A Gentle Introduction to Deep Inference

5. Lecture

Splitting, Context Reduction, and Decomposition

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Another deep inference system

Formulas:

$$A, B ::= a \mid a^\perp \mid 1 \mid \perp \mid A \otimes B \mid A \parr B$$

Negation:

$$a^{\perp\perp} = a \qquad 1^\perp = \perp \qquad (A \otimes B)^\perp = A^\perp \parr B^\perp$$
$$\perp^\perp = 1 \qquad (A \parr B)^\perp = A^\perp \otimes B^\perp$$

Rules:

$$\mathsf{ai}{\downarrow}\,\frac{1}{a^\perp \parr a} \qquad \mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \qquad \equiv\,\frac{A}{B}\ \text{(provided } A \equiv B\text{)} \qquad \mathsf{ai}{\uparrow}\,\frac{a^\perp \otimes a}{\perp}$$

where

$$(A \parr B) \parr C \equiv A \parr (B \parr C) \qquad A \parr B \equiv B \parr C \qquad A \parr \perp \equiv A$$
$$(A \otimes B) \otimes C \equiv A \otimes (B \otimes C) \qquad A \otimes B \equiv B \otimes C \qquad A \otimes 1 \equiv A$$

- This logic is *multiplicative linear logic (MLL)*, and we use the linear logic notation here
  - Jean-Yves Girard: **"Linear Logic".** *Theoretical Computer Science, 1987*
- MLL can be thought of as classical logic without contraction and weakening
- $\equiv$ is the smallest congruence relation closed under associativity, commutativity, and unit-laws for $\parr$ (called *par*) and $\otimes$ (called *tensor*)
- **Exercise 5.1:** Show that $A^{\perp\perp} = A$ for all formulas A.
- We call this proof system SMLS. I.e.

$$\begin{array}{rcl} \mathsf{SMLS} & = & \{\mathsf{ai}{\downarrow}, \mathsf{s}, \equiv, \mathsf{ai}{\uparrow}\} \\ \mathsf{MLS} & = & \{\mathsf{ai}{\downarrow}, \mathsf{s}, \equiv\} \\ \mathsf{SMLS}{\uparrow} & = & \{\mathsf{s}, \equiv, \mathsf{ai}{\uparrow}\} \end{array}$$

- **Exercise 5.2:** Show that the general forms of the rules

$$\mathsf{i}{\downarrow}\,\frac{1}{A^\perp \parr A} \quad \text{and} \quad \mathsf{i}{\uparrow}\,\frac{A^\perp \otimes A}{\perp}$$

  are derivable in MLS (resp. SMLS↑).

Another deep inference system

Rules:

$$\underbrace{\underbrace{\mathsf{ai}{\downarrow}\,\frac{1}{a^\perp \parr a} \qquad \mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \qquad \equiv\,\frac{A}{B}\ \text{(provided } A \equiv B\text{)}}_{\mathsf{MLS}} \qquad \mathsf{ai}{\uparrow}\,\frac{a^\perp \otimes a}{\perp}}_{\mathsf{SMLS}}$$

Theorem (Cut Elimination):
If a formula A is provable in SMLS then it is also provable in MLS.

- **Exercise 5.3:** Show that ai↑ can simulate the cut (use the previous exercise).
- The sequent calculus for MLL is:

$$\mathsf{id}\,\frac{}{\vdash a^\perp, a} \qquad \perp\,\frac{\Gamma}{\vdash \Gamma, \perp} \qquad 1\,\frac{}{\vdash 1}$$

$$\parr\,\frac{\Gamma, A, B}{\vdash \Gamma, A \parr B} \qquad \otimes\,\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}$$

$$\mathsf{cut}\,\frac{\vdash \Gamma, A \quad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}$$

- **Exercise 5.4:** Prove cut elimnation for the sequent calculus for MLL.
- **Exercise 5.5:** Use this to prove cut elimination in deep inference: First show how to translate a $\mathsf{MLL}_{\mathsf{DI}}$ derivation into the sequent calculus, and second, show how a cut-free sequent proof in MLL is translated into a ai↑-free $\mathsf{MLL}_{\mathsf{DI}}$ derivation.

Splitting

Lemma (Splitting):

1. If there is a proof $\begin{array}{c}\delta \,\Vert\, \mathsf{MLS}\\ K \parr (A\otimes B)\end{array}$,

   then there are formulas K_A and K_B and derivations

$$\begin{array}{c}K_A \parr K_B\\ \delta_K \,\Vert\, \mathsf{MLS}\\ K\end{array} \quad\text{and}\quad \begin{array}{c}\delta_A \,\Vert\, \mathsf{MLS}\\ K_A \parr A\end{array} \quad\text{and}\quad \begin{array}{c}\delta_B \,\Vert\, \mathsf{MLS}\\ K_B \parr B\end{array} .$$

2. If there is a proof $\begin{array}{c}\delta \,\Vert\, \mathsf{MLS}\\ K \parr a\end{array}$,

   then there is a derivation

$$\begin{array}{c}a^{\perp}\\ \delta_a \,\Vert\, \mathsf{MLS}\\ K\end{array} .$$

- The idea of splitting is due to
  - Alessio Guglielmi: **"A System of Interaction and Structure".** *ACM Transactions on Computational Logic 1(8), 2007*

  The proof we present here is as in
  - Alessio Guglielmi and Lutz Straßburger: **"A System of Interaction and Structure V: The Exponentials and Splitting".** *Mathematical Structures in Computer Science, 21(3), pp.563–584, 2011*

Splitting (Proof of 1.)

