (B \otimes C)$$

$$A \parr B = B \parr A$$
$$(A \parr B) \parr C = A \parr (B \parr C)$$

$$(A \lhd B) \lhd C = A \lhd (B \lhd C)$$

$$A = A \parr \circ = A \otimes \circ = A \lhd \circ = \circ \lhd A$$

self-dual $\overline{A \lhd B} = \bar{A} \lhd \bar{B}$

non-commutative $A \lhd B \neq B \lhd A$

$$q{\downarrow}\,\frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)}$$

$$q{\downarrow}\;\frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)}$$

$$m\;\frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

Switches decompose identities

$$q\downarrow \frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)}$$

Medials decompose contractions

$$m \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

Switches decompose identities

$$\text{q}\!\downarrow \frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)}$$

$$\text{s}\,\frac{(A \vee B) \wedge C}{(A \wedge C) \vee B}$$

Medials decompose contractions

$$\text{m}\,\frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

Switches decompose identities

$$\text{q}{\downarrow}\,\frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)}$$

Medials decompose contractions

$$\text{s}{\downarrow}\,\frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \qquad \text{s}{\uparrow}\,\frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$\text{m}\,\frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

Switches decompose identities

$$\mathsf{q}{\downarrow}\,\frac{(A \parr B) \triangleleft (C \parr D)}{(A \triangleleft C) \parr (B \triangleleft D)}$$

$$\mathsf{s}{\downarrow}\,\frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)}$$

Medials decompose contractions

$$\mathsf{m}\,\frac{(A \mathbin{\&} B) \oplus (C \mathbin{\&} D)}{(A \oplus C) \mathbin{\&} (B \oplus D)}$$

$$\mathsf{m}_1{\downarrow}\,\frac{(A \parr B) \oplus (C \parr D)}{(A \oplus C) \parr (B \oplus D)}$$

$$\mathsf{m}_2{\downarrow}\,\frac{(A \otimes B) \oplus (C \otimes D)}{(A \oplus C) \otimes (B \oplus D)}$$

$$\mathsf{s}{\downarrow}\,\frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \qquad \mathsf{s}{\uparrow}\,\frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$\mathsf{m}\,\frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

Switches decompose identities

$$q\!\downarrow \frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)}$$

$$p\!\downarrow \frac{!(A \parr B)}{!A \parr ?B} \qquad s\!\downarrow \frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)}$$

Medials decompose contractions

$$m \frac{(A \,\&\, B) \oplus (C \,\&\, D)}{(A \oplus C) \,\&\, (B \oplus D)} \qquad l_1\!\downarrow \frac{?A \oplus ?B}{?(A \oplus B)}$$

$$m_1\!\downarrow \frac{(A \parr B) \oplus (C \parr D)}{(A \oplus C) \parr (B \oplus D)} \qquad l_2\!\downarrow \frac{!A \oplus !B}{!(A \oplus B)}$$

$$m_2\!\downarrow \frac{(A \otimes B) \oplus (C \otimes D)}{(A \oplus C) \otimes (B \oplus D)}$$

$$s\!\downarrow \frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \qquad s\!\uparrow \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$m \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

Subatomic proof theory:

$$\frac{\Phi}{K\{a\}\{\bar{a}\}}$$

Subatomic proof theory :

$$
\begin{tikzcd}[column sep=small]
 & \begin{array}{c} \Phi \\ \hdashline K\{a\}\{\bar{a}\} \end{array} & \\
\begin{array}{c} [\bot/a, \top/\bar{a}]\Phi \\ \hdashline K\{\bot\}\{\top\} \end{array} \arrow[ur] & & \begin{array}{c} [\top/a, \bot/\bar{a}]\Phi \\ \hdashline K\{\top\}\{\bot\} \end{array} \arrow[ul]
\end{tikzcd}
$$

Subatomic proof theory : atoms as self-dual non-commutative connectives whose arguments are their truth value

$$
\begin{tikzcd}
& \dfrac{\Phi'}{K\{\bot a \top\}\{\top a \bot\}} & \\
& \dfrac{\Phi}{K\{a\}\{\bar{a}\}} \arrow[u] & \\
\dfrac{[\bot/a, \top/\bar{a}]\Phi}{K\{\bot\}\{\top\}} \arrow[ur] & & \dfrac{[\top/a, \bot/\bar{a}]\Phi}{K\{\top\}\{\bot\}} \arrow[ul]
\end{tikzcd}
$$

Subatomic proof theory:

atoms as self-dual non-commutative connectives whose arguments are their truth value

Interpretation $[\![\Phi \; a \; \Omega]\!]$ generated by:

$$[\![\bot \, a \, \bot]\!] = \bot \qquad [\![\bot \, a \, \top]\!] = a$$

$$[\![\top \, a \, \bot]\!] = \bar{a} \qquad [\![\top \, a \, \top]\!] = \top$$

$$\begin{tikzcd}[column sep=small]
 & \dfrac{\Phi'}{K\{\bot \, a \, \top\}\{\top \, a \, \bot\}} & \\
 & \dfrac{\Phi}{K\{a\}\{\bar{a}\}} \arrow[u] & \\
\dfrac{[\bot/a, \top/\bar{a}]\,\Phi}{K\{\bot\}\{\top\}} \arrow[ur] & & \dfrac{[\top/a, \bot/\bar{a}]\,\Phi}{K\{\top\}\{\bot\}} \arrow[ul]
\end{tikzcd}$$

