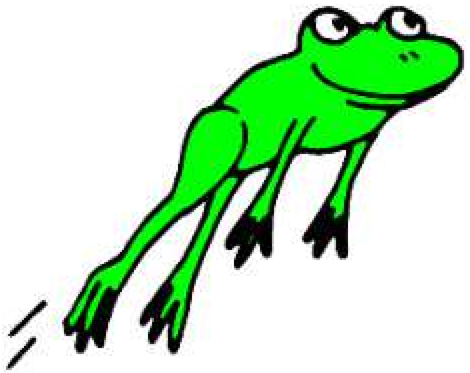
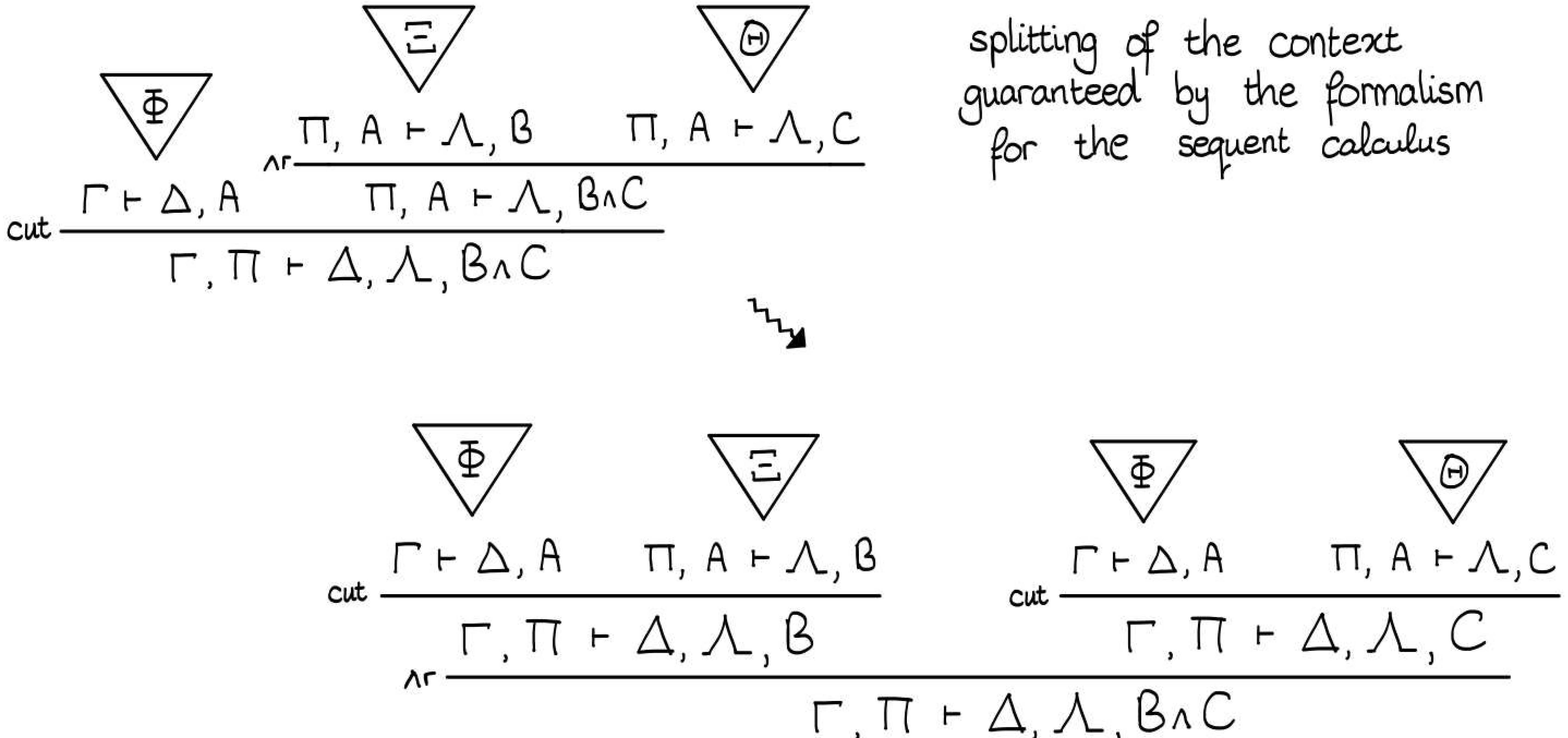
A gentle introduction to deep inference

4. Cut elimination in open deduction for propositional classical logic

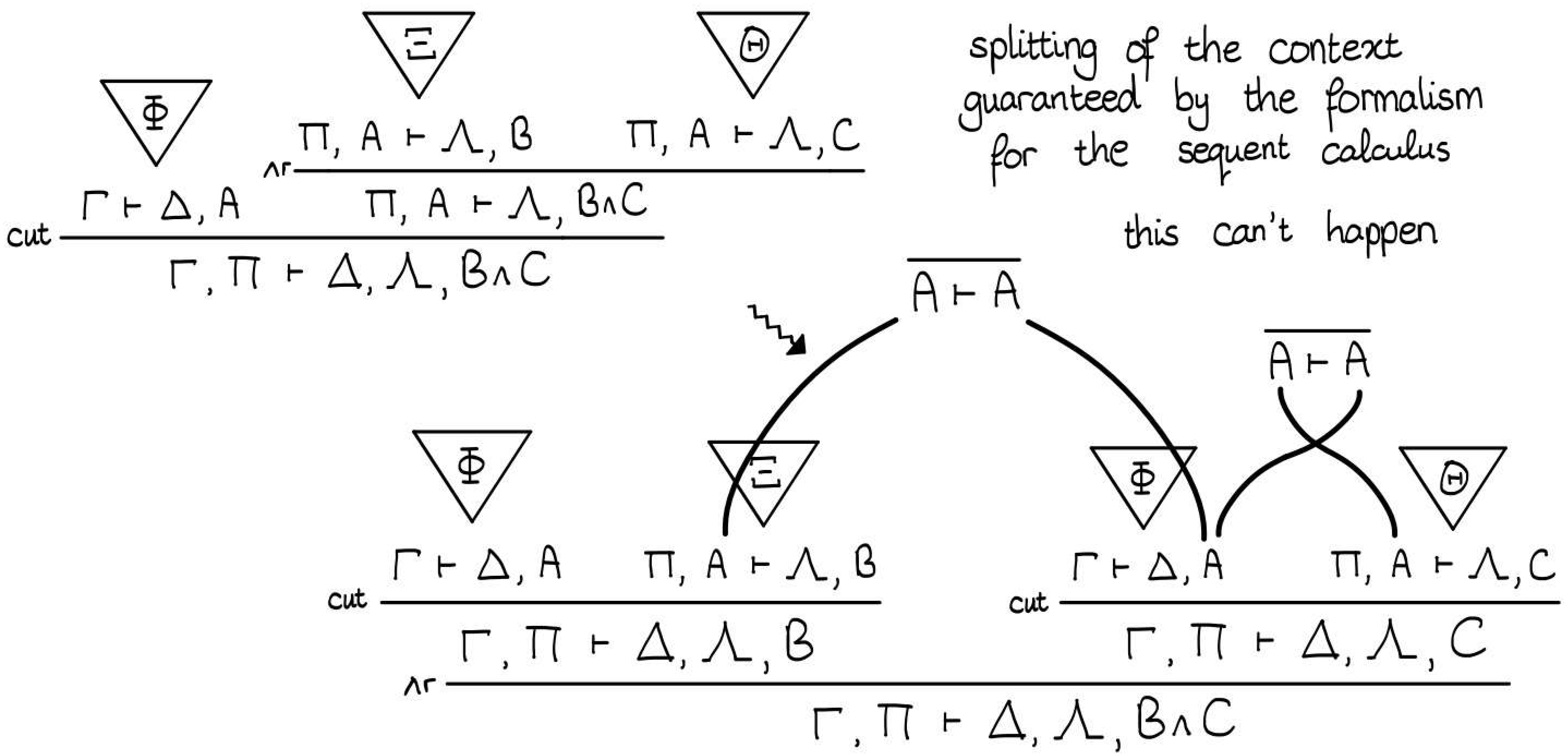


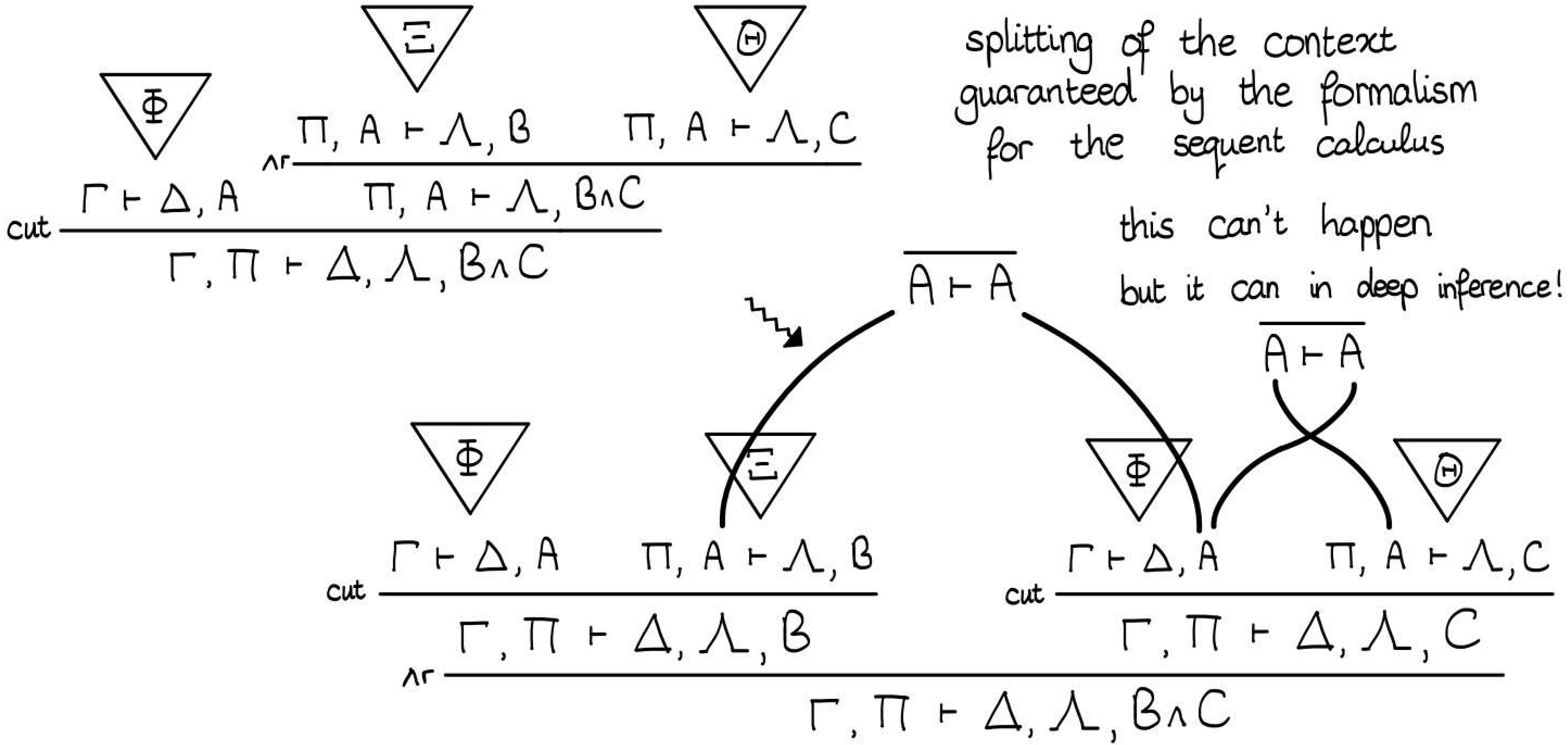
ESSLLI 2025

Victoria Barrett and Lutz Straßburger



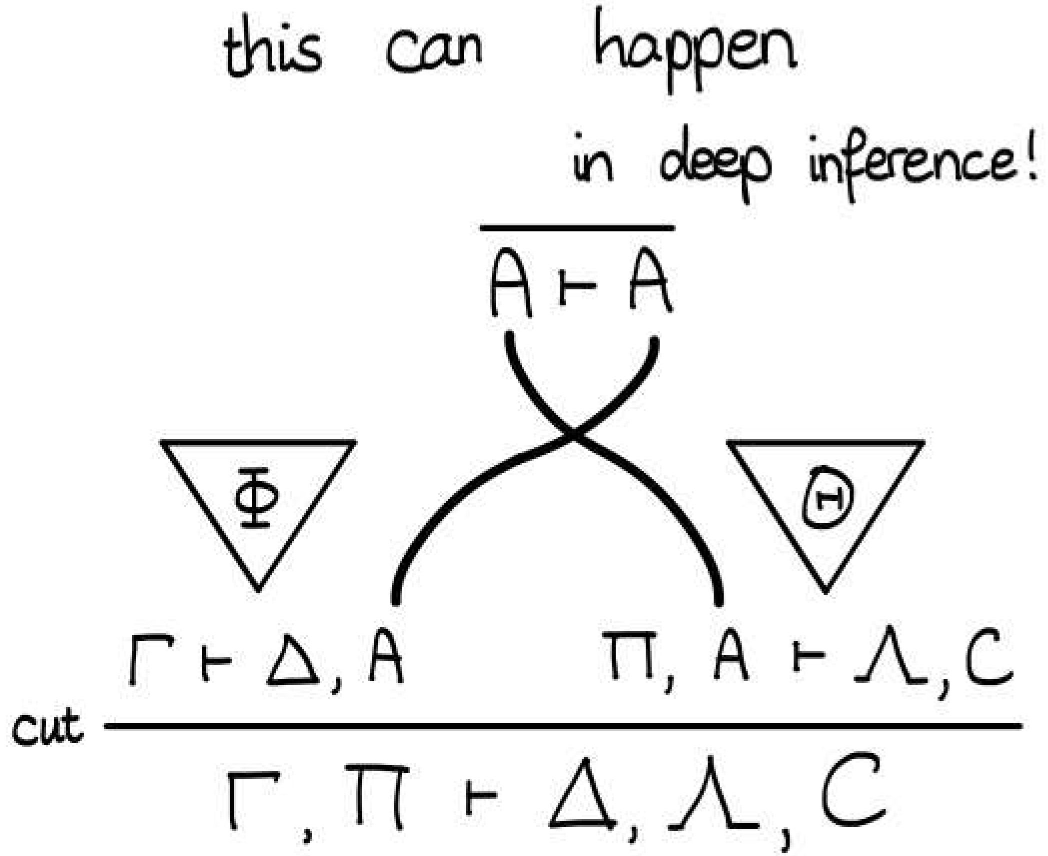
splitting of the context guaranteed by the formalism for the sequent calculus
Φ
Ξ
Θ
Γ ⊢ Δ, A
Π, A ⊢ Λ, B
Π, A ⊢ Λ, C
∧r
Π, A ⊢ Λ, B∧C
cut
Γ, Π ⊢ Δ, Λ, B∧C
Γ ⊢ Δ, A
Π, A ⊢ Λ, B
cut
Γ, Π ⊢ Δ, Λ, B
Γ ⊢ Δ, A
Π, A ⊢ Λ, C
cut
Γ, Π ⊢ Δ, Λ, C
∧r
Γ, Π ⊢ Δ, Λ, B∧C


Φ
Ξ
Θ
Π, A ⊢ Λ, B
Π, A ⊢ Λ, C
∧r
Γ ⊢ Δ, A
Π, A ⊢ Λ, B∧C
cut
Γ, Π ⊢ Δ, Λ, B∧C
splitting of the context
guaranteed by the formalism
for the sequent calculus
this can't happen
A ⊢ A
A ⊢ A
Φ
Ξ
Φ
Θ
Γ ⊢ Δ, A
Π, A ⊢ Λ, B
cut
Γ, Π ⊢ Δ, Λ, B
Γ ⊢ Δ, A
Π, A ⊢ Λ, C
cut
Γ, Π ⊢ Δ, Λ, C
∧r
Γ, Π ⊢ Δ, Λ, B∧C



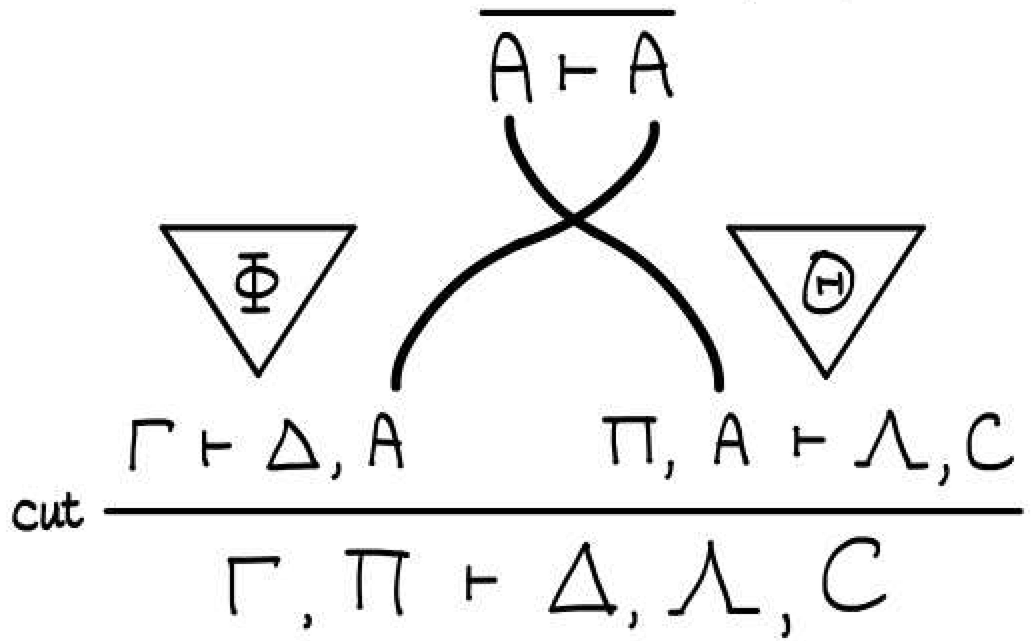
splitting of the context guaranteed by the formalism for the sequent calculus
this can't happen
but it can in deep inference!
Φ
Ξ
Θ
Γ ⊢ Δ, A
Π, A ⊢ Λ, B
Π, A ⊢ Λ, C
∧r
Π, A ⊢ Λ, B∧C
cut
Γ, Π ⊢ Δ, Λ, B∧C
A ⊢ A
A ⊢ A
Γ, Π ⊢ Δ, Λ, B
Γ, Π ⊢ Δ, Λ, C

