# A Gentle Introduction to Deep Inference

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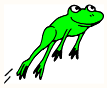



These are the slides an notes for the course *"A Gentle Introduction to Deep Inference"* given at ESSLLI 2025 in Bochum, Germany.

## Overview

1. Some 20th century proof formalisms
2. Open deduction: A 21st century proof formalism
3. Cut elimination in the sequent calculus
4. Cut elimination for classical logic in deep inference
5. Cut elimination via splitting and decomposition
6. Atomic flows
7. Combinatorial proofs
8. What is proof complexity?
9. Comparing different proof systems
10. Other proof compression mechanisms
11. Subatomic proof theory
12. Open problems



## Overview

Day 1: What is a proof system? What is a proof formalism? From 20th century proof theory to 21st century proof theory.

Day 2: Proof Normalization

Day 3: Graphical Proof Representations

Day 4: Proof Complexity

Day 5: The Future of Deep Inference

A Gentle Introduction to Deep Inference

# 1. Lecture

# Some 20th century proof formalisms

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## Modus Ponens

$$\mathsf{mp}\ \frac{A \qquad A \to B}{B}$$



- if we have $A$ and from $A$ follows $B$, then we can conclude $B$

## Syntax for Formulas

*Atoms:*

$$a, b, c, \ldots$$

*Formulas:*

$$A, B ::= \bot \mid \top \mid a \mid \neg A \mid A \vee B \mid A \wedge B \mid A \to B$$



- That way of defining syntax is called Backus-Naur-Form (BNF)

## Syntax for Formulas (Alternative)

Formulas in *negation normal form (nnf)*:

$$A, B ::= a \mid \bar{a} \mid A \vee B \mid A \wedge B \mid \bot \mid \top$$

defining negation for all formulas:

$$\bar{\bar{a}} = a \qquad \overline{A \vee B} = \bar{A} \wedge \bar{B} \qquad \overline{A \wedge B} = \bar{A} \vee \bar{B} \qquad \bar{\bot} = \top \qquad \bar{\top} = \bot$$

defining other connectives:

$$A \to B = \bar{A} \vee B \qquad A \leftrightarrow B = (A \to B) \wedge (B \to A)$$



- The equations for defining negation are also called *De Morgan dualities*
- **Exercise 1.1:** Prove that $\bar{\bar{A}} = A$ for all formulas $A$.
- There are many other equivalent ways of defining formulas.
- **Exercise 1.2:** Define $\wedge$ and $\vee$ in terms of $\to$ and $\neg$.

## A Hilbert system for classical logic

$$A \to (B \to A) \qquad (A \wedge B) \to A$$
$$(A \to (B \to C)) \to (A \to B) \to A \to C \qquad (A \wedge B) \to B$$
$$A \to (A \vee B) \qquad A \to (B \to (A \wedge B))$$
$$B \to (A \vee B) \qquad \bot \to A$$
$$(A \to C) \to (B \to C) \to ((A \vee B) \to C) \qquad \neg\neg A \to A$$

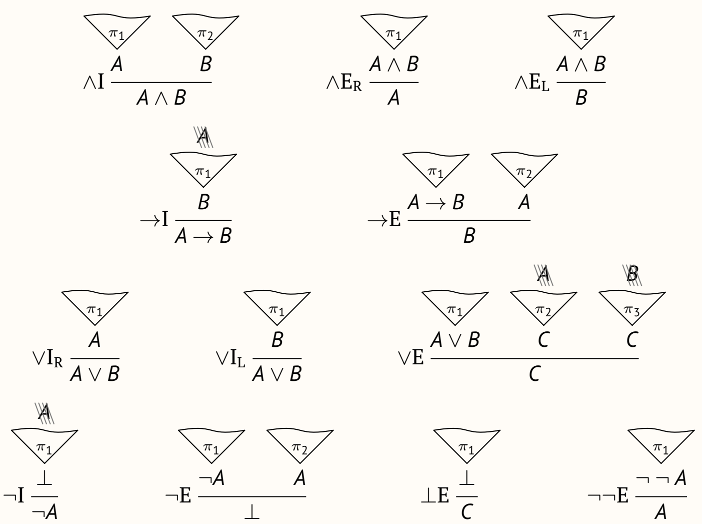
$$\text{mp}\ \frac{A \quad A \to B}{B}$$

