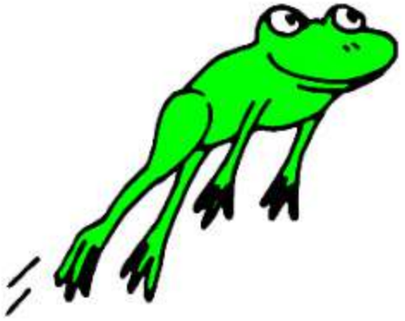
A gentle introduction to deep inference

2. Open deduction : a 21st century proof formalism



ESSLLI 2025

Victoria Barrett and Lutz Straßburger

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Complexity : theorems have small proofs

Proof search : it's easy to find proofs

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \bot \mid A \vee B \mid A \wedge B$$

Derivations

$$\pi ::= \frac{\pi_1 \quad \overset{k \geq 0}{\cdots} \quad \pi_k}{A_1, \ldots, A_m \vdash B_1, \ldots, B_n} \qquad m \geq 0, \quad n \geq 0$$

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inference rules operate on derivation-level connectives, not formula-level

Open deduction

only one level of connective

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$$\begin{array}{c} A \\ \square \\ B \end{array}$$

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$$\begin{array}{cc} A & C \\ \square & \square \\ B & D \end{array}$$

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$$\begin{array}{c} A \\ \square \\ B \end{array} \vee \begin{array}{c} C \\ \square \\ D \end{array}$$

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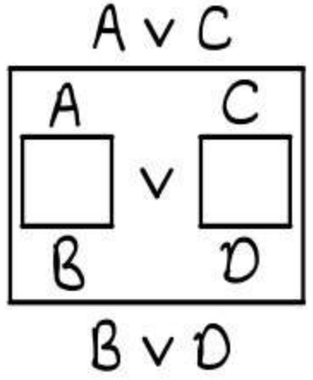

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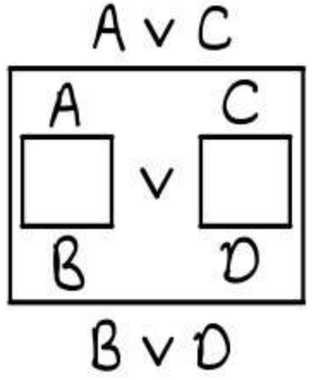

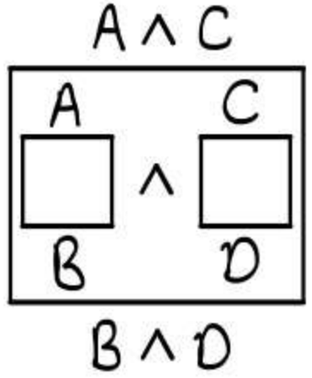

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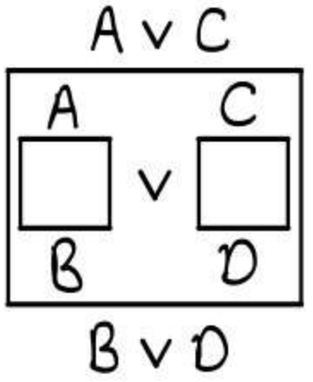

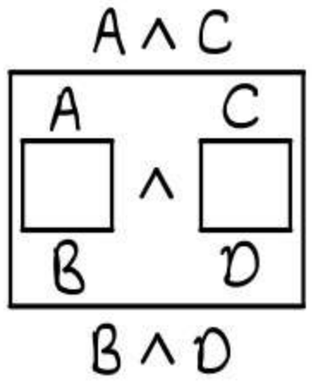

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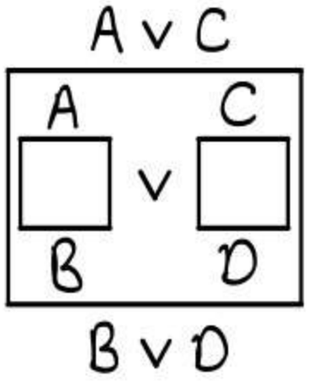