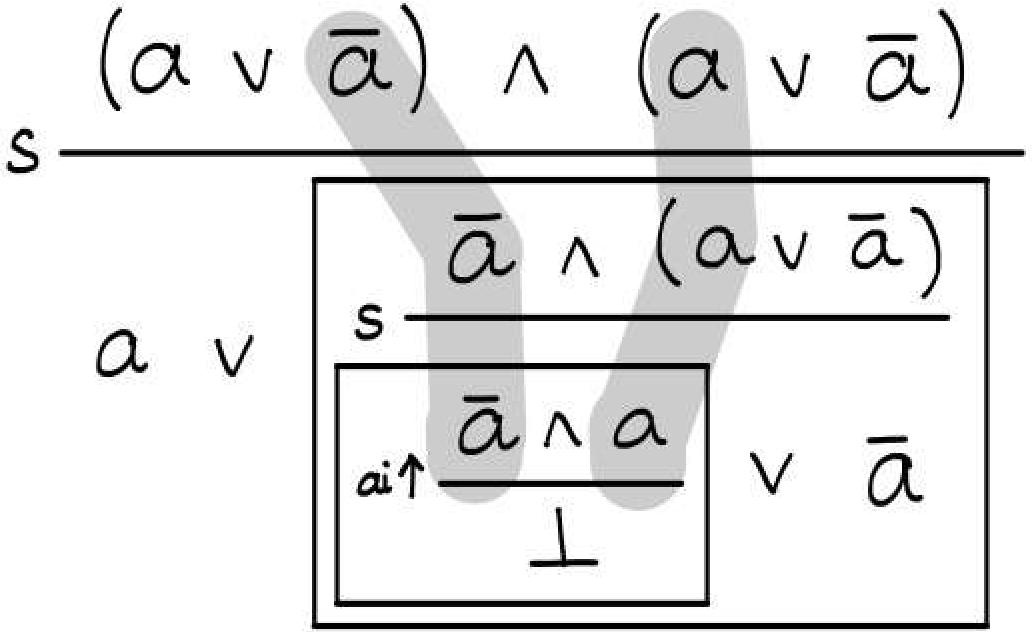
$$\text{ai}\uparrow \frac{\bar{a} \wedge a}{\bot}$$



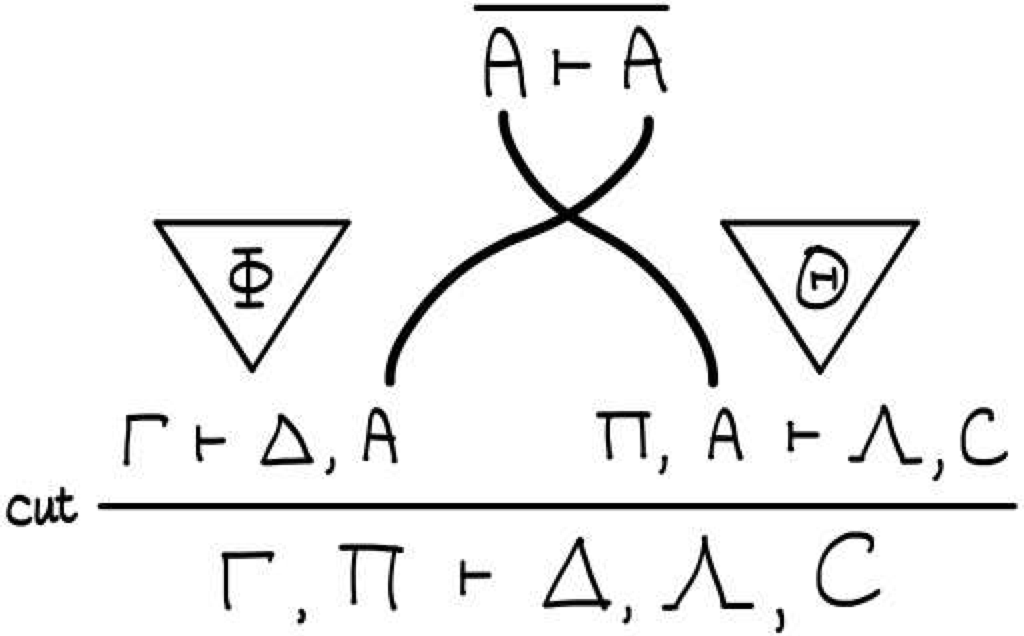
$$\text{s}\ \frac{\bar{a} \wedge (a \vee \bar{a})}{\boxed{\text{ai}\uparrow\ \dfrac{\bar{a} \wedge a}{\bot}} \vee \bar{a}}$$

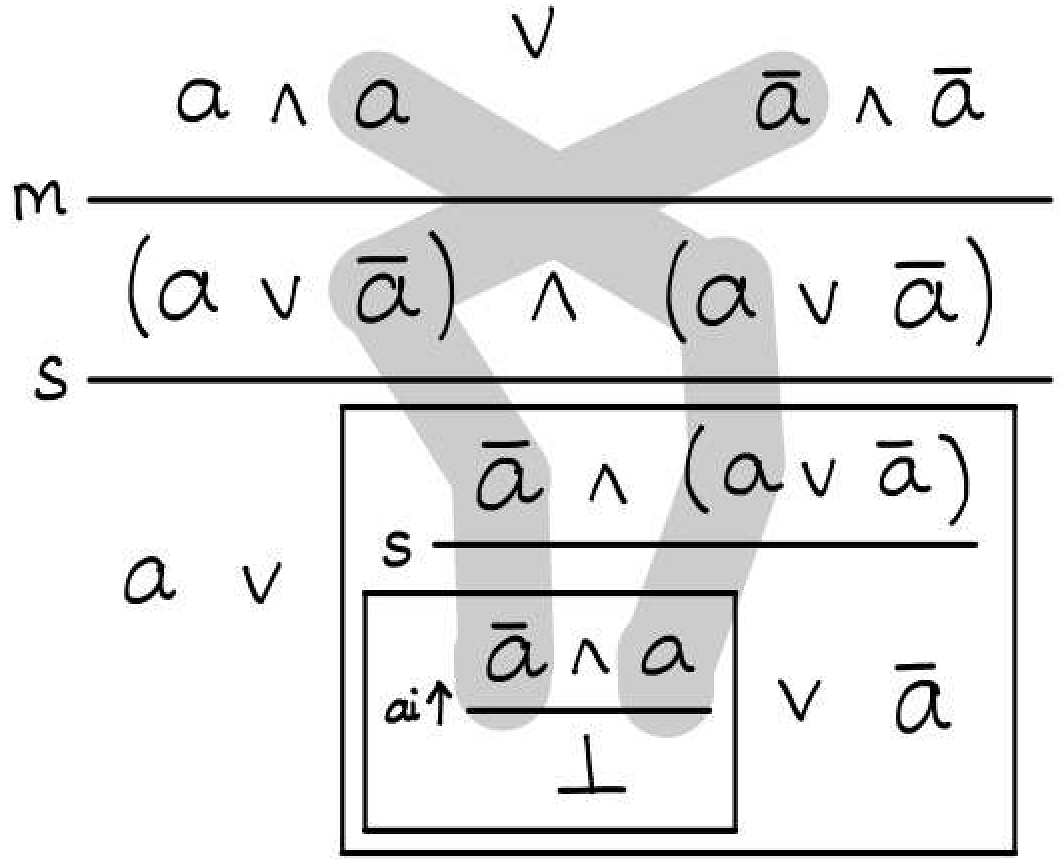
this can happen in deep inference!



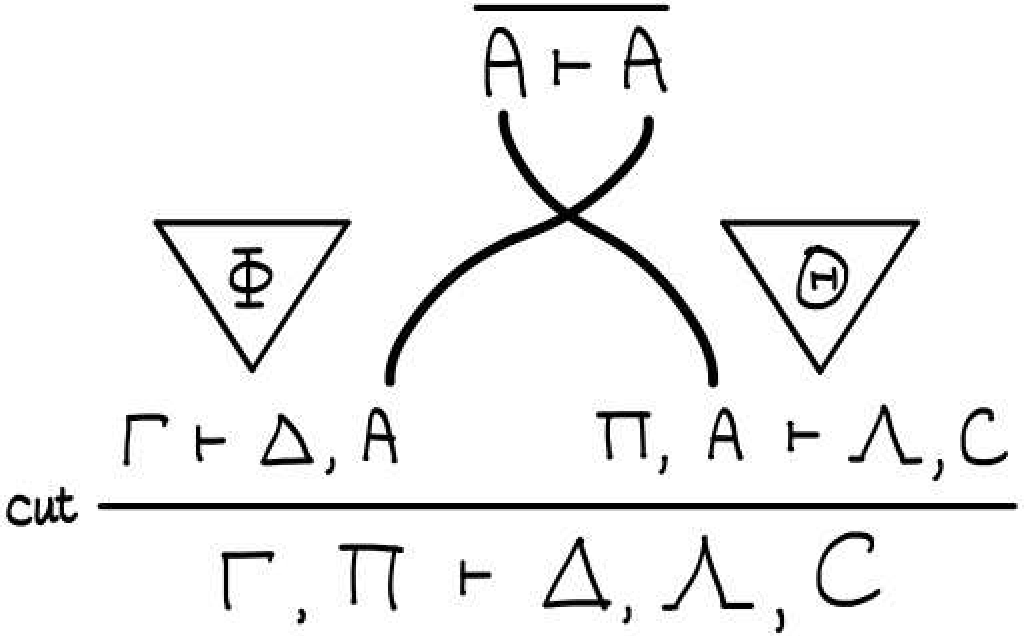


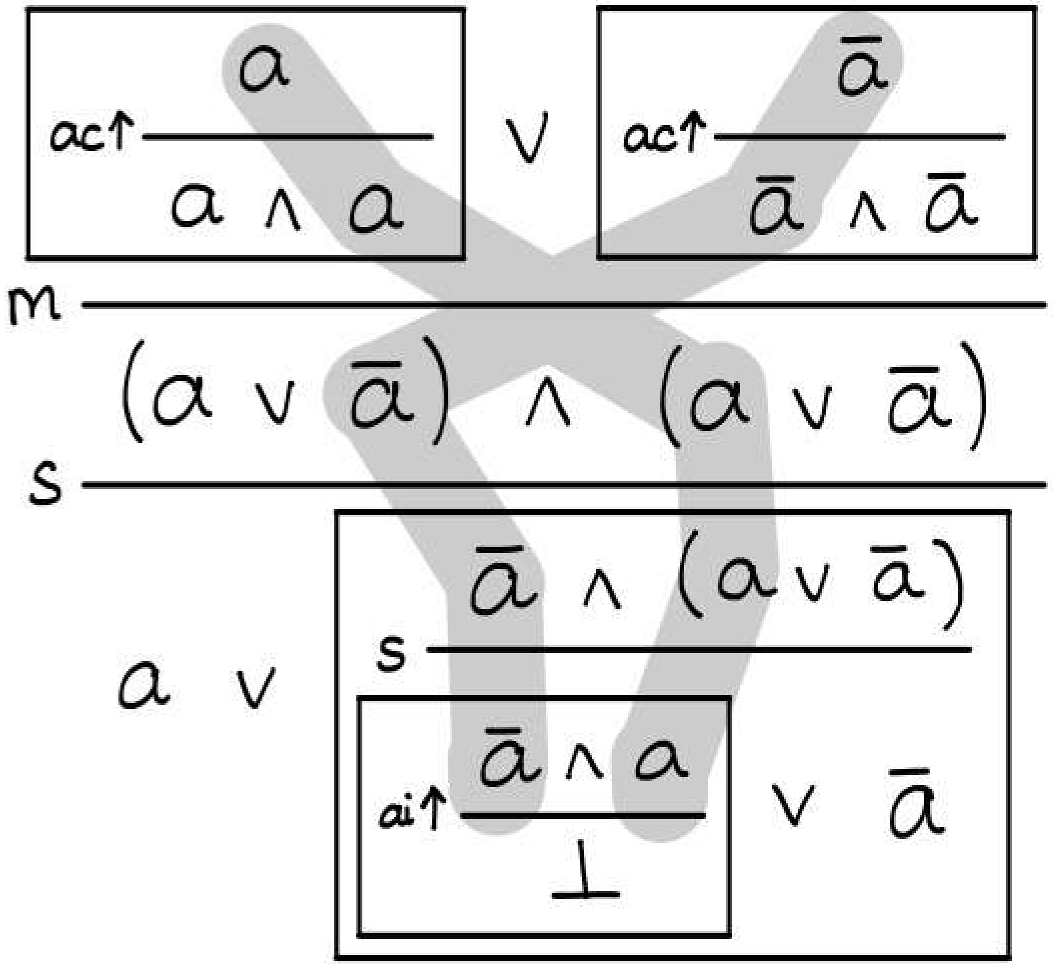
this can happen in deep inference!



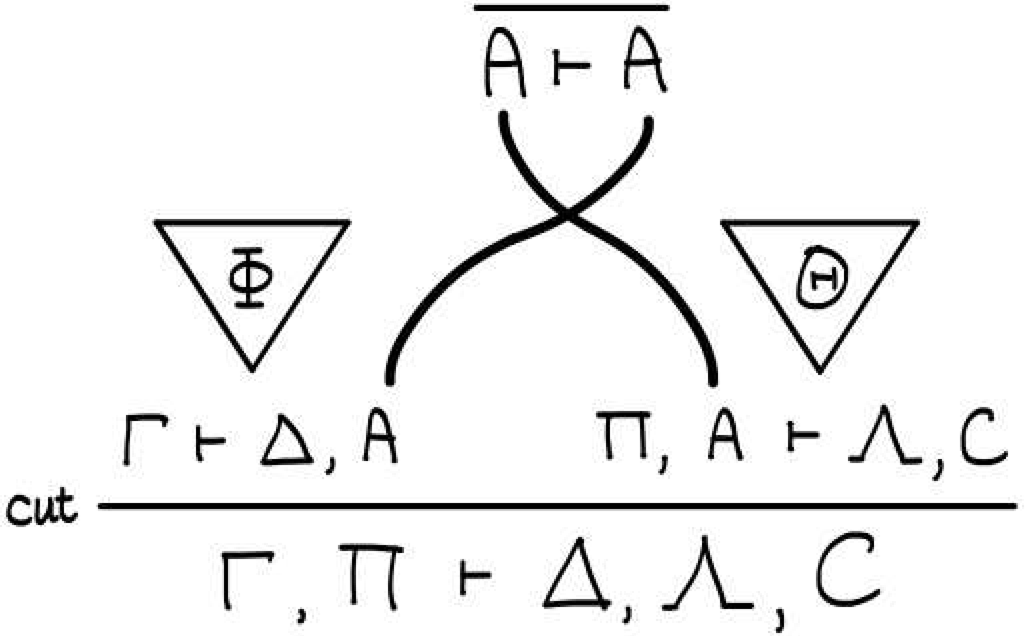


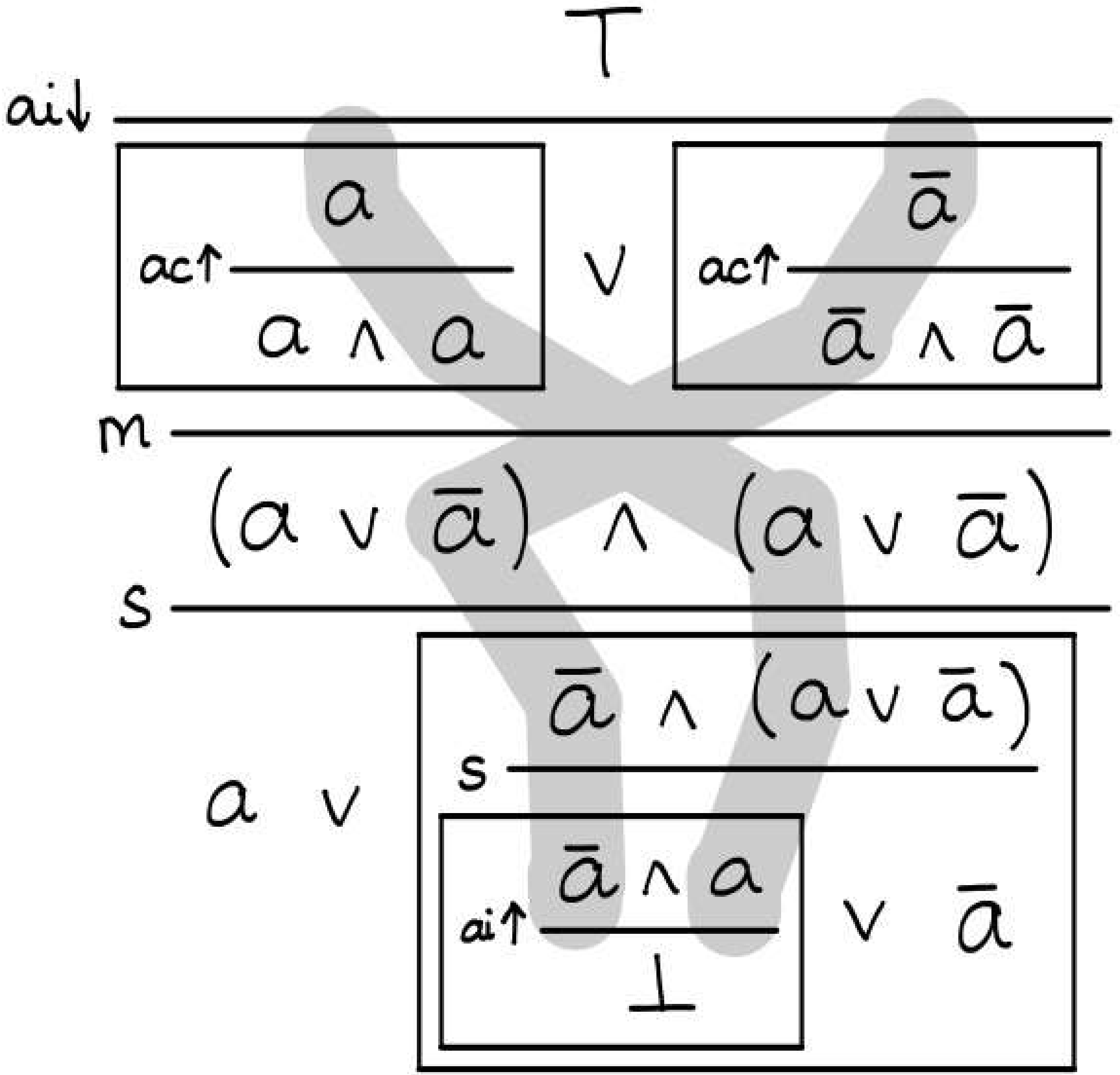
this can happen in deep inference!



