A Gentle Introduction to Deep Inference

3. Lecture

Sequent calculus cut elimination

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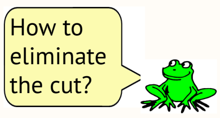
Recall: One-sided sequent calculus

$$\mathsf{id}\,\frac{}{\vdash \bar{a}, a} \qquad \top\,\frac{}{\vdash \top}$$

$$\vee_1\,\frac{\vdash \Theta, A}{\vdash \Theta, A \vee B} \qquad \vee_2\,\frac{\vdash \Theta, B}{\vdash \Theta, A \vee B} \qquad \wedge\,\frac{\vdash \Theta, A \quad \vdash \Theta, B}{\vdash \Theta, A \wedge B}$$

$$\mathsf{weak}\,\frac{\vdash \Theta}{\vdash \Theta, A} \qquad \mathsf{cont}\,\frac{\vdash \Theta, A, A}{\vdash \Theta, A}$$

$$\mathsf{cut}\,\frac{\vdash \Theta, A \quad \vdash \bar{A}, \Lambda}{\vdash \Theta, \Lambda}$$



- There are many different sequent calculi for classical logic. We show here only one (because the sequent calculus is not the main topic of this course).
- Technically, we've shown two. Yesterday we treated sequents as lists of formulas and had the exch-rule in the system. Here we drop the exch-rule and consider sequents as multisets of formulas.
- Also note that we have the id-rule in its atomic form

Basic idea of cut elimination

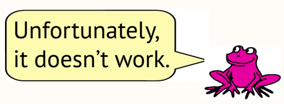
Reduce the cut. Proceed by induction on the cut formula.

- Base case:

$$\mathsf{cut}\,\frac{\mathsf{id}\,\frac{}{\vdash \bar{a}, a} \quad \overset{\pi}{\vdash \bar{a}, \Delta}}{\vdash \bar{a}, \Delta} \quad \rightsquigarrow \quad \overset{\pi}{\vdash \bar{a}, \Delta}$$

- Inductive case:

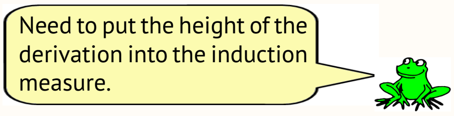
$$\mathsf{cut}\,\frac{\wedge\,\frac{\overset{\pi_1}{\vdash \Gamma, A} \quad \overset{\pi_2}{\vdash \Gamma, B}}{\vdash \Gamma, A \wedge B} \quad \vee_1\,\frac{\overset{\pi_3}{\vdash \bar{A}, \Delta}}{\vdash \bar{A} \vee \bar{B}, \Delta}}{\vdash \Gamma, \Delta} \quad \rightsquigarrow \quad \mathsf{cut}\,\frac{\overset{\pi_1}{\vdash \Gamma, A} \quad \overset{\pi_3}{\vdash \bar{A}, \Delta}}{\vdash \Gamma, \Delta}$$



- The *cut formula* is the formula A introduced by the cut.

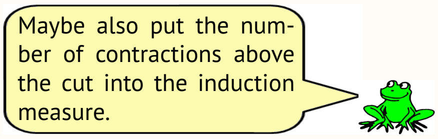
First problem: trivial rule permutations

$$\dfrac{\dfrac{\overset{\pi_1}{\vdash C, \Gamma, A \wedge B}}{\vdash C \vee D, \Gamma, A \wedge B}\,\vee_1 \qquad \overset{\pi_2}{\vdash \bar{A} \vee \bar{B}, \Delta}}{\vdash C \vee D, \Gamma, \Delta}\,\text{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\pi_1}{\vdash C, \Gamma, A \wedge B} \qquad \overset{\pi_2}{\vdash \bar{A} \vee \bar{B}, \Delta}}{\vdash C, \Gamma, \Delta}\,\text{cut}}{\vdash C \vee D, \Gamma, \Delta}\,\vee_1$$



