A Gentle Introduction to Deep Inference Handout for 4. Lecture - Cut Elimination in Deep Inference for Propositional Classical Logic

Victoria Barrett and Lutz Straßburger

July 2025

1 Recall:

1.1 Proof system SKS

The proof system SKS [2, 3] consists of the following inference rules:

- The ***structural rules***:

$$\textit{atomic}\left\{\begin{array}{ccc} \mathsf{ai}{\downarrow}\,\dfrac{\top}{a \vee \bar{a}} & \mathsf{ac}{\downarrow}\,\dfrac{a \vee a}{a} & \mathsf{aw}{\downarrow}\,\dfrac{\bot}{a} \\ \textit{identity} & \textit{contraction} & \textit{weakening} \\[1ex] \mathsf{ai}{\uparrow}\,\dfrac{a \wedge \bar{a}}{\bot} & \mathsf{ac}{\uparrow}\,\dfrac{a}{a \wedge a} & \mathsf{aw}{\uparrow}\,\dfrac{a}{\top} \\ \textit{cut} & \textit{cocontraction} & \textit{coweakening} \end{array}\right\}$$

- The ***logical rules***:

$$\begin{array}{cc} \mathsf{s}\,\dfrac{A \wedge (B \vee C)}{(A \wedge B) \vee C} & \mathsf{m}\,\dfrac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)} \\ \textit{switch} & \textit{medial} \end{array}$$

- An equivalence on formulae, defined to be the minimal equivalence relation generated by the following:

$$\begin{array}{cc} A \wedge \top = A & A \wedge B = B \wedge A \\ A \vee \bot = A & A \vee B = B \vee A \\[1ex] \top \vee \top = \top & A \wedge (B \wedge C) = (A \wedge B) \wedge C \\ \bot \wedge \bot = \bot & A \vee (B \vee C) = (A \vee B) \vee C \end{array}$$

We write $=\frac{A}{B}$ whenever B can be reached from A via this equivalence relation.

A ***proof*** in SKS is a derivation in SKS with premise $\top$. And a formula A is ***provable*** in SKS if there is an SKS proof with conclusion A.

1.2 Proof system SKSg

The proof system SKSg relaxes the condition that the structural rules should be atomic. It consists of the following inference rules:

- The ***structural rules***:

$$\mathsf{i}{\downarrow}\,\frac{\top}{A \vee \bar{A}} \quad \mathsf{c}{\downarrow}\,\frac{A \vee A}{A} \quad \mathsf{w}{\downarrow}\,\frac{\bot}{A}$$

identity *contraction* *weakening*

$$\mathsf{i}{\uparrow}\,\frac{A \wedge \bar{A}}{\bot} \quad \mathsf{c}{\uparrow}\,\frac{A}{A \wedge A} \quad \mathsf{w}{\uparrow}\,\frac{A}{\top}$$

cut *cocontraction* *coweakening*

- The ***logical rule***:

$$\mathsf{s}\,\frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$

switch

- The same equivalence relation as for SKS.

1.3 Proof system KS

The proof system KS is obtained from SKS by removing the *up-rules*; that is, the cut, cocontraction, and coweakening rules.

We are going to show that any proof in SKS can be transformed to one in KS.

2 Some transformations

2.1 Eliminating cocontraction and coweakening

Cocontraction and coweakening can both be derived from their dual rule, switch, cut, identity, and =. The case for coweakening is as follows:

$$
\mathsf{aw}{\uparrow}\,\frac{a}{\top}
\quad\longmapsto\quad
\begin{array}{rc}
= & \dfrac{a}{a \wedge \boxed{=\dfrac{\top}{\bot \vee \top}}} \\[2ex]
\mathsf{s} & \overline{\boxed{\mathsf{ai}{\uparrow}\,\dfrac{a \wedge \boxed{\mathsf{aw}{\downarrow}\,\dfrac{\bot}{\bar{a}}}}{\bot}} \vee \top} \\[2ex]
= & \overline{\top}
\end{array}
$$

Exercise 1: Show that $\mathsf{ac}{\uparrow}$ can be derived from $\mathsf{ac}{\downarrow}$, $\mathsf{i}{\uparrow}$, $\mathsf{i}{\downarrow}$, s and $=$.