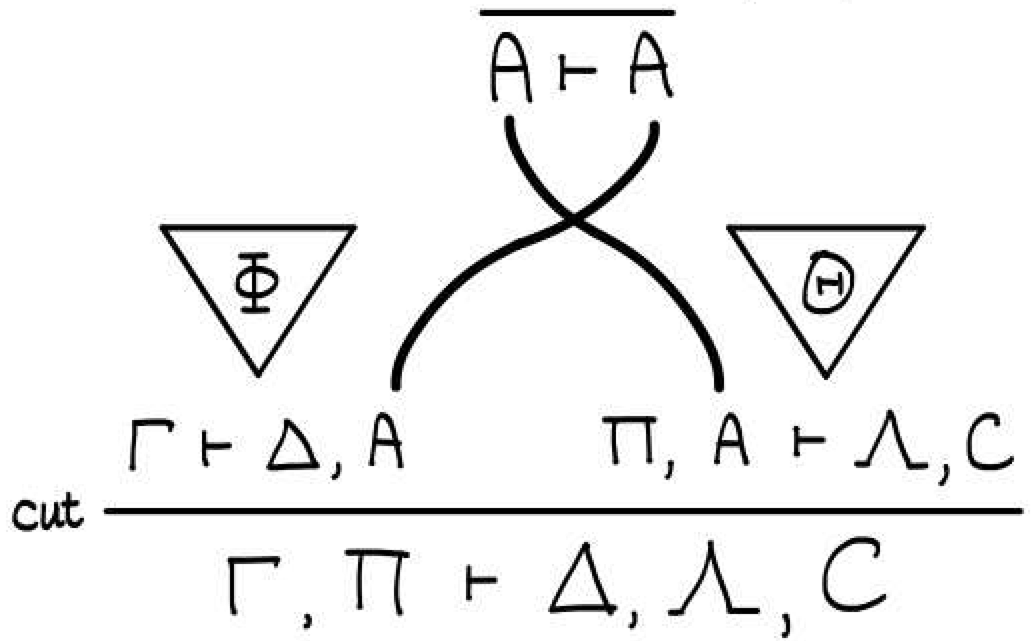


this can happen in deep inference!





this can happen in deep inference!



System SKS

$$\text{ai}{\downarrow}\,\frac{\top}{a \vee \bar{a}}$$ identity

$$\text{ac}{\downarrow}\,\frac{a \vee a}{a}$$ contraction

$$\text{aw}{\downarrow}\,\frac{\bot}{a}$$ weakening

$$\text{ai}{\uparrow}\,\frac{a \wedge \bar{a}}{\bot}$$ cut

$$\text{ac}{\uparrow}\,\frac{a}{a \wedge a}$$ cocontraction

$$\text{aw}{\uparrow}\,\frac{a}{\top}$$ coweakening

$$\text{m}\,\frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$ medial

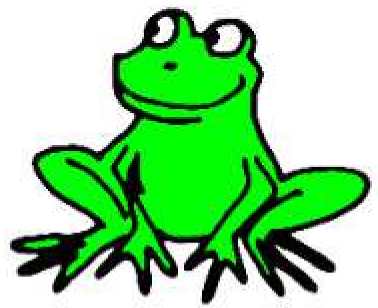
$$\text{s}\,\frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$ switch

equality $=\frac{A}{B}$ for:

$$A \wedge B = B \wedge A$$
$$(A \wedge B) \wedge C = A \wedge (B \wedge C)$$

$$A \vee B = B \vee A$$
$$(A \vee B) \vee C = A \vee (B \vee C)$$

$$A \vee \bot = A = \top \wedge A$$
$$\bot \vee \bot = \bot$$
$$\top = \top \wedge \top$$



System SKS

$$\mathsf{ai}\!\downarrow \frac{\top}{a \vee \bar{a}}$$
identity

$$\mathsf{ac}\!\downarrow \frac{a \vee a}{a}$$
contraction

$$\mathsf{aw}\!\downarrow \frac{\bot}{a}$$
weakening

$$\mathsf{ai}\!\uparrow \frac{a \wedge \bar{a}}{\bot}$$
cut

$$\mathsf{ac}\!\uparrow \frac{a}{a \wedge a}$$
cocontraction

$$\mathsf{aw}\!\uparrow \frac{a}{\top}$$
coweakening

$$\mathsf{m}\ \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$
medial

$$\mathsf{s}\ \frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$
switch

equality $= \frac{A}{B}$ for :

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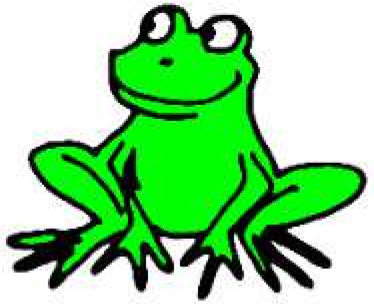

System SKS

$$\text{ai}\!\downarrow \frac{\top}{a \vee \bar{a}}$$
identity

$$\text{ac}\!\downarrow \frac{a \vee a}{a}$$
contraction

$$\text{aw}\!\downarrow \frac{\bot}{a}$$
weakening

$$\text{ai}\!\uparrow \frac{a \wedge \bar{a}}{\bot}$$
cut

$$\text{ac}\!\uparrow \frac{a}{a \wedge a}$$
cocontraction

$$\text{aw}\!\uparrow \frac{a}{\top}$$
coweakening

$$\text{m} \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$
medial

$$\text{s} \frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$
switch

equality $= \frac{A}{B}$ for:

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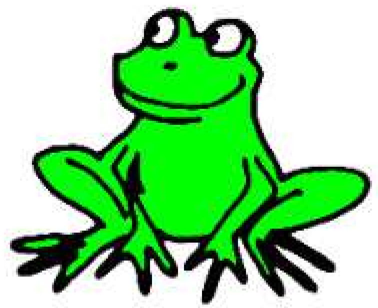

$$a_w \uparrow \frac{a}{T}$$

$$aw\uparrow \frac{a}{T} \longrightarrow \frac{a}{T}$$

$$= \frac{a}{a \;\; \wedge \qquad\qquad T}$$

$$aw\uparrow \frac{a}{T} \longrightarrow$$

T

$$= \frac{a}{a \;\wedge\; = \frac{\top}{\bot \vee \top}}$$

$$aw\uparrow \frac{a}{\top} \longrightarrow$$

$$\top$$

$$aw\uparrow \frac{a}{\top} \longrightarrow$$

$$= \frac{a}{}$$

$$a \wedge \quad = \frac{\top}{\bot \vee \top}$$

$$s \frac{}{(a \wedge \quad \bot) \vee \top}$$

$$\top$$

$$\mathsf{aw}{\uparrow}\frac{a}{\top} \quad \longrightarrow \quad \mathsf{s}\frac{\displaystyle =\frac{a}{a \wedge \; =\frac{\top}{\bot \vee \top}}}{\left(a \wedge \mathsf{aw}{\downarrow}\frac{\bot}{\bar{a}}\right) \vee \top}$$

$$\top$$

$$\mathsf{aw}\!\uparrow\frac{a}{\top} \quad\longrightarrow\quad =\frac{a}{\mathsf{s}\,\dfrac{a \wedge {=}\dfrac{\top}{\bot \vee \top}}{\mathsf{ai}\!\uparrow\dfrac{a \wedge \mathsf{aw}\!\downarrow\dfrac{\bot}{\bar{a}}}{\bot} \vee \top}}$$

$$\mathsf{aw}{\uparrow}\,\frac{a}{\top} \quad\longrightarrow\quad = \frac{\mathsf{s}\,\dfrac{=\dfrac{a}{a \wedge \left(= \dfrac{\top}{\bot \vee \top}\right)}}{\mathsf{ai}{\uparrow}\,\dfrac{a \wedge \mathsf{aw}{\downarrow}\,\dfrac{\bot}{\bar{a}}}{\bot} \vee \top}}{\top}$$

System KS + cut

$$\text{ai}\!\downarrow \frac{\top}{a \vee \bar{a}}$$
identity

$$\text{ac}\!\downarrow \frac{a \vee a}{a}$$
contraction

$$\text{aw}\!\downarrow \frac{\bot}{a}$$
weakening

$$\text{ai}\!\uparrow \frac{a \wedge \bar{a}}{\bot}$$
cut

We can eliminate $\text{ac}\!\uparrow$, $\text{aw}\!\uparrow$ using their dual rules with switch, cut, identity and =

$$\text{m} \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$
medial

$$\text{s} \frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$
switch

equality $= \frac{A}{B}$ for :

$$A \wedge B = B \wedge A$$
$$(A \wedge B) \wedge C = A \wedge (B \wedge C)$$

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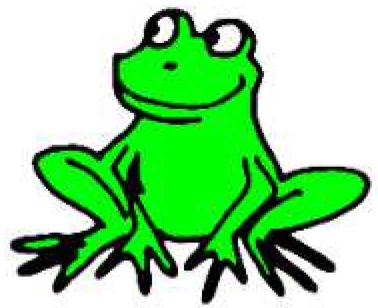

System KS

$$\mathsf{ai}{\downarrow}\ \frac{\top}{a \vee \bar{a}}$$
identity

$$\mathsf{ac}{\downarrow}\ \frac{a \vee a}{a}$$
contraction

$$\mathsf{aw}{\downarrow}\ \frac{\bot}{a}$$
weakening

$$\mathsf{m}\ \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$
medial

$$\mathsf{s}\ \frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$
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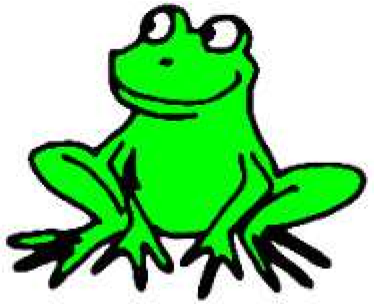


Open deduction

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\pi}}{\boxed{\Delta}}$$

Open deduction

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \dfrac{\boxed{\pi}}{\boxed{\Delta}}$$

$$\mathsf{m}\,\dfrac{\mathsf{ai}\!\downarrow\,\dfrac{1}{\boxed{\mathsf{ac}\!\uparrow\,\dfrac{a}{a \wedge a}} \vee \boxed{\mathsf{ac}\!\uparrow\,\dfrac{\bar{a}}{\bar{a} \wedge \bar{a}}}}}{(a \vee \bar{a}) \wedge (a \vee \bar{a})}$$

Open deduction

$$\boxed{\Pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\Pi} \vee \boxed{\Delta} \mid \boxed{\Pi} \wedge \boxed{\Delta} \mid \dfrac{\boxed{\Pi}}{\boxed{\Delta}}$$

we can sequentialise derivations

$$
\text{m}\,\dfrac{\text{ai}\!\downarrow\dfrac{1}{\boxed{\text{ac}\!\uparrow\dfrac{a}{a \wedge a}} \vee \boxed{\text{ac}\!\uparrow\dfrac{\bar{a}}{\bar{a} \wedge \bar{a}}}}}{(a \vee \bar{a}) \wedge (a \vee \bar{a})}
\quad = \quad
\begin{array}{c}
\text{ai}\!\downarrow\dfrac{1}{a \vee \bar{a}} \\
\text{- - - - - - - - - - - - - - -} \\
\boxed{\text{ac}\!\uparrow\dfrac{a}{a \wedge a}} \vee \bar{a} \\
\text{- - - - - - - - - - - - - - -} \\
\text{m}\,\dfrac{(a \vee \bar{a}) \vee \boxed{\text{ac}\!\uparrow\dfrac{\bar{a}}{\bar{a} \wedge \bar{a}}}}{(a \vee \bar{a}) \wedge (a \vee \bar{a})}
\end{array}
$$

Open deduction

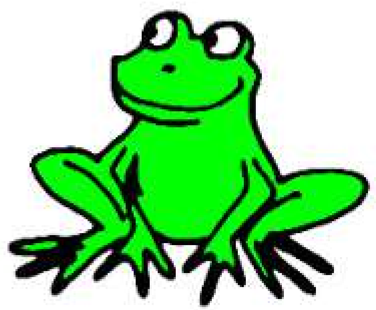
$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \dfrac{\boxed{\pi}}{\boxed{\Delta}}$$

$$\mathsf{ai}{\downarrow}\dfrac{1}{\mathsf{m}\dfrac{\boxed{\mathsf{ac}{\uparrow}\dfrac{a}{a \wedge a}} \vee \boxed{\mathsf{ac}{\uparrow}\dfrac{\bar{a}}{\bar{a} \wedge \bar{a}}}}{(a \vee \bar{a}) \wedge (a \vee \bar{a})}} \quad = \quad \begin{array}{c} \mathsf{ai}{\downarrow}\dfrac{1}{a \vee \bar{a}} \\ \hdashline K\left\{\mathsf{ac}{\uparrow}\dfrac{a}{a \wedge a}\right\} \\ \hdashline \mathsf{m}\dfrac{(a \vee \bar{a}) \vee \boxed{\mathsf{ac}{\uparrow}\dfrac{\bar{a}}{\bar{a} \wedge \bar{a}}}}{(a \vee \bar{a}) \wedge (a \vee \bar{a})} \end{array}$$

where $K\{\,\} = \{\,\} \vee \bar{a}$

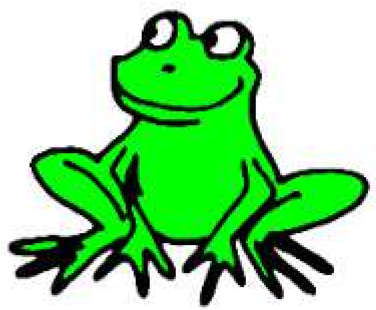
$K\{\,\}$ is a formula context : a formula with a hole

We can make deep cuts shallow in KS + cut :



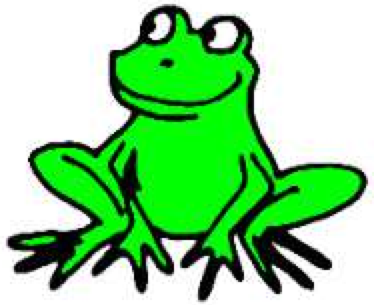
$$\Phi \quad = \quad \begin{array}{c} \Theta \\ \hdashline K\left\{ \mathrm{ai}\!\uparrow \dfrac{a \wedge \bar{a}}{\bot} \right\} \\ \hdashline \Omega \end{array}$$

We can make deep cuts shallow in KS + cut :



$$\Phi = \begin{array}{c} \Theta \\ \hdashline K\left\{ \mathrm{ai}\!\uparrow \dfrac{a \wedge \bar{a}}{\bot} \right\} \\ \hdashline \Omega \end{array} \quad \longrightarrow \quad \Phi' = \begin{array}{c} \Theta' \\ \hdashline \mathrm{ai}\!\uparrow \dfrac{a \wedge \bar{a}}{\bot} \vee C \\ \hdashline \Omega' \end{array}$$

We can make deep cuts shallow in KS + cut :

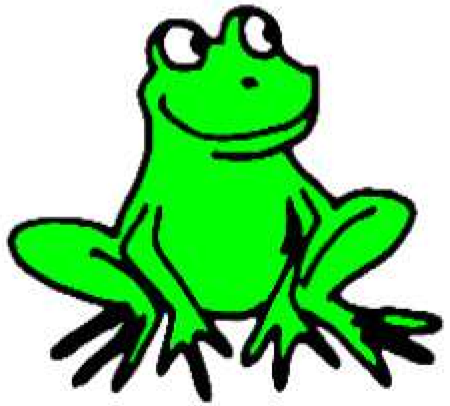


$$\Phi = \begin{array}{c} \Theta \\ \hdashline K\left\{ \mathsf{ai}\uparrow \dfrac{a \wedge \bar{a}}{\bot} \right\} \\ \hdashline \Omega \end{array} \qquad \longrightarrow \qquad \Phi' = \begin{array}{c} \Theta' \\ \hdashline \mathsf{ai}\uparrow \dfrac{a \wedge \bar{a}}{\bot} \vee C \\ \hdashline \Omega' \end{array}$$