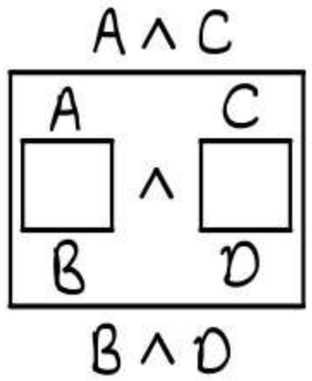

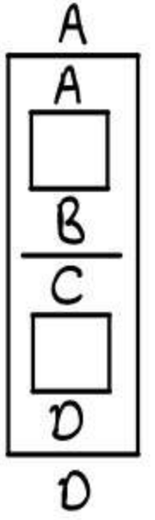

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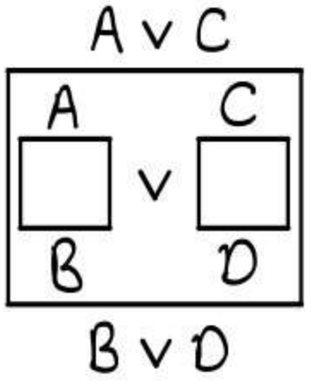

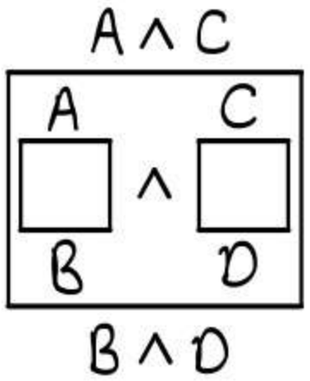

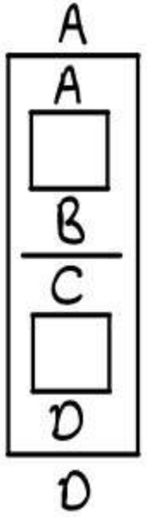

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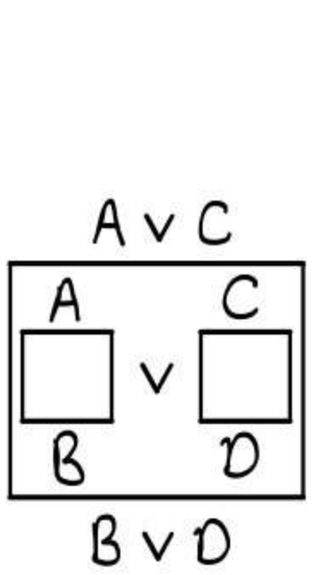

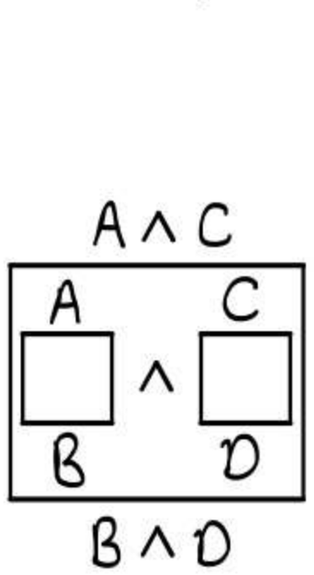

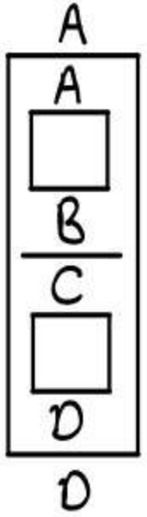

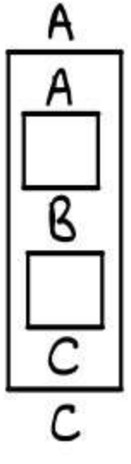

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$$\frac{A \vee C}{B \vee D} \qquad \frac{A \wedge C}{B \wedge D} \qquad \frac{A}{D} \qquad \frac{A}{C}$$

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$$\Gamma, \Theta \vdash \Delta, \Lambda$$

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$$B, C \vdash D, A \qquad\qquad E, A \vdash F$$

$$B, C, E \vdash D, F$$

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$$= \frac{(\bar{B} \vee \bar{C}) \vee (D \vee A)}{((\bar{B} \vee \bar{C}) \vee D) \vee A} \quad \wedge \quad = \frac{(\bar{E} \vee \bar{A}) \vee F}{\bar{A} \vee (\bar{E} \vee F)}$$

$$(\bar{B} \vee \bar{C} \vee \bar{E}) \vee (D \vee F)$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \bot \mid A \vee B \mid A \wedge B$$