2.2 Reducing the context around a cut

We can express a derivation in which a cut $\mathsf{ai}{\uparrow}\,\dfrac{a \wedge \bar{a}}{\bot}$ appears as

$$
\Phi = \begin{array}{c} \Psi \\ \cdots\cdots\cdots \\ K\left\{ \mathsf{ai}{\uparrow}\,\dfrac{a \wedge \bar{a}}{\bot} \right\} \\ \cdots\cdots\cdots \\ \Theta \end{array}
$$

We call $K\{\ \}$ a **context**, and it represents a formula with a hole, which can be filled by a derivation. $K\{a \wedge \bar{a}\}$ and $K\{\bot\}$ are both formulae, and the cut rule can be applied deep inside the formula. $K\{a \wedge \bar{a}\}$ is the conclusion of Ψ and $K\{\bot\}$ is the premise of Θ; the dotted line composition is a meta-level notation which allows us to focus on a particular part of a derivation.

To simplify our cut elimination procedure, we would like to transform this into a derivation where the cut appears as shallowly as possible. That is, we would like to transform this into a derivation

$$
\Phi' = \begin{array}{c} \Psi \\ \cdots\cdots\cdots \\ K\{a \wedge \bar{a}\} \\ \| \\ \boxed{\boxed{\mathsf{ai}{\uparrow}\,\dfrac{a \wedge \bar{a}}{\bot}} \vee R} \\ \| \\ K\{\bot\} \\ \cdots\cdots\cdots \\ \Theta \end{array}
$$

for some formula R determined by $K\{\ \}$. We say that the cut is *shallow* in Φ' and call $\{\ \} \vee C$ a *shallow disjunctive context* in Φ'. Note that we need to mention the whole derivation Φ' here: what is a shallow disjunctive context in one derivation isn't in another.

There are three inductive cases to consider:

1. $K\{\ \} = K'\{\{\ \} \wedge D\}$
2. $K\{\ \} = K'\{(\{\ \} \vee C) \wedge D\}$
3. $K\{\ \} = K'\{(\{\ \} \vee C) \vee D\}$

In the first case, we perform the following local transformation:

$$\boxed{\mathsf{ai}\!\uparrow \frac{a \wedge \bar{a}}{\bot}} \wedge D \qquad \longmapsto \qquad \boxed{= \dfrac{(a \wedge \bar{a}) \wedge \boxed{\mathsf{w}\!\uparrow \dfrac{D}{\top}}}{= \dfrac{\mathsf{ai}\!\uparrow \dfrac{a \wedge \bar{a}}{\bot}}{\bot \wedge \boxed{\mathsf{w}\!\downarrow \dfrac{\bot}{R}}}}}$$

Note that this produces a non-atomic weakening and coweakening; we will need to continue by using w↑, i↑, i↓, s, and = to eliminate the coweakening, and then use aw↓, ai↓, s, m, and = to eliminate the non-atomic weakenings, identity, and cut. This transformation can be done locally, and (importantly for our proof by induction) does not produce any further instances of ai↑ whose parent connective is a $\wedge$.

In the second case, we perform the following local transformation:

$$\left(\boxed{\mathsf{ai}\!\uparrow \frac{a \wedge \bar{a}}{\bot}} \vee C\right) \wedge D \qquad \longmapsto \qquad \boxed{\mathsf{s}\, \dfrac{((a \wedge \bar{a}) \vee C) \wedge D}{= \dfrac{\boxed{\mathsf{ai}\!\uparrow \dfrac{a \wedge \bar{a}}{\bot}} \vee (C \wedge D)}{\boxed{= \dfrac{C}{\bot \vee C}} \wedge D}}}$$

In the third case, we simply apply the commutativity of $\vee$.

Exercise 2: Using the transformations given above, give a full proof that any SKS proof can be transformed into one in which all of the cuts are shallow.

2.3 Propagating a new formula up through an atom

This transformation is essential to our procedure for eliminating the cuts. You can refer to the lecture slides to see a further example of how this procedure works.

Given a KS proof Π of a formula $K\{a\}$, we are able to trace the history of that atom through the proof. This trace goes up through linear rules and equality rules, branches out at contractions, and terminates at weakenings and identities.

Moreover, we are able to replace every occurrence of the atom in that trace by the unit $\top$ in such a way that the proof does not get broken. Note that it is crucial here that we are not replacing every occurrence of a in Π, but only those in the trace of the occurrence of a in the conclusion that we are focussing on.