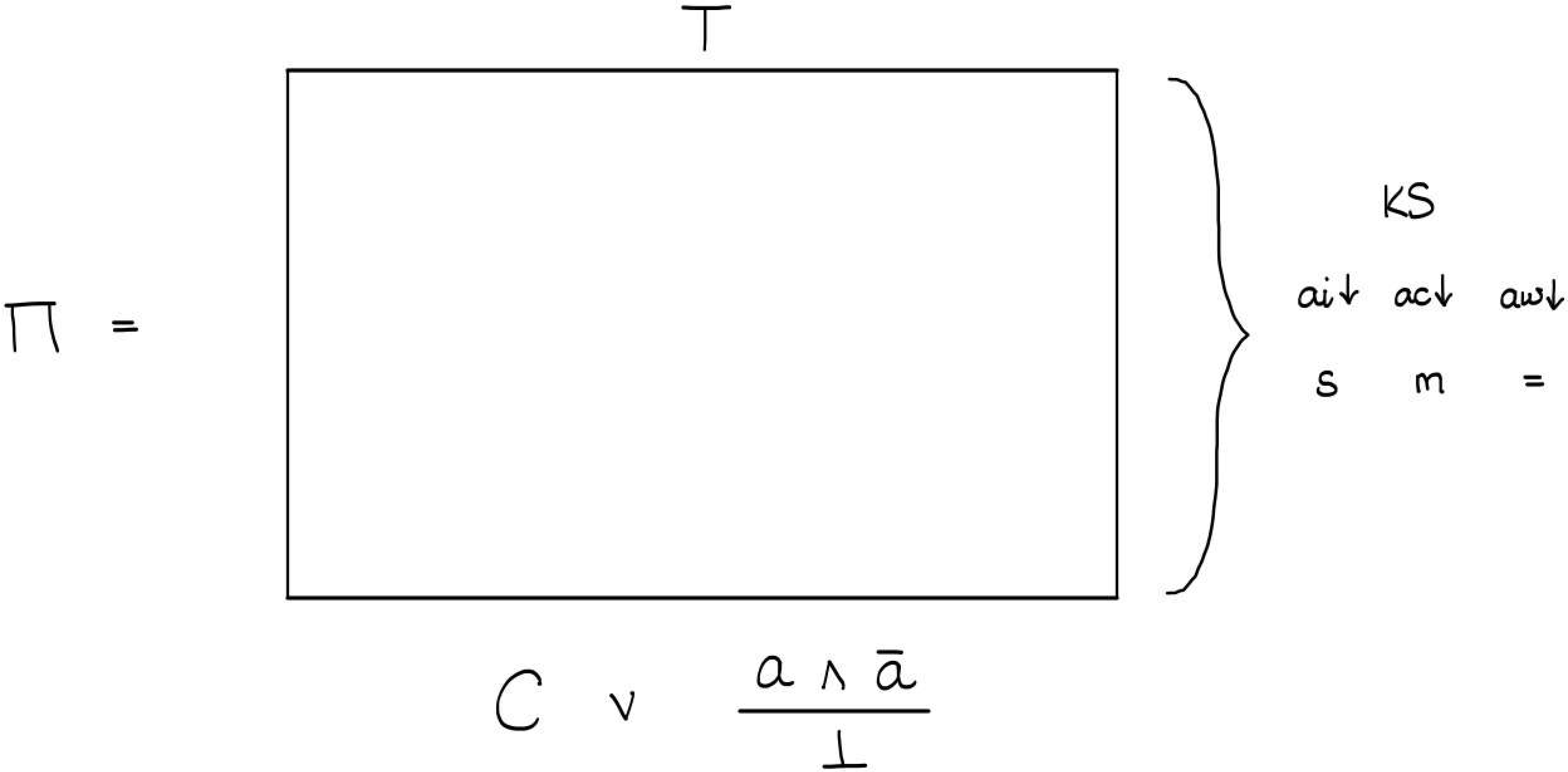
Proof is by induction on the structure of K{ } – try it!

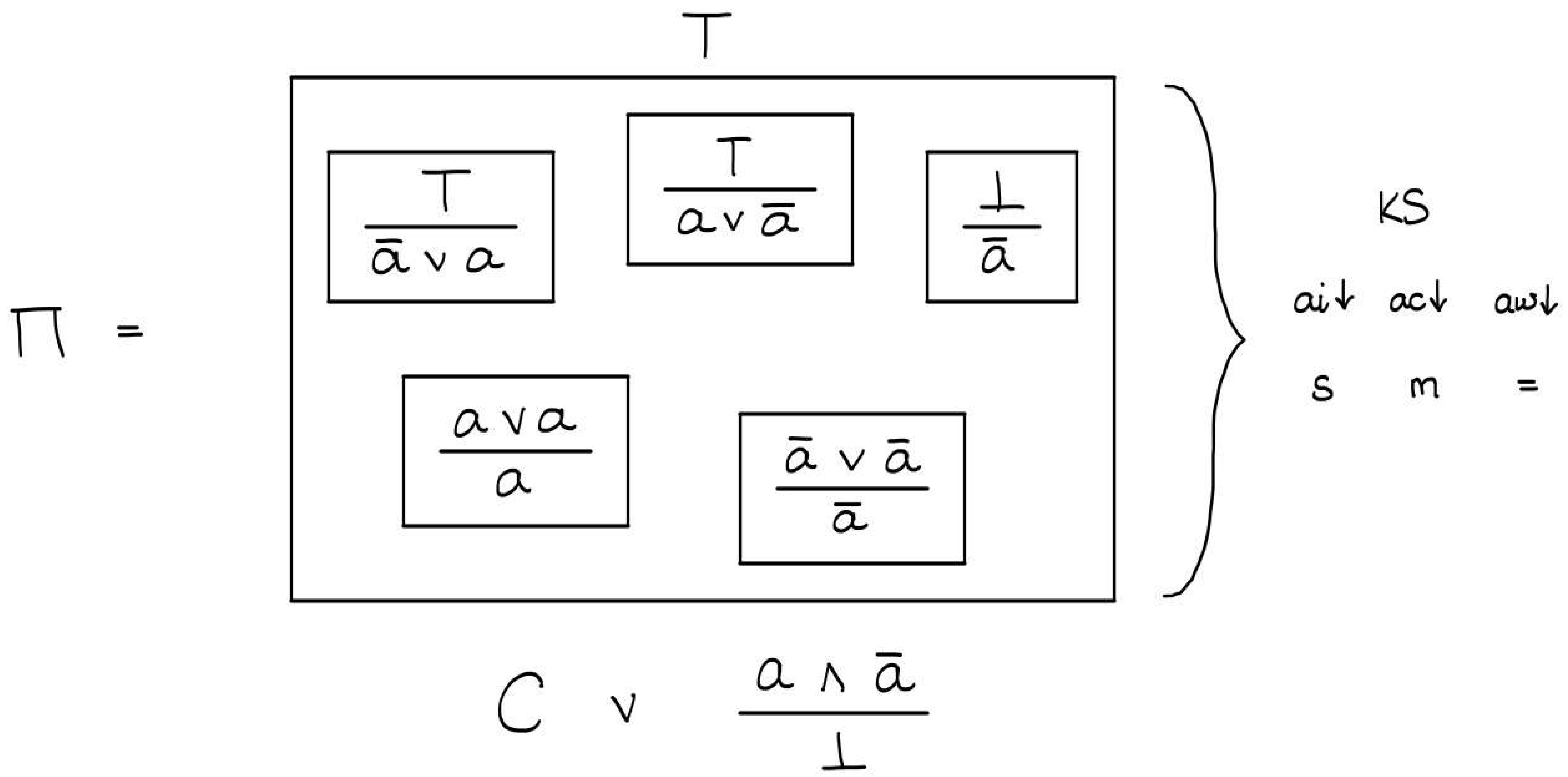
Any proof Φ in KS + cut can be transformed to

$$\Phi \longrightarrow \Psi = \begin{array}{c} \Pi \\ \hline\hline \mathsf{ai}\!\uparrow\dfrac{a \wedge \bar{a}}{\bot} \vee C \\ \hline\hline \Omega \end{array}$$

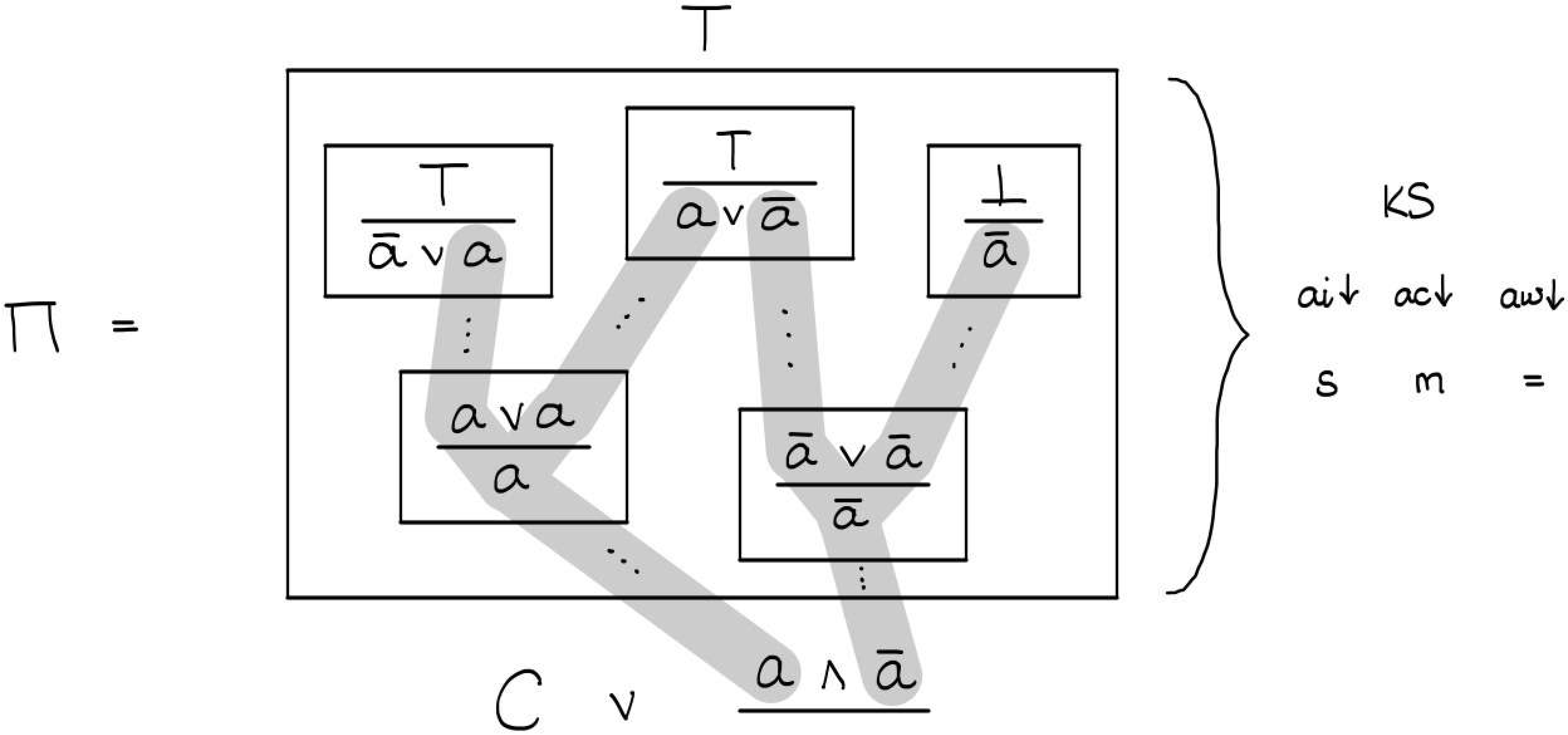


where Π has no cuts

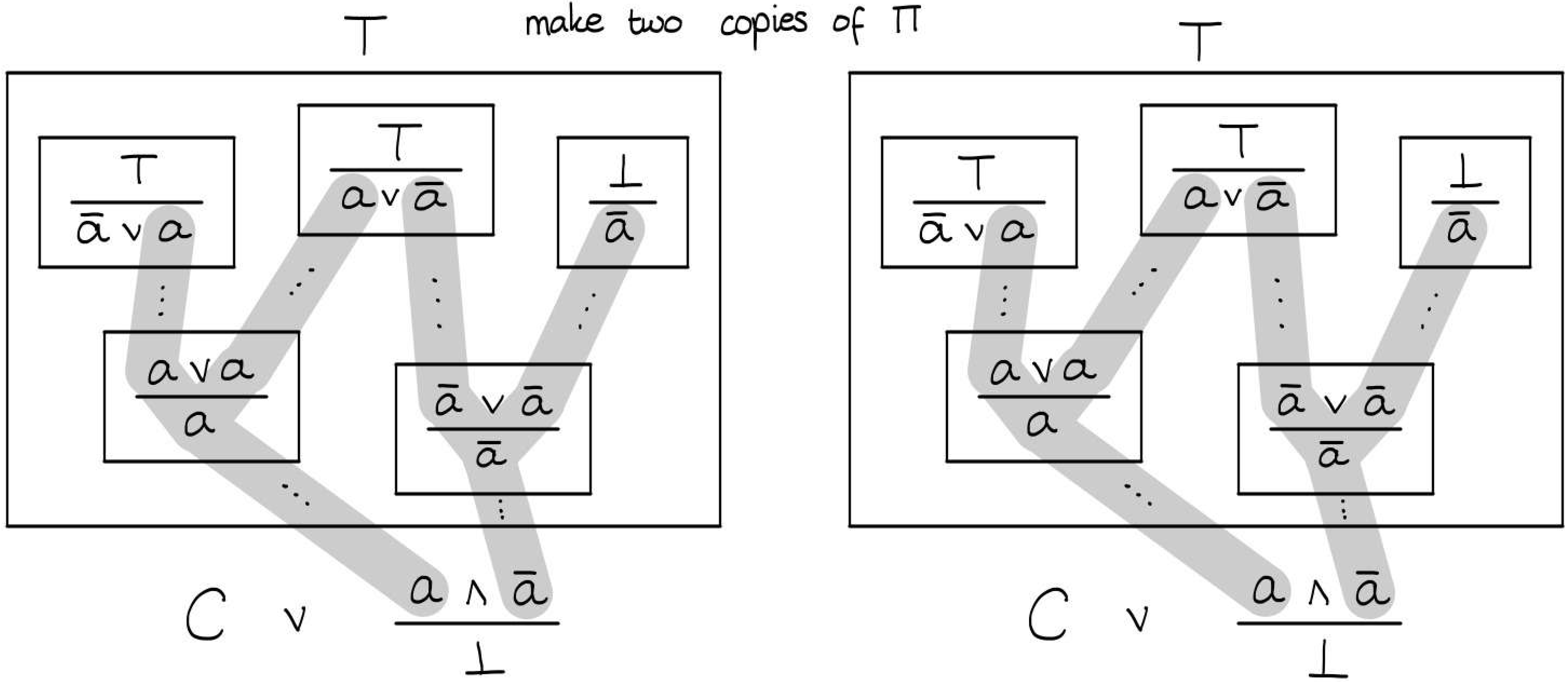

Π =
T
KS
ai↓ ac↓ aw↓
s m =
C ∨
a ∧ ā
⊥

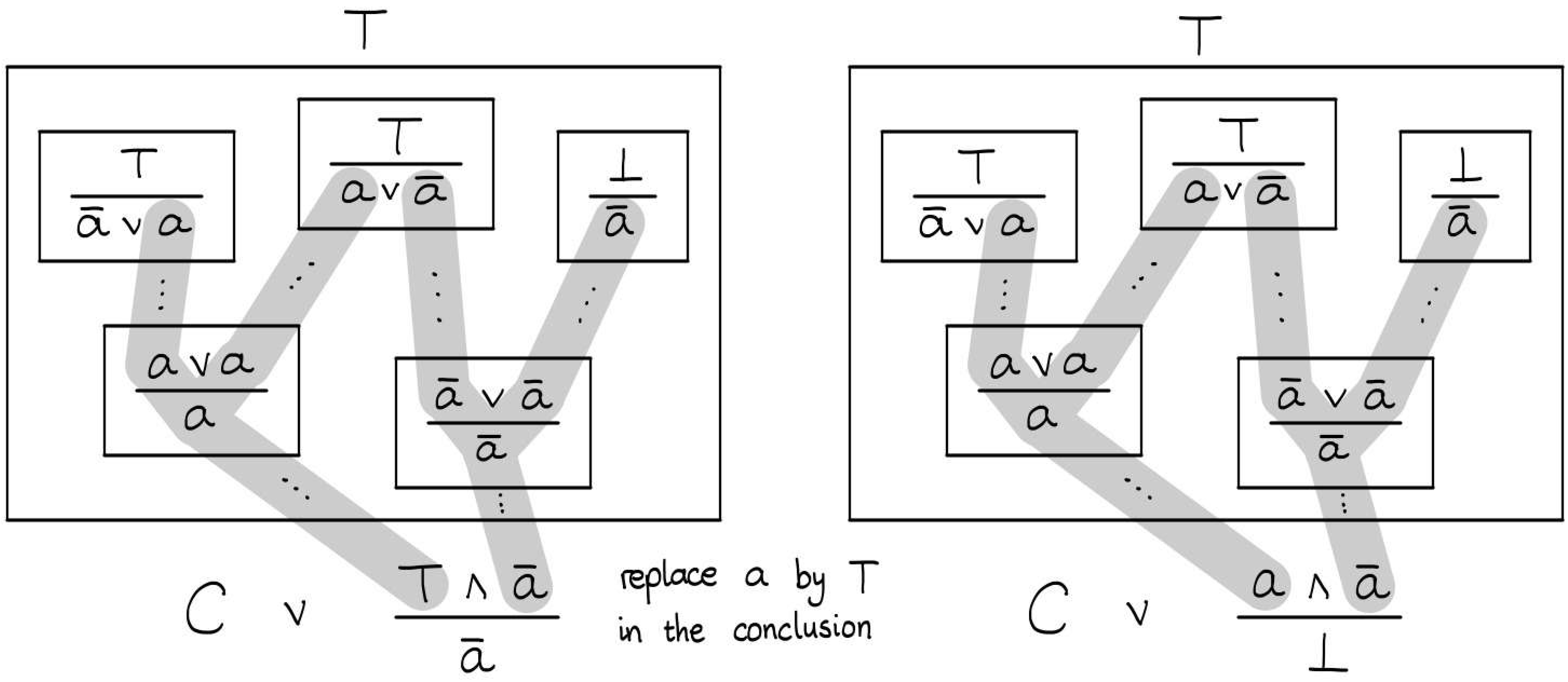


Π =
⊤
⊤ / ā ∨ a
⊤ / a ∨ ā
⊥ / ā
a ∨ a / a
ā ∨ ā / ā
KS
ai↓ ac↓ aw↓
s m =
C ∨ a ∧ ā / ⊥

