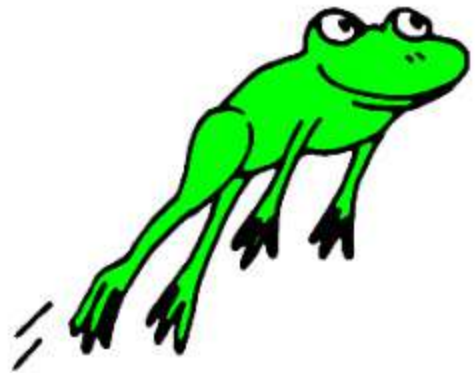


A gentle introduction to deep inference

9. Comparing complexity of different proof systems



ESSLLI 2025

Victoria Barrett and Lutz Straßburger

analytic GS1p

$$\text{id} \frac{}{\vdash A, \bar{A}}$$

$$\text{t} \frac{}{\vdash \top}$$

$$\vee \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}$$

$$\wedge \frac{\vdash \Gamma, A \quad \vdash B, \perp}{\vdash \Gamma, A \wedge B, \perp}$$

$$\omega \frac{\Gamma}{\vdash \Gamma, A}$$

$$\text{c} \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}$$

GS1p

$$\text{cut} \frac{\vdash \Gamma, A \quad \vdash \bar{A}, \perp}{\vdash \Gamma, \perp}$$

Γ is a multiset of formulae, not a list

analytic GS1p

$$\text{id} \frac{}{\vdash A, \bar{A}}$$

$$\text{t} \frac{}{\vdash \top}$$

$$\vee \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}$$

$$\wedge \frac{\vdash \Gamma, A \quad \vdash B, \Lambda}{\vdash \Gamma, A \wedge B, \Lambda}$$

$$\omega \frac{\Gamma}{\vdash \Gamma, A}$$

$$\text{c} \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}$$

GS1p

$$\text{cut} \frac{\vdash \Gamma, A \quad \vdash \bar{A}, \Lambda}{\vdash \Gamma, \Lambda}$$

Γ is a multiset of formulae, not a list

Robustness for Gentzen systems:

$$\text{GS1p} \xleftrightarrow{\text{p}} \text{LK}$$

$$\text{GS1p} \setminus \{\text{cut}\} \xleftrightarrow{\text{p}} \text{LK} \setminus \{\text{cut}\}$$

analytic GS1p

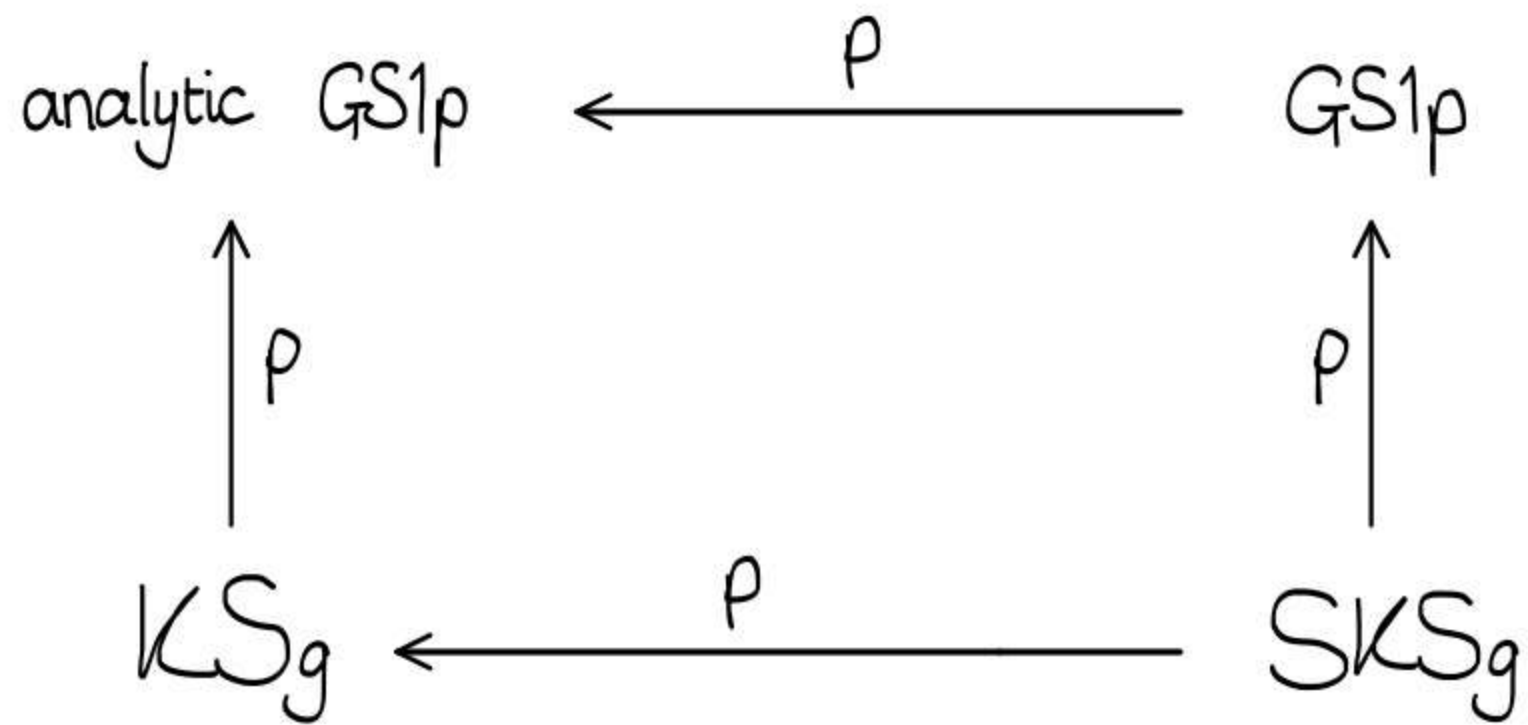
GS1p

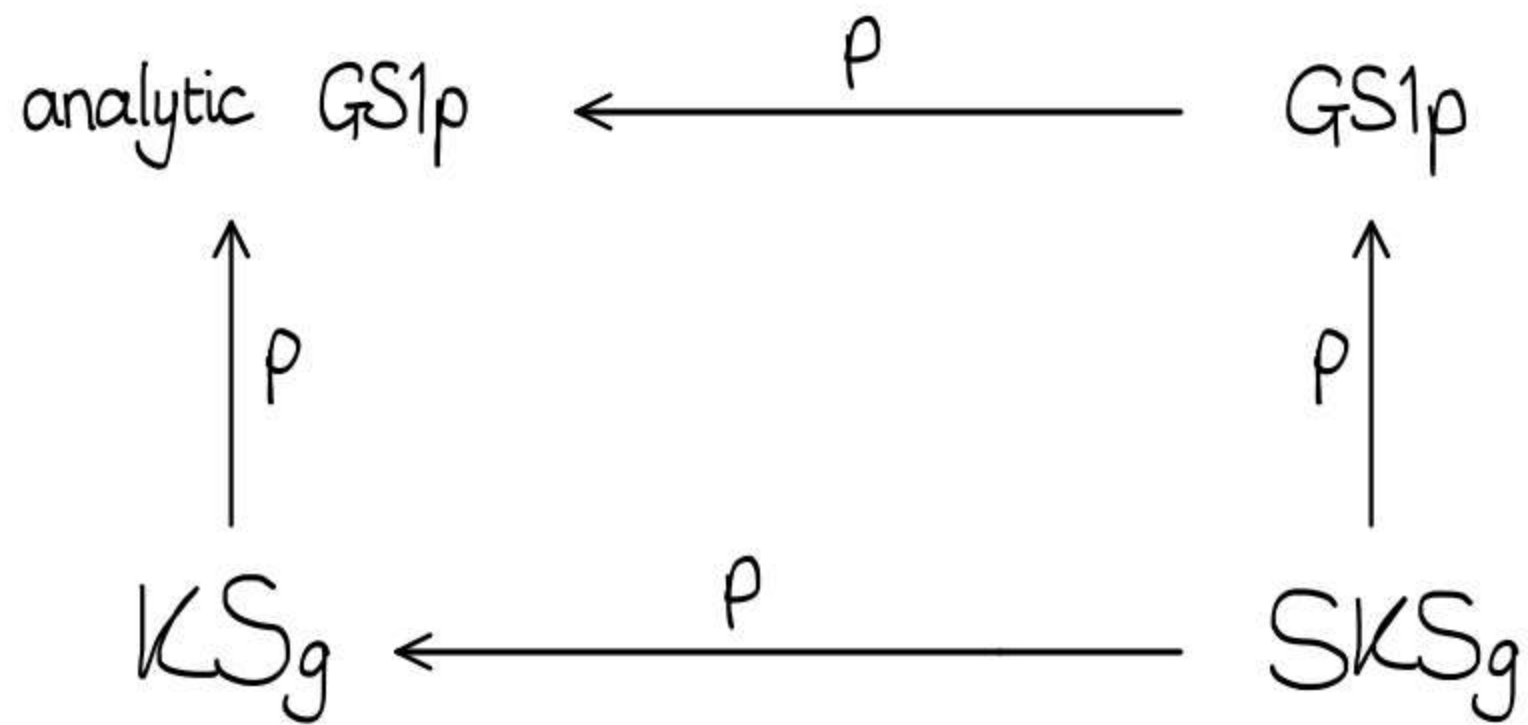
$$\text{analytic GS1p} \xleftarrow{\rho} \text{GS1p}$$

$$\text{analytic GS1p} \xleftarrow{\rho} \text{GS1p}$$

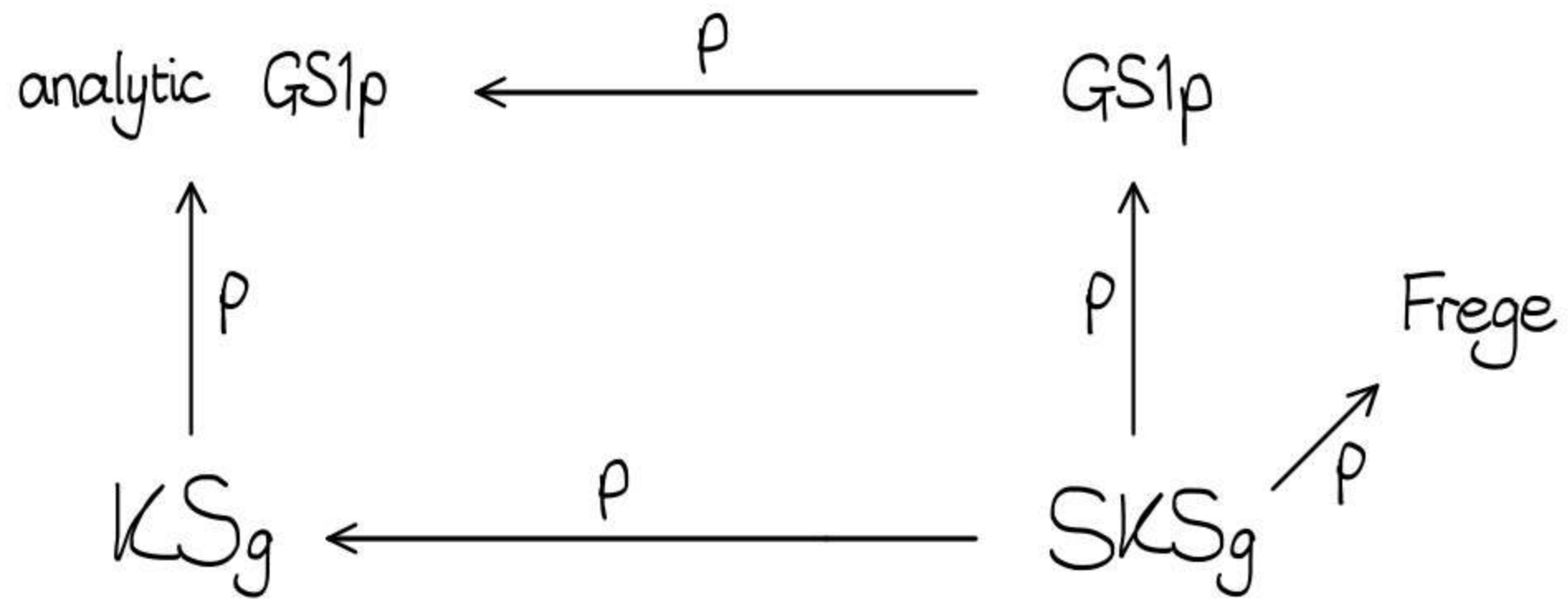
KS_g

SKS_g





Frege



System Frege

Formulae: $A, B ::= a \mid \bar{a} \mid A \vee B \mid A \wedge B$

Negation: $\overline{A \wedge B} = \bar{A} \vee \bar{B} \quad \overline{A \vee B} = \bar{A} \wedge \bar{B}$