**Theorem:** A formula $A$ is provable in the Hilbert system if and only if it is valid.



- Hilbert systems have very few rules (here only modus ponens) and many axioms
- There are many different equivalent Hilbert systems for classical propositional logic.
- They go back to at least
  - David Hilbert: **"Die logischen Grundlagen der Mathematik".** *Mathematische Annalen, 88:151–165, 1922*
- **Exercise 1.3:** Prove Pierce's law $((A \to B) \to A) \to A$ in the Hilbert system.
- **Exercise 1.4:** Show soundness: first show that every axiom is valid, and then show that modus ponens preserves validity.

## Natural deduction for classical logic






- Many rules, no axioms
- Meaning of the rules:
  - $\wedge$I: This rule is called $\wedge$-introduction, because it introduces an $\wedge$ in the conclusion. It says: if there is a proof of $A$ and a proof of $B$, then we can form a proof of $A \wedge B$ which has as assumptions the union of the assumptions of the proofs of $A$ and $B$.
  - $\to$I: This rule is called $\to$-introduction, because introduces an $\to$. It says that if we can prove $B$ under the assumption $A$, then we can prove $A \to B$ without that assumption. The notation $A$ (struck out) simply says that $A$ had been removed from the list of assumptions.
  - $\to$E: This rule is called $\to$-elimination, because it eliminates an $\to$. It is exactly the same as modus ponens.
- **Exercise 1.5:** Find similar explanations for the other rules.
- Natural deduction has already been investigated by
  - Gerhard Gentzen: **"Untersuchungen über das logische Schließen I".** *Mathematische Zeitschrift, 39:176–210, 1935*

  but it is probably older.

## Natural deduction for classical logic

**Theorem:** A formula $A$ is provable in natural deduction if and only if it is valid.

**Example:**

$$\dfrac{\dfrac{A\lor(B\land C) \quad \dfrac{\dfrac{A}{A\lor B}\,{\lor}\mathrm{I_R} \quad \dfrac{A}{A\lor C}\,{\lor}\mathrm{I_R}}{(A\lor B)\land(A\lor C)}\,{\land}\mathrm{I} \quad \dfrac{\dfrac{\dfrac{B\land C}{B}\,{\land}\mathrm{E_R}}{A\lor B}\,{\lor}\mathrm{I_R} \quad \dfrac{\dfrac{B\land C}{C}\,{\land}\mathrm{E_L}}{A\lor C}\,{\lor}\mathrm{I_R}}{(A\lor B)\land(A\lor C)}\,{\land}\mathrm{I}}{(A\lor B)\land(A\lor C)}\,{\lor}\mathrm{E}}{(A\lor(B\land C))\to((A\lor B)\land(A\lor C))}\,{\to}\mathrm{I}$$



- Informally, we can read this proof as follows: We want to prove $(A\lor(B\land C))\to((A\lor B)\land(A\lor C))$. We assume $A\lor(B\land C)$. There are two cases: We have $A$ or we have $B\land C$. In the first case we can conclude $A\lor B$ as well as $A\lor C$, and therefore also $(A\lor B)\land(A\lor C)$. In the second case we can conclude $B$ and $C$, and therefore also $A\lor B$ as well as $A\lor C$, from which we get $(A\lor B)\land(A\lor C)$. We have therefore shown $(A\lor B)\land(A\lor C)$ from the assumption $A\lor(B\land C)$, and we can conclude $(A\lor(B\land C))\to((A\lor B)\land(A\lor C))$.
- **Exercise 1.6:** Prove also the other implication.
- **Exercise 1.7:** Prove the axioms of the Hilbert system in the natural deduction system.