$$\mathsf{ai}\!\downarrow \frac{\top}{b \vee \bar{b}}$$

$$\mathsf{ac}\!\downarrow \frac{a \vee a}{a}$$

$$\mathsf{ac}\!\uparrow \frac{b}{b \wedge b}$$

$$[\![\bot\, a\, \bot]\!] = \bot \qquad [\![\bot\, a\, \top]\!] = a$$

$$\mathsf{ai}\!\uparrow \frac{a \wedge \bar{a}}{\bot}$$

$$[\![\top\, a\, \bot]\!] = \bar{a} \qquad [\![\top\, a\, \top]\!] = \top$$

$$\frac{\top}{(\bot\, b\, \top) \vee (\top\, b\, \bot)} \longrightarrow \mathsf{ai}\!\downarrow \frac{\top}{b \vee \bar{b}}$$

$$\mathsf{ac}\!\downarrow \frac{a \vee a}{a}$$

$$\mathsf{ac}\!\uparrow \frac{b}{b \wedge b}$$

$$\mathsf{ai}\!\uparrow \frac{a \wedge \bar{a}}{\bot}$$

$$[\![\bot\, a\, \bot]\!] = \bot \qquad [\![\bot\, a\, \top]\!] = a$$

$$[\![\top\, a\, \bot]\!] = \bar{a} \qquad [\![\top\, a\, \top]\!] = \top$$

$$\frac{(\bot \vee \top) \; b \; (\top \vee \bot)}{(\bot b \top) \vee (\top b \bot)} \longrightarrow$$

$$\mathsf{ai}{\downarrow}\,\frac{\top}{b \vee \bar{b}}$$

$$\mathsf{ac}{\downarrow}\,\frac{a \vee a}{a}$$

$$\mathsf{ac}{\uparrow}\,\frac{b}{b \wedge b}$$

$$\mathsf{ai}{\uparrow}\,\frac{a \wedge \bar{a}}{\bot}$$

$$[\![\bot a \bot]\!] = \bot \qquad [\![\bot a \top]\!] = a$$

$$[\![\top a \bot]\!] = \bar{a} \qquad [\![\top a \top]\!] = \top$$

$$\frac{(\bot \vee \top)\ b\ (\top \vee \bot)}{(\bot b \top) \vee (\top b \bot)} \longrightarrow \mathsf{ai}{\downarrow}\,\frac{\top}{b \vee \bar{b}}$$

$$\frac{(\bot a \top) \vee (\bot a \top)}{\bot a \top} \longrightarrow \mathsf{ac}{\downarrow}\,\frac{a \vee a}{a}$$

$$\mathsf{ac}{\uparrow}\,\frac{b}{b \wedge b}$$

$$[\![\bot a \bot]\!] = \bot \qquad [\![\bot a \top]\!] = a$$

$$[\![\top a \bot]\!] = \bar{a} \qquad [\![\top a \top]\!] = \top$$

$$\mathsf{ai}{\uparrow}\,\frac{a \wedge \bar{a}}{\bot}$$

$$\frac{(\bot \vee \top)\ b\ (\top \vee \bot)}{(\bot b \top) \vee (\top b \bot)} \longrightarrow \ \mathrm{ai}\!\downarrow \frac{\top}{b \vee \bar{b}}$$

$$\frac{(\bot a \top) \vee (\bot a \top)}{(\bot \vee \bot)\ a\ (\top \vee \top)} \longrightarrow \ \mathrm{ac}\!\downarrow \frac{a \vee a}{a}$$

$$[\![\bot a \bot]\!] = \bot \qquad [\![\bot a \top]\!] = a$$

$$[\![\top a \bot]\!] = \bar{a} \qquad [\![\top a \top]\!] = \top$$

$$\frac{(\bot \vee \top)\ b\ (\top \vee \bot)}{(\bot b \top) \vee (\top b \bot)} \longrightarrow \mathsf{ai}{\downarrow}\,\frac{\top}{b \vee \bar{b}}$$

$$\frac{(\bot a \top) \vee (\bot a \top)}{(\bot \vee \bot)\ a\ (\top \vee \top)} \longrightarrow \mathsf{ac}{\downarrow}\,\frac{a \vee a}{a}$$

$$\frac{(\bot \wedge \bot)\ b\ (\top \wedge \top)}{(\bot b \top) \wedge (\bot b \top)} \longrightarrow \mathsf{ac}{\uparrow}\,\frac{b}{b \wedge b}$$

$$\frac{(\bot a \top) \wedge (\top a \bot)}{(\bot \wedge \top)\ a\ (\top \wedge \bot)} \longrightarrow \mathsf{ai}{\uparrow}\,\frac{a \wedge \bar{a}}{\bot}$$

$$[\![\bot a \bot]\!] = \bot \qquad [\![\bot a \top]\!] = a$$

$$[\![\top a \bot]\!] = \bar{a} \qquad [\![\top a \top]\!] = \top$$

$$\frac{\top}{b \vee \top} \longleftarrow \frac{(\bot \vee \top)\; b\; (\top \vee \top)}{(\bot\, b\, \top) \vee (\top\, b\, \top)} \longrightarrow \mathsf{ai}{\downarrow}\,\frac{\top}{b \vee \bar{b}}$$

$$\frac{(\bot\, a\, \top) \vee (\bot\, a\, \top)}{(\bot \vee \bot)\; a\; (\top \vee \top)} \longrightarrow \mathsf{ac}{\downarrow}\,\frac{a \vee a}{a}$$

$$\frac{(\bot \wedge \bot)\; b\; (\top \wedge \top)}{(\bot\, b\, \top) \wedge (\bot\, b\, \top)} \longrightarrow \mathsf{ac}{\uparrow}\,\frac{b}{b \wedge b}$$