Proof: By induction on the lexicographic pair $\langle |K \parr (A\otimes B)|\, ,\, |\delta| \rangle$
Some cases:

$$\text{(i)}\ \begin{array}{c}\delta' \,\Vert\, \mathsf{MLS}\\ \mathsf{r}\,\dfrac{K'}{K} \parr (A\otimes B)\end{array} \qquad \text{(ii)}\ \begin{array}{c}\delta' \,\Vert\, \mathsf{MLS}\\ K \parr (\mathsf{r}\,\dfrac{A'}{A} \otimes B)\end{array} \qquad \text{(iii)}\ \begin{array}{c}\delta' \,\Vert\, \mathsf{MLS}\\ K \parr (A \otimes \mathsf{r}\,\dfrac{B'}{B})\end{array}$$

In case (i), apply IH to δ' and get:

$$\mathsf{r}\,\dfrac{\begin{array}{c}K_A \parr K_B\\ \delta_{K'} \,\Vert\, \mathsf{MLS}\\ K'\end{array}}{K} \quad\text{and}\quad \begin{array}{c}\delta_A \,\Vert\, \mathsf{MLS}\\ K_A \parr A\end{array} \quad\text{and}\quad \begin{array}{c}\delta_B \,\Vert\, \mathsf{MLS}\\ K_B \parr B\end{array} .$$

Cases (ii) and (iii) are similar.

- The size $|A|$ of a formula A is the number of symbols in it, and the size $|\delta|$ of a derivation δ is the number of inference rule instances in it.
- **Exercise 5.6:** Complete cases (ii) and (iii).

Splitting (Proof of 1.)

Another case: (iv) $\begin{array}{c}\delta' \,\Vert\, \mathsf{MLS}\\ K_1 \parr \mathsf{s}\,\dfrac{A_1\otimes B_1 \otimes (K_2 \parr (A_2\otimes B_2))}{K_2 \parr (A_1\otimes A_2\otimes B_1\otimes B_2)}\end{array}$

By IH, there are L_1 and L_2 such that

$$\begin{array}{c}L_1 \parr L_2\\ \delta_{K_1} \,\Vert\, \mathsf{MLS}\\ K_1\end{array} \quad\text{and}\quad \begin{array}{c}\delta_1 \,\Vert\, \mathsf{MLS}\\ L_1 \parr (A_1\otimes B_1)\end{array} \quad\text{and}\quad \begin{array}{c}\delta_2 \,\Vert\, \mathsf{MLS}\\ L_2 \parr K_2 \parr (A_2\otimes B_2)\end{array} .$$

Applying IH again to δ_1 and to δ_2:

$$\begin{array}{c}K_{A_1} \parr K_{B_1}\\ \delta_{L_1} \,\Vert\, \mathsf{MLS}\\ L_1\end{array} \quad \begin{array}{c}\delta_{A_1} \,\Vert\, \mathsf{MLS}\\ K_{A_1} \parr A_1\end{array} \quad \begin{array}{c}\delta_{B_1} \,\Vert\, \mathsf{MLS}\\ K_{B_1} \parr B_1\end{array} \quad\text{and}\quad \begin{array}{c}K_{A_2} \parr K_{B_2}\\ \delta_{L_2} \,\Vert\, \mathsf{MLS}\\ L_2 \parr K_2\end{array} \quad \begin{array}{c}\delta_{A_2} \,\Vert\, \mathsf{MLS}\\ K_{A_2} \parr A_2\end{array} \quad \begin{array}{c}\delta_{B_2} \,\Vert\, \mathsf{MLS}\\ K_{B_2} \parr B_2\end{array}$$

Putting things together: $K_A = K_{A_1} \parr K_{A_2}$ and $K_B = K_{B_1} \parr K_{B_2}$

