A Gentle Introduction to Deep Inference

10. Lecture

Other Proof Compression Mechanisms

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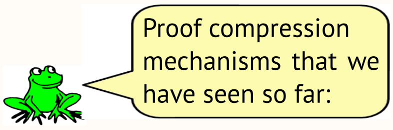


- cut
- cocontraction

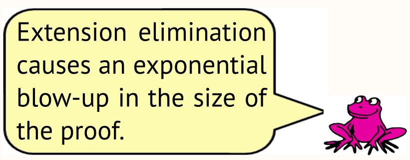
Extension in Frege systems

- *Tseitin extension:*
  add n fresh variables $a_1, \ldots, a_n$ and additional axioms

  $$a_i \leftrightarrow A_i$$

  such that for all $i \in \{1, \ldots, n\}$:
  the variable a_i does not occur in $A_1, \ldots, A_i$

- *extended Frege system* (xFrege)*:*
  A Frege system with extension axioms



- **Exercise 10.1:** Show that extended Frege systems p-simulate Frege systems.
- It is not known whether Frege systems p-simulate extended Frege systems.
- **Exercise 10.2:** Show how extension can be eliminated and why this is exponential. (Hint: the extension variables a_i are replaced by the formulas A_i.)

Extension in deep inference (first version)

- add n fresh variables $a_1, \ldots, a_n$ and additional axioms $a_i \leftrightarrow A_i$ such that for all $i \in \{1, \ldots, n\}$ the variable a_i does not occur in $A_1, \ldots, A_i$
- Define *system* xSKS by changing notion of proof

$$\begin{array}{c} \pi \Big\| \mathsf{xSKS} \\ B \end{array} \quad = \quad \begin{array}{c} (\bar{a}_1 \vee A_1) \wedge (\bar{A}_1 \vee a_1) \wedge \cdots \wedge (\bar{a}_n \vee A_n) \wedge (\bar{A}_n \vee a_n) \\ \pi \Big\| \mathsf{SKS} \\ B \end{array}$$

- **Theorem:**
  xSKS is p-eqivalent to every extended Frege system.

Observation:
The presence of
cut is crucial

- **Exercise 10.3:** Prove that theorem. (Hint: The proof is very similar to the p-eqivalence of SKS and Frege systems.)
- You can also read the proof here:
  - Paola Bruscoli and Alessio Guglielmi: **"On the Proof Complexity of Deep Inference".** *ACM ToCL 10 (2:14) 2009*

Extension in deep inference (second version)

- add n fresh variables $a_1, \ldots, a_n$ and additional axioms $a_i \leftrightarrow A_i$ such that for all $i \in \{1, \ldots, n\}$ the variable a_i does not occur in $A_1, \ldots, A_i$
- Define *system* eSKS adding for each extension axiom the rules

$$\mathsf{ext}{\downarrow}\,\frac{a_i}{A_i} \qquad \text{and} \qquad \mathsf{ext}{\downarrow}\,\frac{\bar{a}_i}{\bar{A}_i}$$

$$\begin{array}{lcl} \mathsf{eSKS} & = & \mathsf{SKS} + \mathsf{ext}{\downarrow} \\ \mathsf{eKS} & = & \mathsf{KS} + \mathsf{ext}{\downarrow} \end{array}$$

Observation:
This is indepen-
dent from **cut**

- **Theorem:**
  eSKS and xSKS are p-eqivalent.

- **Exercise 10.4:** Prove the theorem. (Hint: Look at the following two slides.)
- Or at
  - Lutz Straßburger: **"Extension without Cut".** *APAL 163(12), 2012*

  where eSKS and eKS have been introduced.

