A Gentle Introduction to Deep Inference

12. Lecture

Open Problems

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Proof search in deep inference

- We have seen yesterday:
  In deep inference systems proofs can be much shorter than in traditional systems.
- But:
  More non-determinism because of the flexibility of the inference rules.
  This make proof search inefficient.
- Can we control this non-determinism to find the “short” proofs more efficiently?

The only work done in this direction so far is

- Ozan Kahramanoğulları: **“Reducing Nondeterminism in the Calculus of Structures”.** *LPAR 2006*
- Ozan Kahramanoğulları: **“Ingredients of a Deep Inference Theorem Prover”.** *Workshop on Classical Logic and Computation, 2008*
- Ozan Kahramanoğulları: **“Deep inference for proof search”.** *Workshop on Structures and Deduction, 2009*
- Nicolas Guenot, Kaustuv Chaudhuri, Lutz Straßburger: **“The Focused Calculus of Structures”.** *Proceedings of CSL 2011*

Proof normalization in deep inference

- We have seen on Tuesday:
  Two different normalization methods in deep inference systems.
- Question 1:
  Are there more?
- Question 2:
  Is there a computational content (under the proofs-as-programs paradigm)?

Some work in this direction is coming from the atomic λ-calculus

- Kai Brünnler and Richard McKinley: **“An Algorithmic Interpretation of a Deep Inference System”.** *LPAR 2008*
- Tom Gundersen, Willem Heijltjes, and Michel Parigot: **“ Atomic lambda-calculus: a typed lambda-calculus with explicit sharing”.** *LICS 2013*
- Tom Gundersen, Willem Heijltjes, and Michel Parigot: **“A Proof of Strong Normalisation for the Typed Atomic Lambda-Calculus”.** *LPAR 2013*
- David Sherratt, Willem Heijltjes, Tom Gundersen, and Michel Parigot: **“Spinal atomic lambda-calculus”.** *FoSSaCS 2020*

Deep inference proof theory for your favorite logic

- Quantifiers, first-order logic, higher-order logic
- Intuitionistic logics, intermediate logics
- Substructural logics
- Modal logics
- Fixpoints
- ...

Existing work on first-order logic in deep inference:

- Kai Brünnler: **"Cut Elimination inside a Deep Inference System for Classical Predicate Logic".** *Studia Logica*
- Ben Ralph: **"Modular Normalisation of Classical Proofs".** *PhD Thesis, University of Bath, 2018*
- Cameron Allett: **"Non-Elementary Compression of First-Order Proofs in Deep Inference Using Epsilon-Terms".** *LICS 2024*

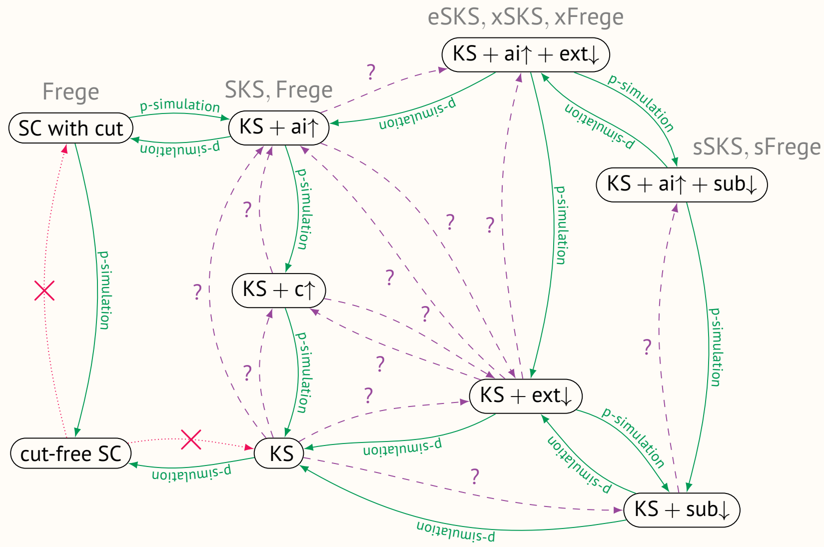
Existing work on intuitionistic logic in deep inference:

- Alwen Tiu: **"A Local System for Intuitionistic Logic".** *LPAR 2006*
- Matteo Acclavio and Lutz Straßburger: **"Intuitionistic BV".** *TABLEAUX 2025*