$$\equiv\dfrac{(K_{A_1} \parr K_{A_2}) \parr (K_{B_1} \parr K_{B_2})}{\equiv\dfrac{\begin{array}{c}K_{A_1} \parr K_{B_1}\\ \delta_{L_1} \,\Vert\, \mathsf{MLS}\\ L_1\end{array} \parr \begin{array}{c}K_{A_2} \parr K_{B_2}\\ \delta_{L_2} \,\Vert\, \mathsf{MLS}\\ L_2 \parr K_2\end{array}}{\begin{array}{c}L_1 \parr L_2\\ \delta_{K_1} \,\Vert\, \mathsf{MLS}\\ K_1\end{array} \parr K_2}} \qquad \mathsf{s}\,\dfrac{\begin{array}{c}\delta_{A_1} \,\Vert\, \mathsf{MLS}\\ K_{A_1} \parr A_1\end{array} \otimes \begin{array}{c}\delta_{A_2} \,\Vert\, \mathsf{MLS}\\ K_{A_2} \parr A_2\end{array}}{K_{A_1} \parr \mathsf{s}\,\dfrac{A_1 \otimes (K_{A_2} \parr A_2)}{K_{A_2} \parr (A_1\otimes A_2)}} \qquad \mathsf{s}\,\dfrac{\begin{array}{c}\delta_{B_1} \,\Vert\, \mathsf{MLS}\\ K_{B_1} \parr B_1\end{array} \otimes \begin{array}{c}\delta_{B_2} \,\Vert\, \mathsf{MLS}\\ K_{B_2} \parr B_2\end{array}}{K_{B_1} \parr \mathsf{s}\,\dfrac{B_1 \otimes (K_{B_2} \parr B_2)}{K_{B_2} \parr (B_1\otimes B_2)}}$$

Splitting (Proof of 1.)

Another case: (v) $$\begin{array}{c}\delta' \,\Vert\, \mathsf{MLS}\\ K_1 \parr \mathsf{s}\,\dfrac{K_3 \otimes (K_2 \parr K_4 \parr (A\otimes B))}{K_2 \parr (K_3\otimes K_4)\parr (A\otimes B)}\end{array}$$

By IH, there are L_1 and L_2 such that

$$\begin{array}{c}L_1 \parr L_2\\ \delta_{K_1} \,\Vert\, \mathsf{MLS}\\ K_1\end{array} \qquad \begin{array}{c}\delta_1 \,\Vert\, \mathsf{MLS}\\ L_1 \parr K_3\end{array} \qquad \begin{array}{c}\delta_2 \,\Vert\, \mathsf{MLS}\\ L_2 \parr K_2 \parr K_4 \parr (A\otimes B)\end{array}$$

We apply the IH again to δ_2:

$$\begin{array}{c}K_A \parr K_B\\ \delta_{L_2} \,\Vert\, \mathsf{MLS}\\ L_2 \parr K_2 \parr K_4\end{array} \qquad \begin{array}{c}\delta_A \,\Vert\, \mathsf{MLS}\\ K_A \parr A\end{array} \qquad \begin{array}{c}\delta_B \,\Vert\, \mathsf{MLS}\\ K_B \parr B\end{array}$$

- **Exercise 5.7:** Complete this case by showing the derivation
$$\begin{array}{c}K_A \parr K_B\\ \delta_K \,\Vert\, \mathsf{MLS}\\ K_1 \parr K_2 \parr K_3 \otimes K_3\end{array}$$
- **Exercise 5.8:** Complete the proof of the first half of the splitting lemma by either
  - showing the missing cases, or
  - arguing that there are no missing cases.

Splitting (Proof of 2.)

There is only one non-trivial case:

$$\begin{array}{c}\delta' \,\Vert\, \mathsf{MLS}\\ K_1 \parr \mathsf{s}\,\dfrac{K_3 \otimes (K_2 \parr K_4 \parr a)}{K_2 \parr ((K_3\otimes K_4)\parr a)}\end{array}$$

By Point 1., there are L_1 and L_2 and

$$\begin{array}{c}L_1 \parr L_2\\ \delta_{K_1} \,\Vert\, \mathsf{MLS}\\ K_1\end{array} \qquad \begin{array}{c}\delta_1 \,\Vert\, \mathsf{MLS}\\ L_1 \parr K_3\end{array} \qquad \begin{array}{c}\delta_2 \,\Vert\, \mathsf{MLS}\\ L_2 \parr K_2 \parr K_4 \parr a\end{array}$$

We apply the IH to δ_2:

$$\begin{array}{c}a^{\perp}\\ \delta_3 \,\Vert\, \mathsf{MLS}\\ L_2 \parr K_2 \parr K_4\end{array}$$

Putting everything together:

$$\mathsf{s}\,\dfrac{\equiv\,\dfrac{\begin{array}{c}a^{\perp}\\ \delta_3 \,\Vert\, \mathsf{MLS}\\ L_2 \parr K_2 \parr K_4\end{array}}{L_2 \parr \mathsf{s}\,\dfrac{\begin{array}{c}\delta_1 \,\Vert\, \mathsf{MLS}\\ L_1 \parr K_3\end{array} \otimes (K_2 \parr K_4)}{L_1 \parr (K_3 \otimes (K_2 \parr K_4))}}}{\begin{array}{c}L_1 \parr L_2\\ \delta_{K_1} \,\Vert\, \mathsf{MLS}\\ K_1\end{array} \parr \mathsf{s}\,\dfrac{K_3 \otimes (K_2 \parr K_4)}{K_2 \parr (K_3 \otimes K_4)}}$$

- **Exercise 5.9:** Show the trivial case(s).
- This completes the proof of the splitting lemma.
- To use it for cut elimination, we have to be able to use it in an arbitrary context $F\{\ \}$. Contexts of the form $K \parr \{\ \}$ are called *shallow*.