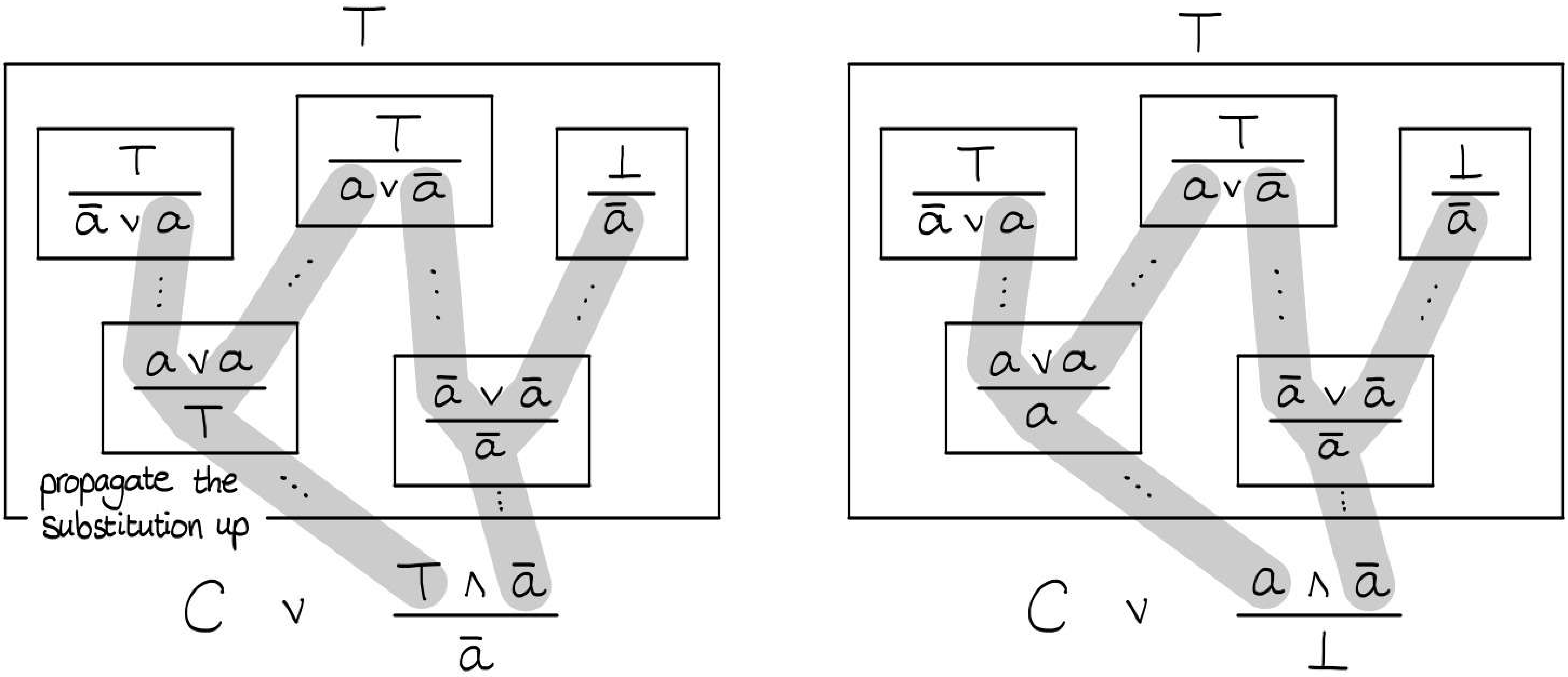


Π =
⊤
⊤
ā ∨ a
⊤
a ∨ ā
⊥
ā
a ∨ a
a
ā ∨ ā
ā
C ∨ a ∧ ā
KS
ai↓ ac↓ aw↓
s m =

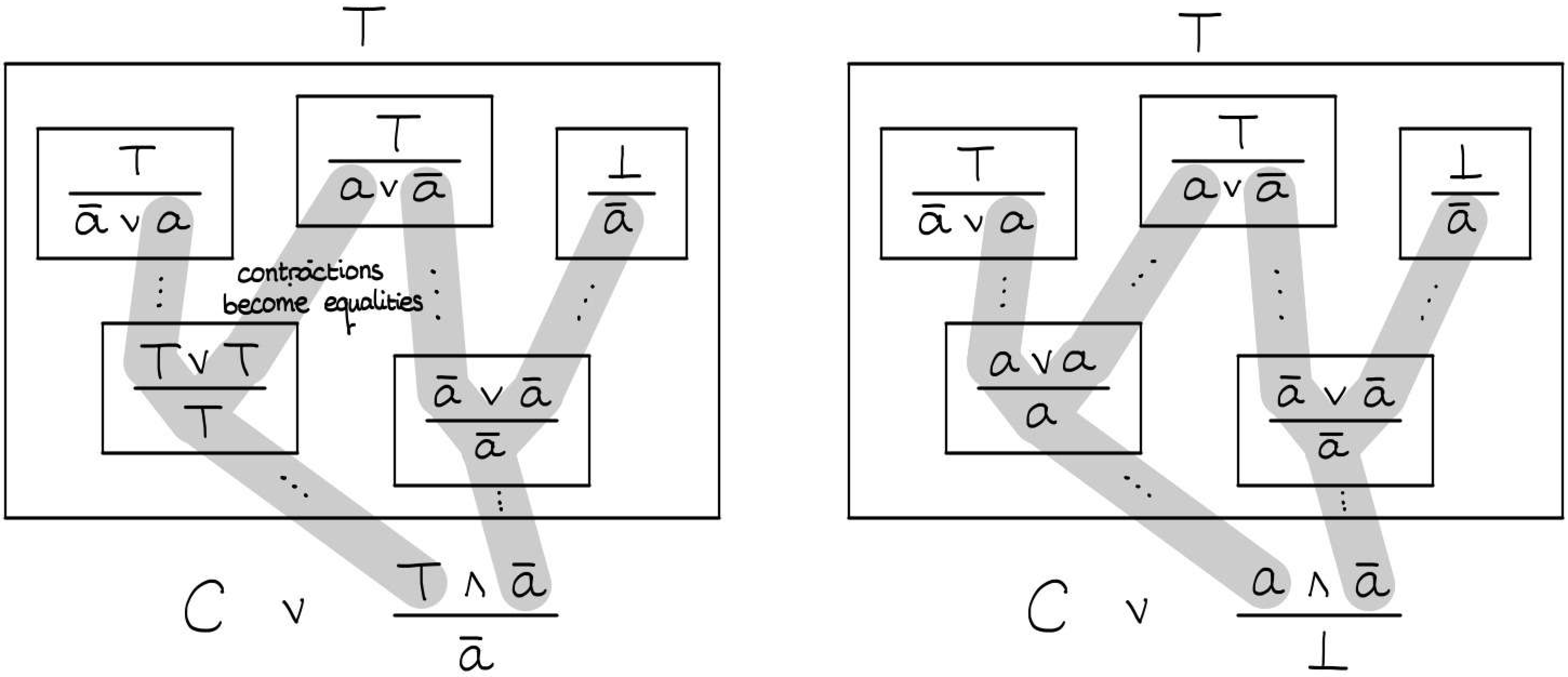

make two copies of Π
⊤
⊤
ā ∨ a
⊤
a ∨ ā
⊥
ā
a ∨ a
a
ā ∨ ā
ā
C ∨
a ∧ ā
⊥
⊤
⊤
ā ∨ a
⊤
a ∨ ā
⊥
ā
a ∨ a
a
ā ∨ ā
ā
C ∨
a ∧ ā
⊥

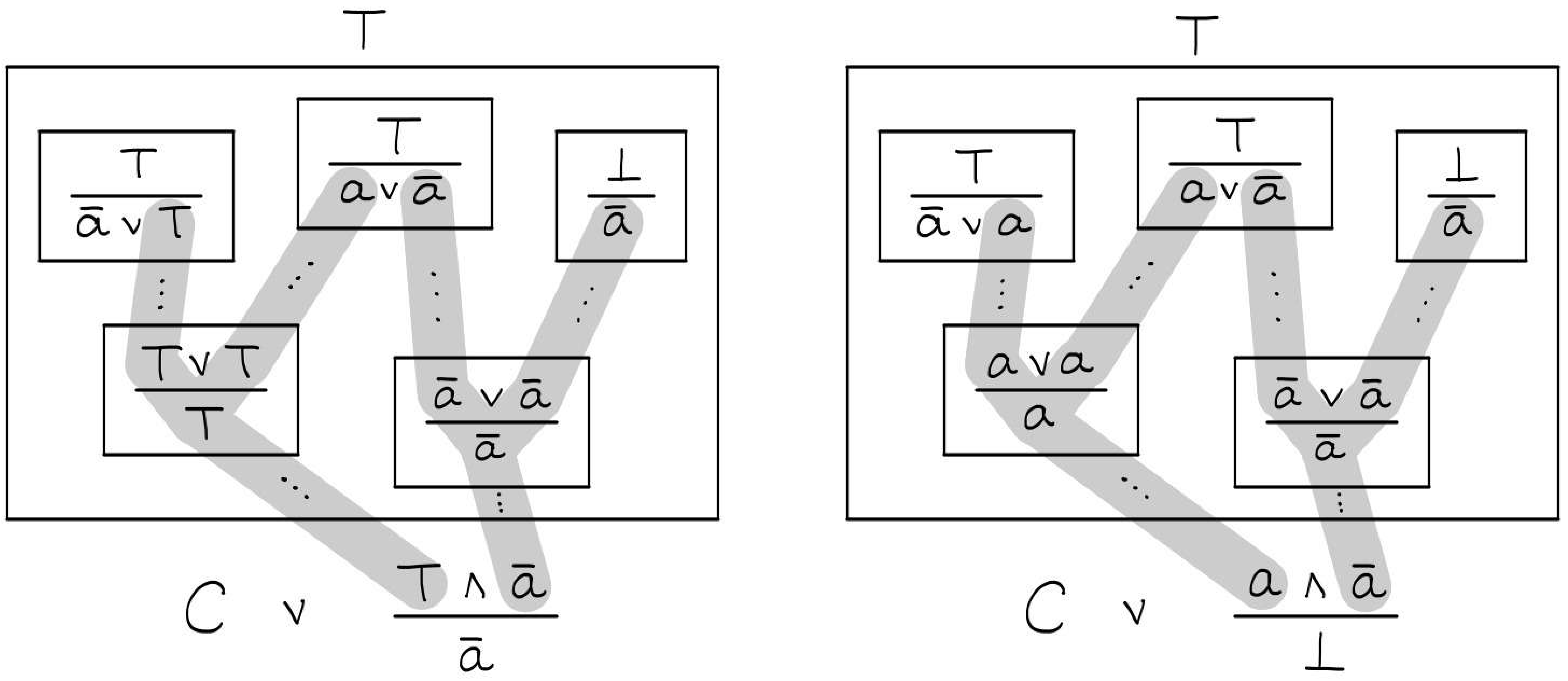


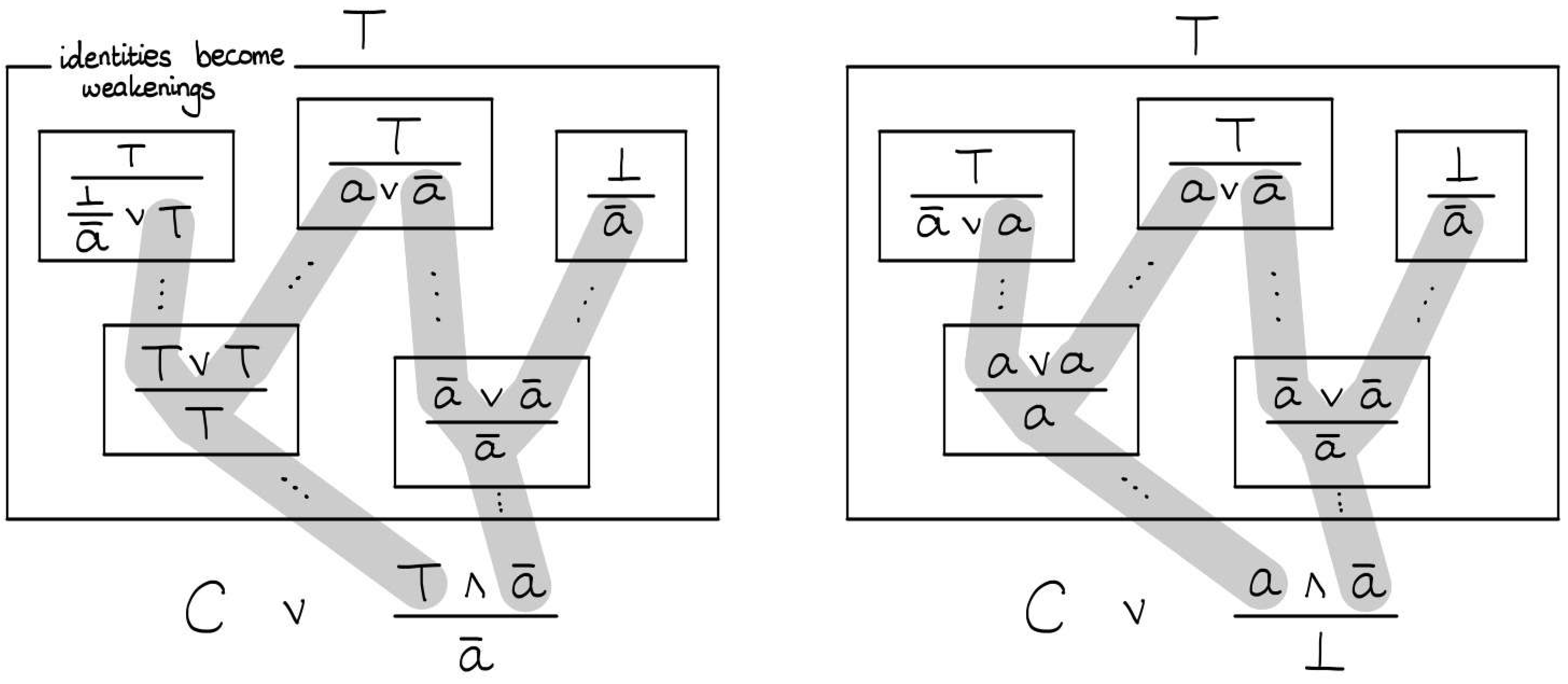
⊤
⊤
ā ∨ a
⊤
a ∨ ā
⊥
ā
a ∨ a
a
ā ∨ ā
ā
C ∨ ⊤ ∧ ā
ā
replace a by ⊤
in the conclusion
⊤
⊤
ā ∨ a
⊤
a ∨ ā
⊥
ā
a ∨ a
a
ā ∨ ā
ā
C ∨ a ∧ ā
⊥



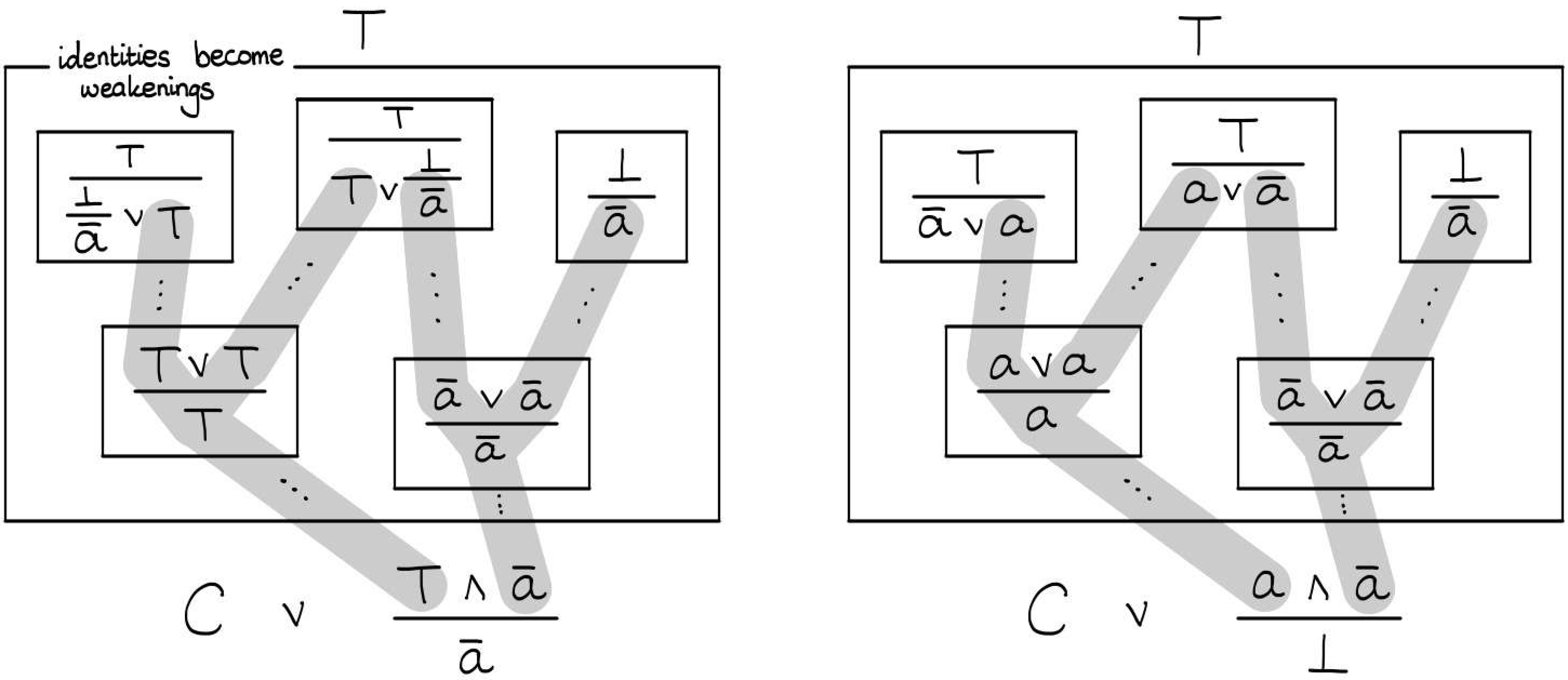
⊤
⊤ / ā ∨ a
⊤ / a ∨ ā
⊥ / ā
a ∨ a / ⊤
ā ∨ ā / ā
propagate the substitution up
C ∨ ⊤ ∧ ā / ā
⊤
⊤ / ā ∨ a
⊤ / a ∨ ā
⊥ / ā
a ∨ a / a
ā ∨ ā / ā
C ∨ a ∧ ā / ⊥