Proofs are lists of formulae $A_1, \dots, A_n$ such that each A_k is either an axiom or follows from previous formulae by the rule modus ponens

$$\text{mp} \frac{A \quad \bar{A} \vee B}{B}$$

System Frege

Formulae: $A, B ::= a \mid \bar{a} \mid A \vee B \mid A \wedge B$

Negation: $\overline{A \wedge B} = \bar{A} \vee \bar{B} \quad \overline{A \vee B} = \bar{A} \wedge \bar{B}$

Proofs are lists of formulae $A_1, \dots, A_n$ such that each A_k is either an axiom or follows from previous formulae by the rule modus ponens

$$\text{mp} \frac{A \quad \bar{A} \vee B}{B}$$

Compression is achieved by
dag-like structure and
efficiency of modus ponens

Frege

A_1

$\vdots$

$A_k = B$

$\vdots$

$A_l = \bar{B} \vee C$

$\vdots$

$A_n = C$

$\prod SKS_g$

$A_1 \wedge \dots \wedge B \wedge \dots \wedge (\bar{B} \vee C) \wedge \dots \wedge A_{n-1}$

Frege

$$\begin{array}{l} A_1 \\ \vdots \\ A_k = B \\ \vdots \\ A_l = \bar{B} \vee C \\ \vdots \\ A_n = C \end{array}$$

$\prod \text{SKS}_g$

$$A_1 \wedge \dots \wedge \text{c}\uparrow \frac{B}{B \wedge B} \wedge \dots \wedge \text{c}\uparrow \frac{(\bar{B} \vee C)}{(\bar{B} \vee C) \wedge (\bar{B} \vee C)} \wedge \dots \wedge A_{n-1}$$

Frege

$$\begin{array}{l}
 A_1 \\
 \vdots \\
 A_k = B \\
 \vdots \\
 A_l = \bar{B} \vee C \\
 \vdots \\
 A_n = C
 \end{array}$$

$\prod$ SKSg

$$\begin{array}{c}
 A_1 \wedge \dots \wedge \text{c}\uparrow \frac{B}{B \wedge B} \wedge \dots \wedge \text{c}\uparrow \frac{(\bar{B} \vee C)}{(\bar{B} \vee C) \wedge (\bar{B} \vee C)} \wedge \dots \wedge A_{n-1} \\
 \hline
 A_1 \wedge \dots \wedge A_{n-1} \wedge (B \wedge (\bar{B} \vee C))
 \end{array}$$

Frege

$$\begin{array}{l}
 A_1 \\
 \vdots \\
 A_k = B \\
 \vdots \\
 A_l = \bar{B} \vee C \\
 \vdots \\
 A_n = C
 \end{array}$$

$\prod \text{SKS}_g$

$$\begin{array}{c}
 A_1 \wedge \dots \wedge \text{c}\uparrow \frac{B}{B \wedge B} \wedge \dots \wedge \text{c}\uparrow \frac{(\bar{B} \vee C)}{(\bar{B} \vee C) \wedge (\bar{B} \vee C)} \wedge \dots \wedge A_{n-1} \\
 \hline
 A_1 \wedge \dots \wedge A_{n-1} \wedge \boxed{
 \begin{array}{c}
 \text{s} \frac{B \wedge (\bar{B} \vee C)}{B \wedge \bar{B}} \\
 \text{i}\uparrow \frac{B \wedge \bar{B}}{\perp} \vee C \\
 \hline
 C
 \end{array}
 }
 \end{array}$$

Frege

cocontraction imitates
dag-ness

$\prod$ SKSg

A_1
 $\vdots$
 $A_k = B$
 $\vdots$
 $A_l = \bar{B} \vee C$
 $\vdots$
 $A_n = C$

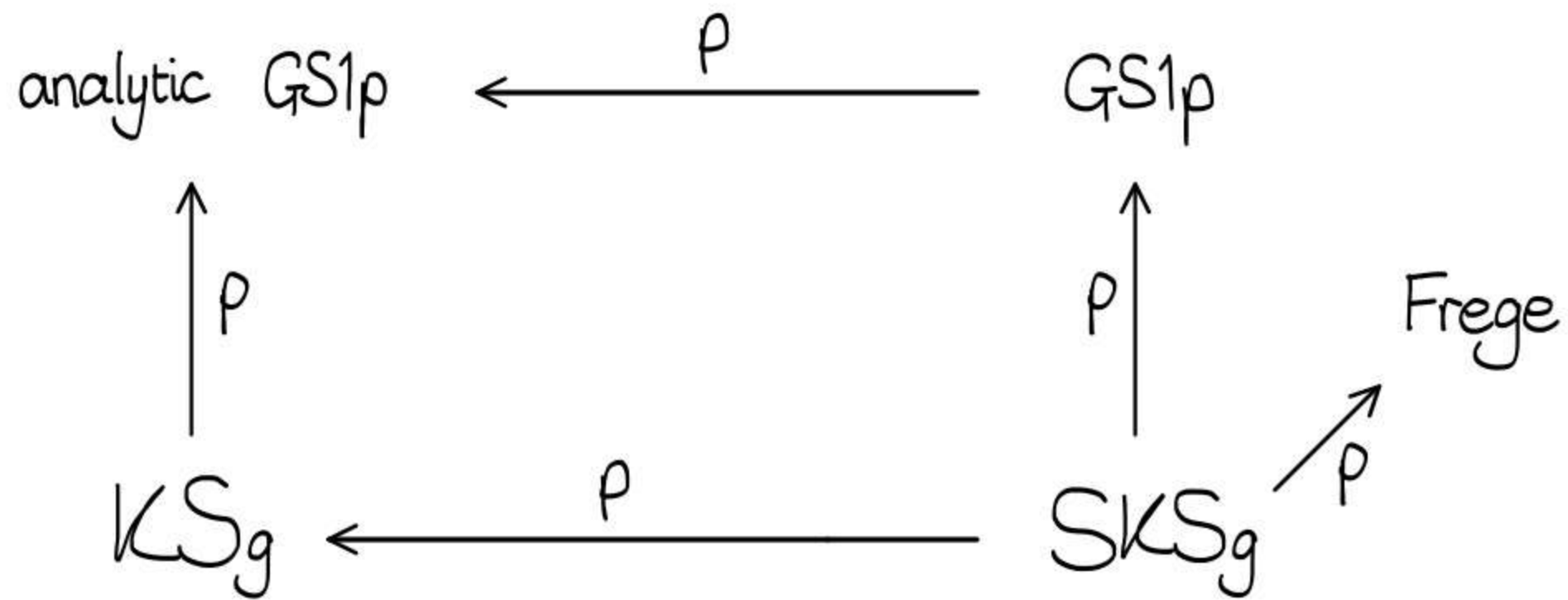
$$A_1 \wedge \dots \wedge c\uparrow \frac{B}{B \wedge B} \wedge \dots \wedge c\uparrow \frac{(\bar{B} \vee C)}{(\bar{B} \vee C) \wedge (\bar{B} \vee C)} \wedge \dots \wedge A_{n-1}$$