Context Reduction

Lemma (Context Reduction): Let A be a formula and $F\{\ \}$ be a context. If there is a derivation

$$\begin{array}{c}\delta \,\Vert\, \mathsf{MLS}\\ F\{A\}\end{array}$$

then there is a formula K, such that for all X, we have

$$\begin{array}{c}K \parr X\\ \delta_X \,\Vert\, \mathsf{MLS}\\ F\{X\}\end{array} \quad \text{and} \quad \begin{array}{c}\delta_A \,\Vert\, \mathsf{MLS}\\ K \parr A\end{array}\ .$$

Proof: By induction on $F\{\ \}$.

Context Reduction (Proof)

By induction on $F\{\ \}$.

- Case 1: $F\{\ \} = L \parr \{\ \}$ is a shallow context.
  Then $K = L$ and $\delta_A = \delta$ and δ_X is trivial
- Case 2: $F\{\ \} = L_1 \parr (L_2 \otimes F'\{\ \})$ for some L_1 and L_2.
  Apply splitting. Get:

$$\begin{array}{c} L_3 \parr L_4 \\ \delta_1 \Vert \text{MLS} \\ L_1 \end{array} \qquad \begin{array}{c} \delta_2 \Vert \text{MLS} \\ L_3 \parr L_2 \end{array} \qquad \begin{array}{c} \delta' \Vert \text{MLS} \\ L_4 \parr F'\{A\} \end{array}$$

Apply IH to δ'. Get:

$$\begin{array}{c} K \parr X \\ \delta'_X \Vert \text{MLS} \\ L_4 \parr F'\{X\} \end{array} \quad \text{and} \quad \begin{array}{c} \delta_A \Vert \text{MLS} \\ K \parr A \end{array}$$

Put everything together.

- **Exercise 5.10:** Put everything together, to show the derivation

$$\begin{array}{c} K \parr X \\ \delta_X \Vert \text{MLS} \\ F\{X\} \end{array}$$

Cut elimination for SMLS

Lemma (Reduction Lemma): If $\begin{array}{c} \Vert \text{MLS} \\ F\{a \otimes \bar{a}\} \end{array}$ then $\begin{array}{c} \Vert \text{MLS} \\ F\{\bot\} \end{array}$.

Proof: By context reduction we have:

$$\begin{array}{c} K \parr \bot \\ \delta_1 \Vert \text{MLS} \\ F\{\bot\} \end{array} \quad \text{and} \quad \begin{array}{c} \delta_2 \Vert \text{MLS} \\ K \parr (a \otimes a^\bot) \end{array}$$

Apply splitting to δ_2:

$$\begin{array}{c} K_a \parr K_{a^\bot} \\ \delta_3 \Vert \text{MLS} \\ K \end{array} \qquad \begin{array}{c} \delta_4 \Vert \text{MLS} \\ K_a \parr a \end{array} \qquad \begin{array}{c} \delta_5 \Vert \text{MLS} \\ K_{a^\bot} \parr a^\bot \end{array}$$

and again:

$$\begin{array}{c} a^\bot \\ \delta_6 \Vert \text{MLS} \\ K_a \end{array} \qquad \begin{array}{c} a \\ \delta_7 \Vert \text{MLS} \\ K_{a^\bot} \end{array}$$

Put $\delta_6, \delta_7, \delta_3, \delta_1$ together to get a proof of $F\{\bot\}$ □

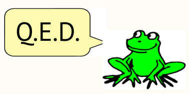
- **Exercise 5.11:** Put $\delta_6, \delta_7, \delta_3, \delta_1$ together to get

$$\begin{array}{c} \Vert \text{MLS} \\ F\{\bot\} \end{array} .$$

Cut elimination for SMLS

Theorem (Cut ELimination): If $\begin{array}{c} \delta \Vert \text{MLS+ai}\uparrow \\ A \end{array}$ then $\begin{array}{c} \delta' \Vert \text{MLS} \\ A \end{array}$.

Proof: By induction on the number of occurrences of ai↑ in δ', using the Reduction Lemma.