Extension in deep inference

- add n fresh variables $a_1, \ldots, a_n$ and additional axioms $a_i \leftrightarrow A_i$ such that for all $i \in \{1, \ldots, n\}$ the variable a_i does not occur in $A_1, \ldots, A_i$
- eSKS p-simulates xSKS:

$$\begin{array}{c} \Big\| \mathsf{ai}{\downarrow} \\ (\bar{a}_1 \vee a_1) \wedge (\bar{a}_1 \vee a_1) \wedge \cdots \wedge (\bar{a}_n \vee a_n) \wedge (\bar{a}_n \vee a_n) \\ \Big\| \mathsf{ext}{\downarrow} \\ (\bar{a}_1 \vee A_1) \wedge (\bar{A}_1 \vee a_1) \wedge \cdots \wedge (\bar{a}_n \vee A_n) \wedge (\bar{A}_n \vee a_n) \\ \Big\| \mathsf{SKS} \\ B \end{array}$$

Extension in deep inference

- add n fresh variables $a_1, \ldots, a_n$ and additional axioms $a_i \leftrightarrow A_i$ such that for all $i \in \{1, \ldots, n\}$ the variable a_i does not occur in $A_1, \ldots, A_i$
- xSKS p-simulates eSKS:

$$\begin{array}{c}\Vert \text{eSKS} \\ B\end{array} \quad \rightsquigarrow \quad \begin{array}{c}(\bar{a}_1 \vee A_1) \wedge (\bar{A}_1 \vee a_1) \wedge \cdots \wedge (\bar{a}_n \vee A_n) \wedge (\bar{A}_n \vee a_n) \\ \Vert \text{eSKS} \\ \mathsf{w}{\uparrow}\dfrac{(\bar{a}_1 \vee A_1) \wedge (\bar{A}_1 \vee a_1) \wedge \cdots \wedge (\bar{a}_n \vee A_n) \wedge (\bar{A}_n \vee a_n) \wedge B}{B}\end{array}$$

The instances of ext↓ are removed as follows:

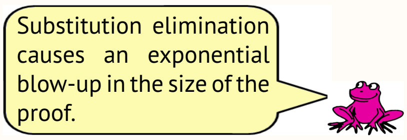
$$\mathsf{ext}{\downarrow}\dfrac{\cdots \wedge (\bar{a}_i \vee A_i) \wedge \cdots \wedge F\{a_i\}}{\cdots \wedge (\bar{a}_i \vee A_i) \wedge \cdots \wedge F\{A_i\}} \quad \rightsquigarrow \quad \begin{array}{c}\mathsf{c}{\uparrow}\dfrac{\cdots \wedge (\bar{a}_i \vee A_i) \wedge \cdots \wedge F\{a_i\}}{\cdots \wedge (\bar{a}_i \vee A_i) \wedge (\bar{a}_i \vee A_i) \wedge \cdots \wedge F\{a_i\}} \\ \Vert \{\mathsf{s}\} \\ \mathsf{s}\dfrac{\cdots \wedge (\bar{a}_i \vee A_i) \wedge \cdots \wedge F\{a_i \wedge (\bar{a}_i \vee A_i)\}}{\mathsf{ai}{\uparrow}\dfrac{\cdots \wedge (\bar{a}_i \vee A_i) \wedge \cdots \wedge F\{(a_i \wedge \bar{a}_i) \vee A_i)\}}{\cdots \wedge (\bar{a}_i \vee A_i) \wedge \cdots \wedge F\{A_i\}}}\end{array}$$

Substitution in Frege systems

- $\sigma\colon \mathcal{A} \to \mathcal{F}$ such that $\sigma(a) = a$ for almost all $a \in \mathcal{A}$
- σA is the formula obtained from A by replacing every atom occurrence a in A by $\sigma(a)$ and $\bar{a}$ by $\overline{\sigma(a)}$.
- we can add substitution to Frege systems, by adding the *substitution rule*

$$\mathsf{sub}\,\dfrac{A}{\sigma A}$$

Then we get *Frege systems with substitution* (sFrege).



- **Exercise 10.5:** Show that Frege systems with substitution p-simulate Frege systems.
- It is not known whether Frege systems p-simulate Frege systems with substitution.
- **Exercise 10.6:** Show how substitution can be eliminated and why this is exponential.

