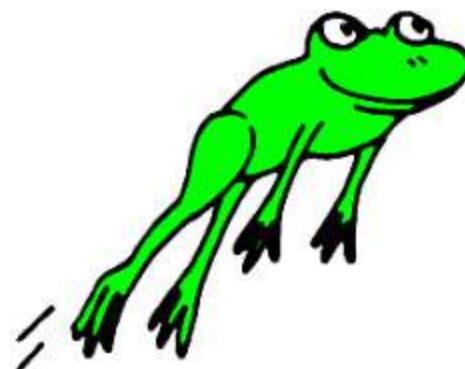


A gentle introduction to deep inference

2. Open deduction :
a 21st century proof formalism



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What makes a proof formalism good?

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Normalisation : proofs have useful normal forms and normalisation procedures are simple

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Normalisation : proofs have useful normal forms and normalisation procedures are simple

Semantics : proofs are represented with little syntactic bureaucracy and it's easy to tell when two proofs are the same

Complexity : theorems have small proofs

Proof search : it's easy to find proofs

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\begin{array}{c} \triangleleft \\ \Pi_1 \\ \vdots \\ \triangleleft \\ \Pi_k \end{array} \quad \begin{array}{c} k \geq 0 \\ \dots \\ k \geq 0 \end{array}}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

$m \geq 0 \quad n \geq 0$

Open deduction

Sequent calculus

Formulae

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Derivations

$$\frac{\begin{array}{c} \triangleleft \\ \Pi_1 \\ \vdots \\ \triangleleft \\ \Pi_k \end{array} \quad \frac{}{A_1, \dots, A_m \vdash B_1, \dots, B_n}}_{\substack{m \geq 0 \\ n \geq 0}} A_1, \dots, A_m \vdash B_1, \dots, B_n$$

$$\text{cut} \frac{\Gamma \vdash \gamma, A \quad \Theta, A \vdash \lambda}{\Gamma, \Theta \vdash \gamma, \lambda}$$

Open deduction

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\begin{array}{c} \triangleleft \\ \Pi_1 \\ \vdots \\ \triangleleft \\ \Pi_k \end{array} \quad \frac{m \geq 0 \quad n \geq 0}{A_1, \dots, A_m \vdash B_1, \dots, B_n}}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

$$\text{cut} \frac{\Gamma \vdash \gamma, A \vee B \quad \Theta, A \wedge C \vdash \lambda}{\Gamma, \Theta, C \vdash \gamma, B, \lambda}$$

Open deduction

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\begin{array}{c} \triangleleft \\ \Pi_1 \\ \vdots \\ \triangleleft \\ \Pi_k \end{array} \quad \frac{m \geq 0}{A_1, \dots, A_m} \quad \frac{n \geq 0}{B_1, \dots, B_n}}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

inference rules operate on derivation-level connectives, not formula - level

Open deduction

only one level
of connective

Derivations

$$\boxed{\Pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\Pi} \vee \boxed{\Delta} \mid \boxed{\Pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\Pi}}{\boxed{\Delta}}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\vdash ::= \frac{\text{Derivations} \quad \begin{array}{c} \triangleleft \\ \pi_1 \\ \vdots \\ \triangleleft_{l_k} \end{array} \quad A_1, \dots, A_m \vdash B_1, \dots, B_n}{m \geq 0 \qquad n \geq 0}$$

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$$\begin{array}{c} A \\ \square \\ B \end{array}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\vdash ::= \frac{\text{Derivations} \quad \begin{array}{c} \triangleleft \\ \Pi_1 \\ \vdots \\ \triangleleft_{l_k} \end{array} \quad \dots \quad \begin{array}{c} \triangleleft \\ \Pi_{l_k} \end{array}}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

$m \geq 0$ $n \geq 0$

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$$\begin{array}{cc} A & C \\ \boxed{ } & \boxed{ } \\ B & D \end{array}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\vdash ::= \frac{\text{Derivations}}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

$\overline{\Pi_1} \quad \dots \quad \overline{\Pi_k}$

$m \geq 0 \qquad \qquad n \geq 0$

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$$\boxed{\Pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\Pi} \vee \boxed{\Delta} \mid \boxed{\Pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\Pi}}{\boxed{\Delta}}$$

$$\begin{array}{c} A \\ \hline B \end{array} \vee \begin{array}{c} C \\ \hline D \end{array}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\begin{array}{c} \triangleleft \\ \vdots \\ \triangleleft \end{array} \quad \begin{array}{c} k \geq 0 \\ \dots \\ k \geq 0 \end{array}}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

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$$\boxed{\Pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\Pi} \vee \boxed{\Delta} \mid \boxed{\Pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\Pi}}{\boxed{\Delta}}$$

$$\frac{\begin{array}{c} A \vee C \\ \hline \begin{array}{c} A \\ B \end{array} \quad \begin{array}{c} C \\ D \end{array} \\ \hline B \vee D \end{array}}{B \vee D}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\text{Derivations } \pi_1, \dots, \pi_k}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