contractions
become equalities

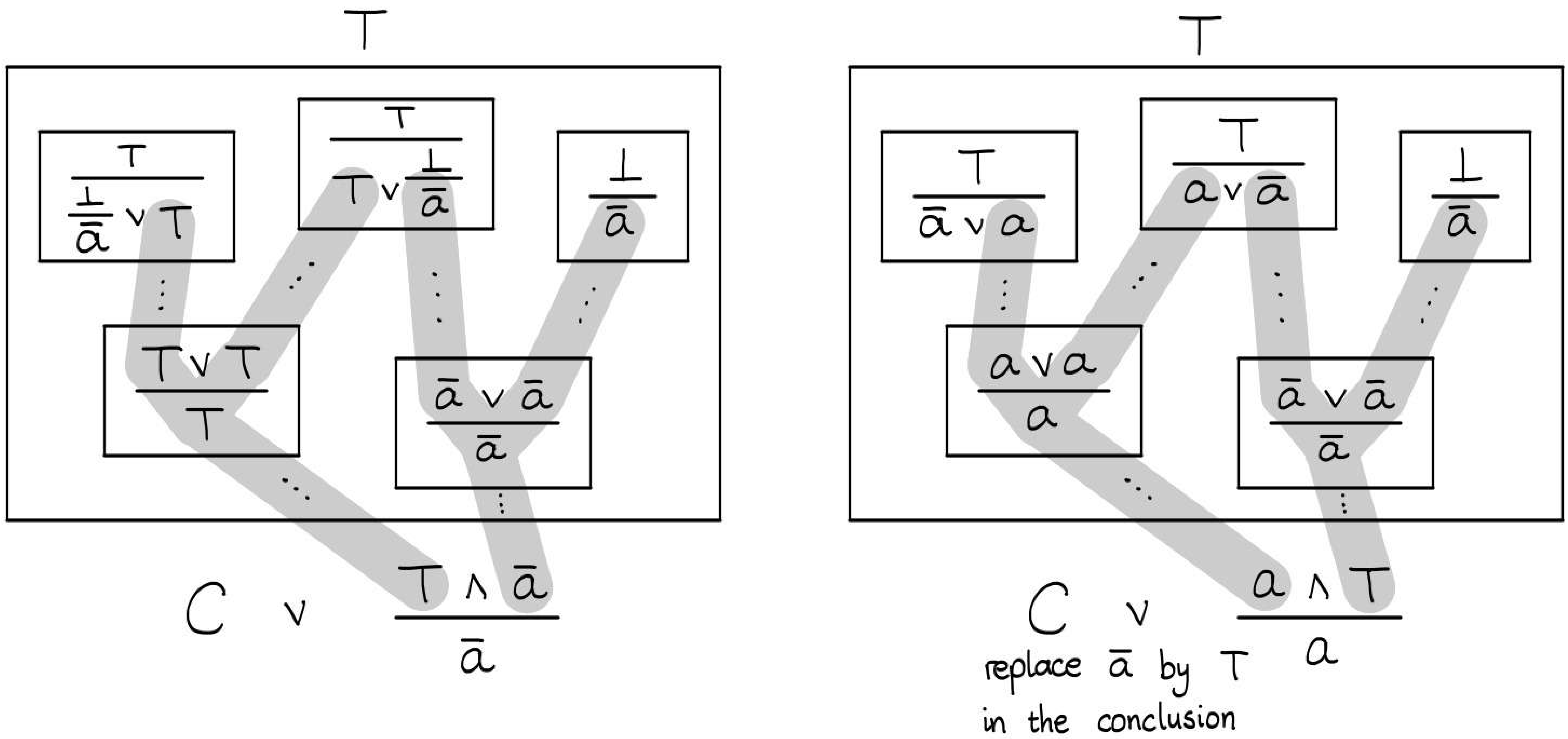

⊤
⊤ / ā ∨ ⊤
⊤ / a ∨ ā
⊥ / ā
⊤ ∨ ⊤ / ⊤
ā ∨ ā / ā
C ∨ ⊤ ∧ ā / ā
⊤
⊤ / ā ∨ a
⊤ / a ∨ ā
⊥ / ā
a ∨ a / a
ā ∨ ā / ā
C ∨ a ∧ ā / ⊥