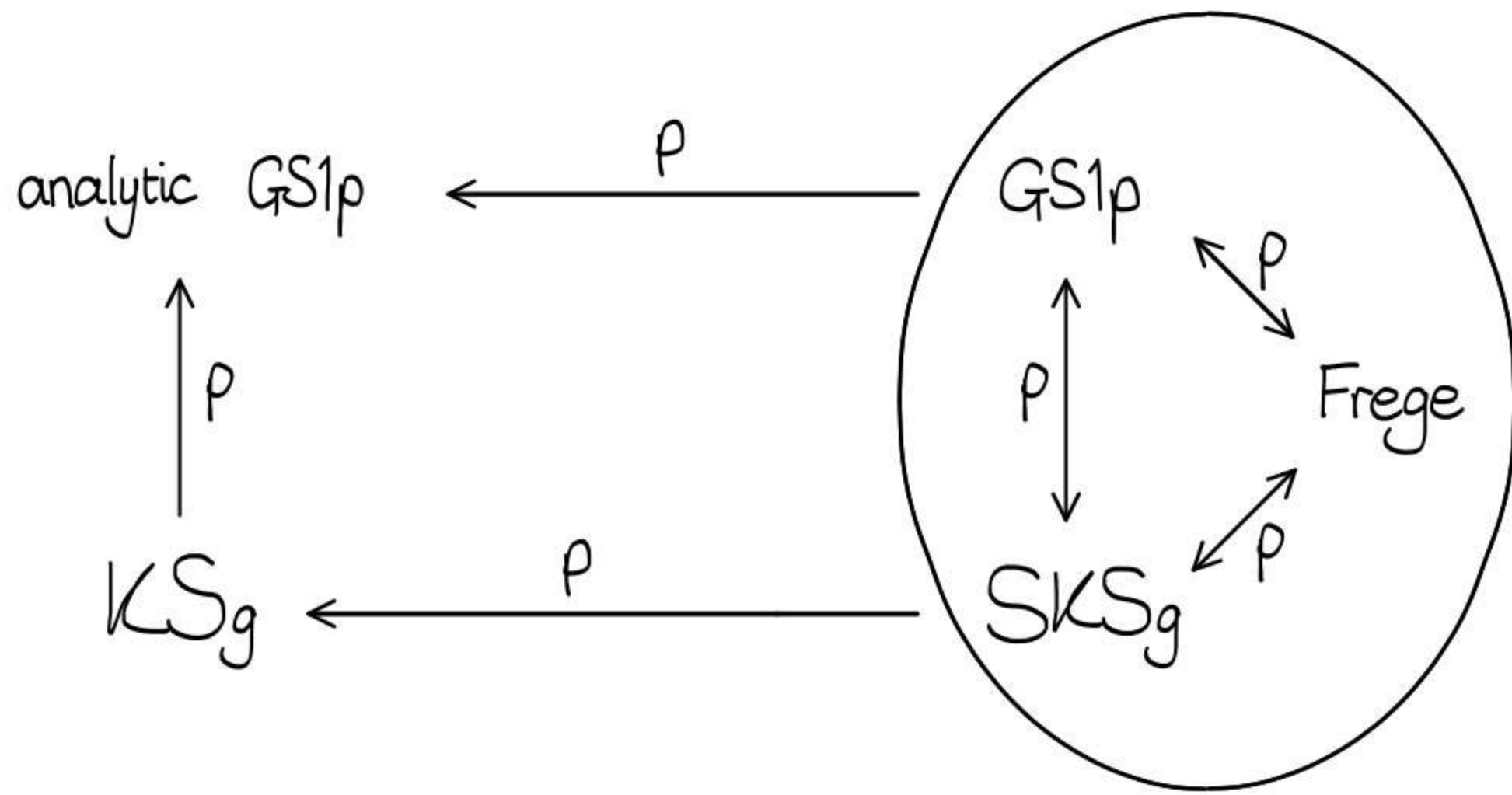
=

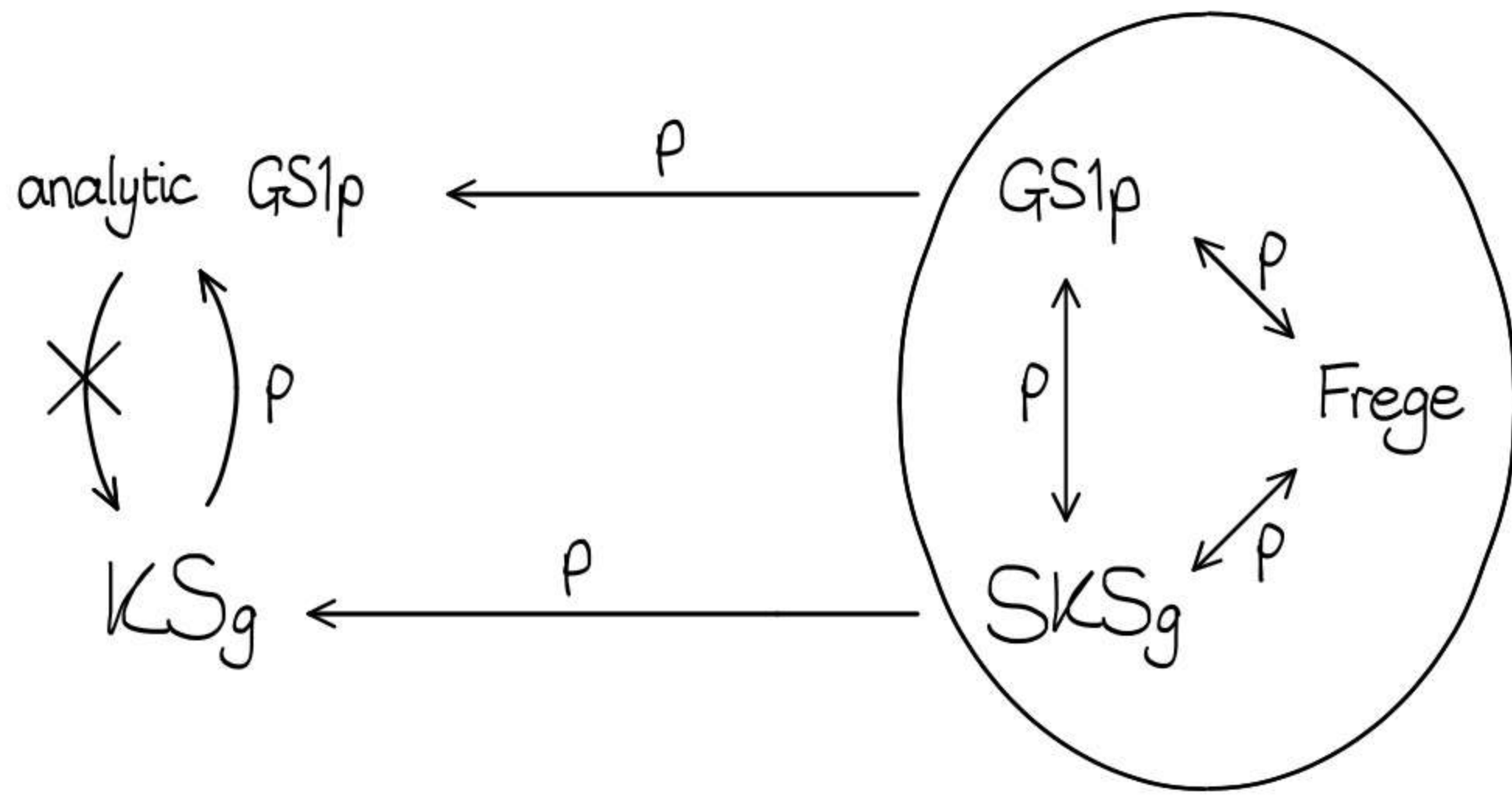
$$A_1 \wedge \dots \wedge A_{n-1} \wedge$$

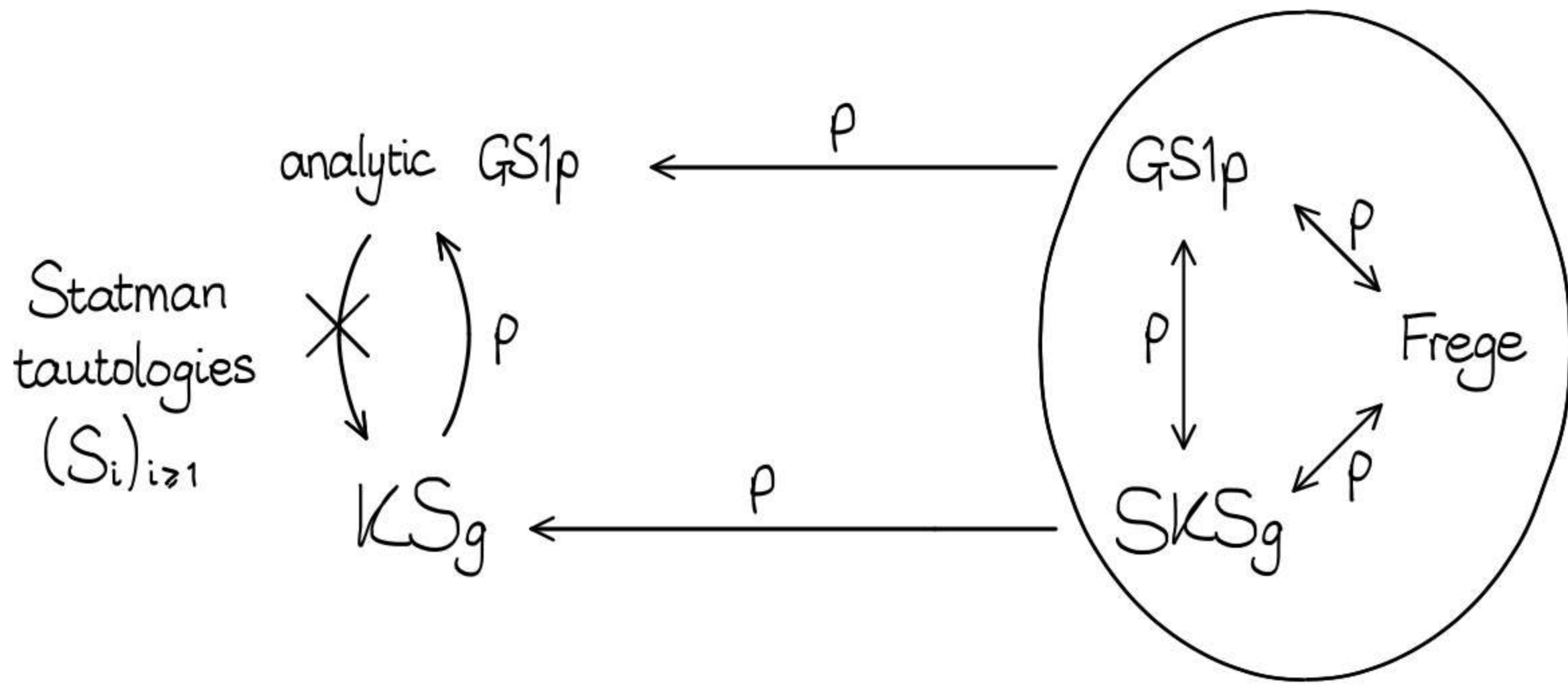
cut imitates
modus ponens

$ \begin{array}{c} s \frac{B \wedge (\bar{B} \vee C)}{B \wedge \bar{B}} \\ i\uparrow \frac{B \wedge \bar{B}}{\perp} \vee C \\ \hline C \end{array} $

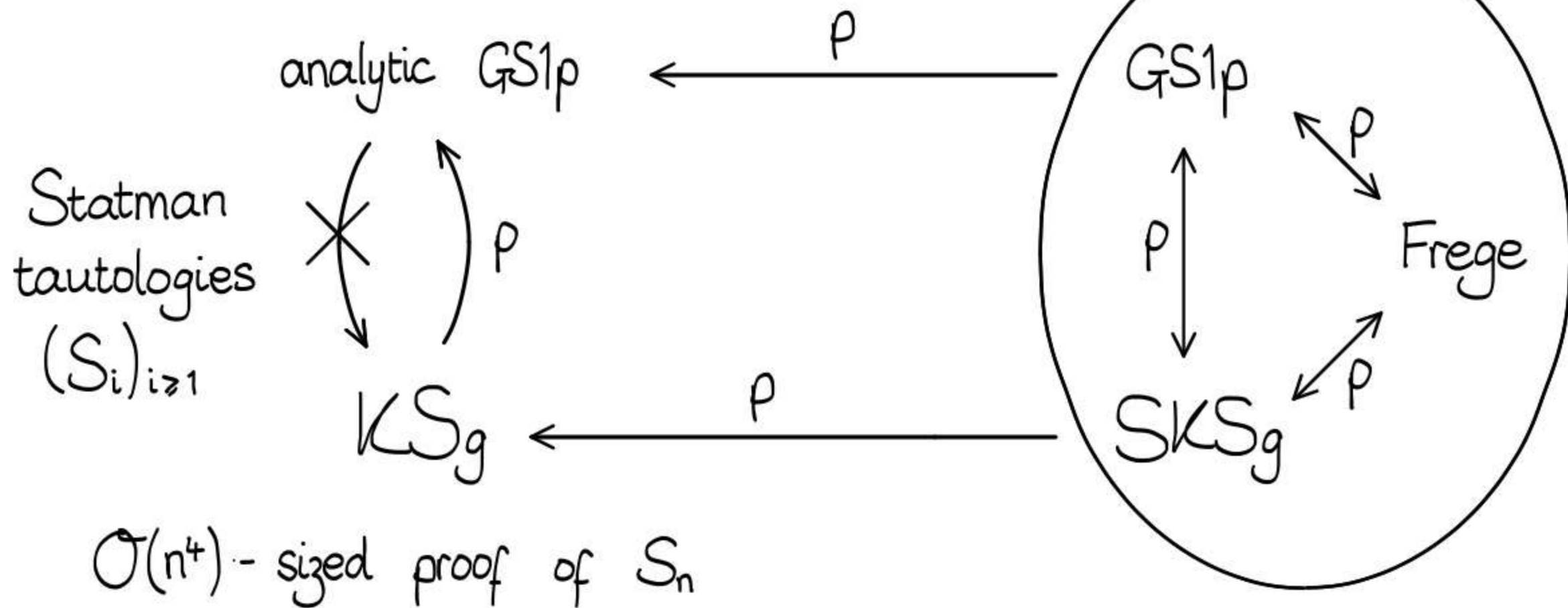








$O(2^n)$ - sized proof of S_n



$$\alpha_1 = \bar{p}_1$$

$$\beta_1 = \bar{q}_1$$

$$\alpha_1 = \bar{p}_1$$

$$\alpha_{i+1} = \left(\bigwedge_{j=1}^i (p_j \vee q_j) \right) \wedge \bar{p}_{i+1}$$

$$\beta_1 = \bar{q}_1$$

$$\beta_{i+1} = \left(\bigwedge_{j=1}^i (p_j \vee q_j) \right) \wedge \bar{q}_{i+1}$$