π

$\pi_1 \quad \dots \quad \pi_k$

$m \geq 0 \quad n \geq 0$

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$A \vee C$

A	C		
B	\vee	D	
D			

$B \vee D$

$A \wedge C$

A	C		
B	\wedge	D	
D			

$B \wedge D$

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\vdash $\pi_1 \quad \dots \quad \pi_k$

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$$\begin{array}{c} A \vee C \\ \hline \boxed{A} \quad \boxed{C} \\ \boxed{B} \quad \boxed{D} \\ B \vee D \end{array}$$

$$\begin{array}{c} A \wedge C \\ \hline \boxed{A} \quad \boxed{C} \\ \boxed{B} \quad \boxed{D} \\ B \wedge D \end{array}$$

$$\begin{array}{c} A \\ \hline \boxed{B} \\ B \\ \\ C \\ \hline \boxed{D} \end{array}$$

Sequent calculus

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$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

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$$\vdash ::= \frac{\text{Derivations } \pi_1, \dots, \pi_k}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

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$$\begin{array}{c} A \vee C \\ \hline \boxed{A} \vee \boxed{C} \\ \boxed{B} \vee \boxed{D} \end{array}$$

$$\begin{array}{c} A \wedge C \\ \hline \boxed{A} \wedge \boxed{C} \\ \boxed{B} \wedge \boxed{D} \end{array}$$

$$\begin{array}{c} A \\ \hline \boxed{A} \\ \boxed{B} \\ \hline \boxed{C} \\ \boxed{D} \end{array}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\vdash ::= \frac{\text{Derivations } \pi_1, \dots, \pi_k}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

\vdash $\pi_1 \quad \dots \quad \pi_k$

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$$\begin{array}{c} A \vee C \\ \hline \boxed{A} \vee \boxed{C} \\ B \quad D \end{array}$$

$B \vee D$

$$\begin{array}{c} A \wedge C \\ \hline \boxed{A} \wedge \boxed{C} \\ B \quad D \end{array}$$

$B \wedge D$

$$\begin{array}{c} A \\ \hline \boxed{A} \\ B \\ \hline \boxed{B} \\ C \\ \hline \boxed{C} \\ D \end{array}$$

$$\begin{array}{c} A \\ \hline \boxed{A} \\ B \\ \hline \boxed{B} \\ C \end{array}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\vdash ::= \frac{\text{Derivations } \pi_1, \dots, \pi_k}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

\vdash $\pi_1 \quad \dots \quad \pi_k$

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$$\begin{array}{c} A \vee C \\ \hline \boxed{A} \vee \boxed{C} \\ \boxed{B} \quad \boxed{D} \end{array}$$

$B \vee D$

$$\begin{array}{c} A \wedge C \\ \hline \boxed{A} \wedge \boxed{C} \\ \boxed{B} \quad \boxed{D} \end{array}$$

$B \wedge D$

$$\begin{array}{c} A \\ \hline \boxed{A} \\ \boxed{B} \\ \hline \boxed{C} \\ \boxed{D} \end{array}$$

$$\begin{array}{c} A \\ \hline \boxed{A} \\ \boxed{B} \\ \hline \boxed{C} \end{array}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\vdash ::= \frac{\text{Derivations } \pi_1, \dots, \pi_k}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

\vdash $\pi_1 \quad \dots \quad \pi_k$

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Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \frac{\pi}{\Delta}$$

$$A \vee C$$

$$\parallel$$

$$B \vee D$$

$$A \wedge C$$

$$\parallel$$

$$B \wedge D$$

$$A \parallel D$$

$$A \parallel C$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\vdash ::= \frac{\text{Derivations } \pi_1, \dots, \pi_k}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

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Open deduction

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$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\pi}}{\boxed{\Delta}}$$

$$A \vee C$$

$$\Delta \parallel$$

$$B \vee D$$

$$A \wedge C$$

$$\Delta \parallel S$$

$$B \wedge D$$

$$A \parallel D$$

$$A \parallel C$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\vdash ::= \frac{\text{Derivations } \pi_1, \dots, \pi_k}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

\vdash $\pi_1 \quad \dots \quad \pi_k$

$m \geq 0 \quad n \geq 0$

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$$A \vee C$$

$$\Delta \parallel$$

$$B \vee D$$

$$A \wedge C$$

$$\Delta \parallel S$$

$$B \wedge D$$

$$\top \parallel$$

$$D \parallel$$

$$A \parallel$$

$$C \parallel$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\vdash ::= \frac{\text{Derivations } \pi_1, \dots, \pi_k}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