$$\frac{(\bot\, a\, \top) \wedge (\top\, a\, \bot)}{(\bot \wedge \top)\; a\; (\top \wedge \bot)} \longrightarrow \mathsf{ai}{\uparrow}\,\frac{a \wedge \bar{a}}{\bot}$$

$$[\![\bot\, a\, \bot]\!] = \bot \qquad [\![\bot\, a\, \top]\!] = a$$

$$[\![\top\, a\, \bot]\!] = \bar{a} \qquad [\![\top\, a\, \top]\!] = \top$$

$$\frac{b}{b \vee \bot} \qquad \frac{\top}{b \vee \top} \quad \longleftarrow \quad \frac{(\bot \vee \bot)\; b\; (\top \vee \bot)}{(\bot\, b\, \top) \vee (\bot\, b\, \bot)} \quad \longrightarrow \quad \mathsf{ai}{\downarrow}\,\frac{\top}{b \vee \bar{b}}$$

$$\frac{(\bot\, a\, \top) \vee (\bot\, a\, \top)}{(\bot \vee \bot)\; a\; (\top \vee \top)} \quad \longrightarrow \quad \mathsf{ac}{\downarrow}\,\frac{a \vee a}{a}$$

$$\frac{(\bot \wedge \bot)\; b\; (\top \wedge \top)}{(\bot\, b\, \top) \wedge (\bot\, b\, \top)} \quad \longrightarrow \quad \mathsf{ac}{\uparrow}\,\frac{b}{b \wedge b}$$

$$\llbracket \bot\, a\, \bot \rrbracket = \bot \qquad \llbracket \bot\, a\, \top \rrbracket = a$$

$$\llbracket \top\, a\, \bot \rrbracket = \bar{a} \qquad \llbracket \top\, a\, \top \rrbracket = \top$$

$$\frac{(\bot\, a\, \top) \wedge (\top\, a\, \bot)}{(\bot \wedge \top)\; a\; (\top \wedge \bot)} \quad \longrightarrow \quad \mathsf{ai}{\uparrow}\,\frac{a \wedge \bar{a}}{\bot}$$

$$\frac{b}{b \vee b} \qquad \frac{b}{b \vee \bot} \qquad \frac{\top}{b \vee \top} \longleftarrow \frac{(\bot \vee \bot)\; b \;(\top \vee \top)}{(\bot\, b\, \top) \vee (\bot\, b\, \top)} \longrightarrow \mathsf{ai}{\downarrow}\,\frac{\top}{b \vee \bar{b}}$$

$$\frac{(\bot\, a\, \top) \vee (\bot\, a\, \top)}{(\bot \vee \bot)\; a \;(\top \vee \top)} \longrightarrow \mathsf{ac}{\downarrow}\,\frac{a \vee a}{a}$$

$$\frac{(\bot \wedge \bot)\; b \;(\top \wedge \top)}{(\bot\, b\, \top) \wedge (\bot\, b\, \top)} \longrightarrow \mathsf{ac}{\uparrow}\,\frac{b}{b \wedge b}$$

$$[\![\bot\, a\, \bot]\!] = \bot \qquad [\![\bot\, a\, \top]\!] = a$$

$$[\![\top\, a\, \bot]\!] = \bar{a} \qquad [\![\top\, a\, \top]\!] = \top$$

$$\frac{(\bot\, a\, \top) \wedge (\top\, a\, \bot)}{(\bot \wedge \top)\; a \;(\top \wedge \bot)} \longrightarrow \mathsf{ai}{\uparrow}\,\frac{a \wedge \bar{a}}{\bot}$$

$$\frac{b}{b \vee b} \qquad \frac{b}{b \vee \bot} \qquad \frac{\top}{b \vee \top} \quad \longleftarrow \quad \frac{(A \vee B)\; b\; (C \vee D)}{(A\, b\, C) \vee (B\, b\, D)} \quad \longrightarrow \quad \mathsf{ai}{\downarrow}\,\frac{\top}{b \vee \bar{b}}$$

$$\frac{a \vee \bar{a}}{\top} \qquad \frac{\top \vee \top}{\top} \qquad \frac{\bot \vee \bot}{\bot}$$

$$\frac{\bot \vee \top}{\top} \qquad \frac{\bot \vee a}{a} \qquad \frac{a \vee \top}{\top} \quad \longleftarrow \quad \frac{(A\, a\, B) \vee (C\, a\, D)}{(A \vee C)\; a\; (B \vee D)} \quad \longrightarrow \quad \mathsf{ac}{\downarrow}\,\frac{a \vee a}{a}$$

$$\frac{(\bot \wedge \bot)\; b\; (\top \wedge \top)}{(\bot\, b\, \top) \wedge (\bot\, b\, \top)} \quad \longrightarrow \quad \mathsf{ac}{\uparrow}\,\frac{b}{b \wedge b}$$

$$[\![\bot\, a\, \bot]\!] = \bot \qquad [\![\bot\, a\, \top]\!] = a$$

$$[\![\top\, a\, \bot]\!] = \bar{a} \qquad [\![\top\, a\, \top]\!] = \top$$

$$\frac{(\bot\, a\, \top) \wedge (\top\, a\, \bot)}{(\bot \wedge \top)\; a\; (\top \wedge \bot)} \quad \longrightarrow \quad \mathsf{ai}{\uparrow}\,\frac{a \wedge \bar{a}}{\bot}$$

$$\mathrm{aw}{\downarrow}\,\frac{\bot}{a}$$

$$\mathrm{aw}{\uparrow}\,\frac{a}{\top}$$

$$[\![\bot\, a\, \bot]\!] = \bot \qquad [\![\bot\, a\, \top]\!] = a$$

$$[\![\top\, a\, \bot]\!] = \bar{a} \qquad [\![\top\, a\, \top]\!] = \top$$

$$\frac{\bot}{\bot\ a\ \top} \longrightarrow \text{aw}{\downarrow}\,\frac{\bot}{a}$$