Derivations

$$\pi ::= \frac{\pi_1 \quad k \geq 0 \quad \cdots \quad \pi_k}{A_1, \ldots, A_m \vdash B_1, \ldots, B_n} \qquad m \geq 0, \; n \geq 0$$

$$\text{cut}\,\frac{\Gamma \vdash \Upsilon, A \qquad \Theta, A \vdash \Lambda}{\Gamma, \Theta \vdash \Upsilon, \Lambda}$$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\pi}}{\boxed{\Delta}}$$

$$= \frac{(\bar{B} \vee \bar{C}) \vee (D \vee A)}{((\bar{B} \vee \bar{C}) \vee D) \vee A} \quad \wedge \quad = \frac{(\bar{E} \vee \bar{A}) \vee F}{\bar{A} \vee (\bar{E} \vee F)}$$

$$(\bar{B} \vee \bar{C} \vee \bar{E}) \vee (D \vee F)$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \bot \mid A \vee B \mid A \wedge B$$

Derivations

$$\pi ::= \dfrac{\pi_1 \quad \overset{k \geq 0}{\cdots} \quad \pi_k}{A_1, \ldots, A_m \vdash B_1, \ldots, B_n} \qquad m \geq 0, \; n \geq 0$$

$$\text{cut}\,\dfrac{\Gamma \vdash \Upsilon, A \qquad \Theta, A \vdash \Lambda}{\Gamma, \Theta \vdash \Upsilon, \Lambda}$$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \dfrac{\boxed{\pi}}{\boxed{\Delta}}$$

$$\text{s}\,\dfrac{\overset{=}{((\bar{B} \vee \bar{C}) \vee D) \vee A} \quad \wedge \quad \overset{=}{\bar{A} \vee (\bar{E} \vee F)}}{((\bar{B} \vee \bar{C}) \vee D) \vee (A \wedge (\bar{A} \vee (\bar{E} \vee F)))}$$

$$\overset{=}{(\bar{B} \vee \bar{C} \vee \bar{E}) \vee (D \vee F)}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \bot \mid A \vee B \mid A \wedge B$$

Derivations

$\pi ::=$

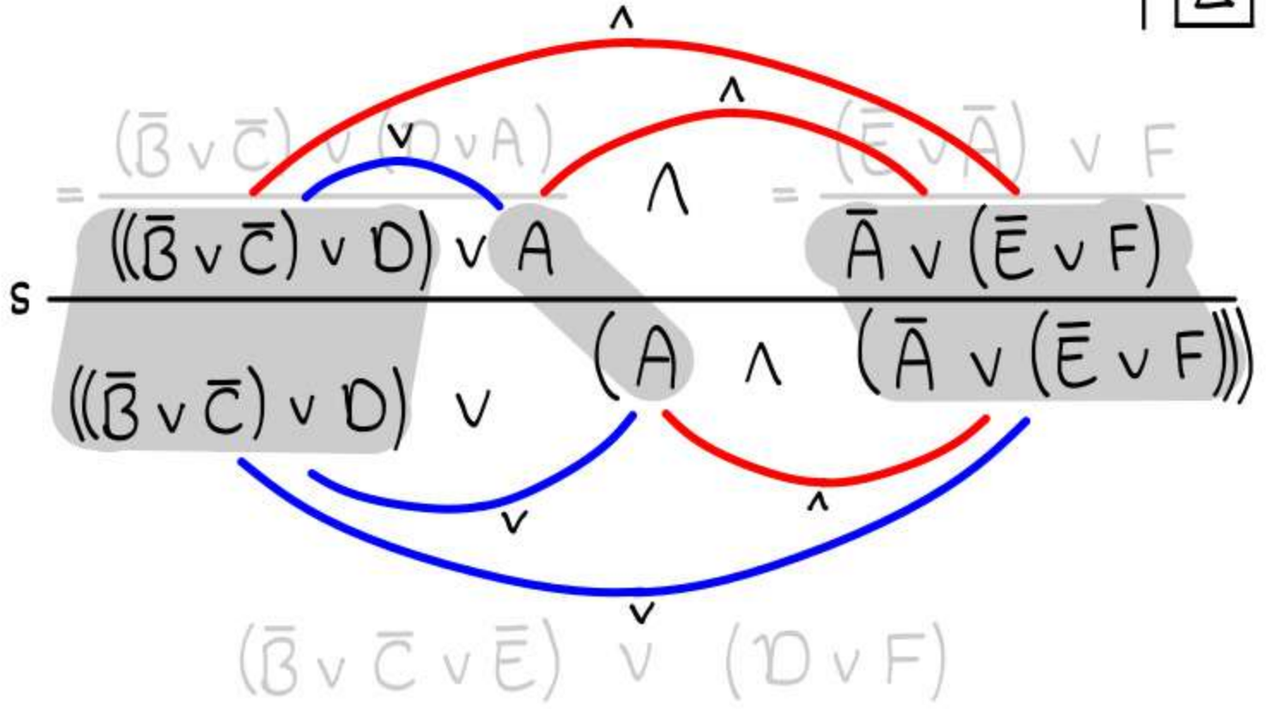
$$\frac{\pi_1 \quad \overset{k \geq 0}{\cdots} \quad \pi_k}{A_1, \ldots, A_m \vdash B_1, \ldots, B_n} \qquad m \geq 0, \quad n \geq 0$$