Second problem: contraction

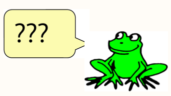
$$\dfrac{\dfrac{\overset{\pi_1}{\vdash \Gamma, A, A}}{\vdash \Gamma, A}\,\text{cont} \qquad \overset{\pi_2}{\vdash \bar{A}, \Delta}}{\vdash \Gamma, \Delta}\,\text{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdash \Gamma, A, A} \qquad \overset{\pi_2}{\vdash \bar{A}, \Delta}}{\vdash \Gamma, \Delta, A}\,\text{cut} \qquad \overset{\pi_2}{\vdash \bar{A}, \Delta}}{\vdash \Gamma, \Delta, \Delta}\,\text{cut}}{\vdash \Gamma, \Delta}\,\text{cont}$$



Second problem: contraction

$$\dfrac{\dfrac{\overset{\pi_1}{\vdash \Gamma, A, A}}{\vdash \Gamma, A}\,\text{cont} \qquad \dfrac{\overset{\pi_2}{\vdash \bar{A}, \bar{A}, \Delta}}{\vdash \bar{A}, \Delta}\,\text{cont}}{\vdash \Gamma, \Delta}\,\text{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdash \Gamma, A, A} \qquad \dfrac{\overset{\pi_2}{\vdash \bar{A}, \bar{A}, \Delta}}{\vdash \bar{A}, \Delta}\,\text{cont}}{\vdash \Gamma, \Delta, A}\,\text{cut} \qquad \dfrac{\overset{\pi_2}{\vdash \bar{A}, \bar{A}, \Delta}}{\vdash \bar{A}, \Delta}\,\text{cont}}{\vdash \Gamma, \Delta, \Delta}\,\text{cut}}{\vdash \Gamma, \Delta}\,\text{cont}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdash \Gamma, A, A} \qquad \dfrac{\overset{\pi_1}{\vdash \Gamma, A, A} \qquad \overset{\pi_2}{\vdash \bar{A}, \bar{A}, \Delta}}{\vdash \bar{A}, \Gamma, \Delta, A,}\,\text{cut}}{\vdash \Gamma, \Gamma, \Delta, A, A}\,\text{cut}}{\vdash \Gamma, \Delta, A}\,\text{cont} \qquad \dfrac{\overset{\pi_2}{\vdash \bar{A}, \bar{A}, \Delta}}{\vdash \bar{A}, \Delta}\,\text{cont}}{\vdash \Gamma, \Delta, \Delta}\,\text{cut}}{\vdash \Gamma, \Delta}\,\text{cont}$$



Solution

Gentzen's insight: multi-cut

$$\mathsf{mcut}_{n,m}\ \frac{\vdash \Gamma, \overbrace{A,\dots,A}^{n} \qquad \vdash \overbrace{\bar{A},\dots,\bar{A}}^{m}, \Delta}{\vdash \Gamma, \Delta}$$

$$\mathsf{mcut}_{n,m}\ \frac{\mathsf{cont}\ \dfrac{\overset{\pi_1}{\vdash \Gamma, A, A, \dots, A}}{\vdash \Gamma, A, \dots, A} \qquad \overset{\pi_2}{\vdash \bar{A}, \dots, \bar{A}, \Delta}}{\vdash \Gamma, \Delta} \quad \rightsquigarrow \quad \mathsf{mcut}_{n+1,m}\ \frac{\overset{\pi_1}{\vdash \Gamma, A, \dots, A} \qquad \overset{\pi_2}{\vdash \bar{A}, \dots, \bar{A}, \Delta}}{\vdash \Gamma, \Delta}$$

- in the German original, Gentzen called this rule *Mischung*. Translating that to English would give *mix*. However, there are already different other rules that are also called *mix*, and therefore, we are not using this name for Gentzen's rule here.
- For all the details, see
  - Gerhard Gentzen: **"Untersuchungen über das logische Schließen I".** *Mathematische Zeitschrift, 39:176–210, 1935*
  - Gerhard Gentzen: **"Untersuchungen über das logische Schließen II".** *Mathematische Zeitschrift, 39:405–431, 1935*

Reduction lemma

Lemma:
Assume, we have a derivation

$$\mathsf{mcut}\ \frac{\overset{\pi_1}{\vdash \Gamma, A, \dots, A} \qquad \overset{\pi_2}{\vdash \bar{A}, \dots, \bar{A}, \Delta}}{\vdash \Gamma, \Delta}$$

where π_1 and π_2 are mcut-free, then there is a mcut-free derivation

$$\overset{\pi'}{\vdash \Gamma, \Delta}$$

Proof:
By induction on the lexicographic pair $\langle |A|, \mathfrak{r}(\pi_1) + \mathfrak{r}(\pi_2) \rangle$ and a case analysis on the bottommost rules in π_1 and π_2.