$$\alpha_1 = \bar{p}_1$$

$$\beta_1 = \bar{q}_1$$

$$\alpha_{i+1} = \left(\bigwedge_{j=1}^i (p_j \vee q_j) \right) \wedge \bar{p}_{i+1}$$

$$\beta_{i+1} = \left(\bigwedge_{j=1}^i (p_j \vee q_j) \right) \wedge \bar{q}_{i+1}$$

$$S_i = \left(\bigvee_{j=1}^i (\alpha_j \wedge \beta_j) \right) \vee (p_i \vee q_i)$$

$$S_1 = \left(\bigvee_{j=1}^1 (\alpha_j \wedge \beta_j) \right) \vee (p_1 \vee q_1)$$

$$\alpha_1 = \bar{p}_1$$

$$\beta_1 = \bar{q}_1$$

$$\alpha_{i+1} = \left(\bigwedge_{j=1}^i (p_j \vee q_j) \right) \wedge \bar{p}_{i+1}$$

$$\beta_{i+1} = \left(\bigwedge_{j=1}^i (p_j \vee q_j) \right) \wedge \bar{q}_{i+1}$$

$$S_i = \left(\bigvee_{j=1}^i (\alpha_j \wedge \beta_j) \right) \vee (p_i \vee q_i)$$

$$S_1 = (\alpha_1 \wedge \beta_1) \vee (p_1 \vee q_1)$$

$$\alpha_1 = \bar{p}_1$$

$$\beta_1 = \bar{q}_1$$

$$\alpha_{i+1} = \left(\bigwedge_{j=1}^i (p_j \vee q_j) \right) \wedge \bar{p}_{i+1}$$

$$\beta_{i+1} = \left(\bigwedge_{j=1}^i (p_j \vee q_j) \right) \wedge \bar{q}_{i+1}$$

$$S_i = \left(\bigvee_{j=1}^i (\alpha_j \wedge \beta_j) \right) \vee (p_i \vee q_i)$$

$$S_1 = (\bar{p}_1 \wedge \bar{q}_1) \vee (p_1 \vee q_1)$$

$$\alpha_1 = \bar{p}_1$$

$$\beta_1 = \bar{q}_1$$

$$\alpha_{i+1} = \left(\bigwedge_{j=1}^i (p_j \vee q_j) \right) \wedge \bar{p}_{i+1}$$

$$\beta_{i+1} = \left(\bigwedge_{j=1}^i (p_j \vee q_j) \right) \wedge \bar{q}_{i+1}$$

$$S_i = \left(\bigvee_{j=1}^i (\alpha_j \wedge \beta_j) \right) \vee (p_i \vee q_i)$$

$$S_1 = (\bar{p}_1 \wedge \bar{q}_1) \vee (p_1 \vee q_1)$$

$$S_2 = \left(\bigvee_{j=1}^2 (\alpha_j \wedge \beta_j) \right) \vee (p_2 \vee q_2)$$

$$\alpha_1 = \bar{p}_1$$

$$\beta_1 = \bar{q}_1$$

$$\alpha_{i+1} = \left(\bigwedge_{j=1}^i (p_j \vee q_j) \right) \wedge \bar{p}_{i+1}$$

$$\beta_{i+1} = \left(\bigwedge_{j=1}^i (p_j \vee q_j) \right) \wedge \bar{q}_{i+1}$$

$$S_i = \left(\bigvee_{j=1}^i (\alpha_j \wedge \beta_j) \right) \vee (p_i \vee q_i)$$

$$S_1 = (\bar{p}_1 \wedge \bar{q}_1) \vee (p_1 \vee q_1)$$

$$S_2 = (\alpha_1 \wedge \beta_1) \vee \left(\alpha_2 \wedge \beta_2 \right) \vee (p_2 \vee q_2)$$

$$\alpha_1 = \bar{p}_1$$

$$\beta_1 = \bar{q}_1$$

$$\alpha_{i+1} = \left(\bigwedge_{j=1}^i (p_j \vee q_j) \right) \wedge \bar{p}_{i+1}$$

$$\beta_{i+1} = \left(\bigwedge_{j=1}^i (p_j \vee q_j) \right) \wedge \bar{q}_{i+1}$$

$$S_i = \left(\bigvee_{j=1}^i (\alpha_j \wedge \beta_j) \right) \vee (p_i \vee q_i)$$

$$S_1 = (\bar{p}_1 \wedge \bar{q}_1) \vee (p_1 \vee q_1)$$

$$S_2 = (\bar{p}_1 \wedge \bar{q}_1) \vee \left(\left(\left(\bigwedge_{j=1}^1 (p_j \vee q_j) \right) \wedge \bar{p}_2 \right) \wedge \left(\left(\bigwedge_{j=1}^1 (p_j \vee q_j) \right) \wedge \bar{q}_2 \right) \right) \vee (p_2 \vee q_2)$$

$$\alpha_1 = \bar{p}_1$$

$$\beta_1 = \bar{q}_1$$

$$\alpha_{i+1} = \left(\bigwedge_{j=1}^i (p_j \vee q_j) \right) \wedge \bar{p}_{i+1}$$

$$\beta_{i+1} = \left(\bigwedge_{j=1}^i (p_j \vee q_j) \right) \wedge \bar{q}_{i+1}$$

$$S_i = \left(\bigvee_{j=1}^i (\alpha_j \wedge \beta_j) \right) \vee (p_i \vee q_i)$$

$$S_1 = (\bar{p}_1 \wedge \bar{q}_1) \vee (p_1 \vee q_1)$$

$$S_2 = (\bar{p}_1 \wedge \bar{q}_1) \vee \left(\left((p_1 \vee q_1) \wedge \bar{p}_2 \right) \wedge \left((p_1 \vee q_1) \wedge \bar{q}_2 \right) \right) \vee (p_2 \vee q_2)$$

$$\alpha_1 = \bar{p}_1$$

$$\beta_1 = \bar{q}_1$$

$$\alpha_{i+1} = \left(\bigwedge_{j=1}^i (p_j \vee q_j) \right) \wedge \bar{p}_{i+1}$$

$$\beta_{i+1} = \left(\bigwedge_{j=1}^i (p_j \vee q_j) \right) \wedge \bar{q}_{i+1}$$

$$S_i = \left(\bigvee_{j=1}^i (\alpha_j \wedge \beta_j) \right) \vee (p_i \vee q_i)$$

$$S_1 = (\bar{p}_1 \wedge \bar{q}_1) \vee (p_1 \vee q_1)$$

$$S_2 = (\bar{p}_1 \wedge \bar{q}_1) \vee \left(\left((p_1 \vee q_1) \wedge \bar{p}_2 \right) \wedge \left((p_1 \vee q_1) \wedge \bar{q}_2 \right) \right) \vee (p_2 \vee q_2)$$

$$S_3 = (\bar{p}_1 \wedge \bar{q}_1) \vee \left(\begin{array}{ccc} \alpha_2 & \wedge & \beta_2 \\ \vee & & \\ \alpha_3 & \wedge & \beta_3 \end{array} \right) \vee (p_3 \vee q_3)$$

$$\alpha_1 = \bar{p}_1$$

$$\beta_1 = \bar{q}_1$$

$$\alpha_{i+1} = \left(\bigwedge_{j=1}^i (p_j \vee q_j) \right) \wedge \bar{p}_{i+1}$$

$$\beta_{i+1} = \left(\bigwedge_{j=1}^i (p_j \vee q_j) \right) \wedge \bar{q}_{i+1}$$

$$S_i = \left(\bigvee_{j=1}^i (\alpha_j \wedge \beta_j) \right) \vee (p_i \vee q_i)$$

$$S_1 = (\bar{p}_1 \wedge \bar{q}_1) \vee (p_1 \vee q_1)$$

$$S_2 = (\bar{p}_1 \wedge \bar{q}_1) \vee \left(((p_1 \vee q_1) \wedge \bar{p}_2) \wedge ((p_1 \vee q_1) \wedge \bar{q}_2) \right) \vee (p_2 \vee q_2)$$

$$S_3 = (\bar{p}_1 \wedge \bar{q}_1) \vee \left(((p_1 \vee q_1) \wedge \bar{p}_2) \wedge ((p_1 \vee q_2) \wedge \bar{q}_2) \right) \\ \vee \left(((p_1 \vee q_1) \wedge (p_2 \vee q_2) \wedge \bar{p}_3) \wedge ((p_1 \vee q_1) \wedge (p_2 \vee q_2) \wedge \bar{q}_3) \right) \vee (p_3 \vee q_3)$$

$$\begin{array}{c}
 \text{id} \frac{}{\vdash A, \bar{A}} \quad \text{t} \frac{}{\vdash \top} \quad \vee \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B} \quad \wedge \frac{\vdash \Gamma, A \quad \vdash B, \perp}{\vdash \Gamma, A \wedge B, \perp} \quad \omega \frac{\Gamma}{\vdash \Gamma, A} \quad \text{c} \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}
 \end{array}$$

$$\begin{aligned}
 S_3 = & (\bar{p}_1 \wedge \bar{q}_1) \vee (((p_1 \vee q_1) \wedge \bar{p}_2) \wedge ((p_1 \vee q_2) \wedge \bar{q}_2)) \\
 & \vee (((p_1 \vee q_1) \wedge (p_2 \vee q_2) \wedge \bar{p}_3) \wedge ((p_1 \vee q_1) \wedge (p_2 \vee q_2) \wedge \bar{q}_3)) \vee (p_3 \vee q_3)
 \end{aligned}$$

$$\text{id} \frac{}{\vdash A, \bar{A}}$$

$$\text{t} \frac{}{\vdash T}$$

$$\vee \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}$$

$$\wedge \frac{\vdash \Gamma, A \quad \vdash B, \perp}{\vdash \Gamma, A \wedge B, \perp}$$

$$\omega \frac{\Gamma}{\vdash \Gamma, A}$$

$$\text{c} \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}$$

$$A = p_1 \vee q_1$$

$$B = p_2 \vee q_2$$

$$\begin{aligned} S_3 = & (\bar{p}_1 \wedge \bar{q}_1) \vee (((p_1 \vee q_1) \wedge \bar{p}_2) \wedge ((p_1 \vee q_2) \wedge \bar{q}_2)) \\ & \vee (((p_1 \vee q_1) \wedge (p_2 \vee q_2) \wedge \bar{p}_3) \wedge ((p_1 \vee q_1) \wedge (p_2 \vee q_2) \wedge \bar{q}_3)) \vee (p_3 \vee q_3) \end{aligned}$$

$$\text{id} \frac{}{\vdash A, \bar{A}}$$

$$\text{t} \frac{}{\vdash T}$$

$$\vee \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}$$

$$\wedge \frac{\vdash \Gamma, A \quad \vdash B, \perp}{\vdash \Gamma, A \wedge B, \perp}$$

$$\omega \frac{\Gamma}{\vdash \Gamma, A}$$

$$\text{c} \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}$$

$$A = p_1 \vee q_1$$

$$B = p_2 \vee q_2$$

$$\vdash \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3$$

$$\text{id} \frac{}{\vdash A, \bar{A}}$$

$$\text{t} \frac{}{\vdash T}$$

$$\vee \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}$$

$$\wedge \frac{\vdash \Gamma, A \quad \vdash B, \perp}{\vdash \Gamma, A \wedge B, \perp}$$

$$\omega \frac{\Gamma}{\vdash \Gamma, A}$$

$$\text{c} \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}$$

$$A = p_1 \vee q_1$$

$$B = p_2 \vee q_2$$

$$\wedge \frac{\vdash (A \wedge B) \wedge \bar{p}_3 \quad \vdash (A \wedge B) \wedge \bar{q}_3}{\vdash \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3}$$