$$\text{cut}\ \frac{\Gamma \vdash \Upsilon, A \qquad \Theta, A \vdash \Lambda}{\Gamma, \Theta \vdash \Upsilon, \Lambda}$$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\pi}}{\boxed{\Delta}}$$



Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \bot \mid A \vee B \mid A \wedge B$$

Derivations

$$\pi ::= \frac{\pi_1 \quad \overset{k \geq 0}{\cdots} \quad \pi_k}{A_1, \ldots, A_m \vdash B_1, \ldots, B_n} \qquad m \geq 0, \; n \geq 0$$

$$\mathrm{cut}\,\frac{\Gamma \vdash \Upsilon, A \qquad \Theta, A \vdash \Lambda}{\Gamma, \Theta \vdash \Upsilon, \Lambda}$$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\pi}}{\boxed{\Delta}}$$

$$s\,\frac{= \dfrac{(\bar{B} \vee \bar{C}) \vee (D \vee A)}{((\bar{B} \vee \bar{C}) \vee D) \vee A} \quad \wedge \quad = \dfrac{(\bar{E} \vee \bar{A}) \vee F}{\bar{A} \vee (\bar{E} \vee F)}}{((\bar{B} \vee \bar{C}) \vee D) \vee (A \wedge (\bar{A} \vee (\bar{E} \vee F)))}$$

$$(\bar{B} \vee \bar{C} \vee \bar{E}) \vee (D \vee F)$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \bot \mid A \vee B \mid A \wedge B$$

Derivations

$$\pi ::= \frac{\pi_1 \quad \overset{k \geq 0}{\cdots} \quad \pi_k}{A_1, \ldots, A_m \vdash B_1, \ldots, B_n} \qquad m \geq 0, \; n \geq 0$$

$$\text{cut}\,\frac{\Gamma \vdash \Upsilon, A \qquad \Theta, A \vdash \Lambda}{\Gamma, \Theta \vdash \Upsilon, \Lambda}$$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\pi}}{\boxed{\Delta}}$$

$$s\frac{= \dfrac{(\bar{B} \vee \bar{C}) \vee (D \vee A)}{((\bar{B} \vee \bar{C}) \vee D) \vee A} \quad \wedge \quad = \dfrac{(\bar{E} \vee \bar{A}) \vee F}{\bar{A} \vee (\bar{E} \vee F)}}{((\bar{B} \vee \bar{C}) \vee D) \vee \; s\dfrac{A \wedge (\bar{A} \vee (\bar{E} \vee F))}{(A \wedge \bar{A})} \vee (\bar{E} \vee F)}$$

$$(\bar{B} \vee \bar{C} \vee \bar{E}) \vee (D \vee F)$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \bot \mid A \vee B \mid A \wedge B$$