- $|A|$ is the *size* of the formula A, that is, the number of symbols occurring in A.
- $\mathfrak{r}(\pi)$ is the *rank* of the derivation π, which the length of the longest path in the derivation tree of π, starting at the root, in which each occurring sequent contains at least one occurrence of the cut formula.
- the *lexicographic ordering* on a pair is defined as follows:

$$\langle x, y \rangle < \langle x', y' \rangle \iff x < x' \quad \text{or} \quad (x = x' \text{ and } y < y')$$

Reduction cases — contraction and weakening

$$\mathsf{mcut}_{n,m}\ \frac{\mathsf{cont}\ \dfrac{\overset{\pi_1'}{\vdash \Gamma, A, A, \dots, A}}{\vdash \Gamma, A, \dots, A} \qquad \overset{\pi_2}{\vdash \bar{A}, \dots, \bar{A}, \Delta}}{\vdash \Gamma, \Delta} \quad \rightsquigarrow \quad \mathsf{mcut}_{n+1,m}\ \frac{\overset{\pi_1'}{\vdash \Gamma, A, \dots, A} \qquad \overset{\pi_2}{\vdash \bar{A}, \dots, \bar{A}, \Delta}}{\vdash \Gamma, \Delta}$$

$$\mathsf{mcut}_{n,m}\ \frac{\mathsf{weak}\ \dfrac{\overset{\pi_1'}{\vdash \Gamma, A, \dots, A}}{\vdash \Gamma, A, A, \dots, A} \qquad \overset{\pi_2}{\vdash \bar{A}, \dots, \bar{A}, \Delta}}{\vdash \Gamma, \Delta} \quad \rightsquigarrow \quad \mathsf{mcut}_{n-1,m}\ \frac{\overset{\pi_1'}{\vdash \Gamma, A, \dots, A} \qquad \overset{\pi_2}{\vdash \bar{A}, \dots, \bar{A}, \Delta}}{\vdash \Gamma, \Delta}$$

$$\mathsf{mcut}_{0,m}\ \frac{\overset{\pi_1}{\vdash \Gamma} \qquad \overset{\pi_2}{\vdash \bar{A}, \dots, \bar{A}, \Delta}}{\vdash \Gamma, \Delta} \quad \rightsquigarrow \quad \mathsf{weak}\ \frac{\overset{\pi_1'}{\vdash \Gamma}}{\vdash \Gamma, \Delta}$$

- **Exercise 3.1:** Show the symmetric cases for π_2.

Reduction cases – commutative cases

$$\mathsf{mcut}_{n,m}\,\dfrac{\mathsf{r}\,\dfrac{\overset{\pi_1'}{\vdash \Gamma', A, \ldots, A}}{\vdash \Gamma, A, \ldots, A} \qquad \overset{\pi_2}{\vdash \bar{A}, \ldots, \bar{A}, \Delta}}{\vdash \Gamma, \Delta} \quad \rightsquigarrow \quad \mathsf{r}\,\dfrac{\mathsf{mcut}_{n,m}\,\dfrac{\overset{\pi_1'}{\vdash \Gamma', A, \ldots, A} \qquad \overset{\pi_2}{\vdash \bar{A}, \ldots, \bar{A}, \Delta}}{\vdash \Gamma', \Delta}}{\vdash \Gamma, \Delta}$$

$$\mathsf{mcut}_{n,m}\,\dfrac{\mathsf{r}\,\dfrac{\overset{\pi_1'}{\vdash \Gamma', A, \ldots, A} \qquad \overset{\pi_1''}{\vdash \Gamma'', A, \ldots, A}}{\vdash \Gamma, A, \ldots, A} \qquad \overset{\pi_2}{\vdash \bar{A}, \ldots, \bar{A}, \Delta}}{\vdash \Gamma, \Delta} \quad \rightsquigarrow$$