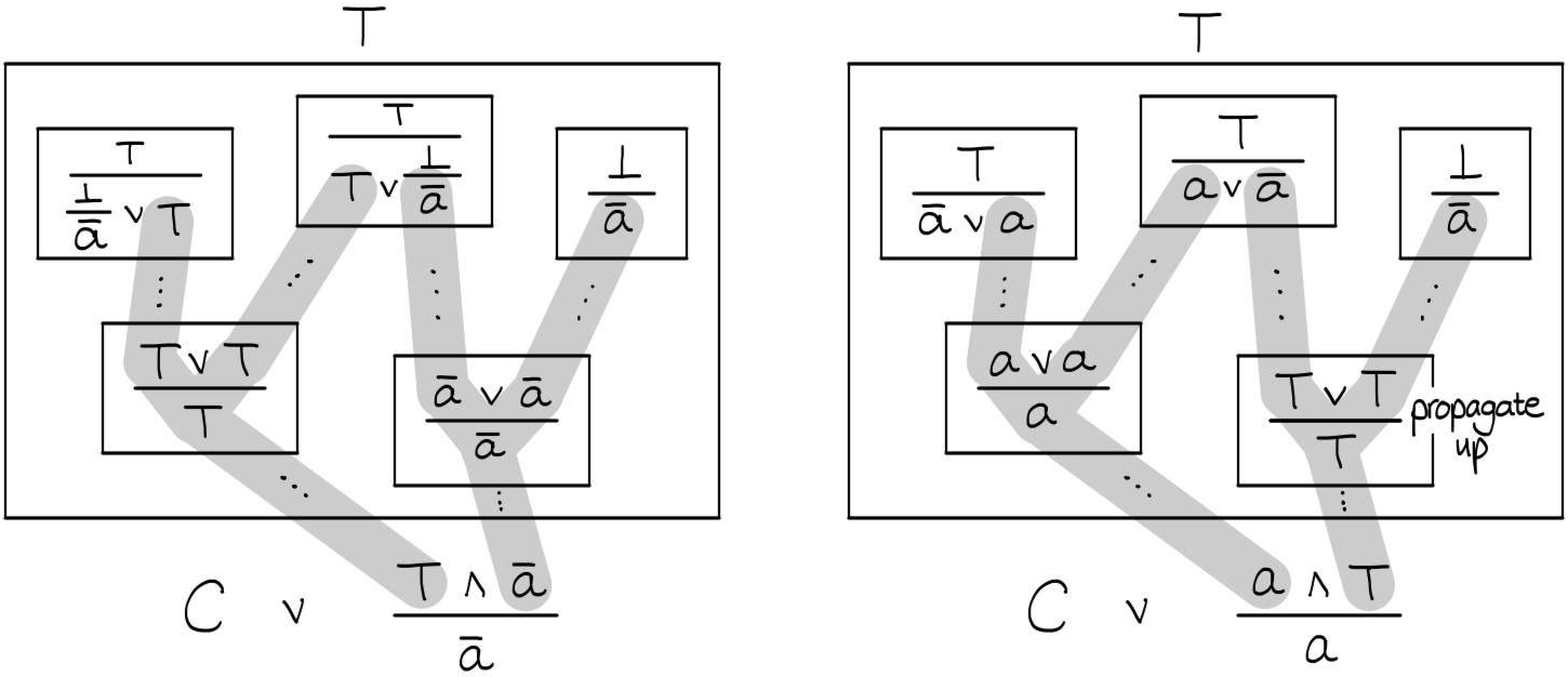
identities become weakenings

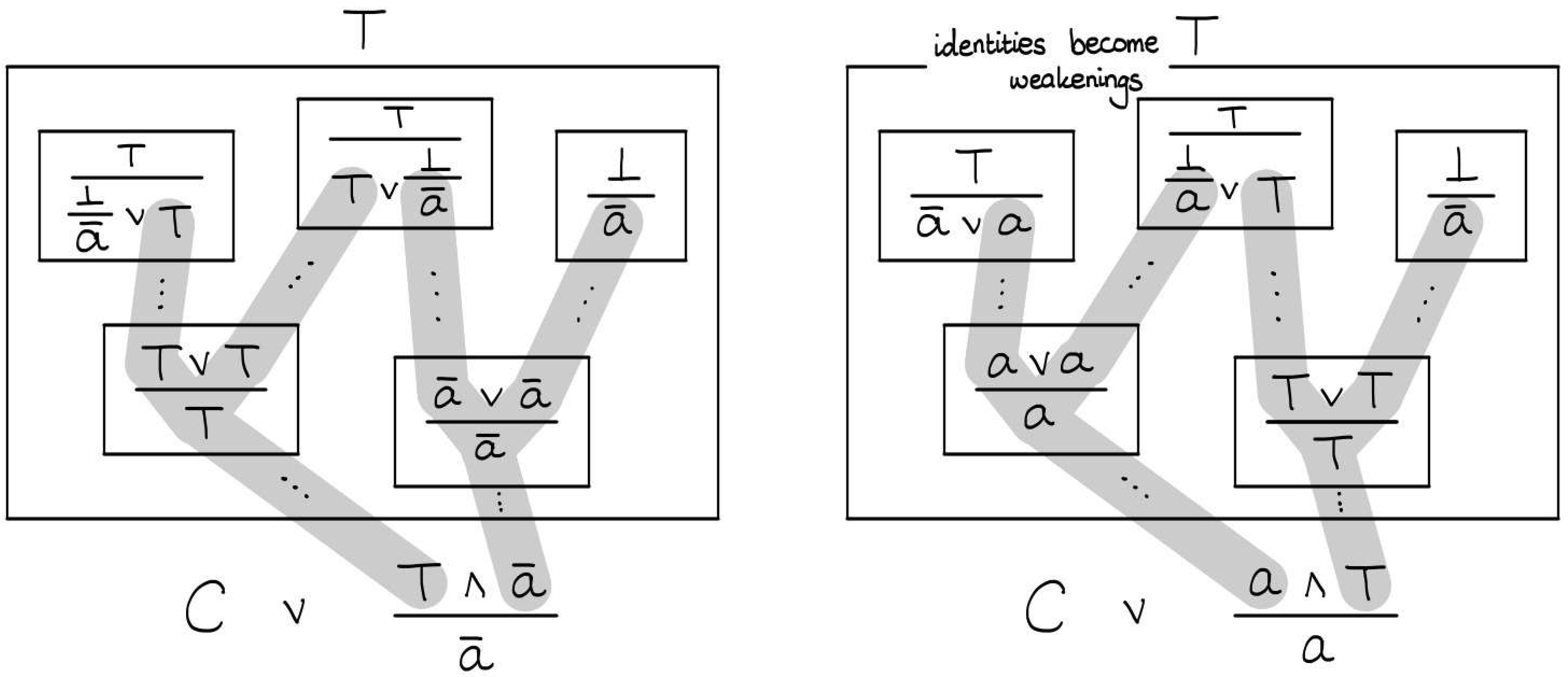


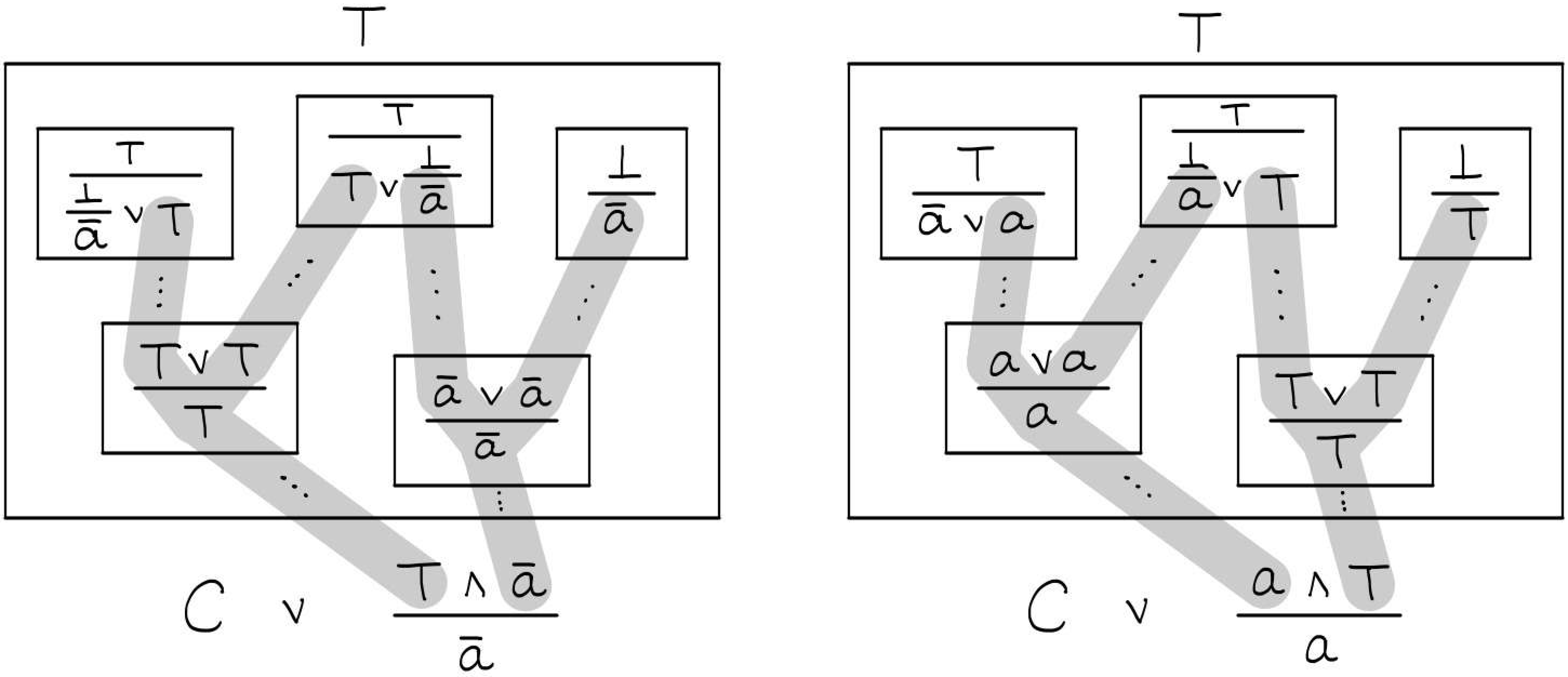
identities become weakenings

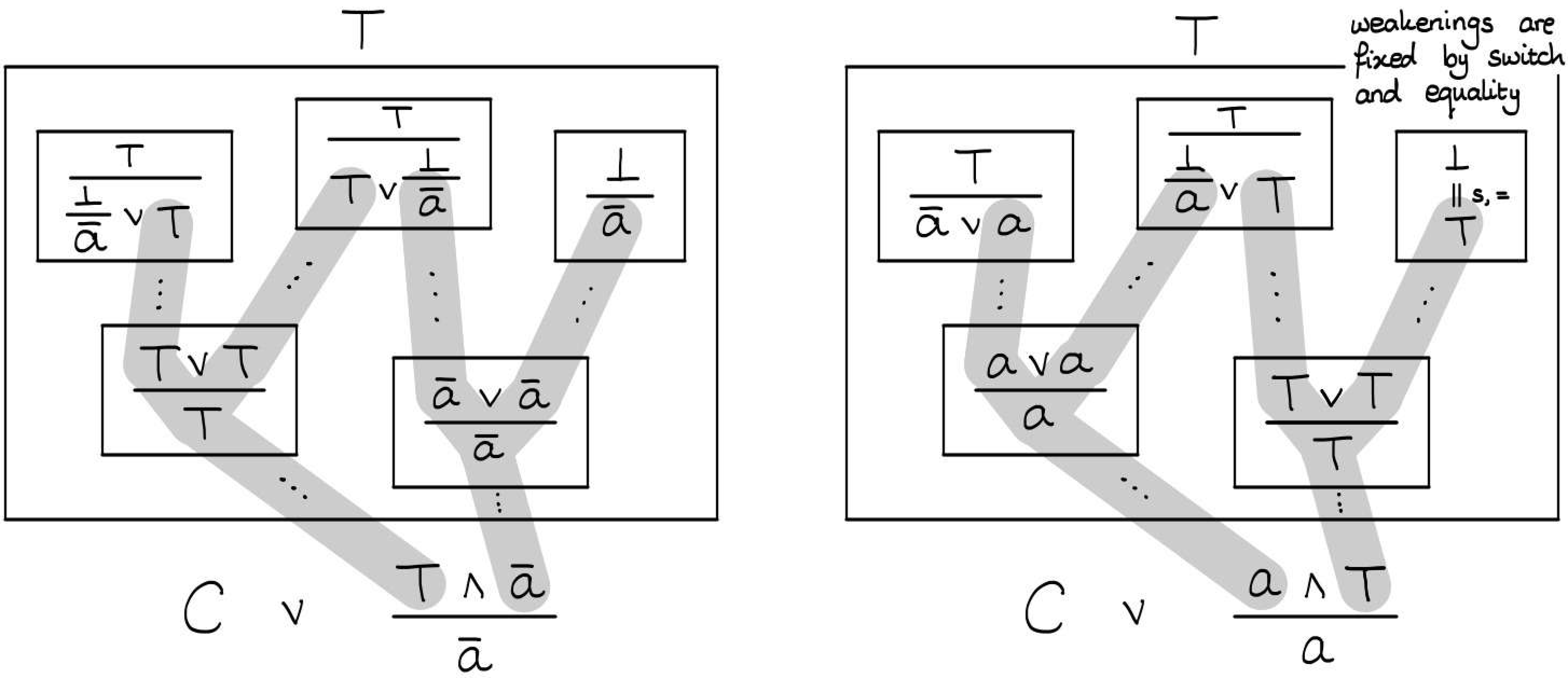


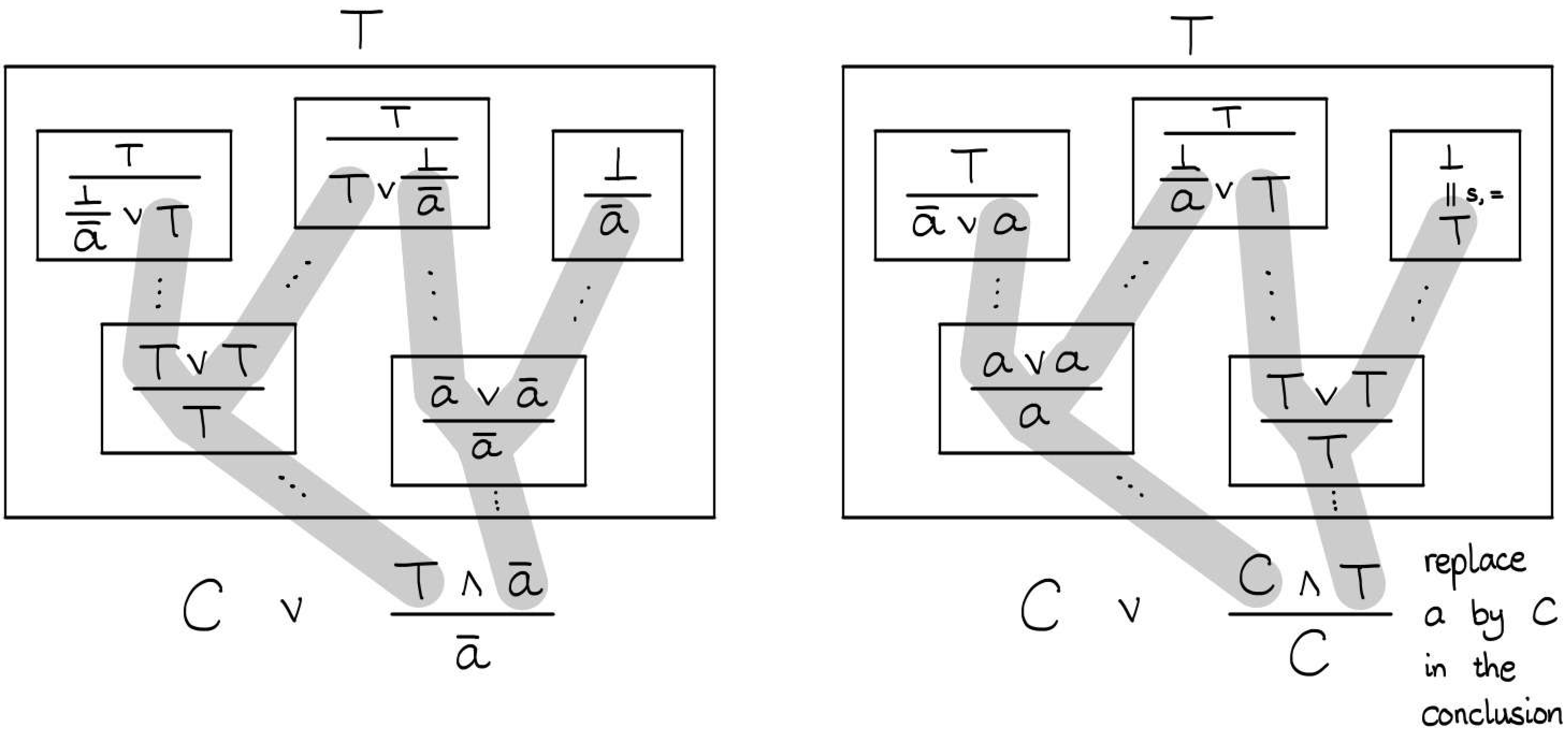
T
T
⊥
ā ∨ T
T
T ∨ ⊥
ā
⊥
ā
T ∨ T
T
ā ∨ ā
ā
C ∨ T ∧ ā
ā
T
T
ā ∨ a
T
a ∨ ā
⊥
ā
a ∨ a
a
ā ∨ ā
ā
C ∨ a ∧ T
a
replace ā by T
in the conclusion