Derivations

$$\pi ::= \frac{\pi_1 \quad \overset{k \geq 0}{\cdots} \quad \pi_k}{A_1, \ldots, A_m \vdash B_1, \ldots, B_n} \qquad m \geq 0, \; n \geq 0$$

$$\mathrm{cut}\,\frac{\Gamma \vdash \Upsilon, A \qquad \Theta, A \vdash \Lambda}{\Gamma, \Theta \vdash \Upsilon, \Lambda}$$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\pi}}{\boxed{\Delta}}$$

$$s\,\frac{= \dfrac{(\bar{B} \vee \bar{C}) \vee (D \vee A)}{((\bar{B} \vee \bar{C}) \vee D) \vee A} \quad \wedge \quad = \dfrac{(\bar{E} \vee \bar{A}) \vee F}{\bar{A} \vee (\bar{E} \vee F)}}{((\bar{B} \vee \bar{C}) \vee D) \vee \; s\,\dfrac{A \wedge (\bar{A} \vee (\bar{E} \vee F))}{i\uparrow \dfrac{A \wedge \bar{A}}{\bot}} \vee (\bar{E} \vee F)}$$

$$(\bar{B} \vee \bar{C} \vee \bar{E}) \vee (D \vee F)$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \bot \mid A \vee B \mid A \wedge B$$

Derivations

$$\pi ::= \frac{\pi_1 \quad k \geq 0 \quad \cdots \quad \pi_k}{A_1, \ldots, A_m \vdash B_1, \ldots, B_n} \qquad m \geq 0, \; n \geq 0$$

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Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\pi}}{\boxed{\Delta}}$$

$$= \frac{(\bar{B} \vee \bar{C}) \vee (D \vee A)}{((\bar{B} \vee \bar{C}) \vee D) \vee A} \; \wedge \; = \frac{(\bar{E} \vee \bar{A}) \vee F}{\bar{A} \vee (\bar{E} \vee F)}$$

$$\mathrm{s}\,\frac{((\bar{B} \vee \bar{C}) \vee D) \vee \mathrm{s}\,\dfrac{A \wedge (\bar{A} \vee (\bar{E} \vee F))}{\mathrm{i}\uparrow \dfrac{A \wedge \bar{A}}{\bot} \vee (\bar{E} \vee F)}}{= \quad (\bar{B} \vee \bar{C} \vee \bar{E}) \vee (D \vee F)}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \bot \mid A \vee B \mid A \wedge B$$

Derivations

$$\pi ::= \frac{\pi_1 \quad \overset{k \geq 0}{\cdots} \quad \pi_k}{A_1, \ldots, A_m \vdash B_1, \ldots, B_n} \qquad m \geq 0, \; n \geq 0$$

$$\text{cut}\,\frac{\Gamma \vdash \Upsilon, A \qquad \Theta, A \vdash \Lambda}{\Gamma, \Theta \vdash \Upsilon, \Lambda}$$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\pi}}{\boxed{\Delta}}$$

$$\text{ir}\,\frac{A \wedge \bar{A}}{\bot}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \bot \mid A \vee B \mid A \wedge B$$

Derivations

$$\pi ::= \dfrac{\pi_1 \quad \overset{k \geqslant 0}{\cdots} \quad \pi_k}{A_1, \ldots, A_m \vdash B_1, \ldots, B_n} \qquad m \geqslant 0, \quad n \geqslant 0$$

$$\mathrm{id}\,\dfrac{}{A \vdash A} \qquad \mathrm{cut}\,\dfrac{\Gamma \vdash \Upsilon, A \quad \Theta, A \vdash \Lambda}{\Gamma, \Theta \vdash \Upsilon, \Lambda}$$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \dfrac{\boxed{\pi}}{\boxed{\Delta}}$$

$$i\downarrow\,\dfrac{\top}{A \vee \bar{A}} \qquad i\uparrow\,\dfrac{A \wedge \bar{A}}{\bot}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \bot \mid A \vee B \mid A \wedge B$$

Derivations

$$\pi ::= \dfrac{\pi_1 \quad \overset{k \geq 0}{\cdots} \quad \pi_k}{A_1, \ldots, A_m \vdash B_1, \ldots, B_n} \qquad m \geq 0, \quad n \geq 0$$

$$\text{cut}\ \dfrac{\Gamma \vdash \Upsilon, A \qquad \Theta, A \vdash \Lambda}{\Gamma, \Theta \vdash \Upsilon, \Lambda}$$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \dfrac{\boxed{\pi}}{\boxed{\Delta}}$$

$$\begin{array}{c} \Gamma \\ \| \\ \Upsilon \vee A \end{array} \qquad \begin{array}{c} \Theta \wedge A \\ \| \\ \Lambda \end{array}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \bot \mid A \vee B \mid A \wedge B$$