Yet another deep inference system

Formulas:

$$A, B ::= a \mid a^{\perp} \mid 1 \mid \perp \mid A \otimes B \mid A \parr B \mid !A \mid ?A$$

Negation:

$$a^{\perp\perp} = a \qquad 1^{\perp} = \perp \qquad (A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp} \qquad (!A)^{\perp} = ?(A^{\perp})$$

$$\perp^{\perp} = 1 \qquad (A \parr b)^{\perp} = A^{\perp} \otimes B^{\perp} \qquad (?A)^{\perp} = !(A^{\perp})$$

Equivalences:

$$(A \parr B) \parr C \equiv A \parr (B \parr C) \qquad A \parr B \equiv B \parr C \qquad A \parr \perp \equiv A$$

$$(A \otimes B) \otimes C \equiv A \otimes (B \otimes C) \qquad A \otimes B \equiv B \otimes C \qquad A \otimes 1 \equiv A$$

- The logic we present here is called *Multiplicative Exponential Linear Logic (MELL)*.
- The sequent calculus for MELL is

$$\mathsf{id}\,\frac{}{\vdash a^{\perp}, a} \qquad \perp\,\frac{\Gamma}{\vdash \Gamma, \perp} \qquad 1\,\frac{}{\vdash 1}$$

$$\parr\,\frac{\Gamma, A, B}{\vdash \Gamma, A \parr B} \qquad \otimes\,\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}$$

$$!\,\frac{\vdash ?B_1, \ldots, ?B_n, A}{\vdash ?B_1, \ldots, ?B_n, !A} \qquad \mathsf{dr}\,\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}$$

$$\mathsf{wk}\,\frac{\vdash \Gamma}{\vdash \Gamma, ?A} \qquad \mathsf{ct}\,\frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}$$

$$\mathsf{cut}\,\frac{\vdash \Gamma, A \quad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}$$

- **Exercise 5.12:** (Hard) Prove cut elimination for the sequent calculus for MELL.

Yet another deep inference system

Rules:

$$\mathsf{ai}{\downarrow}\,\frac{1}{a^{\perp} \parr a} \qquad \equiv\,\frac{A}{B}\ (\text{provided } A \equiv B) \qquad \mathsf{ai}{\uparrow}\,\frac{a^{\perp} \otimes a}{\perp}$$

$$\mathsf{p}{\downarrow}\,\frac{!(A \parr B)}{!A \parr ?B} \qquad \mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \qquad \mathsf{p}{\uparrow}\,\frac{?A \otimes !B}{?(A \otimes B)}$$

$$\mathsf{g}{\downarrow}\,\frac{??A}{?A} \qquad \mathsf{e}{\downarrow}\,\frac{1}{!1} \qquad \mathsf{e}{\uparrow}\,\frac{?\perp}{\perp} \qquad \mathsf{g}{\uparrow}\,\frac{!A}{!!A}$$

$$\mathsf{b}{\downarrow}\,\frac{?A \parr A}{?A} \qquad \mathsf{w}{\downarrow}\,\frac{\perp}{?A} \qquad \mathsf{w}{\uparrow}\,\frac{!A}{1} \qquad \mathsf{b}{\uparrow}\,\frac{!A}{!A \otimes A}$$

System SELS: all rules

System ELS: all ↓-rules + $\{\mathsf{s}, \equiv\}$

Recall: Cut elimination means that the up-fragment is admissible

- We call this system SELS. The system consisting of only the down rules (the ones with a ↓ in the name) together with s and ≡ is called ELS. Both have been studied in
  - Alessio Gugliemi and Lutz Straßburger: **"Non-commutativity and MELL in the Calculus of Structures".** *CSL 2001*
  - Lutz Straßburger: **"Linear Logic and Noncommutativity in the Calculus of Structures".** *PhD Thesis, 2003*
  - Lutz Straßburger: **"MELL in the Calculus of Structures".** *TCS 2003*
- **Exercise 5.13:** Use the previous exercise is to prove cut elimination for SELS: First show how to translate an SELS derivation into the sequent calculus for MELL, and second, show how a cut-free sequent proof in MELL is translated into a ELS-derivation.

Properties of SELS and ELS

Theorem:

The rules $\mathsf{i}{\downarrow}\,\dfrac{1}{A^{\perp} \parr A}$ and $\mathsf{i}{\uparrow}\,\dfrac{A^{\perp} \otimes A}{\perp}$ are derivable in SELS.

Theorem:

- Every rule $\mathsf{r}{\uparrow}$ is derivable in $\{\mathsf{r}{\downarrow}, \mathsf{i}{\downarrow}, \mathsf{i}{\uparrow}, \mathsf{s}, \equiv\}$.
- Every rule $\mathsf{r}{\downarrow}$ is derivable in $\{\mathsf{r}{\uparrow}, \mathsf{i}{\downarrow}, \mathsf{i}{\uparrow}, \mathsf{s}, \equiv\}$.

Theorem:

$$\begin{array}{c} A \\ \| \,\mathsf{SELS} \\ B \end{array} \quad \text{iff} \quad \begin{array}{c} B^{\perp} \\ \| \,\mathsf{SELS} \\ A^{\perp} \end{array} \quad \text{iff} \quad \begin{array}{c} \| \,\mathsf{ELS} \\ A^{\perp} \parr B \end{array}$$

- These properties hold for every well-designed deep inference system. And the proofs are essentially the same for all systems.
- **Exercise 5.14:** Prove these three theorems. (Hint: Cut elimination is not needed.)

Cut elimination for SELS

Theorem: Systems SELS and ELS are equivalent.