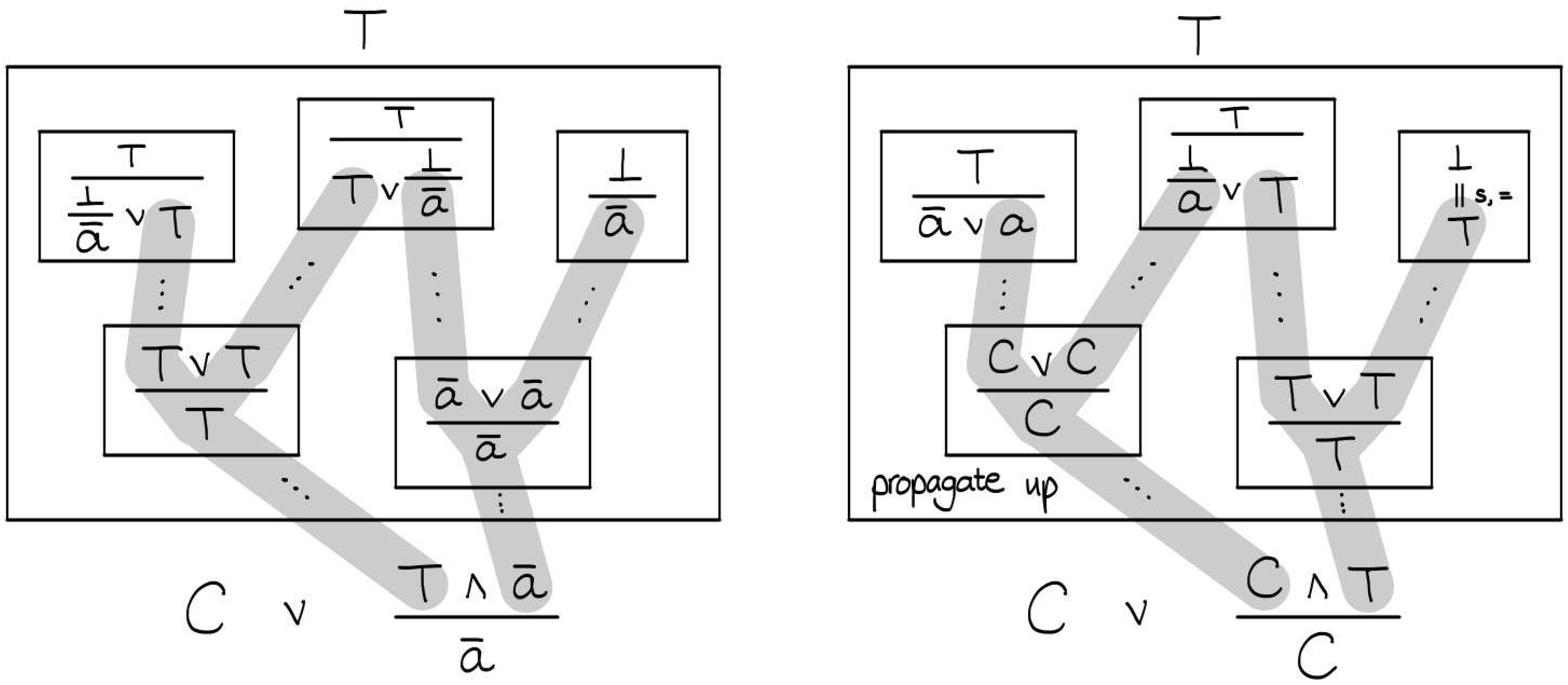

⊤
⊤ ∨ ⊤
⊤
ā ∨ ā
ā
C ∨
⊤ ∧ ā
ā
⊤
a ∨ a
a
⊤ ∨ ⊤
⊤
propagate up
C ∨
a ∧ ⊤
a

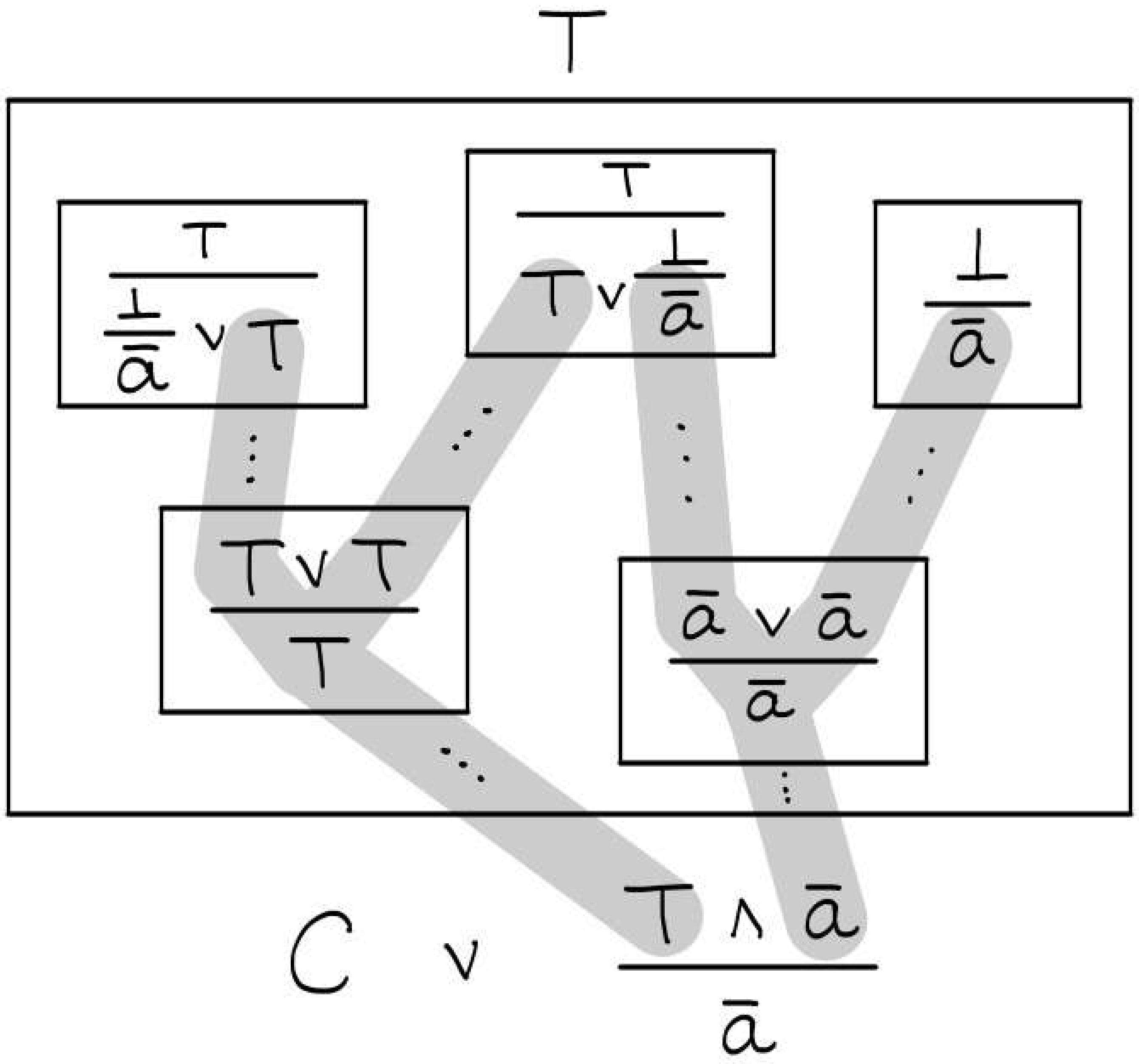

identities become weakenings

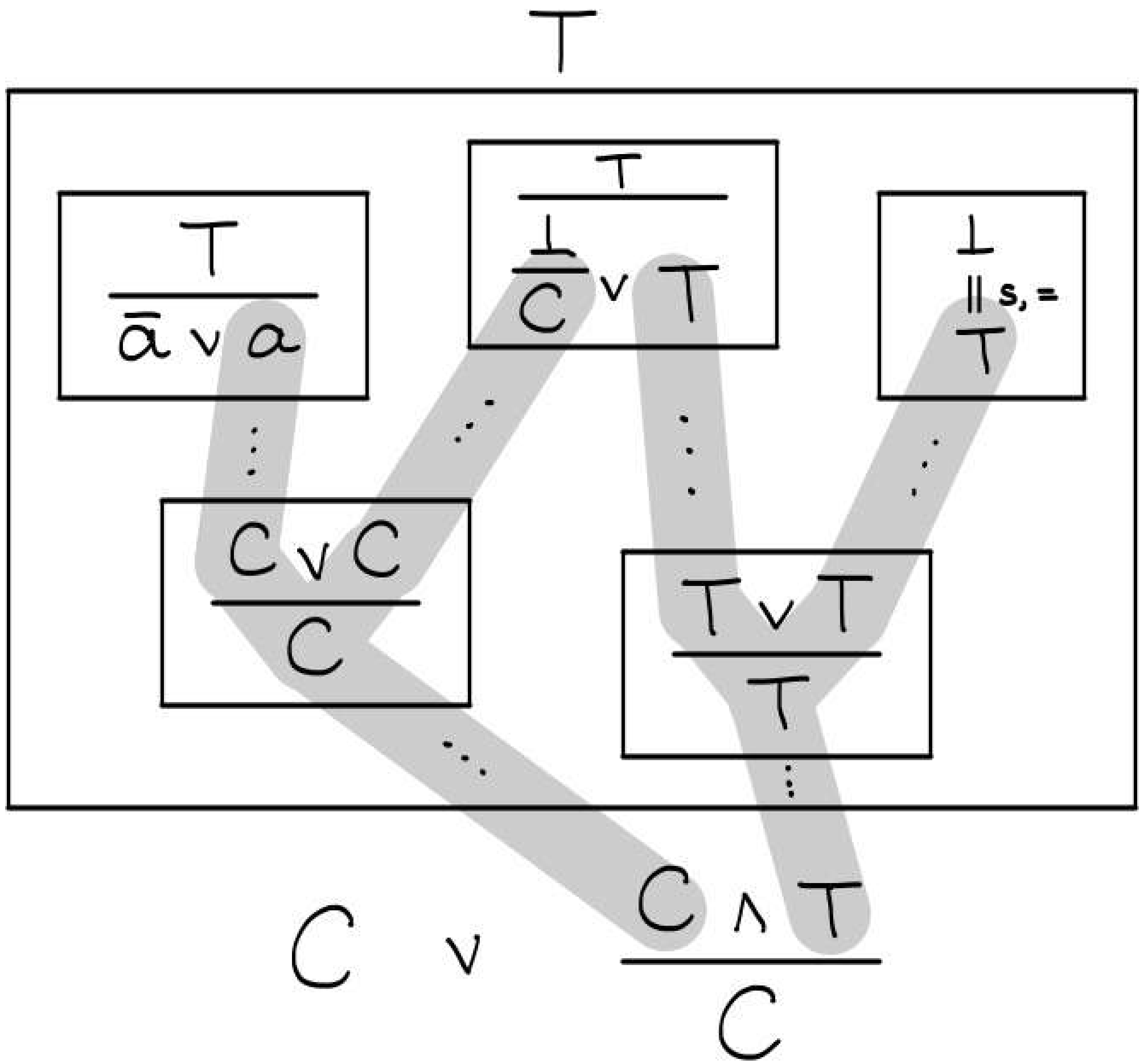

C ∨
C ∨

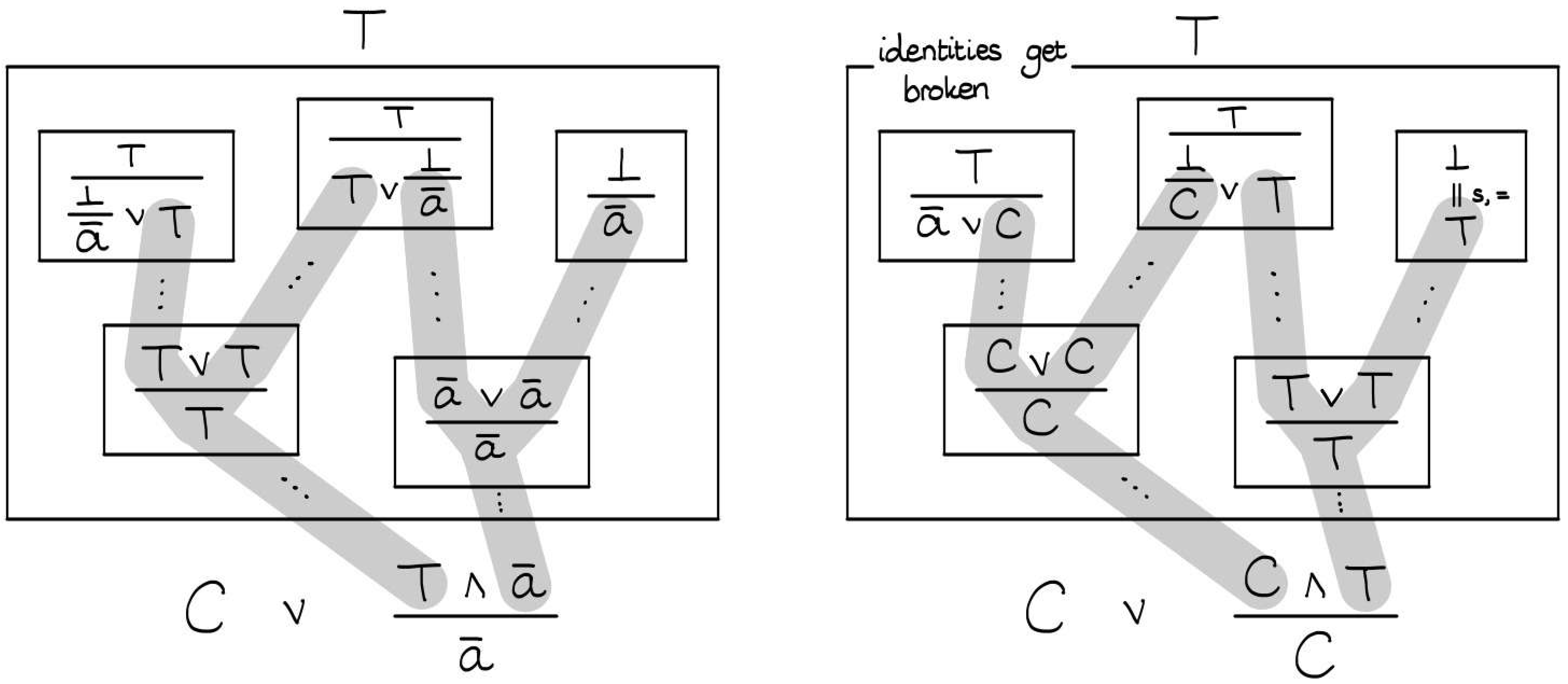

weakenings are fixed by switch and equality


replace a by C in the conclusion

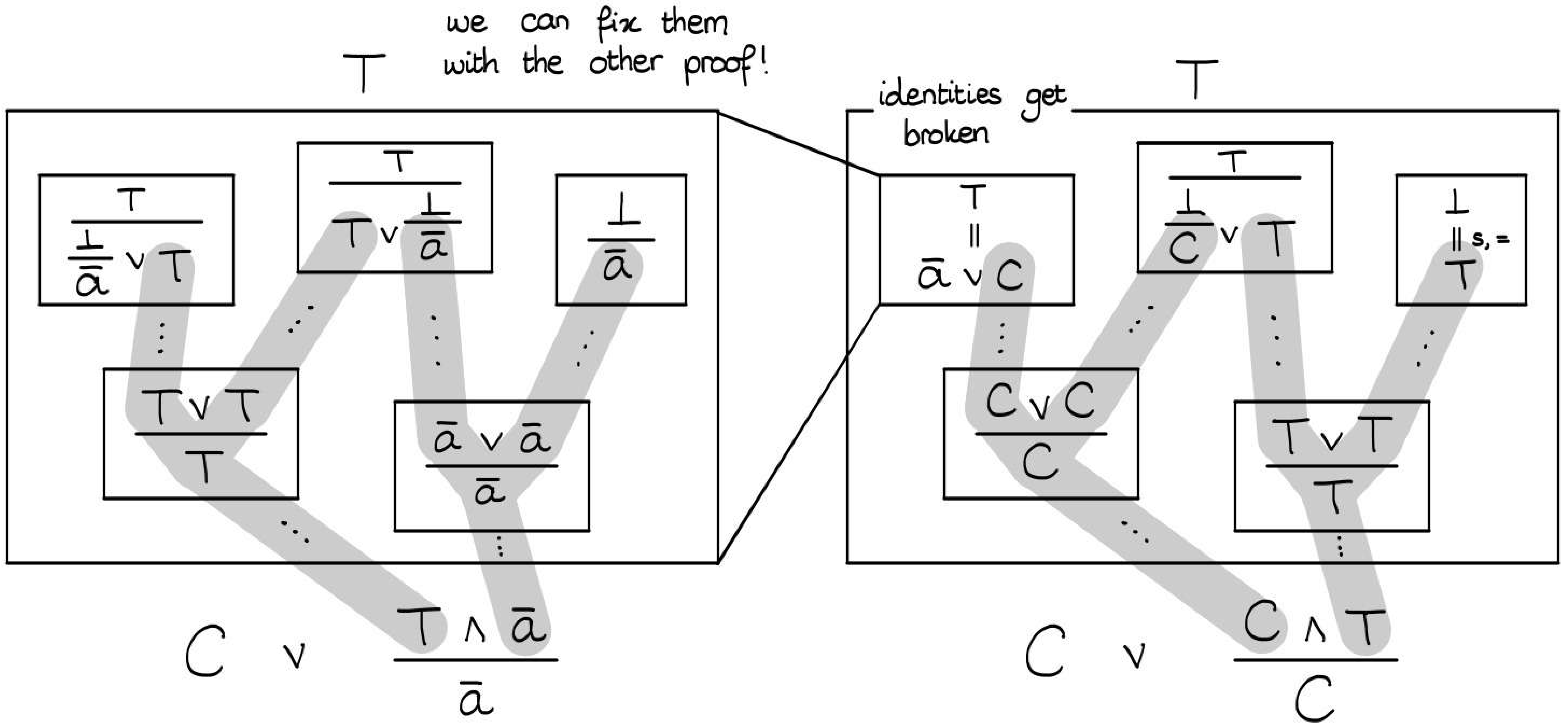

propagate up


⊤
⊤
⊥
ā ∨ ⊤
⊤
⊤ ∨ ⊥
ā
⊥
ā
⊤ ∨ ⊤
⊤
ā ∨ ā
ā
C ∨ ⊤ ∧ ā
ā

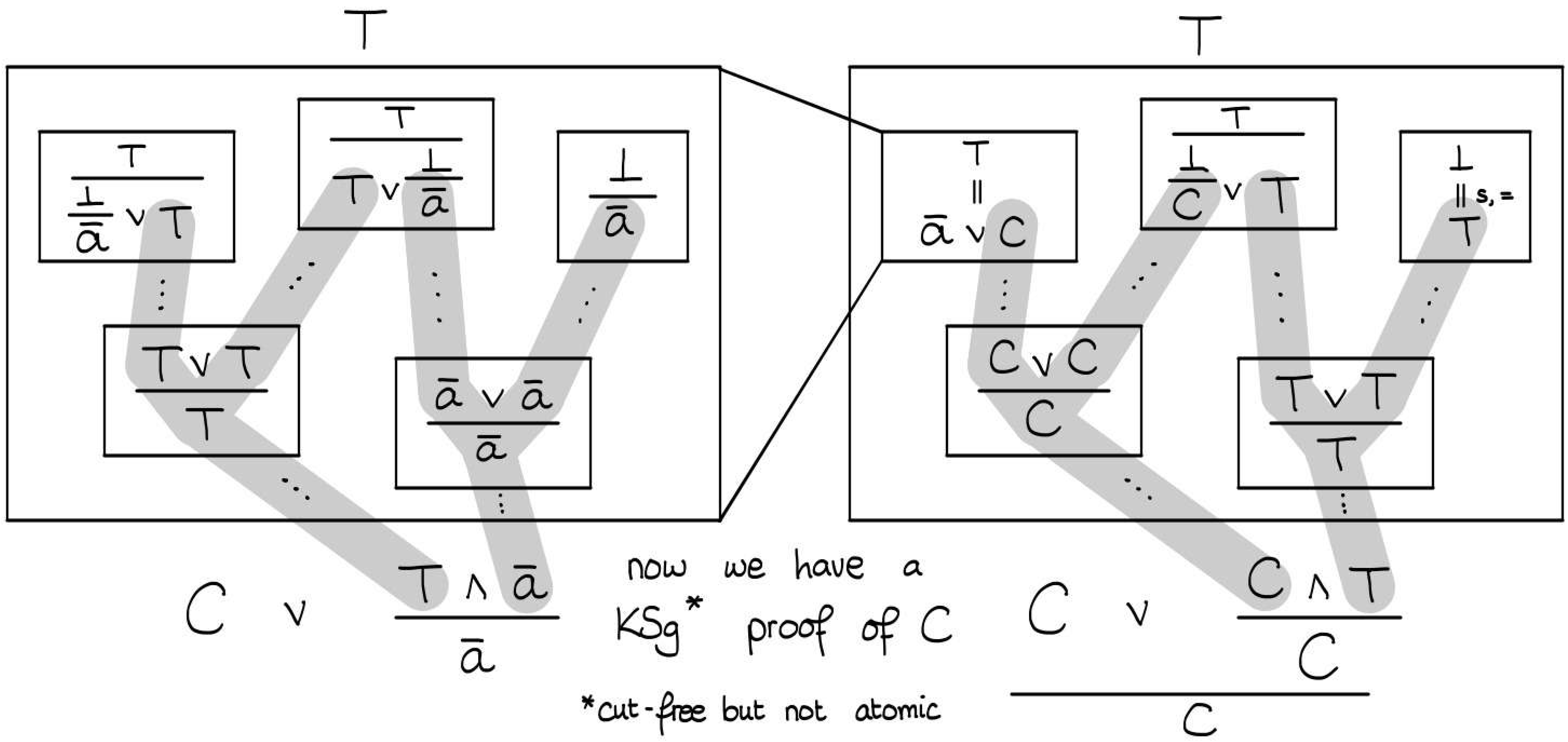

⊤
⊤
ā ∨ a
⊤
⊥
C ∨ ⊤
⊥
‖ s, =
⊤
C ∨ C
C
⊤ ∨ ⊤
⊤
C ∨ C ∧ ⊤
C



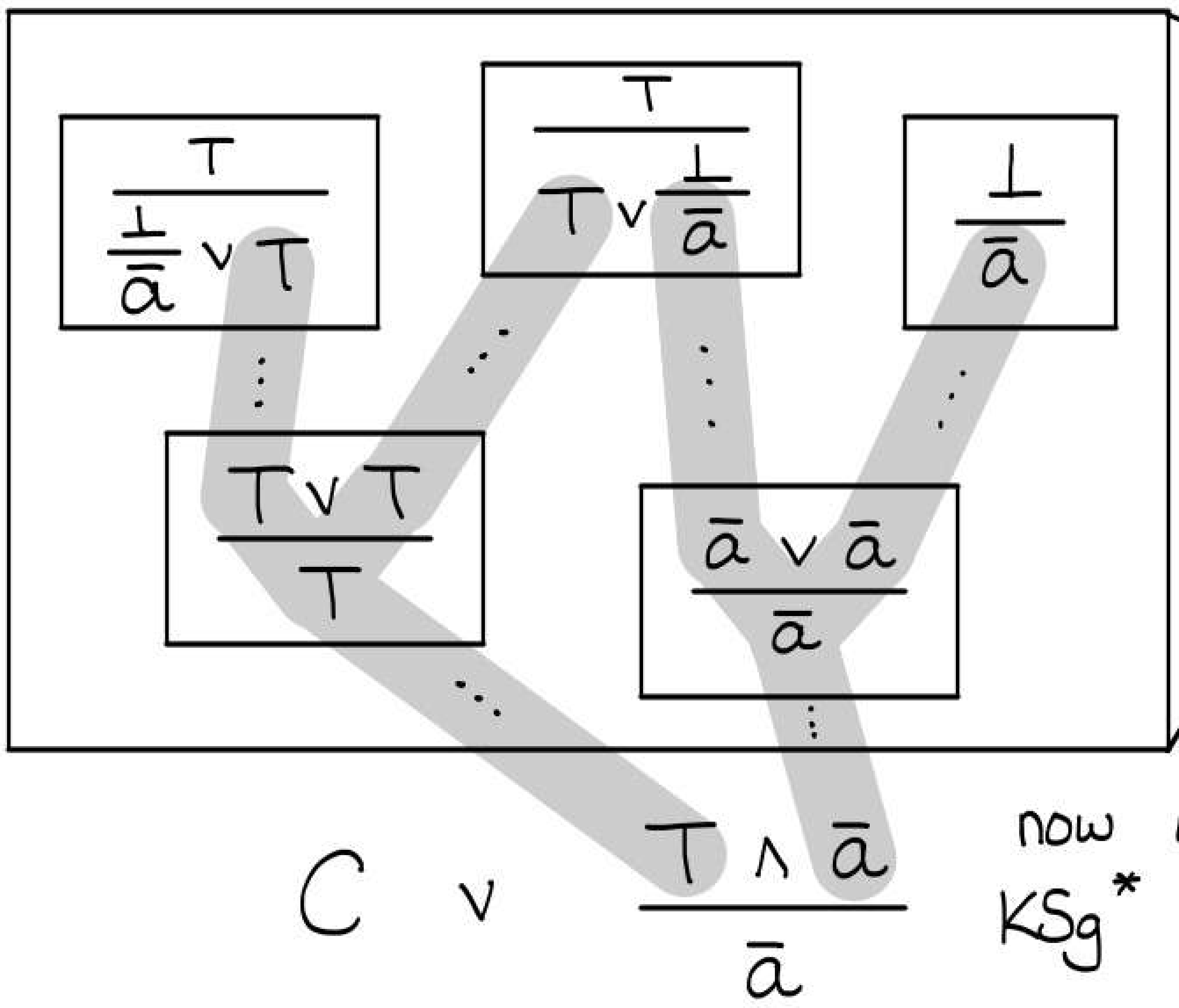
identities get broken



we can fix them
with the other proof!
identities get
broken
s, =

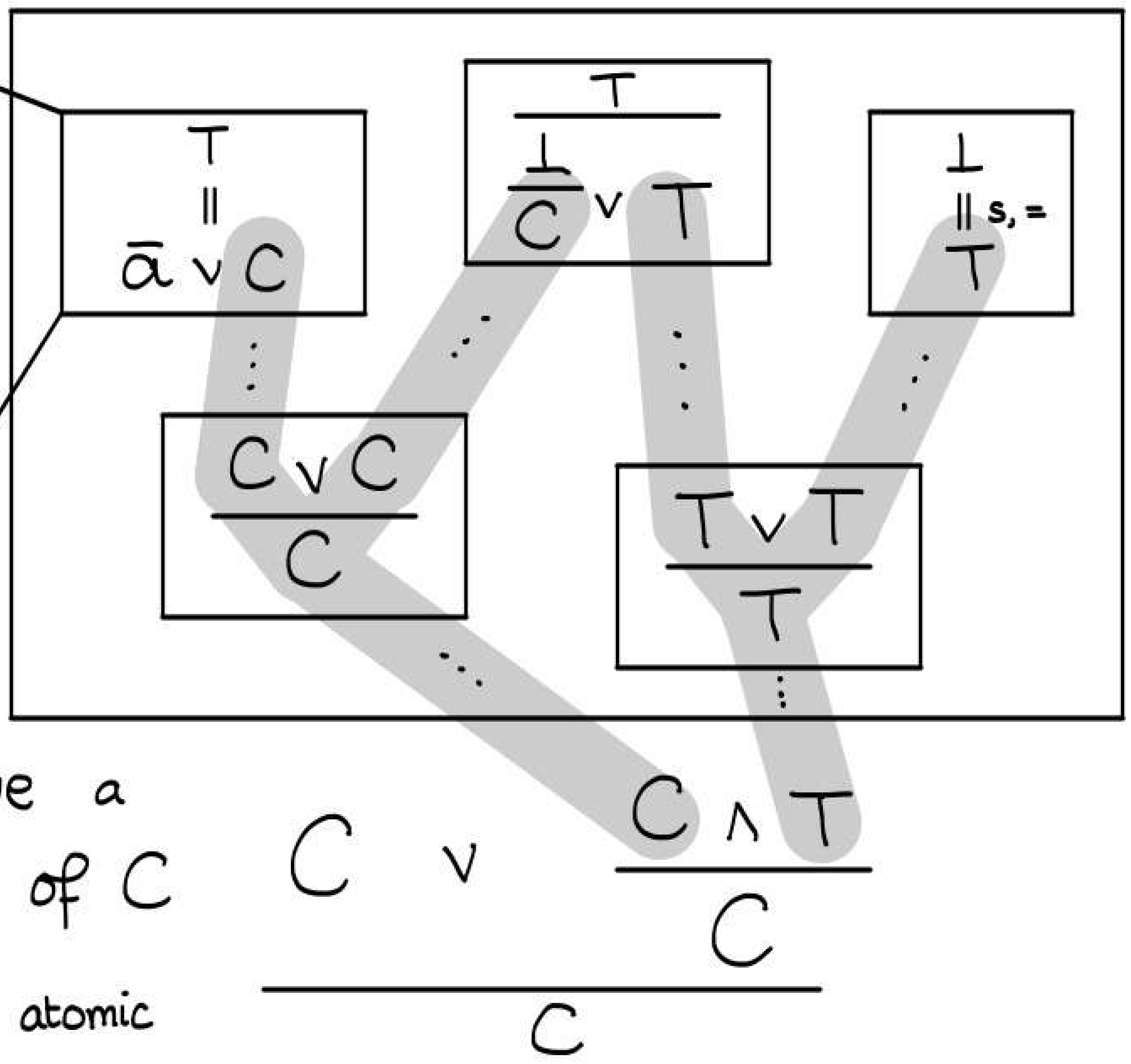

now we have a KSg* proof of C
*cut-free but not atomic

locality breaks cut elimination but we can use locality to fix it

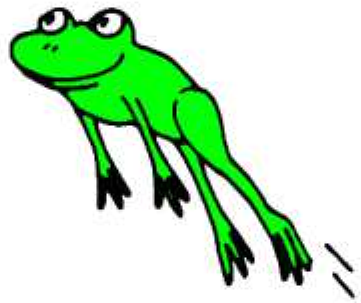


now we have a KSg* proof of C

*cut-free but not atomic

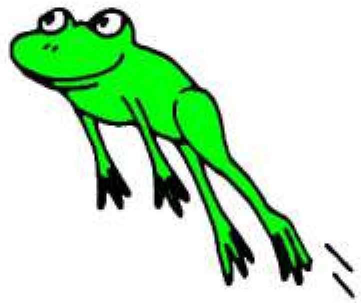


A cut elimination procedure for SKS

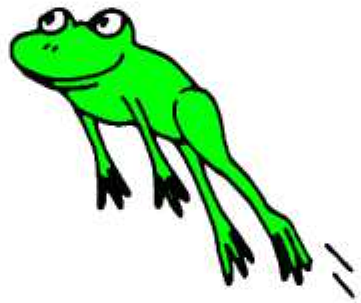


1. Eliminate ac↑, aw↑ using ac↓, aw↓, ai↑, ai↓, s, =
2. Reduce the context around each cut to a disjunctive shallow context
3. Choose the top-most cut and transform the proof above it as shown
4. Go to the next top-most cut and repeat until there are no more cuts

Comparison with sequent calculus



Comparison with sequent calculus



- Simpler induction measures to show that procedures terminate
- Local transformations
- Size increase in both cases is exponential (in general)
- Method tied to classical logic - identities become weakenings and we need to contract C