Existing work on modal logic in deep inference:

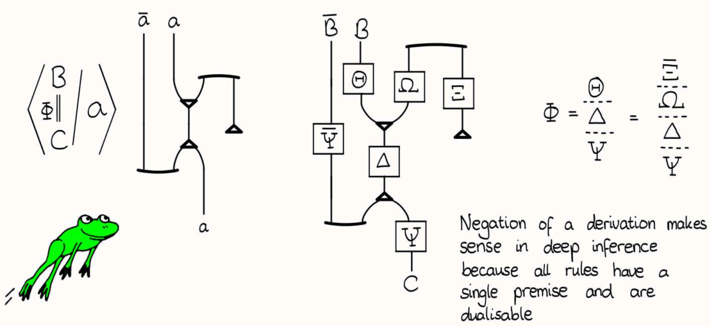
- Robert Hein and Charles Stewart: **"Purity Through Unravelling".** *Proceedings of Structures and Deduction 2005*
- Phiniki Stouppa: **"A Deep Inference System for the Modal Logic S5".** *Studia Logica 85 (2) 2007*

Proof Complexity



Every arrow with a "?" is an open problem.

Formalism B



Decidability of MELL

MELL in the sequent calculus:

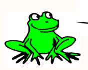
$$\mathsf{id}\,\frac{}{\vdash a^{\perp}, a} \quad \perp\,\frac{\Gamma}{\vdash \Gamma, \perp} \quad 1\,\frac{}{\vdash 1} \quad \parr\,\frac{\Gamma, A, B}{\vdash \Gamma, A \parr B} \quad \otimes\,\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}$$

$$!\,\frac{\vdash ?B_1, \ldots, ?B_n, A}{\vdash ?B_1, \ldots, ?B_n, !A} \quad \mathsf{dr}\,\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A} \quad \mathsf{wk}\,\frac{\vdash \Gamma}{\vdash \Gamma, ?A} \quad \mathsf{ct}\,\frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}$$

MELL in deep inference (System ELS from Tuesday):

$$\mathsf{ai}\!\downarrow\frac{1}{a^{\perp} \parr a} \quad \mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \quad \mathsf{p}\!\downarrow\frac{!(A \parr B)}{!A \parr ?B} \quad \equiv\frac{A}{B}\ \text{(provided } A \equiv B\text{)}$$

$$\mathsf{e}\!\downarrow\frac{1}{!1} \quad \mathsf{g}\!\downarrow\frac{??A}{?A} \quad \mathsf{b}\!\downarrow\frac{?A \parr A}{?A} \quad \mathsf{w}\!\downarrow\frac{\perp}{?A}$$



Is provability in this logic decidable?

It's not necessarily a deep infernce related problem, but we thought it ought to be included here.

For more details on the question, see

- Lutz Straßburger: **"On the Decision Problem for MELL".** *Theoretical Computer Science, Volume 768, Pages 91–98, 2018*
- Ranko Lazic and Sylvain Schmitz: **"Nonelementary Complexities for Branching VASS, MELL, and Extensions".** *ACM ToCL 16(3), 2015*

MLL and ELS (Recall from Tuesday and Wednesday)

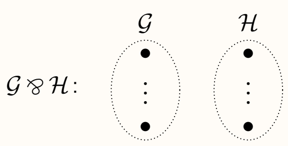
- *Sequent calculus:*

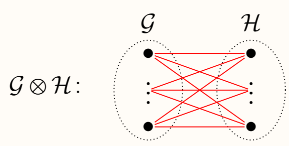
$$\mathbb{I}\,\frac{}{\mathbb{I}} \quad \mathsf{id}\,\frac{}{a, a^{\perp}} \quad \parr\,\frac{\Gamma, A, B}{\Gamma, A \parr B} \quad \otimes\,\frac{\Gamma, A \quad B, \Delta}{\Gamma, A \otimes B, \Delta} \quad \mathsf{mix}\,\frac{\Gamma \quad \Delta}{\Gamma, \Delta}$$

- *Formulas:* $A, B \ ::= \ \mathbb{I} \mid a \mid a^{\perp} \mid A \parr B \mid A \otimes B$
- *Negation:* $\mathbb{I}^{\perp} = \mathbb{I} \qquad (A \parr B)^{\perp} = A \otimes B \qquad (A \otimes B)^{\perp} = A \parr B$
- *Implication:* $A \multimap B \ = \ A^{\perp} \parr B$
- *Sequents:* $\Gamma \ ::= \ A_1, A_2, \ldots, A_n$