\vdash $\pi_1 \quad \dots \quad \pi_k$

$m \geq 0 \quad n \geq 0$

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Open deduction

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Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\pi}}{\boxed{\Delta}}$$

$$A \vee C$$

$$\Delta \parallel$$

$$B \vee D$$

$$A \wedge C$$

$$\Delta \parallel S$$

$$B \wedge D$$

$$\top \parallel D$$

$$A \parallel C$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\begin{array}{c} \triangleleft \\ \Pi_1 \\ \vdots \\ \triangleleft \\ \Pi_k \end{array} \quad \begin{array}{c} k \geq 0 \\ \dots \\ k \geq 0 \end{array}}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

$m \geq 0$ $n \geq 0$

Open deduction

Derivations

$$\boxed{\Pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\Pi} \vee \boxed{\Delta} \mid \boxed{\Pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\Pi}}{\boxed{\Delta}}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\begin{array}{c} \triangleleft \\ \Pi_1 \\ \vdots \\ \triangleleft \\ \Pi_k \end{array} \quad \frac{k \geq 0}{\dots}}{A_1, \dots, A_m \vdash B_1, \dots, B_n} \quad \begin{matrix} m \geq 0 \\ n \geq 0 \end{matrix}$$

$$\text{cut} \frac{\Gamma \vdash \gamma, A \quad \Theta, A \vdash \lambda}{\Gamma, \Theta \vdash \gamma, \lambda}$$

Open deduction

Derivations

$$\boxed{\Pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\Pi} \vee \boxed{\Delta} \mid \boxed{\Pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\Pi}}{\boxed{\Delta}}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\begin{array}{c} \Delta \\ \hline \Gamma_1 \quad \dots \quad \Gamma_k \end{array}}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

$m \geq 0$ $n \geq 0$

$$\text{cut} \frac{\Gamma \vdash \gamma, A \quad \Theta, A \vdash \lambda}{\Gamma, \Theta \vdash \gamma, \lambda}$$

Open deduction

Derivations

$$\frac{\begin{array}{c} \Delta \\ \hline \Pi \end{array}}{\Pi, \Delta ::= a \mid \bar{a} \mid \top \mid \perp \mid \Pi \vee \Delta \mid \Pi \wedge \Delta \mid \frac{\Pi}{\Delta}}$$

$$\Gamma \vdash \gamma, A$$

$$\Theta, A \vdash \lambda$$

$$\Gamma, \Theta \vdash \Delta, \lambda$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\begin{array}{c} \nabla \\ \vdots \\ \nabla_{l_1} \quad \dots \quad \nabla_{l_k} \end{array}}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

$m \geq 0$ $n \geq 0$

$$\text{cut} \frac{\Gamma \vdash \gamma, A \quad \Theta, A \vdash \lambda}{\Gamma, \Theta \vdash \gamma, \lambda}$$

Open deduction

Derivations

$$\boxed{\Pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\Pi} \vee \boxed{\Delta} \mid \boxed{\Pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\Pi}}{\boxed{\Delta}}$$

$$\beta, c \vdash d, a$$

$$e, a \vdash f$$

$$\beta, c, e \vdash d, f$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\begin{array}{c} \nabla \\ \vdots \\ \nabla \end{array} \quad \dots \quad \begin{array}{c} \nabla \\ \vdots \\ \nabla \end{array}}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

$k \geq 0$

$m \geq 0$

$n \geq 0$

$$\text{cut} \frac{\Gamma \vdash \gamma, A \quad \Theta, A \vdash \lambda}{\Gamma, \Theta \vdash \gamma, \lambda}$$

Open deduction

Derivations

$$\boxed{\Pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\Pi} \vee \boxed{\Delta} \mid \boxed{\Pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\Pi}}{\boxed{\Delta}}$$

$$(\bar{B} \vee \bar{C}) \vee (D \vee A)$$

\wedge

$$(\bar{E} \vee \bar{A}) \vee F$$

$$(\bar{B} \vee \bar{C} \vee \bar{E}) \vee (D \vee F)$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\begin{array}{c} \nabla \\ \Pi_1 \\ \vdots \\ \nabla_{l_k} \end{array} \quad \dots \quad \begin{array}{c} \nabla \\ \Pi_1 \\ \vdots \\ \nabla_{l_k} \end{array}}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

$m \geq 0$ $n \geq 0$

$$\text{cut} \frac{\Gamma \vdash \gamma, A \quad \Theta, A \vdash \lambda}{\Gamma, \Theta \vdash \gamma, \lambda}$$

Open deduction

Derivations

$$\boxed{\Pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\Pi} \vee \boxed{\Delta} \mid \boxed{\Pi} \wedge \boxed{\Delta} \mid \frac{\Pi}{\Delta}$$

$$= \frac{(\bar{B} \vee \bar{C}) \vee (D \vee A)}{((\bar{B} \vee \bar{C}) \vee D) \vee A}$$

$$\wedge = \frac{(\bar{E} \vee \bar{A}) \vee F}{\bar{A} \vee (\bar{E} \vee F)}$$

$$(\bar{B} \vee \bar{C} \vee \bar{E}) \vee (D \vee F)$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\begin{array}{c} \nabla \\ \Pi_1 \\ \vdots \\ \nabla_{l_k} \end{array} \quad \dots \quad \begin{array}{c} \nabla \\ \Pi_1 \\ \vdots \\ \nabla_{l_k} \end{array}}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