$$\text{aw}{\uparrow}\,\frac{a}{\top}$$

$$[\![\bot\ a\ \bot]\!] = \bot \qquad [\![\bot\ a\ \top]\!] = a$$

$$[\![\top\ a\ \bot]\!] = \bar{a} \qquad [\![\top\ a\ \top]\!] = \top$$

$$\bot \; a \; \frac{\bot}{\top} \longrightarrow \mathrm{aw}{\downarrow}\,\frac{\bot}{a}$$

$$\mathrm{aw}{\uparrow}\,\frac{a}{\top}$$

$$[\![\bot\, a\, \bot]\!] = \bot \qquad [\![\bot\, a\, \top]\!] = a$$

$$[\![\top\, a\, \bot]\!] = \bar{a} \qquad [\![\top\, a\, \top]\!] = \top$$

$$\perp \quad a \quad \text{mix}\frac{\perp \wedge \top}{\perp \vee \top} \longrightarrow \text{aw}{\downarrow}\frac{\perp}{a}$$

$$\text{aw}{\uparrow}\frac{a}{\top}$$

$$[\![\perp a \perp]\!] = \perp \qquad [\![\perp a \top]\!] = a$$

$$[\![\top a \perp]\!] = \bar{a} \qquad [\![\top a \top]\!] = \top$$

$$\bot \quad a \quad {\scriptstyle \mathrm{mix}}\frac{\bot \wedge \top}{\bot \vee \top} \longrightarrow {\scriptstyle \mathrm{aw}\downarrow}\frac{\bot}{a}$$

$$\llbracket \bot\, a\, \bot \rrbracket = \bot \qquad \llbracket \bot\, a\, \top \rrbracket = a$$

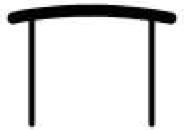
$$\llbracket \top\, a\, \bot \rrbracket = \bar{a} \qquad \llbracket \top\, a\, \top \rrbracket = \top$$

$${\scriptstyle \mathrm{mix}}\frac{\bot \wedge \top}{\bot \vee \top} \quad a \quad \top \longrightarrow {\scriptstyle \mathrm{aw}\uparrow}\frac{a}{\top}$$

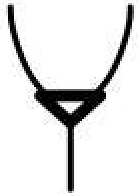
$[\![\Phi \; a \; \Omega]\!]$	$[\![\Omega]\!] = \top$	$[\![\Omega]\!] = \bot$	$[\![\mathrm{pr}\ \Omega]\!] = \bot$ $[\![\mathrm{cn}\ \Omega]\!] = \top$
$[\![\Phi]\!] = \top$	$\top$	$\bar{a}$	$\mathrm{aw}\!\uparrow \dfrac{\bar{a}}{\top}$
$[\![\Phi]\!] = \bot$	a	$\bot$	$\mathrm{aw}\!\downarrow \dfrac{\bot}{a}$
$[\![\mathrm{pr}\ \Phi]\!] = \bot$ $[\![\mathrm{cn}\ \Phi]\!] = \top$	$\mathrm{aw}\!\uparrow \dfrac{a}{\top}$	$\mathrm{aw}\!\downarrow \dfrac{\bot}{\bar{a}}$	$\mathrm{mix}0 \dfrac{\bot}{\top}$

Another deep inference proof system : subatomic SKS

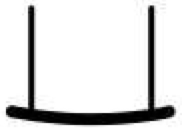
$$\vee b\downarrow\frac{(A \vee B)\; b\; (C \vee D)}{(A\, b\, C) \vee (B\, b\, D)}$$



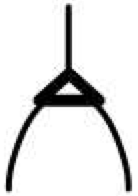
$$a\vee\downarrow\frac{(A\, a\, B) \vee (C\, a\, D)}{(A \vee C)\; a\; (B \vee D)}$$



$$\wedge a\uparrow\frac{(A\, a\, B) \wedge (C\, a\, D)}{(A \wedge C)\; a\; (B \wedge D)}$$



$$b\wedge\uparrow\frac{(A \wedge B)\; b\; (C \wedge D)}{(A\, b\, C) \wedge (B\, b\, D)}$$

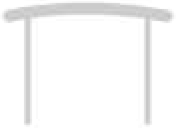


$$=\frac{A}{B}$$

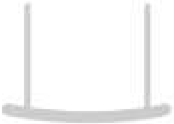
for $=$ as in SKS

$$s\downarrow\frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \qquad s\uparrow\frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)} \qquad m\frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)} \qquad \mathrm{mix}\frac{A \wedge B}{A \vee B}$$

Another deep inference proof system: subatomic SKS

$$\mathsf{vb}\!\downarrow \frac{(A \vee B)\; b\; (C \vee D)}{(A\; b\; C) \vee (B\; b\; D)}$$ 

$$\mathsf{av}\!\downarrow \frac{(A\; a\; B) \vee (C\; a\; D)}{(A \vee C)\; a\; (B \vee D)}$$ 

$$\wedge\mathsf{a}\!\uparrow \frac{(A\; a\; B) \wedge (C\; a\; D)}{(A \wedge C)\; a\; (B \wedge D)}$$ 

$$\mathsf{b}\wedge\!\uparrow \frac{(A \wedge B)\; b\; (C \wedge D)}{(A\; b\; C) \wedge (B\; b\; D)}$$ 

$$= \frac{A}{B}$$

for $=$ as in SKS

$$\mathsf{s}\!\downarrow \frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \qquad \mathsf{s}\!\uparrow \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)} \qquad \mathsf{m} \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)} \qquad \mathsf{mix} \frac{A \wedge B}{A \vee B}$$

$$= \frac{a}{a \vee \bot}$$

$$\mathsf{va}\downarrow \frac{(\bot \vee \bot) \; a \; (\top \vee \bot)}{(\bot a \top) \vee (\bot a \bot)}$$