MLL and ELS (Recall from Tuesday and Wednesday)

- *Operations on graphs:*





- *From formulas to graphs:*

$$[\![\mathbb{I}]\!] = \emptyset \qquad [\![a]\!] = \bullet_a \qquad [\![a^{\perp}]\!] = \bullet_{a^{\perp}}$$

$$[\![A \parr B]\!] = [\![A]\!] \parr [\![B]\!] \qquad [\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!]$$

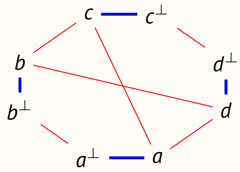
- **Theorem:**

$$[\![A]\!] = [\![B]\!] \quad \Longleftrightarrow \quad A \equiv B$$

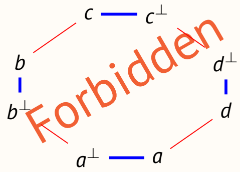
MLL and ELS (Recall from Tuesday and Wednesday)

Example:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{a^\perp, a}\,\text{ax} \qquad \dfrac{}{b^\perp, b}\,\text{ax}}{a^\perp \otimes b^\perp, b \parr a}\,\otimes}{a^\perp \otimes b^\perp, b \parr a}\,\parr \qquad \dfrac{\dfrac{\dfrac{}{d, d^\perp}\,\text{ax} \qquad \dfrac{}{c, c^\perp}\,\text{ax}}{d, c, c^\perp \otimes d^\perp}\,\otimes}{d \parr c, c^\perp \otimes d^\perp}\,\parr}{a^\perp \otimes b^\perp, (b \parr a) \otimes (d \parr c), c^\perp \otimes d^\perp}\,\otimes$$



$$a^\perp \otimes b^\perp, (b \otimes c) \parr (a \otimes d), c^\perp \otimes d^\perp$$



Theorem:

An RB-cograph is the translation of a sequent proof iff there is no chordless æ-cycle.

MLL and ELS (Recall from Tuesday and Wednesday)

ELS Rules:

$$\text{ai}\,\dfrac{\mathbb{I}}{a \parr a^\perp} \qquad \text{s}\,\dfrac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \qquad \equiv\,\dfrac{A}{A'}\ \text{(where } A \equiv A'\text{)}$$

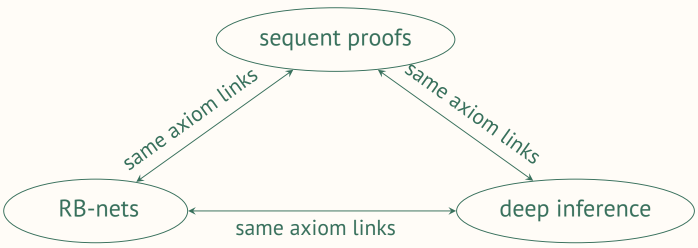
Equivalences:

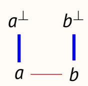
$$A \parr \mathbb{I} \equiv A \qquad A \parr B \equiv B \parr A \qquad A \parr (B \parr C) \equiv (A \parr B) \parr C$$
$$A \otimes \mathbb{I} \equiv A \qquad A \otimes B \equiv B \otimes A \qquad A \otimes (B \otimes C) \equiv (A \otimes B) \otimes C$$

MLL and ELS (Recall from Tuesday and Wednesday)

Theorems:

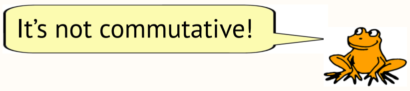
$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash a, a^\perp}\,\text{ax} \qquad \dfrac{}{\vdash b, b^\perp}\,\text{ax}}{\vdash a^\perp b, b^\perp, a^\perp}\,\otimes}{\vdash a \otimes b, b^\perp \parr a^\perp}\,\parr}{\vdash a \otimes b \parr b^\perp \parr a^\perp}\,\parr$$





$$\dfrac{\dfrac{\dfrac{\dfrac{\mathbb{I}}{a \parr a^\perp}\,\text{ai}}{(a \otimes \mathbb{I}) \parr a^\perp}\,\equiv}{(a \otimes (b \parr b^\perp)) \parr a^\perp}\,\text{ai}}{a \otimes b \parr b^\perp \parr a^\perp}\,\text{s}$$