$m \geq 0$ $n \geq 0$

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Open deduction

Derivations

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$$= \frac{(\bar{B} \vee \bar{C}) \vee (\mathcal{D} \vee A)}{((\bar{B} \vee \bar{C}) \vee \mathcal{D}) \vee A}$$

$$\wedge = \frac{(\bar{E} \vee \bar{A}) \vee F}{\bar{A} \vee (\bar{E} \vee F)}$$

$$(\bar{B} \vee \bar{C} \vee \bar{E}) \vee (\mathcal{D} \vee F)$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\begin{array}{c} \nabla \\ \vdots \\ \nabla_{l_1} \quad \dots \quad \nabla_{l_k} \end{array}}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

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Derivations

$$\boxed{\Pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\Pi} \vee \boxed{\Delta} \mid \boxed{\Pi} \wedge \boxed{\Delta} \mid \frac{\Pi}{\Delta}$$

$$\frac{s}{\frac{\frac{\frac{(\bar{B} \vee \bar{C}) \vee (\bar{D} \vee A)}{((\bar{B} \vee \bar{C}) \vee \bar{D}) \vee A} \wedge \frac{(\bar{E} \vee \bar{A}) \vee F}{\bar{A} \vee (\bar{E} \vee F)}}{((\bar{B} \vee \bar{C}) \vee \bar{D}) \vee ((A \wedge (\bar{A} \vee (\bar{E} \vee F))))}}$$

$$(\bar{B} \vee \bar{C} \vee \bar{E}) \vee (\bar{D} \vee F)$$

Sequent calculus

Formulae

$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$

Derivations

$$\frac{\vdash \Pi_1, \dots, \Pi_k}{A_1, \dots, A_m \vdash B_1, \dots, B_n} \quad ::=$$

$$\text{cut} \frac{\Gamma \vdash \gamma, A \quad \Theta, A \vdash \Lambda}{\Gamma, \Theta \vdash \gamma, \Lambda}$$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a | \bar{a} | \tau | \perp | \boxed{\pi} \vee \boxed{\Delta} | \boxed{\pi} \wedge \boxed{\Delta} | \frac{\pi}{\Delta}$$

$$s = \frac{\frac{\frac{((\bar{B} \vee \bar{C}) \vee D) \vee A}{((\bar{B} \vee \bar{C}) \vee D) \vee A} \wedge ((A \wedge (\bar{A} \vee (\bar{E} \vee F))) \vee ((\bar{B} \vee \bar{C}) \vee D)) \vee ((\bar{B} \vee \bar{C} \vee \bar{E}) \vee (D \vee F))}{((\bar{B} \vee \bar{C}) \vee D) \vee A} \wedge ((A \wedge (\bar{A} \vee (\bar{E} \vee F))) \vee ((\bar{B} \vee \bar{C}) \vee D)) \vee ((\bar{B} \vee \bar{C} \vee \bar{E}) \vee (D \vee F))}{((\bar{B} \vee \bar{C}) \vee D) \vee A} \wedge ((A \wedge (\bar{A} \vee (\bar{E} \vee F))) \vee ((\bar{B} \vee \bar{C}) \vee D)) \vee ((\bar{B} \vee \bar{C} \vee \bar{E}) \vee (D \vee F))}{((\bar{B} \vee \bar{C}) \vee D) \vee A} \wedge ((A \wedge (\bar{A} \vee (\bar{E} \vee F))) \vee ((\bar{B} \vee \bar{C}) \vee D)) \vee ((\bar{B} \vee \bar{C} \vee \bar{E}) \vee (D \vee F))}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\begin{array}{c} \nabla \\ \vdots \\ \nabla \end{array} \quad \dots \quad \begin{array}{c} \nabla \\ \vdots \\ \nabla \end{array}}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

$m \geq 0$ $n \geq 0$

$$\text{cut} \frac{\Gamma \vdash \gamma, A \quad \Theta, A \vdash \lambda}{\Gamma, \Theta \vdash \gamma, \lambda}$$

Open deduction

Derivations

$$\boxed{\Pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\Pi} \vee \boxed{\Delta} \mid \boxed{\Pi} \wedge \boxed{\Delta} \mid \frac{\Pi}{\Delta}$$

$$\vdash \frac{s}{((\bar{B} \vee \bar{C}) \vee D) \vee A} \quad \wedge \quad = \frac{(\bar{E} \vee \bar{A}) \vee F}{\bar{A} \vee (\bar{E} \vee F)}$$