Derivations

$$\pi ::= \dfrac{\pi_1 \quad k \geqslant 0 \quad \cdots \quad \pi_k}{A_1, \ldots, A_m \vdash B_1, \ldots, B_n} \qquad m \geqslant 0, \quad n \geqslant 0$$

$$\mathrm{cut}\ \dfrac{\Gamma \vdash \Upsilon, A \qquad \Theta, A \vdash \Lambda}{\Gamma, \Theta \vdash \Upsilon, \Lambda}$$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \dfrac{\boxed{\pi}}{\boxed{\Delta}}$$

$$\begin{array}{c} \Gamma \\ \| \\ \Upsilon \vee A \end{array} \qquad \wedge \qquad \Theta$$

$$\begin{array}{c} \Gamma \\ \| \\ \Upsilon \vee A \end{array} \qquad \begin{array}{c} \Theta \wedge A \\ \| \\ \Lambda \end{array}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \bot \mid A \vee B \mid A \wedge B$$

Derivations

$$\pi ::= \dfrac{\pi_1 \quad k \geq 0 \quad \cdots \quad \pi_k}{A_1, \ldots, A_m \vdash B_1, \ldots, B_n} \qquad m \geq 0, \; n \geq 0$$

$$\text{cut}\,\dfrac{\Gamma \vdash \Upsilon, A \qquad \Theta, A \vdash \Lambda}{\Gamma, \Theta \vdash \Upsilon, \Lambda}$$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \dfrac{\boxed{\pi}}{\boxed{\Delta}}$$

$$\Gamma \;\|\; \vee A \qquad\qquad \Theta \wedge A \;\|\; \qquad\qquad \Gamma \;\|\; \wedge\ \Theta$$

$$\Upsilon \vee A \;\|\; \qquad\qquad \Lambda$$

$$\text{s}\,\dfrac{\Upsilon \vee A}{\Upsilon \vee (\Theta \wedge A)}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \bot \mid A \vee B \mid A \wedge B$$

Derivations

$$\pi ::= \dfrac{\pi_1 \quad \overset{k \geq 0}{\cdots} \quad \pi_k}{A_1, \dots, A_m \vdash B_1, \dots, B_n} \qquad m \geq 0, \quad n \geq 0$$

$$\text{cut}\ \dfrac{\Gamma \vdash \Upsilon, A \qquad \Theta, A \vdash \Lambda}{\Gamma, \Theta \vdash \Upsilon, \Lambda}$$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \dfrac{\boxed{\pi}}{\boxed{\Delta}}$$

$$\dfrac{\Gamma}{\Upsilon \vee A} \qquad \dfrac{\Theta \wedge A}{\Lambda} \qquad \text{s}\dfrac{\Gamma \quad \wedge \quad \Theta}{\Upsilon \vee A \qquad\qquad \Upsilon \ \vee \ \dfrac{\Theta \wedge A}{\Lambda}}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \bot \mid A \vee B \mid A \wedge B$$

Derivations

$$\pi ::= \frac{\pi_1 \quad \overset{k \geq 0}{\cdots} \quad \pi_k}{A_1, \ldots, A_m \vdash B_1, \ldots, B_n} \qquad m \geq 0, \quad n \geq 0$$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\pi}}{\boxed{\Delta}}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \bot \mid A \vee B \mid A \wedge B$$

Derivations

$$\pi ::= \frac{\pi_1 \quad \overset{k \geq 0}{\cdots} \quad \pi_k}{A_1, \ldots, A_m \vdash B_1, \ldots, B_n} \qquad m \geq 0, \; n \geq 0$$

$$\text{r1}\,\frac{\vdash A, B, \Gamma}{\vdash C, B, \Gamma} \qquad \text{r2}\,\frac{\vdash C, B, \Gamma}{\vdash C, D, \Gamma} \qquad\qquad \text{r2}\,\frac{\vdash A, B, \Gamma}{\vdash A, D, \Gamma} \qquad \text{r1}\,\frac{\vdash A, D, \Gamma}{\vdash C, D, \Gamma}$$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\pi}}{\boxed{\Delta}}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \bot \mid A \vee B \mid A \wedge B$$