Unit equality inference rules can also be decomposed

$$\mathsf{s}{\downarrow}\,\frac{=\dfrac{a}{a \vee \bot} \;\wedge\; =\dfrac{b}{b \vee \bot}}{(a \wedge b) \vee \; =\dfrac{\bot \vee \bot}{\bot}}$$

$$\mathsf{va}{\downarrow}\,\frac{(\bot \vee \bot)\ a\ (\top \vee \bot)}{(\bot\, a\, \top) \vee (\bot\, a\, \bot)}$$

Unit equality inference rules can also be decomposed

$$\mathsf{s}{\downarrow}\,\dfrac{=\dfrac{a}{a \vee \bot} \wedge =\dfrac{b}{b \vee \bot}}{(a \wedge b) \vee =\dfrac{\bot \vee \bot}{\bot}}$$

$$\mathsf{s}{\downarrow}\,\dfrac{\mathsf{v}a{\downarrow}\,\dfrac{(\bot \vee \bot)\ a\ (\top \vee \bot)}{(\bot a \top) \vee (\bot a \bot)} \wedge \mathsf{v}b{\downarrow}\,\dfrac{(\bot \vee \bot)\ b\ (\top \vee \bot)}{(\bot b \top) \vee (\bot b \bot)}}{((\bot a \top) \wedge (\bot b \top)) \vee ((\bot a \bot) \vee (\bot b \bot))}$$

Unit equality inference rules can also be decomposed

$$\mathsf{s}\!\downarrow\dfrac{=\dfrac{a}{a \vee \bot} \;\wedge\; =\dfrac{b}{b \vee \bot}}{(a \wedge b) \;\vee\; =\dfrac{\bot \vee \bot}{\bot}}$$

$$\overbrace{\mathsf{v}a\!\downarrow\dfrac{(\bot \vee \bot)\; a \;(\top \vee \bot)}{(\bot a \top) \vee (\bot a \bot)} \;\wedge\; \mathsf{v}b\!\downarrow\dfrac{(\bot \vee \bot)\; b \;(\top \vee \bot)}{(\bot b \top) \vee (\bot b \bot)}}^{A^{\bullet \vee \bot}\ :\ \text{every leaf } x \mapsto x \vee \bot}$$

$$\mathsf{s}\!\downarrow\dfrac{\qquad}{\underbrace{((\bot a \top) \wedge (\bot b \top))}_{A} \;\vee\; \underbrace{((\bot a \bot) \vee (\bot b \bot))}_{\check{A}^{\bot}}}$$

A with $\wedge \mapsto \vee$
$\top \mapsto \bot$

Unit equality inference rules can also be decomposed

$$=\frac{A}{A \vee \bot} \quad \longrightarrow \quad \begin{array}{c} A^{\bullet} \vee \bot \\ \| \\ A \vee \check{A}^{\bot} \end{array}$$

Unit equality inference rules
can also be decomposed

$$=\frac{A}{A \vee \bot} \longrightarrow \begin{array}{c} A^{\bullet} \vee \bot \\ \| \\ A \vee \check{A}^{\bot} \end{array}$$

$$\mathrm{ab}{\downarrow}\,\frac{(A\,a\,B)\;b\;(C\,a\,D)}{(A\,b\,C)\;a\;(B\,b\,D)}$$

Unit equality inference rules
can also be decomposed

$$K\left\{ = \frac{A}{A \vee \bot} \right\} \longrightarrow K\left\{ \begin{array}{c} A^{\bullet \vee \bot} \\ \| \\ A \vee \check{A}^{\bot} \end{array} \right\}$$

Unit equality inference rules
can also be decomposed

$$\begin{array}{c} K\left\{ {=}\dfrac{A}{A \vee \bot} \right\} \\ \Phi \, \Big\| \\ H\left\{ {=}\dfrac{B \vee \bot}{B} \right\} \end{array} \longrightarrow K\left\{ \begin{array}{c} A^{\bullet \vee \bot} \\ \Big\| \\ A \vee \check{A}^{\bot} \end{array} \right\}$$

Unit equality inference rules can also be decomposed

$$K\left\{ = \frac{A}{A \vee \bot} \right\} \longrightarrow K\left\{ \begin{array}{c} A^{\bullet \vee \bot} \\ \| \\ A \vee \check{A}^{\bot} \end{array} \right\}$$

$$\Phi \,\|$$

$$H\left\{ = \frac{B \vee \bot}{B} \right\} \longrightarrow H\left\{ \begin{array}{c} B \vee \hat{B}^{\bot} \\ \| \\ B^{\bullet \vee \bot} \end{array} \right\}$$

$$\begin{array}{c} \check{A}^{\hat{B}^{\bot}} \\ \| \; \text{eversion} \\ \hat{B}^{\check{A}^{\bot}} \end{array}$$