$s = \frac{(\bar{B} \vee \bar{C}) \vee (D \vee A)}{((\bar{B} \vee \bar{C}) \vee D) \vee A}$

$$((\bar{B} \vee \bar{C} \vee \bar{E}) \vee (D \vee F))$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\begin{array}{c} \nabla \\ \vdots \\ \nabla \end{array} \quad \dots \quad \begin{array}{c} \nabla \\ \vdots \\ \nabla \end{array}}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

$m \geq 0$ $n \geq 0$

$$\text{cut} \frac{\Gamma \vdash \gamma, A \quad \Theta, A \vdash \lambda}{\Gamma, \Theta \vdash \gamma, \lambda}$$

Open deduction

Derivations

$$\boxed{\Pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\Pi} \vee \boxed{\Delta} \mid \boxed{\Pi} \wedge \boxed{\Delta} \mid \frac{\Pi}{\Delta}$$

$$\frac{s \frac{(\bar{B} \vee \bar{C}) \vee (D \vee A)}{((\bar{B} \vee \bar{C}) \vee D) \vee A} \quad \wedge \quad \frac{(\bar{E} \vee \bar{A}) \vee F}{\bar{A} \vee (\bar{E} \vee F)}}{((\bar{B} \vee \bar{C}) \vee D) \vee \frac{A \wedge (\bar{A} \vee (\bar{E} \vee F))}{(A \wedge \bar{A}) \vee (\bar{E} \vee F)}} \\ (\bar{B} \vee \bar{C} \vee \bar{E}) \vee (D \vee F)$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\Delta ::= \frac{\begin{array}{c} \nabla_{\pi_1} \dots \nabla_{\pi_k} \\ m \geq 0 \quad n \geq 0 \end{array}}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

$$\text{cut} \frac{\Gamma \vdash \gamma, A \quad \Theta, A \vdash \lambda}{\Gamma, \Theta \vdash \gamma, \lambda}$$

Open deduction

Derivations

$$\Pi, \Delta ::= a \mid \bar{a} \mid \top \mid \perp \mid \frac{\Pi \vee \Delta}{\Pi} \mid \frac{\Pi \wedge \Delta}{\Delta}$$

$$\frac{s \frac{(\bar{B} \vee \bar{C}) \vee (D \vee A)}{((\bar{B} \vee \bar{C}) \vee D) \vee A} \wedge \frac{(\bar{E} \vee \bar{A}) \vee F}{\bar{A} \vee (\bar{E} \vee F)}}{((\bar{B} \vee \bar{C}) \vee D) \vee \frac{A \wedge (\bar{A} \vee (\bar{E} \vee F))}{\frac{A \wedge \bar{A}}{\perp} \vee (\bar{E} \vee F)}} \\ ((\bar{B} \vee \bar{C} \vee \bar{E}) \vee (D \vee F)}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\begin{array}{c} \nabla \\ \vdots \\ \nabla \end{array} \quad \dots \quad \begin{array}{c} \nabla \\ \vdots \\ \nabla \end{array}}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

$\pi_1 \quad k \geq 0 \quad \dots \quad \pi_k$

$m \geq 0 \quad n \geq 0$

$$\text{cut} \frac{\Gamma \vdash \gamma, A \quad \Theta, A \vdash \lambda}{\Gamma, \Theta \vdash \gamma, \lambda}$$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \frac{\pi}{\Delta}$$

$$\begin{aligned}
 & = \frac{(\bar{B} \vee \bar{C}) \vee (D \vee A)}{((\bar{B} \vee \bar{C}) \vee D) \vee A} \quad \wedge \quad = \frac{(\bar{E} \vee \bar{A}) \vee F}{\bar{A} \vee (\bar{E} \vee F)} \\
 & \vdash \frac{s}{((\bar{B} \vee \bar{C}) \vee D) \vee A} \quad \vdash \frac{s}{(\bar{A} \vee (\bar{E} \vee F))} \\
 & \quad \vdash \frac{\frac{A \wedge (\bar{A} \vee (\bar{E} \vee F))}{A \wedge \bar{A}}}{\perp} \vee (\bar{E} \vee F) \\
 & = \frac{}{(\bar{B} \vee \bar{C} \vee \bar{E}) \vee (D \vee F)}
 \end{aligned}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\begin{array}{c} \triangleleft \\ \Pi_1 \\ \vdots \\ \triangleleft \\ \Pi_k \end{array} \quad \frac{m \geq 0}{A_1, \dots, A_m} \quad \dots \quad \frac{n \geq 0}{B_1, \dots, B_n}}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

$$\text{cut} \frac{\Gamma \vdash \gamma, A \quad \Theta, A \vdash \lambda}{\Gamma, \Theta \vdash \gamma, \lambda}$$