Derivations

$$\pi ::= \dfrac{\pi_1 \quad \overset{k \geq 0}{\cdots} \quad \pi_k}{A_1, \dots, A_m \vdash B_1, \dots, B_n} \qquad m \geq 0, \quad n \geq 0$$

$$\text{r1}\dfrac{\text{r2}\dfrac{\vdash A, B, \Gamma}{\vdash C, B, \Gamma}}{\vdash C, D, \Gamma} \qquad \text{r2}\dfrac{\text{r1}\dfrac{\vdash A, B, \Gamma}{\vdash A, D, \Gamma}}{\vdash C, D, \Gamma}$$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \dfrac{\boxed{\pi}}{\boxed{\Delta}}$$

$$\dfrac{A}{\boxed{\Delta_1}}{\scriptstyle C} \vee \dfrac{B}{\boxed{\Delta_2}}{\scriptstyle D} \vee G$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \bot \mid A \vee B \mid A \wedge B$$

Derivations

$$\pi ::= \frac{\pi_1 \quad \overset{k \geq 0}{\cdots} \quad \pi_k}{A_1, \ldots, A_m \vdash B_1, \ldots, B_n} \qquad m \geq 0, \quad n \geq 0$$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\pi}}{\boxed{\Delta}}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \bot \mid A \vee B \mid A \wedge B$$

Derivations

$$\pi ::= \frac{\pi_1 \quad k \geq 0 \quad \cdots \quad \pi_k}{A_1, \ldots, A_m \vdash B_1, \ldots, B_n} \qquad m \geq 0, \; n \geq 0$$

$$\text{cont}\ \frac{\vdash A, A, \Gamma}{\vdash A, \Gamma}$$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\pi}}{\boxed{\Delta}}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \bot \mid A \vee B \mid A \wedge B$$

Derivations

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$$\text{cont}\ \frac{\vdash A, A, \Gamma}{\vdash A, \Gamma}$$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\pi}}{\boxed{\Delta}}$$

$$(B \wedge C) \vee (B \wedge C)$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \bot \mid A \vee B \mid A \wedge B$$

Derivations

$$\pi ::= \frac{\pi_1 \quad k \geq 0 \quad \cdots \quad \pi_k}{A_1, \ldots, A_m \vdash B_1, \ldots, B_n} \qquad m \geq 0, \; n \geq 0$$

$$\text{cont}\ \frac{\vdash A, A, \Gamma}{\vdash A, \Gamma}$$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\pi}}{\boxed{\Delta}}$$

$$m\,\frac{(B \wedge C) \vee (B \wedge C)}{(B \vee B) \wedge (C \vee C)}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \bot \mid A \vee B \mid A \wedge B$$

Derivations

$$\pi ::= \frac{\pi_1 \quad k \geq 0 \quad \cdots \quad \pi_k}{A_1, \ldots, A_m \vdash B_1, \ldots, B_n} \qquad m \geq 0, \; n \geq 0$$

$$\text{cont}\ \frac{\vdash A, A, \Gamma}{\vdash A, \Gamma}$$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\pi}}{\boxed{\Delta}}$$

$$m\ \frac{(a \wedge C) \vee (a \wedge C)}{(a \vee a) \wedge (C \vee C)}$$

$$ac\downarrow\ \frac{(a \vee a) \wedge (C \vee C)}{a}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \bot \mid A \vee B \mid A \wedge B$$

Derivations

$$\pi ::= \frac{\pi_1 \quad \overset{k \geq 0}{\cdots} \quad \pi_k}{A_1, \ldots, A_m \vdash B_1, \ldots, B_n} \qquad m \geq 0, \; n \geq 0$$

$$\text{cont}\ \frac{\vdash A, A, \Gamma}{\vdash A, \Gamma}$$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\pi}}{\boxed{\Delta}}$$

$$\begin{array}{l} m\ \dfrac{(a \wedge C) \vee (a \wedge C)}{} \\ ac\downarrow\ \dfrac{(a \vee a) \wedge (C \vee C)}{a} \\ \qquad\qquad \| m, ac\downarrow \\ \qquad\qquad C \end{array}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \bot \mid A \vee B \mid A \wedge B$$