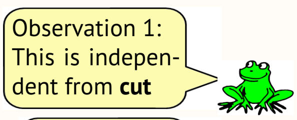
Substitution in deep inference

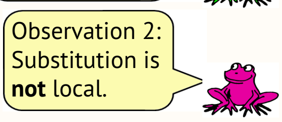
- $\sigma\colon \mathcal{A} \to \mathcal{F}$ such that $\sigma(a) = a$ for almost all $a \in \mathcal{A}$
- σA is the formula obtained from A by replacing every atom occurrence a in A by $\sigma(a)$ and $\bar{a}$ by $\overline{\sigma(a)}$.
- we can add substitution to a deep inference system, by adding the *substitution rule*

$$\mathsf{sub}{\downarrow}\,\dfrac{A}{\sigma A}$$

sSKS	=	SKS + sub↓
sKS	=	KS + sub↓

- **Theorem:** sSKS and sFrege are p-eqivalent.





- "Not local" means that the rule cannot be applied locally, i.e., it has to be appled to the whole formula.
- **Exercise 10.7:** Prove that theorem. (Hint: The proof is very similar to the p-eqivalence of SKS and Frege systems.)
- You can also read the proof here:
  - Paola Bruscoli and Alessio Guglielmi: **"On the Proof Complexity of Deep Inference".** *ACM ToCL 10 (2:14) 2009*

From Extension to Substitution

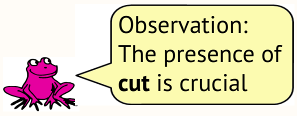
Theorem: sSKS p-simulates xSKS.

Proof: From an xSKS proof of B:

$$\begin{array}{c}(\bar{a}_1 \vee A_1) \wedge (\bar{A}_1 \vee a_1) \wedge \cdots \wedge (\bar{a}_n \vee A_n) \wedge (\bar{A}_n \vee a_n) \\ {\scriptstyle\pi}\Big\Vert{\scriptstyle\mathsf{SKS}} \\ B\end{array}$$

to an sSKS proof of B:

$$\equiv\frac{\mathsf{i}{\downarrow}\dfrac{\top}{\mathsf{sub}{\downarrow}\dfrac{(a_1 \wedge \bar{A}_1) \vee (\bar{a}_1 \wedge A_1)}{\mathsf{i}{\uparrow}\dfrac{A_1 \wedge \bar{A}_1}{\bot} \vee \mathsf{i}{\uparrow}\dfrac{A_1 \wedge \bar{A}_1}{\bot}} \vee \cdots \vee \mathsf{sub}{\downarrow}\dfrac{(a_n \wedge \bar{A}_n) \vee (\bar{a}_n \wedge A_n)}{\mathsf{i}{\uparrow}\dfrac{A_n \wedge \bar{A}_n}{\bot} \vee \mathsf{i}{\uparrow}\dfrac{A_n \wedge \bar{A}_n}{\bot}} \vee \begin{array}{c}(\bar{a}_1 \vee A_1) \wedge (\bar{A}_1 \vee a_1) \wedge \cdots \wedge (\bar{a}_n \vee A_n) \wedge (\bar{A}_n \vee a_n) \\ {\scriptstyle\pi}\Vert{\scriptstyle\mathsf{SKS}} \\ B\end{array}}}{B}$$



- This proof is a simplification of the one presented in
  - Paola Bruscoli and Alessio Guglielmi: **"On the Proof Complexity of Deep Inference".** *ACM ToCL 10 (2:14) 2009*
- **Exercise 10.8:** Show that it follows that sFrege p-simulates xFrege.
- This has been first show in
  - Stephen A. Cook and Robert A. Reckhow: **"The Relative Efficiency of Propositional Proof Systems".** *The Journal of Symboloc Logic 44(1), 1979*

  long before deep inference.
- **Exercise 10.9:** Look at that proof. Which one do you find simpler? the orginal proof by Cook and Reckhow, or the one via deep inference? Or do you think they are both the same proof?

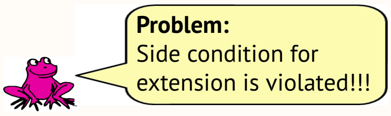
From Substitution to Extension

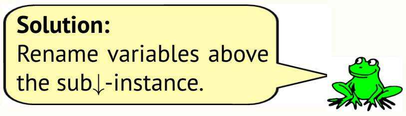
Theorem: eSKS p-simulates sSKS.