Open deduction

Derivations

$$\frac{\Pi, \Delta ::= a \mid \bar{a} \mid \top \mid \perp \mid \frac{\Pi \vee \Delta}{\Pi} \mid \frac{\Pi \wedge \Delta}{\Delta}}{\frac{\Pi}{\Delta}}$$

$$\uparrow \frac{A \wedge \bar{A}}{\perp}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\begin{array}{c} \triangleleft \\ \Pi_1 \\ \vdots \\ \triangleleft \\ \Pi_k \end{array} \quad \frac{m \geq 0}{A_1, \dots, A_m} \quad \dots \quad \frac{n \geq 0}{B_1, \dots, B_n}}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

$$\text{id} \frac{}{A \vdash A}$$

$$\text{cut} \frac{\Gamma \vdash \gamma, A \quad \Theta, A \vdash \lambda}{\Gamma, \Theta \vdash \gamma, \lambda}$$

Open deduction

Derivations

$$\frac{\Pi, \Delta ::= a \mid \bar{a} \mid \top \mid \perp \mid \frac{\Pi \vee \Delta}{\Pi} \mid \frac{\Pi \wedge \Delta}{\Delta}}{\frac{\Pi}{\Delta}}$$

$$\Downarrow \frac{T}{A \vee \bar{A}}$$

$$\Uparrow \frac{A \wedge \bar{A}}{\perp}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\vdash \Pi ::= \frac{\begin{array}{c} \Delta_1, \dots, \Delta_k \\ \hline \Gamma_1, \dots, \Gamma_l \end{array}}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

$m \geq 0$ $n \geq 0$

$$\text{cut} - \frac{\Gamma \vdash \gamma, A \quad \Theta, A \vdash \lambda}{\Gamma, \Theta \vdash \gamma, \lambda}$$

Open deduction

Derivations

$$\boxed{\Pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\Pi} \vee \boxed{\Delta} \mid \boxed{\Pi} \wedge \boxed{\Delta} \mid \frac{\Pi}{\Delta}$$

$$\frac{\Gamma \parallel \Theta \wedge A}{\Gamma \vdash \gamma, A} \quad \frac{\Theta \wedge A \parallel \Lambda}{\Lambda}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\boxed{\Pi} ::= \frac{\begin{array}{c} \triangleleft_{\Pi_1} \dots \triangleright_{\Pi_k} \\ k \geq 0 \end{array}}{A_1, \dots, A_m \vdash B_1, \dots, B_n} \quad m \geq 0 \quad n \geq 0$$

$$\text{cut} \frac{\Gamma \vdash \gamma, A \quad \Theta, A \vdash \lambda}{\Gamma, \Theta \vdash \gamma, \lambda}$$

Open deduction

Derivations

$$\boxed{\Pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\Pi} \vee \boxed{\Delta} \mid \boxed{\Pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\Pi}}{\boxed{\Delta}}$$

$$\frac{\Gamma \parallel \Theta}{\Gamma \wedge \Theta}$$

$$\frac{\Gamma \parallel \Theta \wedge A}{\Gamma \vee A} \quad \frac{\Theta \wedge A \parallel \Lambda}{\Lambda} \quad \frac{\Gamma \parallel \Theta \vee A}{\Gamma \wedge A}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\begin{array}{c} \triangleleft \\ \Gamma_1 \\ \vdots \\ \triangleleft_{l_k} \end{array} \quad \dots \quad \begin{array}{c} \triangleleft \\ \Gamma_{l_k} \end{array}}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

$m \geq 0$ $n \geq 0$

$$\text{cut} \frac{\Gamma \vdash \gamma, A \quad \Theta, A \vdash \lambda}{\Gamma, \Theta \vdash \gamma, \lambda}$$

Open deduction

Derivations

$$\boxed{\Pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\Pi} \vee \boxed{\Delta} \mid \boxed{\Pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\Pi}}{\boxed{\Delta}}$$

$$\frac{\Gamma \quad \parallel \quad \Lambda \quad \Theta}{\Lambda \quad \Theta \wedge A}$$

$$\frac{s \quad \Gamma \vee A}{\Gamma \vee (\Theta \wedge A)}$$

$\Gamma \quad \parallel \quad \Theta \wedge A \quad \parallel \quad \Lambda \quad \Gamma \vee A$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\begin{array}{c} \triangleleft \\ \Pi_1 \\ \vdots \\ \triangleleft \\ \Pi_k \end{array} \quad \frac{k \geq 0}{\dots}}{A_1, \dots, A_m \vdash B_1, \dots, B_n} \quad \frac{m \geq 0 \quad n \geq 0}{\Gamma}$$

$$\text{cut} \frac{\Gamma \vdash \gamma, A \quad \Theta, A \vdash \lambda}{\Gamma, \Theta \vdash \gamma, \lambda}$$