Derivations

$\pi ::=$

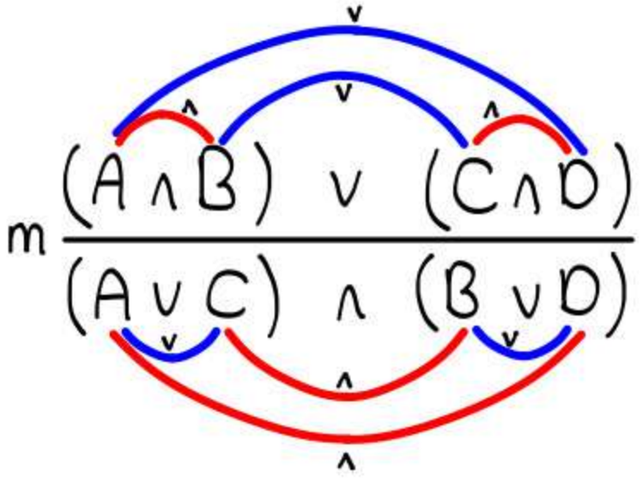
$$\frac{\pi_1 \quad \overset{k \geq 0}{\cdots} \quad \pi_k}{A_1, \ldots, A_m \vdash B_1, \ldots, B_n} \qquad m \geqslant 0, \; n \geqslant 0$$

$$\text{cont}\ \frac{\vdash A, A, \Gamma}{\vdash A, \Gamma}$$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\pi}}{\boxed{\Delta}}$$



Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \bot \mid A \vee B \mid A \wedge B$$

Derivations

$$\pi ::= \frac{\pi_1 \quad k \geq 0 \quad \cdots \quad \pi_k}{A_1, \ldots, A_m \vdash B_1, \ldots, B_n} \qquad m \geq 0, \; n \geq 0$$

$$\mathrm{cont}\ \frac{\vdash A, A, \Gamma}{\vdash A, \Gamma}$$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\pi}}{\boxed{\Delta}}$$

$$m\ \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \bot \mid A \vee B \mid A \wedge B$$

Derivations

$$\pi ::= \frac{\pi_1 \quad \overset{k \geq 0}{\cdots} \quad \pi_k}{A_1, \ldots, A_m \vdash B_1, \ldots, B_n} \qquad m \geq 0, \quad n \geq 0$$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \bot \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\pi}}{\boxed{\Delta}}$$

more symmetry: proofs have a single premise and conclusion, duality of cut and identity restored

bureaucracy of parallel derivations reduced

all structural rules are atomic and all rules can be checked locally

What makes a proof formalism good?

Normalisation : proofs have useful normal forms and normalisation procedures are simple

Semantics : proofs are represented with little syntactic bureaucracy and it's easy to tell when two proofs are the same

Complexity : theorems have small proofs

Proof search : it's easy to find proofs

System SKS

$$\mathrm{ai}{\downarrow}\ \frac{\top}{a \vee \bar{a}} \qquad \text{identity}$$

$$\mathrm{ac}{\downarrow}\ \frac{a \vee a}{a} \qquad \text{contraction}$$

$$\mathrm{aw}{\downarrow}\ \frac{\bot}{a} \qquad \text{weakening}$$

$$\mathrm{ai}{\uparrow}\ \frac{a \wedge \bar{a}}{\bot} \qquad \text{cut}$$

$$\mathrm{ac}{\uparrow}\ \frac{a}{a \wedge a} \qquad \text{cocontraction}$$

$$\mathrm{aw}{\uparrow}\ \frac{a}{\top} \qquad \text{coweakening}$$

$$\mathrm{m}\ \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)} \qquad \text{medial}$$

$$\mathrm{s}\ \frac{A \wedge (B \vee C)}{(A \wedge B) \vee C} \qquad \text{switch}$$

equality $= \frac{A}{B}$ for :

$$A \wedge B = B \wedge A$$
$$(A \wedge B) \wedge C = A \wedge (B \wedge C)$$

$$A \vee B = B \vee A$$
$$(A \vee B) \vee C = A \vee (B \vee C)$$

$$A \vee \bot = A = \top \wedge A$$
$$\bot \vee \bot = \bot$$
$$\top = \top \wedge \top$$