$$\text{id} \frac{}{\vdash A, \bar{A}}$$

$$\text{t} \frac{}{\vdash \top}$$

$$\vee \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}$$

$$\wedge \frac{\vdash \Gamma, A \quad \vdash B, \perp}{\vdash \Gamma, A \wedge B, \perp}$$

$$\omega \frac{\Gamma}{\vdash \Gamma, A}$$

$$\text{c} \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}$$

$$A = p_1 \vee q_1$$

$$B = p_2 \vee q_2$$

$$\wedge \frac{\vdash (A \wedge B) \wedge \bar{p}_3, p_3 \quad \vdash (A \wedge B) \wedge \bar{q}_3, q_3}{\vdash \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3}$$

$$\text{id} \frac{}{\vdash A, \bar{A}}$$

$$\text{t} \frac{}{\vdash \top}$$

$$\vee \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}$$

$$\wedge \frac{\vdash \Gamma, A \quad \vdash B, \perp}{\vdash \Gamma, A \wedge B, \perp}$$

$$\omega \frac{\Gamma}{\vdash \Gamma, A}$$

$$\text{c} \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}$$

$$A = p_1 \vee q_1$$

$$B = p_2 \vee q_2$$

$$\wedge \frac{\vdash (A \wedge B) \wedge \bar{p}_3, p_3}{\vdash \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3}$$

$$\vdash (A \wedge B) \wedge \bar{q}_3, q_3$$

$$\vdash \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3$$

$$\text{id} \frac{}{\vdash A, \bar{A}}$$

$$\text{t} \frac{}{\vdash T}$$

$$\vee \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}$$

$$\wedge \frac{\vdash \Gamma, A \quad \vdash B, \perp}{\vdash \Gamma, A \wedge B, \perp}$$

$$\omega \frac{\Gamma}{\vdash \Gamma, A}$$

$$\text{c} \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}$$

$$A = p_1 \vee q_1$$

$$B = p_2 \vee q_2$$

$$\wedge \frac{\vdash (A \wedge B) \wedge \bar{p}_3, p_3}{}$$

$$\vdash (A \wedge B) \wedge \bar{q}_3, q_3$$

$$\vdash \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3$$

$$\text{id} \frac{}{\vdash A, \bar{A}}$$

$$\text{t} \frac{}{\vdash \top}$$

$$\vee \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}$$

$$\wedge \frac{\vdash \Gamma, A \quad \vdash B, \perp}{\vdash \Gamma, A \wedge B, \perp}$$

$$\omega \frac{\Gamma}{\vdash \Gamma, A}$$

$$\text{c} \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}$$

$$A = p_1 \vee q_1$$

$$B = p_2 \vee q_2$$

$$\wedge \frac{\vdash \bar{A}, (A \wedge B) \wedge \bar{p}_3, p_3 \quad \vdash \bar{A}, (A \wedge B) \wedge \bar{q}_3, q_3}{\vdash \bar{A}, \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3}$$

$$\text{c} \frac{}{\vdash \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3}$$

$$\text{id} \frac{}{\vdash A, \bar{A}}$$

$$\text{t} \frac{}{\vdash \top}$$

$$\vee \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}$$

$$\wedge \frac{\vdash \Gamma, A \quad \vdash B, \perp}{\vdash \Gamma, A \wedge B, \perp}$$

$$\omega \frac{\Gamma}{\vdash \Gamma, A}$$

$$\text{c} \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}$$

$$A = p_1 \vee q_1$$

$$B = p_2 \vee q_2$$

$$\begin{array}{c} \text{c}, \wedge \frac{\vdash \bar{A}, (A \wedge B) \wedge \bar{p}_3, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), p_3 \quad \vdash \bar{A}, (A \wedge B) \wedge \bar{q}_3, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), q_3}{\vdash \bar{A}, \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3} \\ \text{c} \frac{}{\vdash \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3} \end{array}$$

$$\text{id} \frac{}{\vdash A, \bar{A}}$$

$$\text{t} \frac{}{\vdash \top}$$

$$\vee \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}$$

$$\wedge \frac{\vdash \Gamma, A \quad \vdash B, \perp}{\vdash \Gamma, A \wedge B, \perp}$$

$$\omega \frac{\Gamma}{\vdash \Gamma, A}$$

$$\text{c} \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}$$

$$A = p_1 \vee q_1$$

$$B = p_2 \vee q_2$$

$$\text{id} \frac{}{\vdash p_3, \bar{p}_3}$$

$$\vdash \bar{A}, A \wedge B, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)$$

$$\wedge \frac{}{\vdash \bar{A}, (A \wedge B) \wedge \bar{p}_3, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), p_3}$$

$$\vdash \bar{A}, (A \wedge B) \wedge \bar{q}_3, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), q_3$$

$$\text{c}, \wedge \frac{}{\vdash \bar{A}, \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3}$$

$$\text{c} \frac{}{\vdash \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3}$$

$$\text{id} \frac{}{\vdash A, \bar{A}}$$

$$\text{t} \frac{}{\vdash \top}$$

$$\vee \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}$$

$$\wedge \frac{\vdash \Gamma, A \quad \vdash B, \perp}{\vdash \Gamma, A \wedge B, \perp}$$

$$\omega \frac{\Gamma}{\vdash \Gamma, A}$$

$$\text{c} \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}$$

$$A = p_1 \vee q_1$$

$$B = p_2 \vee q_2$$

$$\text{id} \frac{}{\vdash p_3, \bar{p}_3}$$

$$2c, \wedge \frac{\vdash \bar{A}, A \wedge B, A \wedge \bar{p}_2 \quad \vdash \bar{A}, A \wedge B, A \wedge \bar{q}_2}{\vdash \bar{A}, A \wedge B, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)}$$

$$\begin{array}{c} \wedge \\ c, \wedge \end{array} \frac{\vdash \bar{A}, (A \wedge B) \wedge \bar{p}_3, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), p_3 \quad \vdash \bar{A}, (A \wedge B) \wedge \bar{q}_3, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), q_3}{\text{c} \frac{\vdash \bar{A}, \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3}{\vdash \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3}}$$

$$\text{id} \frac{}{\vdash A, \bar{A}}$$

$$\text{t} \frac{}{\vdash \top}$$

$$\vee \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}$$

$$\wedge \frac{\vdash \Gamma, A \quad \vdash B, \perp}{\vdash \Gamma, A \wedge B, \perp}$$

$$\omega \frac{\Gamma}{\vdash \Gamma, A}$$

$$\text{c} \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}$$

$$A = p_1 \vee q_1$$

$$B = p_2 \vee q_2$$

$$\text{id} \frac{}{\vdash \bar{A}, A}$$

$$\vdash A \wedge B, \bar{p}_2$$

$$\wedge \frac{\vdash \bar{A}, A \wedge B, A \wedge \bar{p}_2}{\vdash \bar{A}, A \wedge B, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)}$$

$$\vdash \bar{A}, A \wedge B, A \wedge \bar{q}_2$$

$$\text{id} \frac{}{\vdash p_3, \bar{p}_3}$$

$$2c, \wedge$$

$$\vdash \bar{A}, A \wedge B, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)$$

$$\wedge$$

$$\vdash \bar{A}, (A \wedge B) \wedge \bar{p}_3, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), p_3$$

$$\vdash \bar{A}, (A \wedge B) \wedge \bar{q}_3, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), q_3$$

$$c, \wedge$$

$$\text{c} \frac{\vdash \bar{A}, \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3}{\vdash \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3}$$

$$\text{id} \frac{}{\vdash A, \bar{A}}$$

$$\text{t} \frac{}{\vdash \top}$$

$$\vee \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}$$

$$\wedge \frac{\vdash \Gamma, A \quad \vdash B, \perp}{\vdash \Gamma, A \wedge B, \perp}$$

$$\omega \frac{\Gamma}{\vdash \Gamma, A}$$

$$\text{c} \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}$$

$$A = p_1 \vee q_1$$

$$B = p_2 \vee q_2$$

$$\begin{array}{c} \text{id} \frac{}{\vdash \bar{A}, A} \\ \text{c}, \wedge \frac{}{\vdash \bar{A}, A \wedge B, \bar{p}_2} \\ \text{id} \frac{}{\vdash \bar{p}_3, \bar{p}_3} \\ \wedge \frac{}{\vdash \bar{A}, (A \wedge B) \wedge \bar{p}_3, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), p_3} \\ \text{c}, \wedge \frac{}{\vdash \bar{A}, \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3} \\ \text{c} \frac{}{\vdash \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3} \end{array}$$

$$\text{id} \frac{}{\vdash A, \bar{A}}$$

$$\text{t} \frac{}{\vdash \top}$$

$$\vee \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}$$

$$\wedge \frac{\vdash \Gamma, A \quad \vdash B, \perp}{\vdash \Gamma, A \wedge B, \perp}$$

$$\omega \frac{\Gamma}{\vdash \Gamma, A}$$

$$\text{c} \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}$$

$$A = p_1 \vee q_1$$

$$B = p_2 \vee q_2$$

$$\begin{array}{c} \text{id} \frac{}{\vdash \bar{A}, A} \quad \wedge \frac{\text{id} \frac{}{\vdash \bar{A}, A} \quad \vdash B, \bar{p}_2}{\vdash \bar{A}, A \wedge B, \bar{p}_2} \\ \text{c}, \wedge \frac{\vdash \bar{A}, A \quad \vdash \bar{A}, A \wedge B, \bar{p}_2}{\vdash \bar{A}, A \wedge B, A \wedge \bar{p}_2} \quad \vdash \bar{A}, A \wedge B, A \wedge \bar{q}_2 \\ \text{id} \frac{}{\vdash p_3, \bar{p}_3} \quad 2\text{c}, \wedge \frac{\vdash \bar{A}, A \wedge B, A \wedge \bar{p}_2 \quad \vdash \bar{A}, A \wedge B, A \wedge \bar{q}_2}{\vdash \bar{A}, A \wedge B, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)} \\ \wedge \frac{\vdash p_3, \bar{p}_3 \quad \vdash \bar{A}, A \wedge B, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)}{\vdash \bar{A}, (A \wedge B) \wedge \bar{p}_3, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), p_3} \quad \vdash \bar{A}, (A \wedge B) \wedge \bar{q}_3, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), q_3 \\ \text{c}, \wedge \frac{\vdash \bar{A}, (A \wedge B) \wedge \bar{p}_3, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), p_3 \quad \vdash \bar{A}, (A \wedge B) \wedge \bar{q}_3, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), q_3}{\vdash \bar{A}, \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3} \\ \text{c} \frac{\vdash \bar{A}, \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3}{\vdash \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3} \end{array}$$

$$\text{id} \frac{}{\vdash A, \bar{A}}$$

$$\text{t} \frac{}{\vdash \top}$$

$$\vee \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}$$

$$\wedge \frac{\vdash \Gamma, A \quad \vdash B, \perp}{\vdash \Gamma, A \wedge B, \perp}$$