The transformation is as follows:

$$\rho\,\frac{K\{a\}}{H\{a\}} \qquad\longmapsto\qquad \frac{K\{\top\}}{H\{\top\}}$$

$$\mathsf{ai}{\downarrow}\,\frac{\top}{a \vee \bar{a}} \qquad\longmapsto\qquad =\frac{\top}{\top \vee \boxed{\mathsf{aw}{\downarrow}\,\frac{\bot}{\bar{a}}}}$$

$$\mathsf{ac}{\downarrow}\,\frac{a \vee a}{a} \qquad\longmapsto\qquad =\frac{\top \vee \top}{\top}$$

$$\mathsf{aw}{\downarrow}\,\frac{\bot}{a} \qquad\longmapsto\qquad =\frac{\mathsf{s}\,\dfrac{=\dfrac{\bot}{\bot \wedge \boxed{=\frac{\top}{\top \wedge \top}}}}{\boxed{=\frac{\bot \wedge \top}{\bot}} \vee \top}}{\top}$$

where $\rho \in \{\mathsf{m}, \mathsf{s}, =\}$

We can propagate formulae other than $\top$ up through proofs in this way; however in general this breaks identity rules and so the resulting object is not a proof. If we tried to propagate a formula C in the same way, the case for $\mathsf{ai}{\downarrow}$ would be:

$$\mathsf{ai}{\downarrow}\,\frac{\top}{a \vee \bar{a}} \qquad\longmapsto\qquad X\,\frac{\top}{C \vee \bar{a}}$$

3 The Cut Elimination Procedure

We are given a proof Φ in SKS. We can transform this into a proof in KS as follows:

1. Eliminate $\mathsf{ac}{\uparrow}$ and $\mathsf{aw}{\uparrow}$ by means of their dual rules, switch, $\mathsf{ai}{\downarrow}$, and $\mathsf{ai}{\uparrow}$.
2. Reduce the context around each cut so that each cut appears in a disjunctive shallow context. Call the resulting SKS proof Ψ.

3. Isolate the top-most cut in Ψ to obtain a KS proof Π of $(a \wedge \bar{a}) \vee C$, for some atom a and formula C.

4. Duplicate Π. In one copy, choose the occurrence of a in the conclusion which is to be cut, and replace it by $\top$ as described in Subsection 2.3 to obtain a KS proof $\Pi_{\bar{a}}$ of $C \vee \bar{a}$. In the other copy, choose the occurrence of $\bar{a}$ in the conclusion which is to be cut, and similarly obtain a KS proof Π_a of $C \vee a$.

5. Take the proof Π_a, and replace the atom $\bar{a}$ in its conclusion by C. Propagate this up through the proof. Fix each broken identity $x \frac{\top}{C \vee \bar{a}}$ by replacing it with the proof $\Pi_{\bar{a}}$. This results in a KS proof of $C \vee C$, in which there are as many copies of $\Pi_{\bar{a}}$ as there are identities in the trace of $\bar{a}$ in Π_a.

6. Contract $C \vee C$ using $\mathsf{ac}\!\downarrow$, m, and $=$ to obtain a KS proof Ξ of C. Replace Π by Ξ, and continue from the next top-most cut.

4 Further Reading

The cut elimination procedure that we have seen today was first presented in Kai Brünnler's PhD thesis [2] and also in his paper [1].

Note that there the formalism used is the *calculus of structures*, not open deduction as we have been using. However, they are both self-contained in introducing the calculus of structures, and it should not be difficult to translate between the two formalisms.

References

[1] Kai Brünnler. Atomic cut elimination for classical logic. In M. Baaz and J. A. Makowsky, editors, *CSL 2003*, volume 2803 of *Lecture Notes in Computer Science*, pages 86–97. Springer-Verlag, 2003.

[2] Kai Brünnler. *Deep Inference and Symmetry for Classical Proofs.* PhD thesis, Technische Universität Dresden, 2003.

[3] Alessio Guglielmi, Tom Gundersen, and Michel Parigot. A Proof Calculus Which Reduces Syntactic Bureaucracy. In Christopher Lynch, editor, *21st International Conference on Rewriting Techniques and Applications (RTA)*, volume 6 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 135–150. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2010.