Theorem: The i$\uparrow$ is admissible for ELS.

Theorem: All $\uparrow$-rules of SELS are admissible for ELS.

Three different ways of saying the same thing.

But how to prove it?

- **Exercise 5.15:** Show that these three theorems imply each other.

Systems SELS and ELS

Rules:

$$\mathsf{ai}\!\downarrow \frac{1}{a^\perp \parr a} \qquad \equiv \frac{A}{B} \text{ (provided } A \equiv B) \qquad \mathsf{ai}\!\uparrow \frac{a^\perp \otimes a}{\perp}$$

$$\mathsf{p}\!\downarrow \frac{!(A \parr B)}{!A \parr ?B} \qquad \mathsf{s} \frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \qquad \mathsf{p}\!\uparrow \frac{?A \otimes !B}{?(A \otimes B)}$$

$$\mathsf{g}\!\downarrow \frac{??A}{?A} \qquad \mathsf{e}\!\downarrow \frac{1}{!1} \qquad \mathsf{e}\!\uparrow \frac{?\perp}{\perp} \qquad \mathsf{g}\!\uparrow \frac{!A}{!!A}$$

$$\mathsf{b}\!\downarrow \frac{?A \parr A}{?A} \qquad \mathsf{w}\!\downarrow \frac{\perp}{?A} \qquad \mathsf{w}\!\uparrow \frac{!A}{1} \qquad \mathsf{b}\!\uparrow \frac{!A}{!A \otimes A}$$

System SELS: all rules

System ELS: all $\downarrow$-rules + $\{\mathsf{s}, \equiv\}$

Core: first two lines + $\{\mathsf{e}\!\downarrow, \mathsf{e}\!\uparrow\}$

Splitting for ELS

Lemma (Splitting):

1. If there is a proof $\begin{array}{l} \delta \Big\| \{\mathsf{e}\!\downarrow,\mathsf{ai}\!\downarrow,\mathsf{s},\mathsf{p}\!\downarrow,\equiv\} \\ K \parr (A \otimes B) \end{array}$,

   then there are formulas K_A and K_B and derivations

   $\begin{array}{c} K_A \parr K_B \\ \delta_K \Big\| \{\mathsf{e}\!\downarrow,\mathsf{ai}\!\downarrow,\mathsf{s},\mathsf{p}\!\downarrow,\equiv\} \\ K \end{array}$ and $\begin{array}{l} \delta_A \Big\| \{\mathsf{e}\!\downarrow,\mathsf{ai}\!\downarrow,\mathsf{s},\mathsf{p}\!\downarrow,\equiv\} \\ K_A \parr A \end{array}$ and $\begin{array}{l} \delta_B \Big\| \{\mathsf{e}\!\downarrow,\mathsf{ai}\!\downarrow,\mathsf{s},\mathsf{p}\!\downarrow,\equiv\} \\ K_B \parr B \end{array}$.

2. If there is a proof $\begin{array}{l} \delta \Big\| \{\mathsf{e}\!\downarrow,\mathsf{ai}\!\downarrow,\mathsf{s},\mathsf{p}\!\downarrow,\equiv\} \\ K \parr a \end{array}$,

   then there is a derivation

   $\begin{array}{c} a^\perp \\ \delta_a \Big\| \{\mathsf{e}\!\downarrow,\mathsf{ai}\!\downarrow,\mathsf{s},\mathsf{p}\!\downarrow,\equiv\} \\ K \end{array}$.

- Splitting (and context reduction) holds exactly for the core-fragment of the systems
- And that's why it's called *core-fragment*.

Splitting for ELS (cont.)

3. If there is a proof $\begin{array}{l}\delta \Vert \{\mathsf{e}\downarrow,\mathsf{ai}\downarrow,\mathsf{s},\mathsf{p}\downarrow,\equiv\} \\ K \parr\, !A\end{array}$,

   then there are formulas $K_1, \ldots, K_n$ and derivations

   $$\begin{array}{c} ?K_1 \parr \cdots \parr ?K_n \\ \delta_K \Vert \{\mathsf{e}\downarrow,\mathsf{ai}\downarrow,\mathsf{s},\mathsf{p}\downarrow,\equiv\} \\ K \end{array} \quad \text{and} \quad \begin{array}{l} \delta_A \Vert \{\mathsf{e}\downarrow,\mathsf{ai}\downarrow,\mathsf{s},\mathsf{p}\downarrow,\equiv\} \\ K_1 \parr \cdots \parr K_n \parr A \end{array} .$$

4. If there is a proof $\begin{array}{l}\delta \Vert \{\mathsf{e}\downarrow,\mathsf{ai}\downarrow,\mathsf{s},\mathsf{p}\downarrow,\equiv\} \\ K \parr\, ?A\end{array}$,

   then there is a formula K_A and derivations

   $$\begin{array}{c} !K_A \\ \delta_K \Vert \{\mathsf{e}\downarrow,\mathsf{ai}\downarrow,\mathsf{s},\mathsf{p}\downarrow,\equiv\} \\ K \end{array} \quad \text{and} \quad \begin{array}{l} \delta_A \Vert \{\mathsf{e}\downarrow,\mathsf{ai}\downarrow,\mathsf{s},\mathsf{p}\downarrow,\equiv\} \\ K_A \parr A \end{array} .$$

- **Exercise 5.16:** Prove splitting for ELS.