Unit equality inference rules can also be decomposed

$$
\begin{tikzcd}
K\left\{ \dfrac{A}{A \vee \bot}{=} \right\} \arrow[r] \arrow[d, equal, "\Phi"'] & K\left\{ \begin{array}{c} A^{\bullet \vee \hat{B}^{\bot}} \\ \| \\ A \vee \check{A}^{\hat{B}^{\bot}} \end{array} \right\} \arrow[d, equal, "\Phi'"'] & \begin{array}{c} \check{A}^{\hat{B}^{\bot}} \\ \| \quad \text{eversion} \\ \hat{B}^{\check{A}^{\bot}} \end{array} \arrow[l] \\
H\left\{ \dfrac{B \vee \bot}{B}{=} \right\} \arrow[r] & H\left\{ \begin{array}{c} B \vee \hat{B}^{\check{A}^{\bot}} \\ \| \\ B^{\bullet \vee \check{A}^{\bot}} \end{array} \right\} &
\end{tikzcd}
$$

Unit equality inference rules
can also be decomposed

Another deep inference proof system: subatomic SKS

$$\vee b\!\downarrow \frac{(A \vee B)\ b\ (C \vee D)}{(A\,b\,C) \vee (B\,b\,D)} \qquad a\vee\!\downarrow \frac{(A\,a\,B) \vee (C\,a\,D)}{(A \vee C)\ a\ (B \vee D)} \qquad ab\!\downarrow \frac{(A\,a\,B)\ b\ (C\,a\,D)}{(A\,b\,C)\ a\ (B\,b\,D)}$$

$$\wedge a\!\uparrow \frac{(A\,a\,B) \wedge (C\,a\,D)}{(A \wedge C)\ a\ (B \wedge D)} \qquad b\wedge\!\uparrow \frac{(A \wedge B)\ b\ (C \wedge D)}{(A\,b\,C) \wedge (B\,b\,D)}$$

for all $a, b \in$ Atoms

$$s\!\downarrow \frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)}$$

$$s\!\uparrow \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$m \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)} \qquad \mathsf{mix} \frac{A \wedge B}{A \vee B}$$

$$\frac{(A \vee B) \vee (C \vee D)}{(A \vee C) \vee (B \vee D)}$$

$$\frac{(A \wedge B) \wedge (C \wedge D)}{(A \wedge C) \wedge (B \wedge D)}$$

Normalisation

Semantics

Complexity

Proof search

Normalisation

Semantics : increase in syntax

Complexity

Proof search : increase in non-determinism

Normalisation

Semantics : increase in syntax

Complexity : linear rules → substitution for derivations
→ proof compression

Proof search : increase in non-determinism

Normalisation : a more general theory of normalisation, given by the relations between connectives

Semantics : increase in syntax

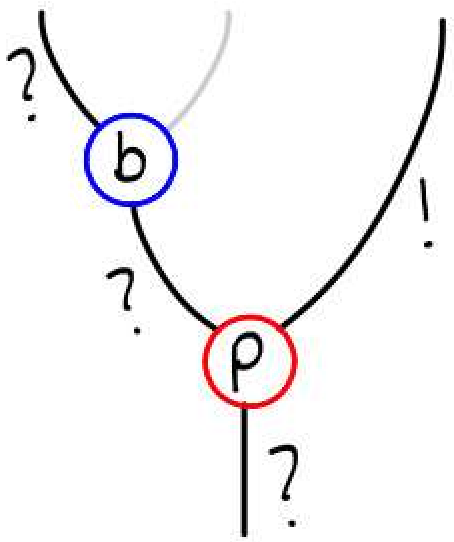
Complexity : linear rules $\rightarrow$ substitution for derivations $\rightarrow$ proof compression

Proof search : increase in non-determinism

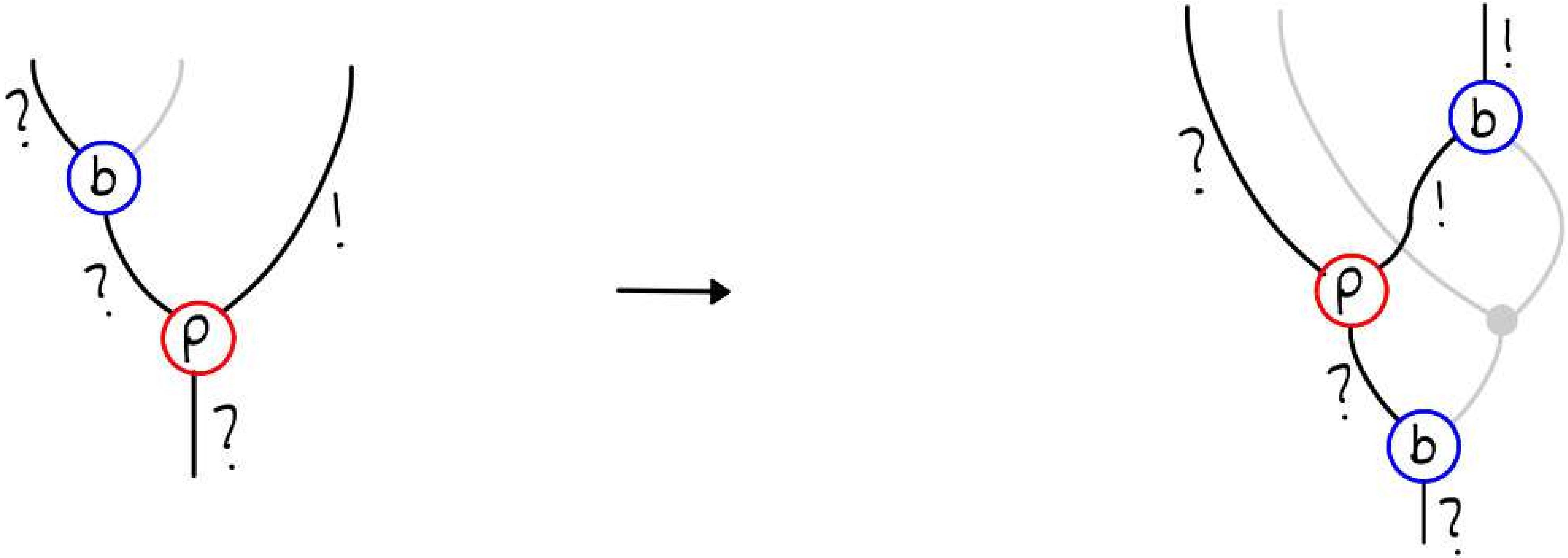
$$
b\downarrow\;\dfrac{\;?A \parr A\;}{?A} \quad \otimes \quad !B
$$