## Sequent calculus (two-sided version)

*Sequents:*

$$A_1, A_2, \ldots, A_n \vdash B_1, B_2, \ldots, B_m$$

Meaning:

$$(A_1\land A_2\land\cdots\land A_m)\to(B_1\lor B_2\lor\cdots\lor B_m)$$

A comma means different things, depending on where it stands



- Gentzen's insight: use structural connectives

## Sequent calculus (two-sided version, rules for LK)

$$\bot\ \dfrac{}{\bot\vdash} \qquad \mathsf{id}\ \dfrac{}{A\vdash A} \qquad \top\ \dfrac{}{\vdash\top}$$

$$\lor\mathrm{L}\ \dfrac{A,\Gamma\vdash\Theta \quad B,\Gamma\vdash\Theta}{A\lor B,\Gamma\vdash\Theta} \qquad \lor\mathrm{R}_1\ \dfrac{\Gamma\vdash\Theta,A}{\Gamma\vdash\Theta,A\lor B} \qquad \lor\mathrm{R}_2\ \dfrac{\Gamma\vdash\Theta,B}{\Gamma\vdash\Theta,A\lor B}$$

$$\land\mathrm{L}_1\ \dfrac{A,\Gamma\vdash\Theta}{A\land B,\Gamma\vdash\Theta} \qquad \land\mathrm{L}_2\ \dfrac{B,\Gamma\vdash\Theta}{A\land B,\Gamma\vdash\Theta} \qquad \land\mathrm{R}\ \dfrac{\Gamma\vdash\Theta,A \quad \Gamma\vdash\Theta,B}{\Gamma\vdash\Theta,A\land B}$$

$$\to\mathrm{L}\ \dfrac{\Gamma\vdash\Theta,A \quad B,\Delta\vdash\Lambda}{A\to B,\Gamma,\Delta\vdash\Theta,\Lambda} \qquad \to\mathrm{R}\ \dfrac{A,\Gamma\vdash\Theta,B}{\Gamma\vdash\Theta,A\to B}$$

$$\neg\mathrm{L}\ \dfrac{\Gamma\vdash\Theta,A}{\neg A,\Gamma\vdash\Theta} \qquad \neg\mathrm{R}\ \dfrac{A,\Gamma\vdash\Theta}{\Gamma\vdash\Theta,\neg A}$$

$$\mathsf{exchL}\ \dfrac{\Delta,B,A,\Gamma\vdash\Theta}{\Delta,A,B,\Gamma\vdash\Theta} \qquad \mathsf{exchR}\ \dfrac{\Gamma\vdash\Theta,B,A,\Lambda}{\Gamma\vdash\Theta,A,B,\Lambda}$$

$$\mathsf{weakL}\ \dfrac{\Gamma\vdash\Theta}{A,\Gamma\vdash\Theta} \qquad \mathsf{contL}\ \dfrac{A,A,\Gamma\vdash\Theta}{A,\Gamma\vdash\Theta} \qquad \mathsf{weakR}\ \dfrac{\Gamma\vdash\Theta}{\Gamma\vdash\Theta,A} \qquad \mathsf{contR}\ \dfrac{\Gamma\vdash\Theta,A,A}{\Gamma\vdash\Theta,A}$$

$$\mathsf{cut}\ \dfrac{\Gamma\vdash\Theta,A \quad A,\Delta\vdash\Lambda}{\Gamma,\Delta\vdash\Theta,\Lambda}$$



- this is Gentzen's sequent calculus LK, introduced in
  - Gerhard Gentzen: **"Untersuchungen über das logische Schließen I".** *Mathematische Zeitschrift, 39:176–210, 1935*
  - Gerhard Gentzen: **"Untersuchungen über das logische Schließen II".** *Mathematische Zeitschrift, 39:405–431, 1935*
- **Exercise 1.8:** Show that all rules preserve validity.
- **Exercise 1.9:** Prove the Hilbert axioms in LK.
- **Exercise 1.10:** Show how modus pones can be simulated using the cut rule.