Context reduction for ELS

Lemma: Let A be a formula and $F\{\ \}$ be a context. If there is a derivation

$$\begin{array}{l} \delta \Vert \{\mathsf{e}\downarrow,\mathsf{ai}\downarrow,\mathsf{s},\mathsf{p}\downarrow,\equiv\} \\ F\{A\} \end{array}$$

then there is a formula K, such that for all X, we have

$$\begin{array}{c} ! \cdots !(K \parr X) \\ \delta_X \Vert \{\mathsf{e}\downarrow,\mathsf{ai}\downarrow,\mathsf{s},\mathsf{p}\downarrow,\equiv\} \\ F\{X\} \end{array} \quad \text{and} \quad \begin{array}{l} \delta_A \Vert \{\mathsf{e}\downarrow,\mathsf{ai}\downarrow,\mathsf{s},\mathsf{p}\downarrow,\equiv\} \\ K \parr A \end{array}$$

Proof: By induction on $F\{\ \}$.

- The number of ! in front of the $K \parr X$ is the modality depth of $F\{\ \}$.
- **Exercise 5.17:** Prove context reduction for ELS.
- **Exercise 5.18:** Use context reduction and splitting to show that $\mathsf{ai}\uparrow$ and $\mathsf{p}\uparrow$ and $\mathsf{e}\uparrow$ are admissible for $\{\mathsf{e}\downarrow, \mathsf{ai}\downarrow, \mathsf{s}, \mathsf{p}\downarrow, \equiv\}$. (Hint: this is very similar to the case of MLS.)

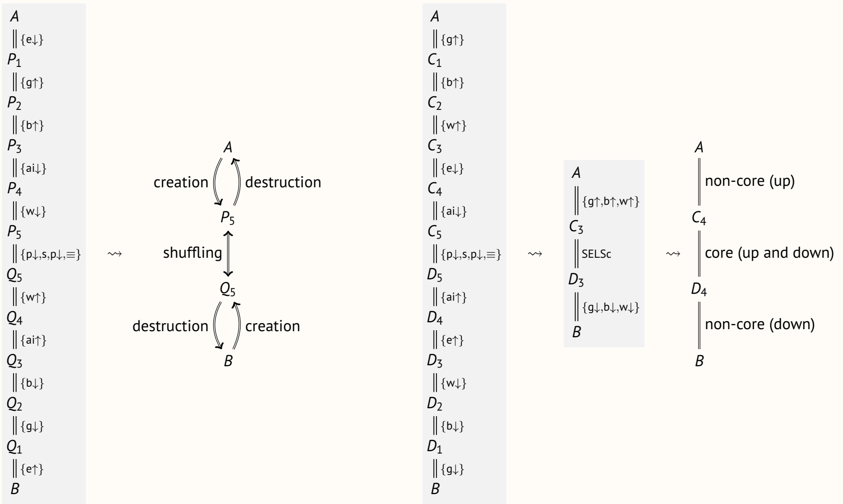
Decomposition

Theorem: For every derivation $\begin{array}{c} A \\ \delta \Vert \mathsf{SELS} \\ B \end{array}$ there are derivations