A New Connective



- *Connectives:*

  par: $\parr$ seq/before: $\lhd$ tensor: $\otimes$

- *Some Implications:*

$$A \otimes B \multimap A \lhd B \qquad A \lhd B \multimap A \parr B$$

- *Some Equivalences:*

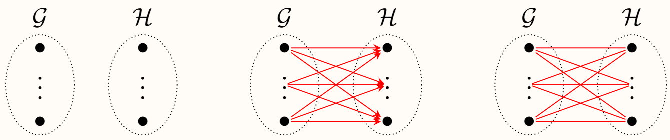
$$(A \lhd B)^{\perp} \equiv A^{\perp} \lhd B^{\perp} \qquad A \lhd B \equiv B \rhd A$$

$$(A \lhd B) \lhd C \equiv A \lhd (B \lhd C) \qquad A \lhd \mathbb{I} \equiv A \equiv \mathbb{I} \lhd A$$

From Formulas to Graphs (refined)

- *Operations on graphs:*

$\mathcal{G} \parr \mathcal{H}$: $\mathcal{G} \lhd \mathcal{H}$: $\mathcal{G} \otimes \mathcal{H}$:



- *From formulas to graphs:*

$$[\![\mathbb{I}]\!] = \emptyset \qquad [\![a]\!] = \bullet_a \qquad [\![a^{\perp}]\!] = \bullet_{a^{\perp}}$$

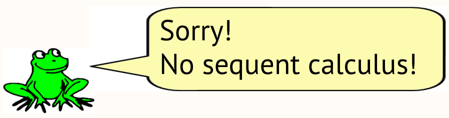
$$[\![A \parr B]\!] = [\![A]\!] \parr [\![B]\!] \qquad [\![A \lhd B]\!] = [\![A]\!] \lhd [\![B]\!] \qquad [\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!]$$

- **Theorem:**

$$[\![A]\!] = [\![B]\!] \iff A \equiv B$$

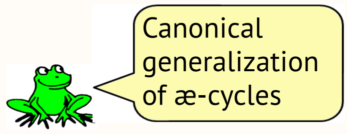
A New Logic: Sequent Calculus Proofs

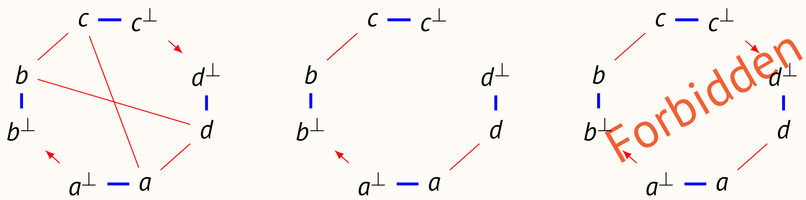
Connectives: $\parr$ (par)
$\lhd$ (seq/before)
$\otimes$ (tensor)



A New Logic: Pomset Logic

Connectives: ⅋ (par)
◁ (seq/before)
⊗ (tensor)





no chordless æ-cycle | no chordless æ-cycle | a chordless æ-cycle

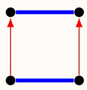
Pomset logic poof: correct RB-net.

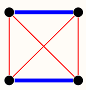
- Pomset logic has been introduced by
  - Christian Retoré: **“Réseaux et Séquents Ordonnés”.** *PhD-Thesis, Université Paris VII, 1993*
  - Christian Retoré: **“Pomset Logic: A Non-Commutative Extension of Classical Linear Logic”.** *TLCA'97, LNCS 1210, 1997*
  - Christian Retoré: **“Pomset Logic as a Calculus of Directed Cographs”.** *Dynamic Perspectives in Logic and Linguistics, 1999 (also available as Inria RR-3714*