$$\mathsf{r}\,\dfrac{\mathsf{mcut}_{n,m}\,\dfrac{\overset{\pi_1'}{\vdash \Gamma', A, \ldots, A} \qquad \overset{\pi_2}{\vdash \bar{A}, \ldots, \bar{A}, \Delta}}{\vdash \Gamma', \Delta} \qquad \mathsf{mcut}_{n,m}\,\dfrac{\overset{\pi_1''}{\vdash \Gamma'', A, \ldots, A} \qquad \overset{\pi_2}{\vdash \bar{A}, \ldots, \bar{A}, \Delta}}{\vdash \Gamma'', \Delta}}{\vdash \Gamma, \Delta}$$

- **Exercise 3.2:** Show the symmetric cases for π_2.

Reduction cases – key cases

$$\mathsf{mcut}_{n,m}\,\dfrac{\vee_1\,\dfrac{\overset{\pi_1'}{\vdash \Gamma, B, B \vee C, \ldots, B \vee C}}{\vdash \Gamma, B \vee C, B \vee C, \ldots, B \vee C} \qquad \overset{\pi_2}{\vdash \bar{B} \wedge \bar{C}, \ldots, \bar{B} \wedge \bar{C}, \Delta}}{\vdash \Gamma, \Delta} \quad \rightsquigarrow$$

$$\mathsf{cont}\,\dfrac{\mathsf{mcut}_{1,m}\,\dfrac{\vee_1\,\dfrac{\mathsf{mcut}_{n-1,m}\,\dfrac{\overset{\pi_1'}{\vdash \Gamma, B, B \vee C, \ldots, B \vee C} \qquad \overset{\pi_2}{\vdash \bar{B} \wedge \bar{C}, \ldots, \bar{B} \wedge \bar{C}, \Delta}}{\vdash \Gamma, B, \Delta}}{\vdash \Gamma, B \vee C, \Delta} \qquad \overset{\pi_2}{\vdash \bar{B} \wedge \bar{C}, \ldots, \bar{B} \wedge \bar{C}, \Delta}}{\vdash \Gamma, \Delta, \Delta}}{\vdash \Gamma, \Delta}$$

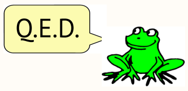
- **Exercise 3.3:** Do the case for $\vee_2$.
- **Exercise 3.4:** Do the case for $\wedge$.
- **Exercise 3.5:** Show the symmetric cases for π_2.

Reduction cases – key cases

$$\mathsf{mcut}_{1,1}\,\dfrac{\wedge\,\dfrac{\overset{\pi_1}{\vdash \Gamma, B} \qquad \overset{\pi_2}{\vdash \Gamma, C}}{\vdash \Gamma, B \wedge C} \qquad \vee_1\,\dfrac{\overset{\pi_3}{\vdash \bar{B}, \Delta}}{\vdash \bar{B} \vee \bar{C}, \Delta}}{\vdash \Gamma, \Delta} \quad \rightsquigarrow \quad \mathsf{mcut}_{1,1}\,\dfrac{\overset{\pi_1}{\vdash \Gamma, B} \qquad \overset{\pi_3}{\vdash \bar{B}, \Delta}}{\vdash \Gamma, \Delta}$$

Reduction cases — base cases

$$\dfrac{\dfrac{}{\vdash \bar{a}, a}\,\mathsf{id} \qquad \begin{matrix}\pi_2 \\ \vdash \bar{a}, \ldots, \bar{a}, \Delta\end{matrix}}{\vdash \bar{a}, \Delta}\,\mathsf{mcut}_{1,m} \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\pi_2 \\ \vdash \bar{a}, \ldots, \bar{a}, \Delta\end{matrix}}{\vdash \bar{a}, \Delta}\,\mathsf{cont}$$



- **Exercise 3.6:** Show the symmetric case where π_2 is a axiom.
- **Exercise 3.7:** What happens in the $\bot - \top$ case?
- **Exercise 3.8:** Are there other missing cases? If yes, show them. If no, explain why there are no missing cases.
- This ends the proof of the reduction lemma.
- **Exercise 3.9:** Prove the cut elimination theorem, using the reduction lemma.