$$\begin{array}{l} A \\ \Vert\{\mathsf{e}\downarrow\} \\ P_1 \\ \Vert\{\mathsf{g}\uparrow\} \\ P_2 \\ \Vert\{\mathsf{b}\uparrow\} \\ P_3 \\ \Vert\{\mathsf{ai}\downarrow\} \\ P_4 \\ \Vert\{\mathsf{w}\downarrow\} \\ P_5 \\ \Vert\{\mathsf{p}\downarrow,\mathsf{s},\mathsf{p}\downarrow,\equiv\} \\ Q_5 \\ \Vert\{\mathsf{w}\uparrow\} \\ Q_4 \\ \Vert\{\mathsf{ai}\uparrow\} \\ Q_3 \\ \Vert\{\mathsf{b}\downarrow\} \\ Q_2 \\ \Vert\{\mathsf{g}\downarrow\} \\ Q_1 \\ \Vert\{\mathsf{e}\uparrow\} \\ B \end{array} \quad \text{and} \quad \begin{array}{l} A \\ \Vert\{\mathsf{g}\uparrow\} \\ X_1 \\ \Vert\{\mathsf{b}\uparrow\} \\ X_2 \\ \Vert\{\mathsf{e}\downarrow\} \\ X_3 \\ \Vert\{\mathsf{w}\downarrow\} \\ X_4 \\ \Vert\{\mathsf{ai}\downarrow\} \\ X_5 \\ \Vert\{\mathsf{p}\downarrow,\mathsf{s},\mathsf{p}\downarrow,\equiv\} \\ Y_5 \\ \Vert\{\mathsf{ai}\uparrow\} \\ Y_4 \\ \Vert\{\mathsf{w}\uparrow\} \\ Y_3 \\ \Vert\{\mathsf{e}\uparrow\} \\ Y_2 \\ \Vert\{\mathsf{b}\downarrow\} \\ Y_1 \\ \Vert\{\mathsf{g}\downarrow\} \\ B \end{array} \quad \text{and} \quad \begin{array}{l} A \\ \Vert\{\mathsf{e}\downarrow\} \\ F_1 \\ \Vert\{\mathsf{g}\uparrow\} \\ F_2 \\ \Vert\{\mathsf{b}\uparrow\} \\ F_3 \\ \Vert\{\mathsf{w}\uparrow\} \\ F_4 \\ \Vert\{\mathsf{ai}\downarrow\} \\ F_5 \\ \Vert\{\mathsf{p}\downarrow,\mathsf{s},\mathsf{p}\downarrow,\equiv\} \\ G_5 \\ \Vert\{\mathsf{ai}\uparrow\} \\ G_4 \\ \Vert\{\mathsf{w}\downarrow\} \\ G_3 \\ \Vert\{\mathsf{b}\downarrow\} \\ G_2 \\ \Vert\{\mathsf{g}\downarrow\} \\ G_1 \\ \Vert\{\mathsf{e}\uparrow\} \\ B \end{array} \quad \text{and} \quad \begin{array}{l} A \\ \Vert\{\mathsf{g}\uparrow\} \\ C_1 \\ \Vert\{\mathsf{b}\uparrow\} \\ C_2 \\ \Vert\{\mathsf{w}\uparrow\} \\ C_3 \\ \Vert\{\mathsf{e}\downarrow\} \\ C_4 \\ \Vert\{\mathsf{ai}\downarrow\} \\ C_5 \\ \Vert\{\mathsf{p}\downarrow,\mathsf{s},\mathsf{p}\downarrow,\equiv\} \\ D_5 \\ \Vert\{\mathsf{ai}\uparrow\} \\ D_4 \\ \Vert\{\mathsf{e}\uparrow\} \\ D_3 \\ \Vert\{\mathsf{w}\downarrow\} \\ D_2 \\ \Vert\{\mathsf{b}\downarrow\} \\ D_1 \\ \Vert\{\mathsf{g}\downarrow\} \\ B \end{array}$$

- These four theorems have been investigated in
  - Alessio Guglielmi and Lutz Straßburger: **"Non-commutativity and MELL in the Calculus of Structures".** *CSL 2001*
  - Lutz Straßburger: **"Linear Logic and Noncommutativity in the Calculus of Structures".** *PhD Thesis, 2003*
  - Lutz Straßburger: **"MELL in the Calculus of Structures".** *TCS 2003*
  - Alessio Guglielmi and Lutz Straßburger: **"A System of Interaction and Structure IV: The Exponentials and Decomposition".** *ACM ToCL 12(4:23), 2011*

Decomposition theorems

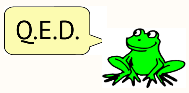


Cut elimination for SELS (finally)

Theorem: If $\begin{array}{c} \Vert \mathsf{SELS} \\ A \end{array}$ then $\begin{array}{c} \Vert \mathsf{ELS} \\ A \end{array}$.

Proof:

$$\begin{array}{c} 1 \\ \Vert \mathsf{SELS} \\ A \end{array} \rightsquigarrow \begin{array}{c} 1 \\ \Vert \{g\uparrow,b\uparrow,w\uparrow\} \\ A'' \\ \Vert \mathsf{SELSc} \\ A' \\ \Vert \{g\downarrow,b\downarrow,w\downarrow\} \\ A \end{array} \rightsquigarrow \begin{array}{c} 1 \\ \Vert \mathsf{SELSc} \\ A' \\ \Vert \{g\downarrow,b\downarrow,w\downarrow\} \\ A \end{array} \rightsquigarrow \begin{array}{c} 1 \\ \Vert \{e\downarrow,ai\downarrow,s,p\downarrow,\equiv\} \\ A' \\ \Vert \{g\downarrow,b\downarrow,w\downarrow\} \\ A \end{array} \rightsquigarrow \begin{array}{c} 1 \\ \Vert \mathsf{ELS} \\ A \end{array}$$



- The first step is decomposition.
- Then, inspecting the rules $e\uparrow, g\uparrow, b\uparrow$ shows that $A'' = 1$.
- Now we have a derivation of A' in SELSc, and we can eliminate $e\uparrow$, $p\uparrow$, and $ai\uparrow$ using context reduction and splitting.
- **Exercise 5.19:** If you did not yet do the previous exercise, do it now.
- Finally, we habe a derivation of A' in $\{e\downarrow, ai\downarrow, s, p\downarrow, \equiv\}$, and therefore a derivation of A in ELS.