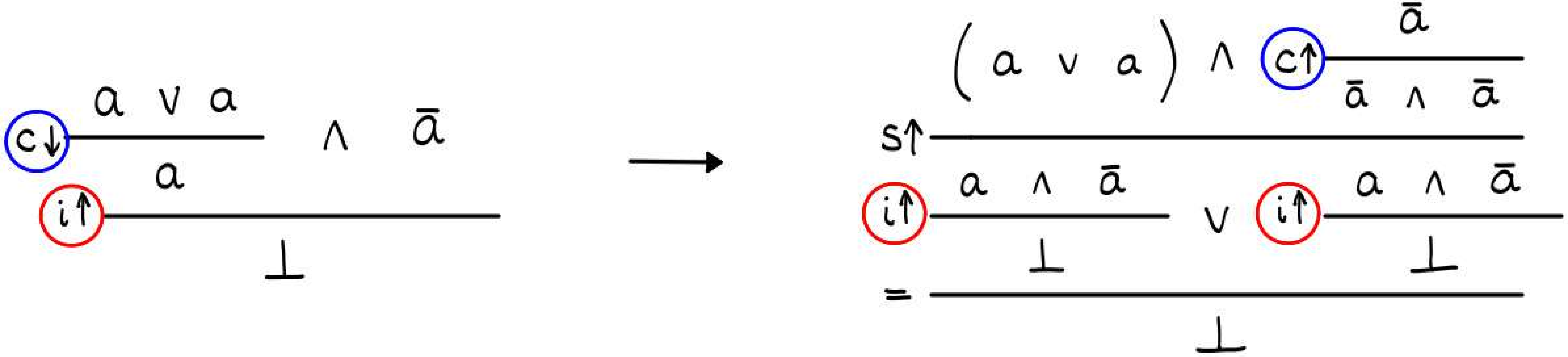
$$
p\uparrow\;\dfrac{\left(b\downarrow\;\dfrac{?A \parr A}{?A}\right) \otimes\; !B}{?(A \otimes B)}
$$

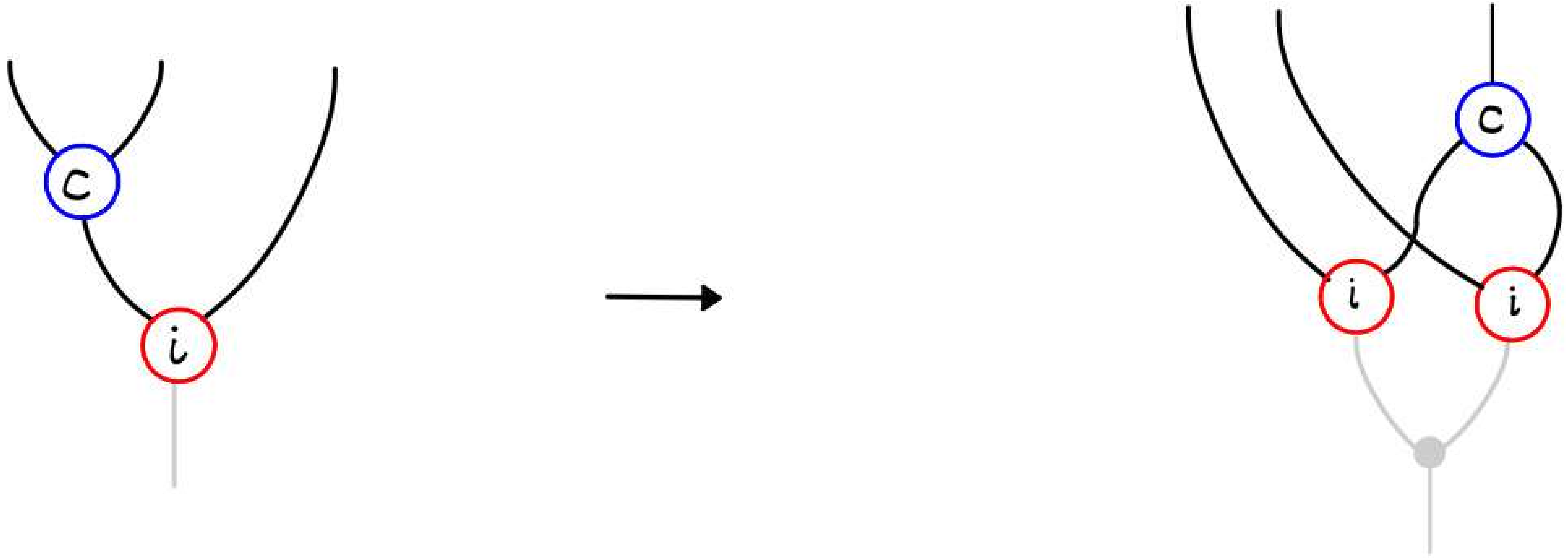
$$\cfrac{\cfrac{?A \parr A}{?A}\; b{\downarrow} \quad \otimes \quad !B}{?(A \otimes B)}\; p{\uparrow}$$



$$
{\scriptstyle p\uparrow}\,\dfrac{{\scriptstyle b\downarrow}\,\dfrac{?A \parr A}{?A} \quad \otimes \quad !B}{?(A \otimes B)}
\qquad \longrightarrow \qquad
{\scriptstyle b\downarrow}\,\dfrac{{\scriptstyle s\uparrow}\,\dfrac{(?A \parr A) \otimes {\scriptstyle b\uparrow}\,\dfrac{!B}{!B \otimes B}}{{\scriptstyle p\uparrow}\,\dfrac{?A \otimes !B}{?(A \otimes B)} \quad \parr \quad (A \otimes B)}}{?(A \otimes B)}
$$