$$\omega \frac{\Gamma}{\vdash \Gamma, A}$$

$$\text{c} \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}$$

$$A = p_1 \vee q_1$$

$$B = p_2 \vee q_2$$

$$\begin{array}{c} \text{id} \frac{}{\vdash \bar{A}, A} \quad \wedge \frac{\text{id} \frac{}{\vdash \bar{A}, A} \quad \vdash p_2 \vee q_2, \bar{p}_2}{\vdash \bar{A}, A \wedge B, \bar{p}_2} \\ \text{c}, \wedge \frac{\text{id} \frac{}{\vdash \bar{A}, A} \quad \vdash \bar{A}, A \wedge B, \bar{p}_2}{\vdash \bar{A}, A \wedge B, A \wedge \bar{p}_2} \quad \vdash \bar{A}, A \wedge B, A \wedge \bar{q}_2 \\ \text{id} \frac{}{\vdash p_3, \bar{p}_3} \quad 2\text{c}, \wedge \frac{\vdash \bar{A}, A \wedge B, A \wedge \bar{p}_2 \quad \vdash \bar{A}, A \wedge B, A \wedge \bar{q}_2}{\vdash \bar{A}, A \wedge B, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)} \\ \wedge \frac{\vdash \bar{A}, (A \wedge B) \wedge \bar{p}_3, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), p_3 \quad \vdash \bar{A}, (A \wedge B) \wedge \bar{q}_3, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), q_3}{\vdash \bar{A}, \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3} \\ \text{c}, \wedge \frac{\vdash \bar{A}, \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3}{\vdash \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3} \end{array}$$

$$\begin{array}{c}
\text{id} \frac{}{\vdash A, \bar{A}} \quad \text{t} \frac{}{\vdash \top} \quad \vee \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B} \quad \wedge \frac{\vdash \Gamma, A \quad \vdash B, \perp}{\vdash \Gamma, A \wedge B, \perp} \quad \omega \frac{\Gamma}{\vdash \Gamma, A} \quad \text{c} \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}
\end{array}$$

$$A = p_1 \vee q_1$$

$$B = p_2 \vee q_2$$

$$\begin{array}{c}
\text{id} \frac{}{\vdash \bar{A}, A} \quad \text{id} \frac{}{\vdash \bar{p}_2, p_2} \quad \vee, \omega \frac{}{\vdash p_2 \vee q_2, \bar{p}_2} \\
\wedge \frac{\text{id} \frac{}{\vdash \bar{A}, A} \quad \text{id} \frac{}{\vdash \bar{p}_2, p_2}}{\vdash \bar{A}, A \wedge B, \bar{p}_2} \\
\text{id} \frac{}{\vdash \bar{A}, A} \quad \wedge \frac{}{\vdash \bar{A}, A \wedge B, \bar{p}_2} \\
\text{c}, \wedge \frac{}{\vdash \bar{A}, A \wedge B, A \wedge \bar{p}_2} \quad \vdash \bar{A}, A \wedge B, A \wedge \bar{q}_2 \\
\text{id} \frac{}{\vdash p_3, \bar{p}_3} \quad 2\text{c}, \wedge \frac{}{\vdash \bar{A}, A \wedge B, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)} \\
\wedge \frac{}{\vdash \bar{A}, (A \wedge B) \wedge \bar{p}_3, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), p_3} \quad \vdash \bar{A}, (A \wedge B) \wedge \bar{q}_3, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), q_3 \\
\text{c}, \wedge \frac{}{\vdash \bar{A}, \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3} \\
\text{c} \frac{}{\vdash \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3}
\end{array}$$

$$\begin{array}{c}
\text{id} \frac{}{\vdash A, \bar{A}} \quad \text{t} \frac{}{\vdash \top} \quad \vee \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B} \quad \wedge \frac{\vdash \Gamma, A \quad \vdash B, \perp}{\vdash \Gamma, A \wedge B, \perp} \quad \text{w} \frac{\Gamma}{\vdash \Gamma, A} \quad \text{c} \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}
\end{array}$$

$$A = p_1 \vee q_1$$

$$B = p_2 \vee q_2$$

$$\begin{array}{c}
\text{id} \frac{}{\vdash \bar{A}, A} \quad \text{id} \frac{}{\vdash \bar{p}_2, p_2} \quad \vee, \text{w} \frac{}{\vdash p_2 \vee q_2, \bar{p}_2} \\
\wedge \frac{\vdash \bar{A}, A \quad \vdash \bar{p}_2, p_2}{\vdash \bar{A}, A \wedge B, \bar{p}_2} \\
\text{id} \frac{}{\vdash \bar{A}, A} \quad \wedge \frac{}{\vdash \bar{A}, A \wedge B, \bar{p}_2} \\
\text{c}, \wedge \frac{}{\vdash \bar{A}, A \wedge B, A \wedge \bar{p}_2} \quad \vdots \\
\text{id} \frac{}{\vdash p_3, \bar{p}_3} \quad 2\text{c}, \wedge \frac{}{\vdash \bar{A}, A \wedge B, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)} \quad \vdots \\
\wedge \frac{}{\vdash \bar{A}, (A \wedge B) \wedge \bar{p}_3, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), p_3} \quad \vdots \\
\text{c}, \wedge \frac{}{\vdash \bar{A}, (A \wedge B) \wedge \bar{q}_3, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), q_3} \\
\text{c} \frac{}{\vdash \bar{A}, \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3} \\
\vdots \\
\text{c} \frac{}{\vdash \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3}
\end{array}$$

$$\begin{array}{c}
 \text{id} \frac{}{\vdash A, \bar{A}} \quad \text{t} \frac{}{\vdash \top} \quad \vee \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B} \quad \wedge \frac{\vdash \Gamma, A \quad \vdash B, \perp}{\vdash \Gamma, A \wedge B, \perp} \quad \text{w} \frac{\Gamma}{\vdash \Gamma, A} \quad \text{c} \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}
 \end{array}$$

$$\begin{array}{l}
 A = p_1 \vee q_1 \\
 B = p_2 \vee q_2
 \end{array}$$

2^3 copies of
this identity

$$\begin{array}{c}
 \text{id} \frac{}{\vdash \bar{A}, A} \quad \text{id} \frac{}{\vdash \bar{A}, A} \quad \text{id} \frac{}{\vdash \bar{p}_2, p_2} \\
 \text{c}, \wedge \frac{}{\vdash \bar{A}, A} \quad \vee, \text{w} \frac{}{\vdash p_2 \vee q_2, \bar{p}_2} \\
 \wedge \frac{}{\vdash \bar{A}, A \wedge B, \bar{p}_2} \quad \vdash \bar{A}, A \wedge B, \bar{p}_2 \\
 \vdash \bar{A}, A \wedge B, A \wedge \bar{p}_2 \quad \vdash \bar{A}, A \wedge B, A \wedge \bar{q}_2 \\
 \text{id} \frac{}{\vdash p_3, \bar{p}_3} \quad 2\text{c}, \wedge \frac{}{\vdash \bar{A}, A \wedge B, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)} \\
 \wedge \frac{}{\vdash \bar{A}, (A \wedge B) \wedge \bar{p}_3, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), p_3} \quad \vdash \bar{A}, (A \wedge B) \wedge \bar{q}_3, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), q_3 \\
 \text{c}, \wedge \frac{}{\vdash \bar{A}, \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3} \\
 \text{c} \frac{}{\vdash \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3}
 \end{array}$$

$$\begin{array}{c}
\text{id} \frac{}{\vdash A, \bar{A}} \quad \text{t} \frac{}{\vdash \top} \quad \vee \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B} \quad \wedge \frac{\vdash \Gamma, A \quad \vdash B, \perp}{\vdash \Gamma, A \wedge B, \perp} \quad \text{w} \frac{\Gamma}{\vdash \Gamma, A} \quad \text{c} \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}
\end{array}$$

$$\begin{array}{l}
A = p_1 \vee q_1 \\
B = p_2 \vee q_2
\end{array}$$

2^3 copies of
this identity

$\therefore$ the proof grows
exponentially with n

$\therefore$

$$\begin{array}{c}
\text{id} \frac{}{\vdash \bar{A}, A} \quad \text{id} \frac{}{\vdash \bar{A}, A} \quad \text{id} \frac{}{\vdash \bar{p}_2, p_2} \\
\text{c}, \wedge \frac{}{\vdash \bar{A}, A} \quad \vee, \text{w} \frac{}{\vdash p_2 \vee q_2, \bar{p}_2} \\
\wedge \frac{}{\vdash \bar{A}, A \wedge B, \bar{p}_2} \\
\text{id} \frac{}{\vdash p_3, \bar{p}_3} \quad 2\text{c}, \wedge \frac{}{\vdash \bar{A}, A \wedge B, A \wedge \bar{p}_2} \quad \vdash \bar{A}, A \wedge B, A \wedge \bar{q}_2 \\
\wedge \frac{}{\vdash \bar{A}, (A \wedge B) \wedge \bar{p}_3, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), p_3} \\
\text{c}, \wedge \frac{}{\vdash \bar{A}, \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3} \\
\text{c} \frac{}{\vdash \bar{A}, (A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2), ((A \wedge B) \wedge \bar{p}_3) \wedge ((A \wedge B) \wedge \bar{q}_3), p_3, q_3}
\end{array}$$

$$i \downarrow \frac{T}{A \vee \bar{A}}$$

$$c \downarrow \frac{A \vee A}{A}$$

$$w \downarrow \frac{\perp}{A}$$

$$s \frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$

$$= \frac{A}{B}$$

$$A = p_1 \vee q_1$$

$$B = p_2 \vee q_2$$

$$i \downarrow \frac{T}{A \vee \bar{A}}$$

$$c \downarrow \frac{A \vee A}{A}$$

$$w \downarrow \frac{\perp}{A}$$

$$s \frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$

$$= \frac{A}{B}$$

$$A = p_1 \vee q_1$$

$$B = p_2 \vee q_2$$

$$S_2 = (\bar{p}_1 \wedge \bar{q}_1) \vee ((A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)) \vee p_2 \vee q_2$$

$$S_3 = \bar{A} \vee ((A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)) \vee ((A \wedge B \wedge \bar{p}_3) \wedge (A \wedge B \wedge \bar{q}_3)) \vee p_3 \vee q_3$$

$$i \downarrow \frac{T}{A \vee \bar{A}}$$

$$c \downarrow \frac{A \vee A}{A}$$

$$w \downarrow \frac{\perp}{A}$$

$$s \frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$

$$= \frac{A}{B}$$

$$A = p_1 \vee q_1$$

$$B = p_2 \vee q_2$$

$$S'_2 = (\bar{p}_2 \wedge \bar{q}_2) \vee ((B \wedge \bar{p}_3) \wedge (B \wedge \bar{q}_3)) \vee p_3 \vee q_3$$

$$S_3 = \bar{A} \vee ((A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)) \vee ((A \wedge B \wedge \bar{p}_3) \wedge (A \wedge B \wedge \bar{q}_3)) \vee p_3 \vee q_3$$

$$i \downarrow \frac{T}{A \vee \bar{A}}$$

$$c \downarrow \frac{A \vee A}{A}$$

$$w \downarrow \frac{1}{A}$$

$$s \frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$

$$= \frac{A}{B}$$

$$A = p_1 \vee q_1$$

$$B = p_2 \vee q_2$$

$$S'_2 = (\bar{p}_2 \wedge \bar{q}_2) \vee ((B \wedge \bar{p}_3) \wedge (B \wedge \bar{q}_3)) \vee p_3 \vee q_3$$

$$S_3 = \bar{A} \vee ((A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)) \vee ((A \wedge B \wedge \bar{p}_3) \wedge (A \wedge B \wedge \bar{q}_3)) \vee p_3 \vee q_3$$

$$i \downarrow \frac{T}{A \vee \bar{A}}$$

$$c \downarrow \frac{A \vee A}{A}$$

$$w \downarrow \frac{\perp}{A}$$

$$s \frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$

$$= \frac{A}{B}$$

$\bar{p}_2$

$\bar{q}_2$

$B \wedge \bar{p}_3$

$B \wedge \bar{q}_3$

$\bar{A}$

$\vee$

$((A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2))$

$\vee$

$((A \wedge B \wedge \bar{p}_3) \wedge (A \wedge B \wedge \bar{q}_3))$

$\vee p_3 \vee q_3$

$$i \downarrow \frac{T}{A \vee \bar{A}}$$

$$c \downarrow \frac{A \vee A}{A}$$

$$w \downarrow \frac{\perp}{A}$$

$$s \frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$

$$= \frac{A}{B}$$

$$\bar{p}_2$$

$$\bar{q}_2$$

$$B \wedge \bar{p}_3$$

$$B \wedge \bar{q}_3$$

$$\vee p_3 \vee q_3$$

$$\bar{A} \quad \vee \quad ((A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)) \quad \vee \quad ((A \wedge B \wedge \bar{p}_3) \wedge (A \wedge B \wedge \bar{q}_3))$$