## Sequent calculus (two-sided version)

**Theorem (Soundness):**
If a formula is provable in LK then it is valid.

**Theorem (Completeness):**
If a formula is valid then it is provable in LK.



- **Exercise 1.11:** Prove these two theorems (use the previous three exercises)

## Sequent calculus (two-sided version)

**Theorem (Gentzen's Hauptsatz):**
If a sequent is provable in LK, then it is also provable without the cut rule.

This theorem is the reason for the invention of the sequent calculus.



- If we look at the rules of LK, then we see that all rules have the *subformula property* (i.e., the formulas in the premise are subformulas of the formulas in the concclusion). This means that we can use LK for *proof search*.

## Sequent calculus (one-sided version)

*One-sided sequents:*

$$\vdash B_1, B_2, \ldots, B_m$$

Meaning:

$$B_1 \vee B_2 \vee \cdots \vee B_m$$

A comma is a disjunction

Translating two-sided sequents into one-sided sequents:

$$A_1, \ldots, A_n \vdash B_1, \ldots, B_m \quad \rightsquigarrow \quad \vdash \bar{A}_1, \ldots, \bar{A}_n, B_1, \ldots, B_m$$



- The idea of a one-side sequent calculus goes back to
  - Kurt Schütte: **"Schlussweisen-Kalküle der Prädikatenlogik".** *Mathematische Annalen, vol. 122, pp. 47–65, 1950*

## Sequent calculus (one-sided version)

$$\text{id}\ \frac{}{\vdash \bar{A}, A} \qquad \top\ \frac{}{\vdash \top}$$

$$\vee_1\ \frac{\vdash \Theta, A}{\vdash \Theta, A \vee B} \qquad \vee_2\ \frac{\vdash \Theta, B}{\vdash \Theta, A \vee B}$$

$$\wedge\ \frac{\vdash \Theta, A \qquad \vdash \Theta, B}{\vdash \Theta, A \wedge B}$$

$$\rightarrow\ \frac{\vdash \Theta, \bar{A}, B}{\vdash \Theta, A \rightarrow B}$$

$$\text{exch}\ \frac{\vdash \Theta, B, A, \Lambda}{\vdash \Theta, A, B, \Lambda}$$

$$\text{weak}\ \frac{\vdash \Theta}{\vdash \Theta, A} \qquad \text{cont}\ \frac{\vdash \Theta, A, A}{\vdash \Theta, A}$$

$$\text{cut}\ \frac{\vdash \Theta, A \qquad \vdash \bar{A}, \Lambda}{\vdash \Theta, \Lambda}$$



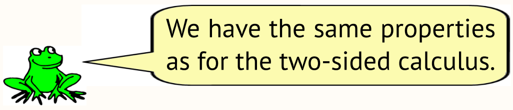
You can do for the one-sided sequent calculus the same exercises you did for the two-sided one:

- **Exercise 1.12:** Show that all rules preserve validity.
- **Exercise 1.13:** Prove the Hilbert axioms.
- **Exercise 1.14:** Show how modus pones can be simulated using the cut rule.x
- **Exercise 1.15:** Show that the general identity rule

  $\text{id}\ \frac{}{\vdash \bar{A}, A}$ can be replaced by the atomic version

  $\text{id}\ \frac{}{\vdash \bar{a}, a}$.

## Sequent calculus (two-sided version)




**Theorem:**
If a formula is provable, then it is valid.

**Theorem:**
If a formula is valid, then it is provable.

**Theorem:**
If a formula is provable with cut, then it is also provable without cut.



**Exercise 1.16:** Prove these three theorems (you can use all the previous ones).