Open deduction

Derivations

$$\boxed{\Pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\Pi} \vee \boxed{\Delta} \mid \boxed{\Pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\Pi}}{\boxed{\Delta}}$$

$$\frac{\Gamma \quad \parallel}{\Lambda} \quad \wedge \quad \Theta$$

$$\frac{\Gamma \quad \parallel}{\gamma \vee A} \quad \frac{\Theta \wedge A \quad \parallel}{\Lambda} \quad \frac{s \quad \Gamma \vee A}{\Theta \wedge A} \quad \frac{\Gamma \quad \parallel}{\gamma \vee \Lambda} \quad \frac{\parallel}{\Lambda}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\begin{array}{c} \triangleleft \\ \Pi_1 \\ \vdots \\ \triangleleft \\ \Pi_k \end{array} \quad \begin{array}{c} k \geq 0 \\ \dots \\ k \geq 0 \end{array}}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

$m \geq 0$ $n \geq 0$

Open deduction

Derivations

$$\boxed{\Pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\Pi} \vee \boxed{\Delta} \mid \boxed{\Pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\Pi}}{\boxed{\Delta}}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\begin{array}{c} \triangleleft \\ \Pi_1 \\ \vdots \\ \triangleleft \\ \Pi_k \end{array} \quad \frac{m \geq 0}{A_1, \dots, A_m}, \frac{n \geq 0}{B_1, \dots, B_n}}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

$$\frac{r_1 \frac{\vdash A, B, \Gamma}{\vdash C, B, \Gamma}}{r_2 \frac{}{\vdash C, D, \Gamma}}$$

$$\frac{r_2 \frac{\vdash A, B, \Gamma}{r_1 \frac{\vdash A, D, \Gamma}{\vdash C, D, \Gamma}}}{}$$

Open deduction

Derivations

$$\frac{\Pi, \Delta ::= a \mid \bar{a} \mid \top \mid \perp \mid \frac{\Pi \vee \Delta}{\Pi} \mid \frac{\Pi \wedge \Delta}{\Delta}}{\frac{\Pi}{\Delta}}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\boxed{\Pi} ::= \frac{\begin{array}{c} \triangle \quad \dots \quad \triangle \\ \Pi_1 \quad \dots \quad \Pi_k \end{array}}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

$$\frac{r_1 \quad \vdash A, B, \Gamma}{\vdash C, B, \Gamma}$$

$$\frac{r_2 \quad \vdash A, B, \Gamma}{\vdash A, D, \Gamma}$$

$$\frac{r_1 \quad r_2}{\vdash C, D, \Gamma}$$

Open deduction

Derivations

$$\boxed{\Pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\Pi} \vee \boxed{\Delta} \mid \boxed{\Pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\Pi}}{\boxed{\Delta}}$$

$$\frac{\begin{array}{c} A \quad B \\ \Delta_1 \quad \Delta_2 \\ C \quad D \end{array}}{G}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\begin{array}{c} \triangleleft \\ \Pi_1 \\ \vdots \\ \triangleleft \\ \Pi_k \end{array} \quad \begin{array}{c} k \geq 0 \\ \dots \\ k \geq 0 \end{array}}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

$m \geq 0$ $n \geq 0$

Open deduction

Derivations

$$\boxed{\Pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\Pi} \vee \boxed{\Delta} \mid \boxed{\Pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\Pi}}{\boxed{\Delta}}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\begin{array}{c} \triangleleft \\ \pi_1 \\ \vdots \\ \triangleleft \\ \pi_k \end{array} \quad \begin{array}{c} k \geq 0 \\ \dots \\ k \geq 0 \end{array}}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

cont $\frac{\vdash A, A, \Gamma}{\vdash A, \Gamma}$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\pi}}{\boxed{\Delta}}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\begin{array}{c} \triangleleft \\ \pi_1 \\ \vdots \\ \triangleleft \\ \pi_k \end{array} \quad \dots \quad \begin{array}{c} \triangleleft \\ \pi_1 \\ \vdots \\ \triangleleft \\ \pi_k \end{array}}{A_1, \dots, \underset{m \geq 0}{A_m} \vdash \underset{n \geq 0}{B_1, \dots, B_n}}$$

cont $\frac{\vdash A, A, \Gamma}{\vdash A, \Gamma}$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\pi}}{\boxed{\Delta}}$$

$$(B \wedge C) \vee (B \wedge C)$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\begin{array}{c} \triangleleft \\ \vdots \\ \triangleleft \end{array} \quad \begin{array}{c} k \geq 0 \\ \dots \\ k \geq 0 \end{array}}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

$$\text{cont } \frac{\vdash A, A, \Gamma}{\vdash A, \Gamma}$$

Open deduction

Derivations

$$\boxed{\Pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\Pi} \vee \boxed{\Delta} \mid \boxed{\Pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\Pi}}{\boxed{\Delta}}$$