$$i \downarrow \frac{T}{A \vee \bar{A}}$$

$$c \downarrow \frac{A \vee A}{A}$$

$$w \downarrow \frac{\perp}{A}$$

$$s \frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$

$$= \frac{A}{B}$$

$$= \frac{\bar{p}_2}{i \downarrow \frac{T}{\bar{A} \vee A} \wedge \bar{p}_2}$$

$$\bar{q}_2$$

$$B \wedge \bar{p}_3$$

$$B \wedge \bar{q}_3$$

$$\vee p_3 \vee q_3$$

$$\bar{A} \quad \vee \quad ((A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)) \quad \vee \quad ((A \wedge B \wedge \bar{p}_3) \wedge (A \wedge B \wedge \bar{q}_3))$$

$$i \downarrow \frac{T}{A \vee \bar{A}}$$

$$c \downarrow \frac{A \vee A}{A}$$

$$w \downarrow \frac{\perp}{A}$$

$$s \frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$

$$= \frac{A}{B}$$

$$= \frac{\bar{p}_2}{i \downarrow \frac{T}{\bar{A} \vee A} \wedge \bar{p}_2}$$

$$s \frac{\bar{A} \vee (A \wedge \bar{p}_2)}{\bar{A} \vee (A \wedge \bar{p}_2)}$$

$$\bar{q}_2$$

$$B \wedge \bar{p}_3$$

$$B \wedge \bar{q}_3$$

$$\vee p_3 \vee q_3$$

$$\bar{A} \vee ((A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)) \vee ((A \wedge B \wedge \bar{p}_3) \wedge (A \wedge B \wedge \bar{q}_3))$$

$$i \downarrow \frac{T}{A \vee \bar{A}}$$

$$c \downarrow \frac{A \vee A}{A}$$

$$w \downarrow \frac{\perp}{A}$$

$$s \frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$

$$= \frac{A}{B}$$

$$= \frac{\bar{p}_2}{i \downarrow \frac{T}{\bar{A} \vee A} \wedge \bar{p}_2} \wedge$$

$$s \frac{\bar{A} \vee (A \wedge \bar{p}_2)}{s}$$

$$= \frac{\bar{q}_2}{i \downarrow \frac{T}{\bar{A} \vee A} \wedge \bar{q}_2} \wedge$$

$$s \frac{\bar{A} \vee (A \wedge \bar{q}_2)}{s}$$

$$B \wedge \bar{p}_3$$

$$B \wedge \bar{q}_3$$

$$\vee p_3 \vee q_3$$

$$\bar{A} \vee ((A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)) \vee ((A \wedge B \wedge \bar{p}_3) \wedge (A \wedge B \wedge \bar{q}_3))$$

$$i \downarrow \frac{T}{A \vee \bar{A}}$$

$$c \downarrow \frac{A \vee A}{A}$$

$$w \downarrow \frac{\perp}{A}$$

$$s \frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$

$$= \frac{A}{B}$$

$$= \frac{\bar{p}_2}{i \downarrow \frac{T}{\bar{A} \vee A} \wedge \bar{p}_2} \wedge i \downarrow \frac{T}{\bar{A} \vee A} \wedge \bar{q}_2$$

$$s \frac{\bar{A} \vee (A \wedge \bar{p}_2)}{s \frac{\bar{A} \vee (A \wedge \bar{q}_2)}$$

$$2s \frac{\bar{A} \vee \bar{A}}{\bar{A} \vee \bar{A} \vee ((A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2))}$$

$$B \wedge \bar{p}_3$$

$$B \wedge \bar{q}_3$$

$$\vee p_3 \vee q_3$$

$$\bar{A} \vee ((A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)) \vee ((A \wedge B \wedge \bar{p}_3) \wedge (A \wedge B \wedge \bar{q}_3))$$

$$\downarrow \frac{T}{A \vee \bar{A}}$$

$$\downarrow \frac{A \vee A}{A}$$

$$\downarrow \frac{\perp}{A}$$

$$s \frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$

$$= \frac{A}{B}$$

$$\begin{array}{l}
 = \frac{\bar{p}_2}{\downarrow \frac{T}{\bar{A} \vee A} \wedge \bar{p}_2} \quad \wedge \quad \downarrow \frac{T}{\bar{A} \vee A} \wedge \bar{q}_2 \\
 s \frac{\bar{A} \vee (A \wedge \bar{p}_2)}{s \frac{\bar{A} \vee (A \wedge \bar{q}_2)}{2s \frac{\bar{A} \vee \bar{A}}{\downarrow \frac{\bar{A} \vee \bar{A}}{\bar{A}} \vee ((A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2))}}
 \end{array}$$

$$B \wedge \bar{p}_3$$

$$B \wedge \bar{q}_3$$

$$\vee p_3 \vee q_3$$

$$\bar{A} \vee ((A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)) \vee ((A \wedge B \wedge \bar{p}_3) \wedge (A \wedge B \wedge \bar{q}_3))$$

$$i \downarrow \frac{T}{A \vee \bar{A}}$$

$$c \downarrow \frac{A \vee A}{A}$$

$$w \downarrow \frac{\perp}{A}$$

$$s \frac{A \wedge (B \vee C)}{(A \wedge B) \vee C} = \frac{A}{B}$$

$$= \frac{\bar{p}_2}{i \downarrow \frac{T}{\bar{A} \vee A} \wedge \bar{p}_2} \wedge i \downarrow \frac{T}{\bar{A} \vee A} \wedge \bar{q}_2$$

$$s \frac{\bar{A} \vee (A \wedge \bar{p}_2)}{2s \frac{\bar{A} \vee \bar{A}}{c \downarrow \frac{\bar{A} \vee \bar{A}}{\bar{A}} \vee ((A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2))}}$$

$$= \frac{\bar{q}_2}{i \downarrow \frac{T}{\bar{A} \vee A} \wedge \bar{q}_2} s \frac{\bar{A} \vee (A \wedge \bar{q}_2)}{2s \frac{\bar{A} \vee \bar{A}}{c \downarrow \frac{\bar{A} \vee \bar{A}}{\bar{A}} \vee ((A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2))}}$$

$\vee$

$$= \frac{B \wedge \bar{p}_3}{i \downarrow \frac{T}{\bar{A} \vee A} \wedge (B \wedge \bar{p}_3)} s \frac{\bar{A} \vee (A \wedge B \wedge \bar{p}_3)}{2s \frac{\bar{A} \vee \bar{A}}{c \downarrow \frac{\bar{A} \vee \bar{A}}{\bar{A}} \vee ((A \wedge B \wedge \bar{p}_3) \wedge (A \wedge B \wedge \bar{q}_3))}}$$

$$= \frac{B \wedge \bar{q}_3}{i \downarrow \frac{T}{\bar{A} \vee A} \wedge (B \wedge \bar{q}_3)} s \frac{\bar{A} \vee (A \wedge B \wedge \bar{q}_3)}{2s \frac{\bar{A} \vee \bar{A}}{c \downarrow \frac{\bar{A} \vee \bar{A}}{\bar{A}} \vee ((A \wedge B \wedge \bar{p}_3) \wedge (A \wedge B \wedge \bar{q}_3))}}$$

$\vee p_3 \vee q_3$

$$\bar{A} \vee ((A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)) \vee ((A \wedge B \wedge \bar{p}_3) \wedge (A \wedge B \wedge \bar{q}_3))$$

$$i\downarrow \frac{T}{A \vee \bar{A}}$$

$$c\downarrow \frac{A \vee A}{A}$$

$$w\downarrow \frac{\perp}{A}$$

$$s \frac{A \wedge (B \vee C)}{(A \wedge B) \vee C} = \frac{A}{B}$$

$$\begin{aligned}
 &= \frac{\bar{p}_2}{i\downarrow \frac{T}{\bar{A} \vee A} \wedge \bar{p}_2} \wedge i\downarrow \frac{T}{\bar{A} \vee A} \wedge \bar{q}_2 \quad \vee \quad = \frac{\bar{q}_2}{i\downarrow \frac{T}{\bar{A} \vee A} \wedge (B \wedge \bar{p}_3)} \wedge i\downarrow \frac{T}{\bar{A} \vee A} \wedge (B \wedge \bar{q}_3) \\
 &\quad s \frac{\bar{A} \vee (A \wedge \bar{p}_2)}{s \frac{\bar{A} \vee (A \wedge \bar{q}_2)}{2s \frac{\bar{A} \vee \bar{A}}{c\downarrow \frac{\bar{A} \vee \bar{A}}{\bar{A}} \vee ((A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2))}} \quad \vee \quad = \frac{B \wedge \bar{p}_3}{i\downarrow \frac{T}{\bar{A} \vee A} \wedge (B \wedge \bar{p}_3)} \wedge i\downarrow \frac{T}{\bar{A} \vee A} \wedge (B \wedge \bar{q}_3) \\
 &\quad s \frac{\bar{A} \vee (A \wedge B \wedge \bar{p}_3)}{s \frac{\bar{A} \vee (A \wedge B \wedge \bar{q}_3)}{2s \frac{\bar{A} \vee \bar{A}}{c\downarrow \frac{\bar{A} \vee \bar{A}}{\bar{A}} \vee ((A \wedge B \wedge \bar{p}_3) \wedge (A \wedge B \wedge \bar{q}_3))}} \quad \vee \quad p_3 \vee q_3 \\
 &= \frac{\bar{A} \vee \bar{A}}{c\downarrow \frac{\bar{A} \vee \bar{A}}{\bar{A}} \vee ((A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)) \vee ((A \wedge B \wedge \bar{p}_3) \wedge (A \wedge B \wedge \bar{q}_3))}
 \end{aligned}$$