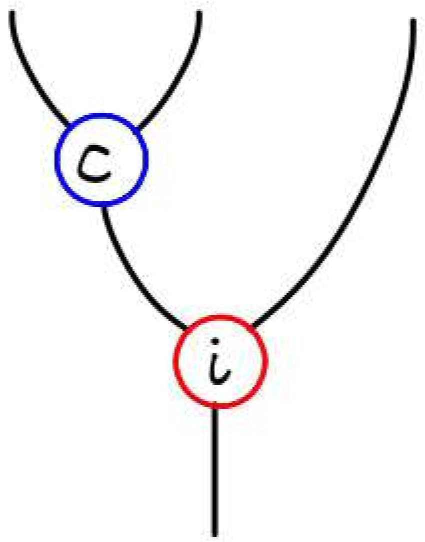


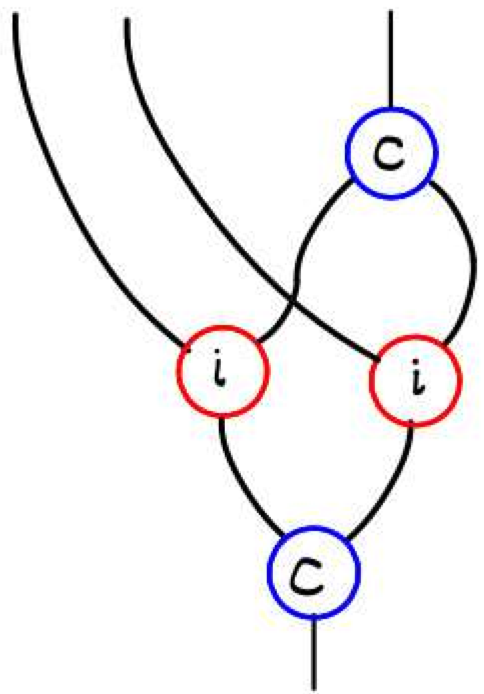

c↓
a ∨ a
∧ ā
i↑
a
⊥
(a ∨ a) ∧
c↑
ā
ā ∧ ā
s↑
i↑
a ∧ ā
⊥
∨
i↑
a ∧ ā
⊥
=
⊥


c
i
c
i
i

$$
\mathsf{i}\!\uparrow\frac{\mathsf{c}\!\downarrow\dfrac{\bot a \top \vee \bot a \top}{(\bot\vee\bot)\,a\,(\top\vee\top)} \wedge \top a \bot}{((\bot\vee\bot)\wedge\top)\; a\; ((\top\vee\top)\wedge\bot)}
$$

$$
\mathsf{c}\!\downarrow\frac{\mathsf{s}\!\uparrow\dfrac{(\bot a\top \vee \bot a \top) \wedge \mathsf{c}\!\uparrow\dfrac{(\top\wedge\top)\,a\,(\bot\wedge\bot)}{\top a\bot \wedge \top a \bot}}{\mathsf{i}\!\uparrow\dfrac{\bot a\top \wedge \top a\bot}{(\bot\wedge\top)\,a\,(\top\wedge\bot)} \vee \mathsf{i}\!\uparrow\dfrac{\bot a\top \wedge \top a\bot}{(\bot\wedge\top)\,a\,(\top\wedge\bot)}}}{((\bot\wedge\top)\vee(\bot\wedge\top))\; a\; ((\top\wedge\bot)\vee(\top\wedge\bot))}
$$





Subatomic SKS

$$\mathsf{vb}{\downarrow}\,\frac{(A \vee B)\; b\; (C \vee D)}{(A\; b\; C) \vee (B\; b\; D)}$$

$$\mathsf{av}{\downarrow}\,\frac{(A\; a\; B) \vee (C\; a\; D)}{(A \vee C)\; a\; (B \vee D)}$$

$$\mathsf{ab}{\downarrow}\,\frac{(A\; a\; B)\; b\; (C\; a\; D)}{(A\; b\; C)\; a\; (B\; b\; D)}$$

$$\wedge\mathsf{a}{\uparrow}\,\frac{(A\; a\; B) \wedge (C\; a\; D)}{(A \wedge C)\; a\; (B \wedge D)}$$

$$\mathsf{b}\wedge{\uparrow}\,\frac{(A \wedge B)\; b\; (C \wedge D)}{(A\; b\; C) \wedge (B\; b\; D)}$$

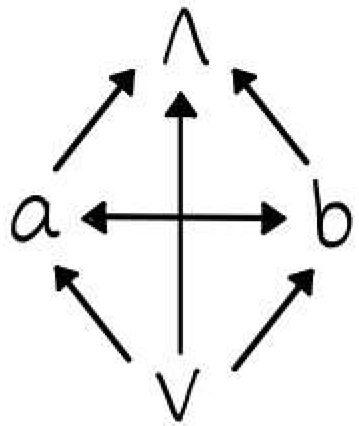
for all $a, b \in$ Atoms

$$\mathsf{s}{\downarrow}\,\frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)}$$

$$\mathsf{s}{\uparrow}\,\frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$\mathsf{m}\,\frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

Medials



Switches