Proof:
Idea: replace one substitution by many extensions

$$\begin{array}{c}\vdots \\ {\scriptstyle\pi}\Vert \\ \cdots\ \mathsf{sub}{\downarrow}\dfrac{a \vee (b \wedge c) \vee \bar{a}}{(a \wedge c) \vee (b \wedge (a \vee c)) \vee \bar{a} \vee \bar{c}}\ \cdots \\ \Vert \\ \vdots\end{array} \quad\rightsquigarrow\quad \begin{array}{c}\vdots \\ {\scriptstyle\pi}\Vert \\ \cdots\ \mathsf{ext}{\downarrow}\dfrac{a}{a \wedge c} \vee (b \wedge \mathsf{ext}{\downarrow}\dfrac{c}{a \vee c}) \vee \mathsf{ext}{\downarrow}\dfrac{\bar{a}}{\bar{a} \vee \bar{c}}\ \cdots \\ \Vert \\ \vdots\end{array}$$

$\sigma = \{a \mapsto a \wedge c, c \mapsto a \vee c\}$ $\qquad$ $a \leftrightarrow a \wedge c, \quad c \leftrightarrow a \vee c$





- This proof has first been shown in
  - Lutz Straßburger: **"Extension without Cut".** *APAL 163(12), 2012*
- **Exercise 10.10:** Show that it follows that xFrege p-simulates sFrege.
- This has been first show in
  - Jan Krajíček and Pavel Pudlák: **"Propositional Proof Systems, the Consistency of First Order Theories and the Complexity of Computations".** *JSL 54(2), 1989*

  (it was an open problem for 10 years)
- **Exercise 10.11:** Look at that proof. Which one do you find simpler? the orginal proof by Krajíček and Pudlák, or the one via deep inference? Or do you think they are both the same proof?

From Substitution to Extension (and back)

Theorem: eKS p-simulates sKS.

Proof:
The previous proof does not use cuts.

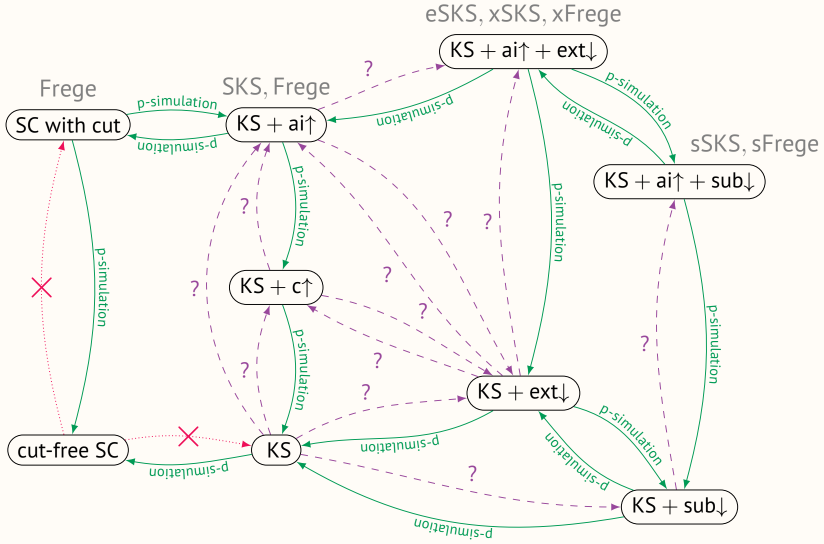
Theorem: sKS p-simulates eKS.

Proof:
We need some clever book-keeping and variable renaming to ensure that the condition on the extension variables is satisfied.

That sKS p-simulates eKS has been shown in

- Novak Novakovic and Lutz Straßburger: **"On the Power of Substitution in the Calculus of Structures".** *ACM Trans. Comput. Log. 16(3), 2015*

Summary



eSKS, xSKS, xFrege
KS + ai↑ + ext↓
Frege
SKS, Frege
SC with cut
KS + ai↑
sSKS, sFrege
KS + ai↑ + sub↓
KS + c↑
KS + ext↓
cut-free SC
KS
KS + sub↓
p-simulation
?