$$i\downarrow \frac{T}{A \vee \bar{A}}$$

$$c\downarrow \frac{A \vee A}{A}$$

$$w\downarrow \frac{1}{A}$$

$$s \frac{A \wedge (B \vee C)}{(A \wedge B) \vee C} = \frac{A}{B}$$

compression from applying identity where we need it, not outside

$$\begin{aligned}
 & \left(\begin{aligned} & \text{---} \frac{\bar{p}_2}{\text{---}} \\ & i\downarrow \frac{T}{\bar{A} \vee A} \wedge \bar{p}_2 \\ & s \frac{\text{---}}{\bar{A} \vee (A \wedge \bar{p}_2)} \\ & 2s \frac{\text{---}}{\bar{A} \vee \bar{A}} \\ & c\downarrow \frac{\bar{A} \vee \bar{A}}{\bar{A}} \vee ((A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)) \\ & = \text{---} \\ & c\downarrow \frac{\bar{A} \vee \bar{A}}{\bar{A}} \vee ((A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)) \end{aligned} \right) \wedge \left(\begin{aligned} & \text{---} \frac{\bar{q}_2}{\text{---}} \\ & i\downarrow \frac{T}{\bar{A} \vee A} \wedge \bar{q}_2 \\ & s \frac{\text{---}}{\bar{A} \vee (A \wedge \bar{q}_2)} \\ & 2s \frac{\text{---}}{\bar{A} \vee \bar{A}} \\ & c\downarrow \frac{\bar{A} \vee \bar{A}}{\bar{A}} \vee ((A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)) \end{aligned} \right) \vee \left(\begin{aligned} & \text{---} \frac{B \wedge \bar{p}_3}{\text{---}} \\ & i\downarrow \frac{T}{\bar{A} \vee A} \wedge (B \wedge \bar{p}_3) \\ & s \frac{\text{---}}{\bar{A} \vee (A \wedge B \wedge \bar{p}_3)} \\ & 2s \frac{\text{---}}{\bar{A} \vee \bar{A}} \\ & c\downarrow \frac{\bar{A} \vee \bar{A}}{\bar{A}} \vee ((A \wedge B \wedge \bar{p}_3) \wedge (A \wedge B \wedge \bar{q}_3)) \end{aligned} \right) \wedge \left(\begin{aligned} & \text{---} \frac{B \wedge \bar{q}_3}{\text{---}} \\ & i\downarrow \frac{T}{\bar{A} \vee A} \wedge (B \wedge \bar{q}_3) \\ & s \frac{\text{---}}{\bar{A} \vee (A \wedge B \wedge \bar{q}_3)} \\ & 2s \frac{\text{---}}{\bar{A} \vee \bar{A}} \\ & c\downarrow \frac{\bar{A} \vee \bar{A}}{\bar{A}} \vee ((A \wedge B \wedge \bar{p}_3) \wedge (A \wedge B \wedge \bar{q}_3)) \end{aligned} \right) \vee p_3 \vee q_3 \\
 & = \text{---} \\
 & c\downarrow \frac{\bar{A} \vee \bar{A}}{\bar{A}} \vee ((A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)) \vee ((A \wedge B \wedge \bar{p}_3) \wedge (A \wedge B \wedge \bar{q}_3))
 \end{aligned}$$

$$i \downarrow \frac{T}{A \vee \bar{A}}$$

$$c \downarrow \frac{A \vee A}{A}$$

$$w \downarrow \frac{1}{A}$$

$$s \frac{A \wedge (B \vee C)}{(A \wedge B) \vee C} = \frac{A}{B}$$

$$\text{width} = O\left(\sum_{i=1}^{n-1} |\alpha_i| + |\beta_i|\right) = O(n^2)$$

$$\begin{aligned}
 &= \frac{\bar{p}_2}{i \downarrow \frac{T}{\bar{A} \vee A} \wedge \bar{p}_2} \wedge i \downarrow \frac{T}{\bar{A} \vee A} \wedge \bar{q}_2 \\
 &\quad s \frac{\bar{A} \vee (A \wedge \bar{p}_2)}{\bar{A} \vee (A \wedge \bar{q}_2)} \vee \\
 &\quad 2s \frac{\bar{A} \vee \bar{A}}{\bar{A}} \vee ((A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)) \\
 &= \frac{\bar{q}_2}{i \downarrow \frac{T}{\bar{A} \vee A} \wedge \bar{q}_2} \vee \\
 &\quad s \frac{\bar{A} \vee (A \wedge \bar{q}_2)}{\bar{A} \vee (A \wedge \bar{p}_3)} \vee \\
 &\quad 2s \frac{\bar{A} \vee \bar{A}}{\bar{A}} \vee ((A \wedge \bar{p}_3) \wedge (A \wedge \bar{q}_3)) \\
 &= \frac{B \wedge \bar{p}_3}{i \downarrow \frac{T}{\bar{A} \vee A} \wedge (B \wedge \bar{p}_3)} \wedge i \downarrow \frac{T}{\bar{A} \vee A} \wedge (B \wedge \bar{q}_3) \\
 &\quad s \frac{\bar{A} \vee (A \wedge B \wedge \bar{p}_3)}{\bar{A} \vee (A \wedge B \wedge \bar{q}_3)} \vee \\
 &\quad 2s \frac{\bar{A} \vee \bar{A}}{\bar{A}} \vee ((A \wedge B \wedge \bar{p}_3) \wedge (A \wedge B \wedge \bar{q}_3)) \\
 &= \frac{B \wedge \bar{q}_3}{i \downarrow \frac{T}{\bar{A} \vee A} \wedge (B \wedge \bar{q}_3)} \wedge i \downarrow \frac{T}{\bar{A} \vee A} \wedge (B \wedge \bar{p}_3) \\
 &\quad s \frac{\bar{A} \vee (A \wedge B \wedge \bar{q}_3)}{\bar{A} \vee (A \wedge B \wedge \bar{p}_3)} \vee \\
 &\quad 2s \frac{\bar{A} \vee \bar{A}}{\bar{A}} \vee ((A \wedge B \wedge \bar{q}_3) \wedge (A \wedge B \wedge \bar{p}_3)) \\
 &= \frac{\bar{A} \vee \bar{A}}{\bar{A}} \vee ((A \wedge \bar{p}_2) \wedge (A \wedge \bar{q}_2)) \vee ((A \wedge B \wedge \bar{p}_3) \wedge (A \wedge B \wedge \bar{q}_3))
 \end{aligned}$$

$$\text{height} = O(n)$$

$$\vee p_3 \vee q_3$$

$$i \downarrow \frac{T}{A \vee \bar{A}}$$

$$c \downarrow \frac{A \vee A}{A}$$

$$w \downarrow \frac{\perp}{A}$$

$$s \frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$

$$= \frac{A}{B}$$

$$i \downarrow \frac{T}{(\bar{p}_n \wedge \bar{q}_n) \vee (p_n \vee q_n)}$$

$$\Phi_2 \parallel$$

$$\vdots$$

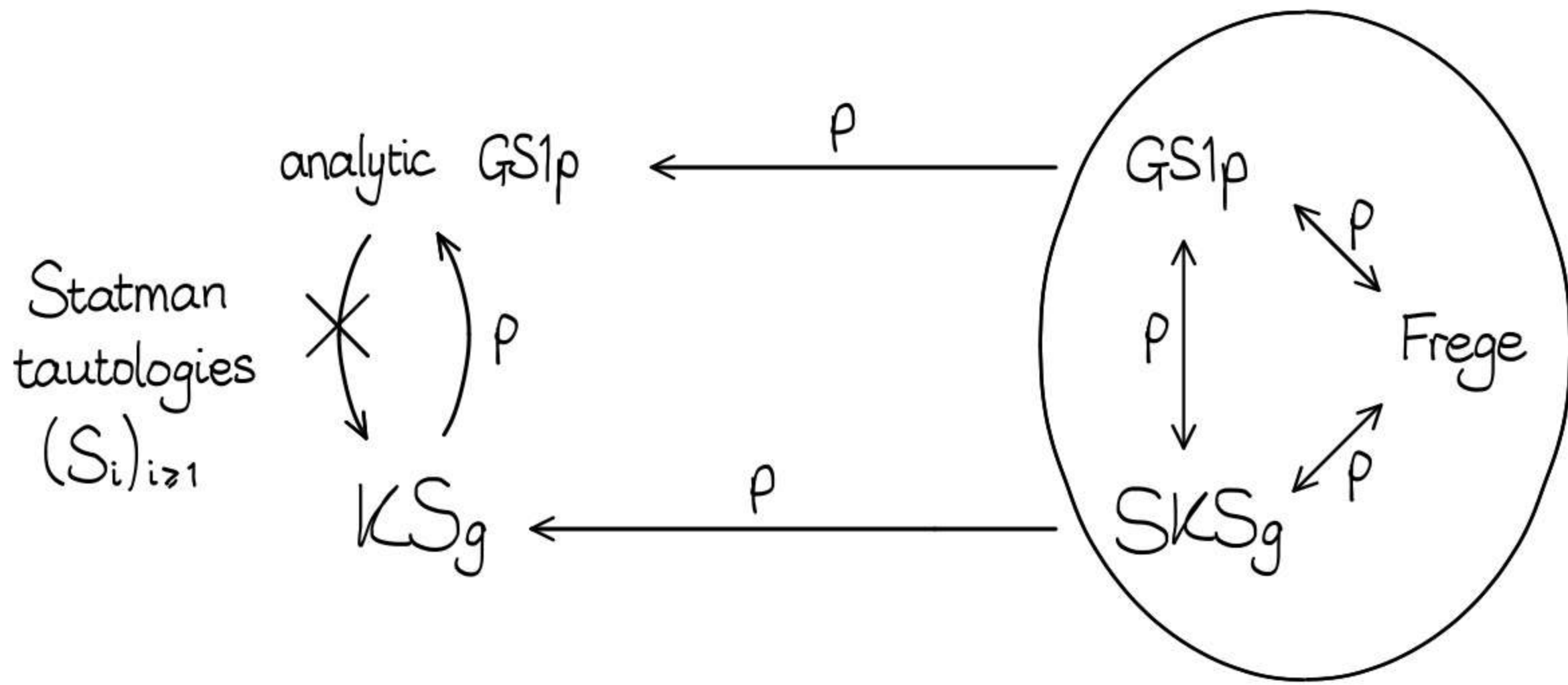
$$S'_{n-1}$$

$$\Phi_n \parallel$$

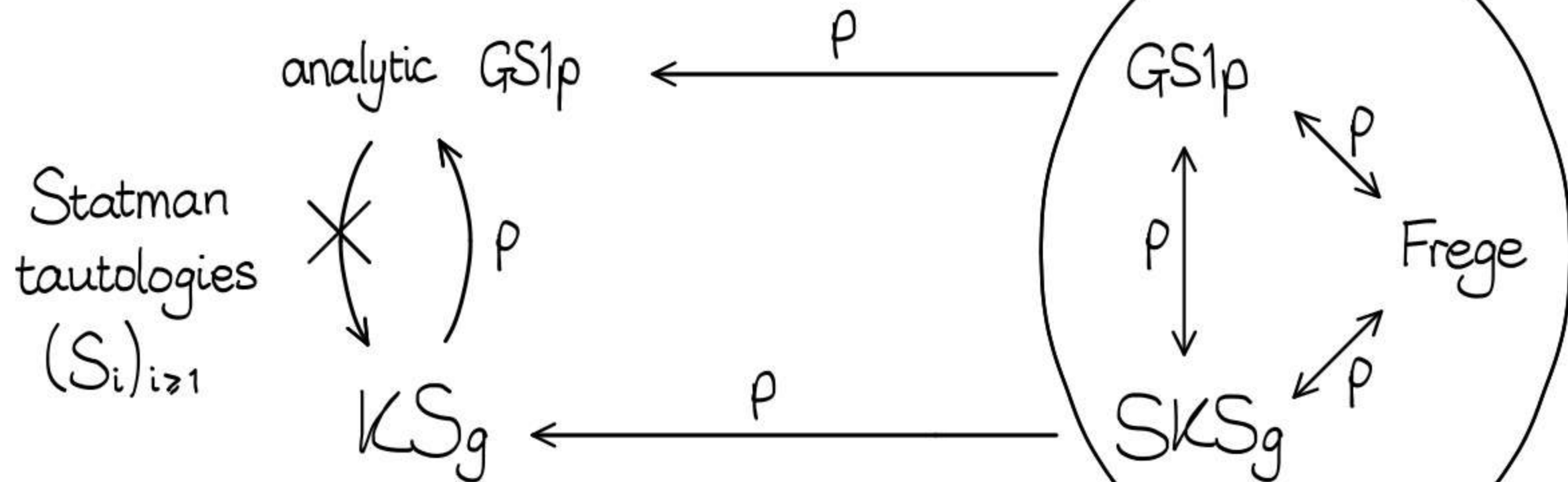
$$S_n$$

size of Φ_n is $O(n^3)$

$$O(n^4)$$



size explosion because we have to
duplicate the context to get pieces
where we need them



compression because we can
create material where we like