$$m \frac{(B \wedge C) \vee (B \wedge C)}{(B \vee B) \wedge (C \vee C)}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\begin{array}{c} \triangleleft \\ \pi_1 \\ \vdots \\ \triangleleft \\ \pi_k \end{array}}{A_1, \dots, \underset{m \geq 0}{A_m} \vdash \underset{n \geq 0}{B_1, \dots, B_n}}$$

cont $\frac{\vdash A, A, \Gamma}{\vdash A, \Gamma}$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \frac{\pi}{\Delta}$$

$$\frac{\text{act} \quad \begin{array}{c} m \frac{(a \wedge C) \vee (a \wedge C)}{(a \vee a) \wedge (C \vee C)} \\ \downarrow \end{array}}{a}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\begin{array}{c} \triangleleft \\ \pi_1 \\ \vdots \\ \triangleleft \\ \pi_k \end{array}}{A_1, \dots, \underset{m \geq 0}{A_m} \vdash \underset{n \geq 0}{B_1, \dots, B_n}}$$

$$\text{cont } \frac{\vdash A, A, \Gamma}{\vdash A, \Gamma}$$

Open deduction

Derivations

$$\boxed{\pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\pi} \vee \boxed{\Delta} \mid \boxed{\pi} \wedge \boxed{\Delta} \mid \frac{\pi}{\Delta}$$

$$\frac{\begin{array}{c} m \frac{(a \wedge C) \vee (a \wedge C)}{(a \vee a) \wedge (C \vee C)} \\ \text{act} \end{array}}{\begin{array}{c} a \\ ||m, \text{act} \\ C \end{array}}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\vdash \Pi ::= \frac{\text{Derivations } \Pi_1, \dots, \Pi_k}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

$\vdash \Pi$

$\vdash \Pi_1 \quad \dots \quad \vdash \Pi_k$

$m \geq 0 \quad n \geq 0$

cont

$$\frac{\vdash A, A, \Gamma}{\vdash A, \Gamma}$$

Open deduction

Derivations

$$\boxed{\Pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\Pi} \vee \boxed{\Delta} \mid \boxed{\Pi} \wedge \boxed{\Delta} \mid \frac{\Pi}{\Delta}$$

$$m \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\vdash \Pi ::= \frac{\text{Derivations } \Pi_1, \dots, \Pi_k}{A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

$\vdash \Pi$

$\vdash \Pi_1 \quad \dots \quad \vdash \Pi_k$

$m \geq 0 \quad n \geq 0$

cont

$$\frac{\vdash A, A, \Gamma}{\vdash A, \Gamma}$$

Open deduction

Derivations

$$\boxed{\Pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\Pi} \vee \boxed{\Delta} \mid \boxed{\Pi} \wedge \boxed{\Delta} \mid \frac{\Pi}{\Delta}$$

$$m \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

Sequent calculus

Formulae

$$A, B ::= a \mid \bar{a} \mid \top \mid \perp \mid A \vee B \mid A \wedge B$$

Derivations

$$\frac{\text{Derivations}}{\Gamma \vdash A_1, \dots, A_m \vdash B_1, \dots, B_n}$$

$\Gamma_i \quad i \geq 0 \quad \dots \quad \Gamma_k$

Open deduction

Derivations

$$\boxed{\Pi}, \boxed{\Delta} ::= a \mid \bar{a} \mid \top \mid \perp \mid \boxed{\Pi} \vee \boxed{\Delta} \mid \boxed{\Pi} \wedge \boxed{\Delta} \mid \frac{\boxed{\Pi}}{\boxed{\Delta}}$$

more symmetry: proofs have a single premise and conclusion, duality of cut and identity restored

bureaucracy of parallel derivations reduced
all structural rules are atomic and
all rules can be checked locally

What makes a proof formalism good?

Normalisation : proofs have useful normal forms and normalisation procedures are simple

Semantics : proofs are represented with little syntactic bureaucracy and it's easy to tell when two proofs are the same

Complexity : theorems have small proofs

Proof search : it's easy to find proofs

System SKS

$$ai\downarrow \frac{T}{a \vee \bar{a}}$$

identity

$$ac\downarrow \frac{a \vee a}{a}$$

contraction

$$aw\downarrow \frac{\perp}{a}$$

weakening

$$ai\uparrow \frac{a \wedge \bar{a}}{\perp}$$

cut

$$ac\uparrow \frac{a}{a \wedge a}$$

cocontraction

$$aw\uparrow \frac{a}{T}$$

coweakening

$$m \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

medial

$$s \frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$

switch

equality $\vdash \frac{A}{B}$ for :

$$A \wedge B = B \wedge A$$

$$(A \wedge B) \wedge C = A \wedge (B \wedge C)$$

$$A \vee B = B \vee A$$

$$(A \vee B) \vee C = A \vee (B \vee C)$$

$$A \vee \perp = A = T \wedge A$$

$$\perp \vee \perp = \perp$$

$$T = T \wedge T$$