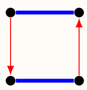
A New Logic: Pomset Logic

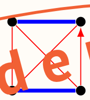
Examples:

correct proofs

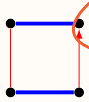

$(a \lhd b) ⅋ (a^\perp \lhd b^\perp)$
no chordless æ-cycle

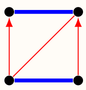

$(a ⅋ a^\perp) \otimes (b ⅋ b^\perp)$
no chordless æ-cycle


$(b \lhd a) ⅋ (a^\perp \lhd b^\perp)$
a chordless æ-cycle


no underlying formula

Forbidden


$(a \otimes b) ⅋ (a^\perp \lhd b^\perp)$
a chordless æ-cycle


no underlying formula

A New Logic: BV

Formulas:

$$A, B ::= a \mid a^\perp \mid \circ \mid A \otimes B \mid A ⅋ B \mid A \lhd B$$

Negation:

$$a^{\perp\perp} = a \quad \circ^\perp = \circ \quad (A \otimes B)^\perp = A^\perp ⅋ B^\perp \quad (A ⅋ B)^\perp = A^\perp \otimes B^\perp \quad (A \lhd B)^\perp = A^\perp \lhd B^\perp$$

Rules for BV and SBV:

$$\mathsf{ai}{\downarrow}\,\frac{\circ}{a^\perp ⅋ a} \qquad \mathsf{s}\,\frac{A \otimes (B ⅋ C)}{(A \otimes B) ⅋ C} \qquad \equiv\,\frac{A}{B}\ \text{(provided } A \equiv B\text{)} \qquad \mathsf{ai}{\uparrow}\,\frac{a^\perp \otimes a}{\perp}$$

$$\mathsf{q}{\downarrow}\,\frac{(A ⅋ C) \lhd (B ⅋ D)}{(A \lhd B) ⅋ (C \lhd D)} \qquad \mathsf{q}{\uparrow}\,\frac{(A \lhd C) \otimes (B \lhd D)}{(A \otimes B) \lhd (C \otimes D)}$$

where

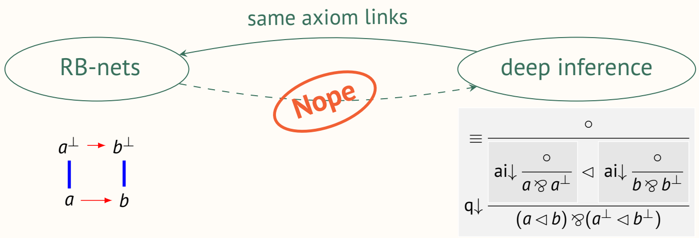
$$(A ⅋ B) ⅋ C \equiv A ⅋ (B ⅋ C) \qquad A ⅋ B \equiv B ⅋ C \qquad A ⅋ \circ \equiv A$$
$$(A \otimes B) \otimes C \equiv A \otimes (B \otimes C) \qquad A \otimes B \equiv B \otimes C \qquad A \otimes \circ \equiv A$$
$$(A \lhd B) \lhd C \equiv A \lhd (B \lhd C) \qquad A \lhd \circ \equiv A \equiv \circ \lhd A$$

- SBV = {ai↓, s, ≡, q↓, q↑, ai↑}
- BV = {ai↓, s, ≡, q↓}
- These two systems have been introduced by
  - Alessio Guglielmi: **“A Calculus of Order and Interaction”.** *TU Dresden, Technical report WV-99-04, 1999*
  - Alessio Guglielmi: **“A System of Interaction and Structure”.** *ACM ToCL 1(8), 2007*
- And this logic was reason why deep infernce has been intruduced.

BV and Pomset logic

Theorem:

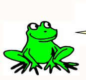
Pomset Logic $\supsetneq$ System BV



correctness is **coNP**-complete — correctness is in **P**

provability is $\Sigma_2^{\mathbf{p}}$-complete — provability is **NP**-complete

BV and Pomset logic



- Which one is the “right” logic, BV or pomset logic?
- Are there more logics with $\lhd$ between $\parr$ and $\otimes$?
- Can we have a correctness criterion for BV?
- …
- Adding ! and ? (the modalities of linear logic) to BV (or pomset logic) make it undecidable. But what about a self-dual modality for $\lhd$?

- Difference of BV and pomset logic:
  - Lê Thành Dũng Nguyễn and Lutz Straßburger: **“BV and Pomset Logic are not the same”.** *Proceedings of CSL 2022*
- **NP**-completeness of BV:
  - Ozan Kahramanoğulları: **“System BV is NP-complete”.** *Annals of Pure and Applied Logic, 2007*
- $\Sigma_2^{\mathbf{p}}$-completeness of pomset logic:
  - Lê Thành Dũng Nguyễn and Lutz Straßburger: **“A System of Interaction and Structure III: The Complexity of BV and Pomset Logic”.** *Logical Methods in Computer Science 19(4), 2023*