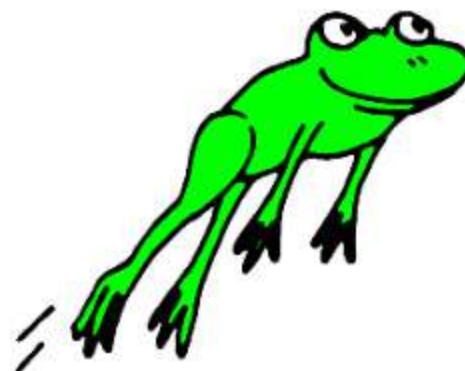


A gentle introduction to deep inference

6. Atomic flows



ESSLLI 2025

Victoria Barrett and Lutz Straßburger

From derivations to flows

$$\frac{\frac{\frac{a}{\text{act} \uparrow a \wedge a} \vee \frac{\bar{a}}{\text{act} \uparrow \bar{a} \wedge \bar{a}}}{m \frac{}{(a \vee \bar{a}) \wedge (\bar{a} \vee \bar{a})}}}{s \frac{}{\frac{\bar{a} \wedge (a \vee \bar{a})}{\frac{s \frac{}{\frac{\bar{a} \wedge a}{\text{ai} \uparrow \perp}} \vee \frac{a}{T}}}{a \vee}}$$

From derivations to flows

$$\frac{\frac{m}{(a \vee \bar{a}) \wedge (\bar{a} \vee a)} \quad s \frac{s}{\frac{a \vee \frac{a \wedge \bar{a}}{\perp}}{\bar{a} \wedge (a \vee \bar{a})}}}{v \frac{act \frac{\bar{a}}{\bar{a} \wedge \bar{a}}}{\bar{a} \wedge \bar{a}}} \quad v \frac{act \frac{a}{T}}{a}}$$

Diagram illustrating the derivation:

- The top part shows a flow diagram with a node labeled a . An incoming arrow from the left is labeled $act \uparrow$. Two outgoing arrows point downwards, both labeled a .
- To the right of the node is a vertical bar (v) followed by a box containing \bar{a} above a horizontal line, with $act \uparrow$ above it.
- Below this is another box containing $\bar{a} \wedge \bar{a}$ below the horizontal line.
- A horizontal line with a break separates the top from the middle section.
- The middle section starts with m above a horizontal line, followed by $(a \vee \bar{a}) \wedge (\bar{a} \vee a)$ below the line.
- Below this is another horizontal line with a break.
- The bottom section starts with s above a horizontal line, followed by $\bar{a} \wedge (a \vee \bar{a})$ below the line.
- Below this is another horizontal line with a break.
- The bottom-most section starts with $a \vee$ above a box containing $\bar{a} \wedge a$ above a horizontal line, with $act \uparrow$ above it.
- Below this is another box containing \perp below the horizontal line.
- Following this is a vertical bar (v) followed by a box containing $\frac{a}{T}$.

From derivations to flows

$$\frac{\frac{m}{(a \vee \bar{a}) \wedge (\bar{a} \vee a)} \quad s \frac{s}{\frac{a \vee \bar{a} \wedge (a \vee \bar{a})}{\frac{ai \uparrow \bar{a} \wedge a}{\perp} \vee \frac{a}{T}}}}{v \left[\begin{array}{c} \bar{a} \\ \hline a \wedge \bar{a} \end{array} \right]}$$

Diagram illustrating the derivation:

- The top part shows a flow diagram with a node labeled a . An incoming edge from the left is labeled a , and an outgoing edge to the right is also labeled a . A curved arrow labeled \wedge connects the two a nodes. Above the node, there is a small circle with a diagonal line through it, and above that, the word "act".
- To the right of the node is a vertical bar labeled v .
- Below the node is a rectangular box containing the formula \bar{a} at the top, followed by a horizontal line, and then $\bar{a} \wedge a$ below it.
- The bottom part shows a horizontal line with two segments:
 - The left segment is labeled m above and s below.
 - The right segment is labeled s above and $a \vee \bar{a} \wedge (a \vee \bar{a})$ below.
- Below the bottom line is another rectangular box:
 - Left side: $a \vee$ above and $ai \uparrow \bar{a} \wedge a$ below.
 - Right side: \perp below and $\vee \frac{a}{T}$ below.

From derivations to flows

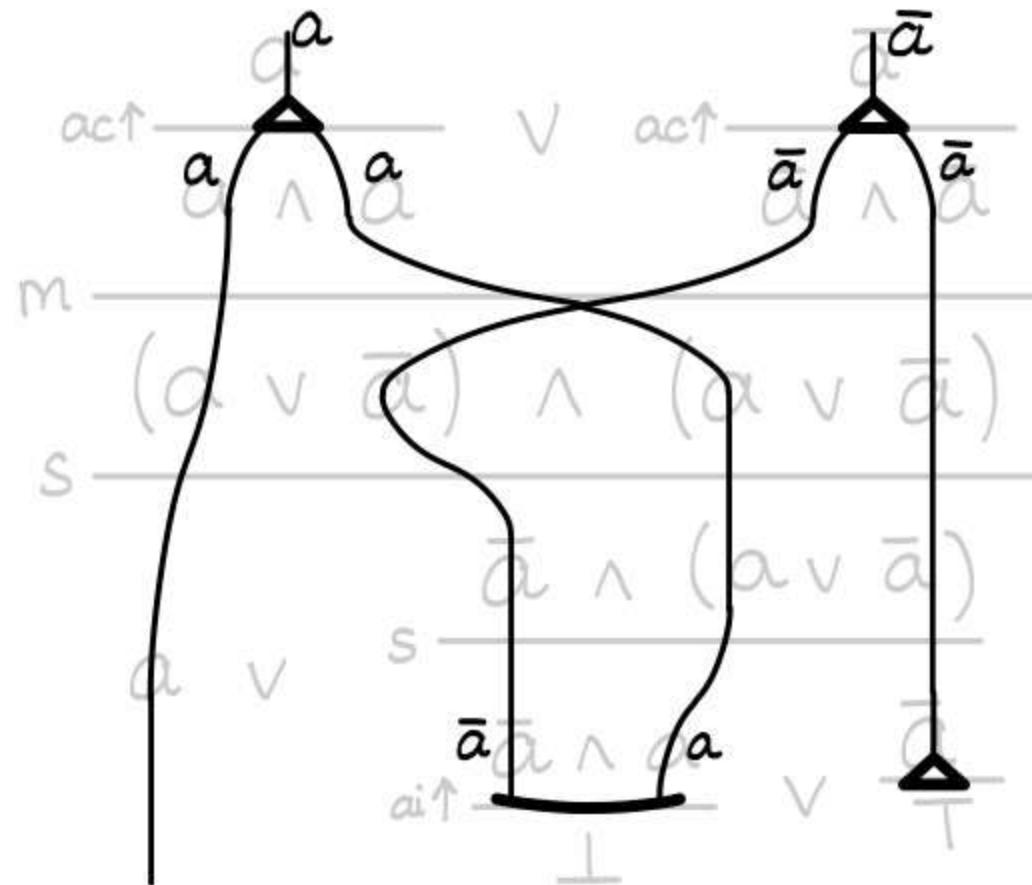
$$\frac{\frac{m}{(a \vee \bar{a}) \wedge (\bar{a} \vee a)} \quad s \frac{s}{\bar{a} \wedge (a \vee \bar{a})}}{a \vee \frac{\bar{a} \vdash \bot}{\perp} \quad \frac{\bar{a} \vdash \top}{\top}}$$

Diagram illustrating the derivation:

- The top part shows a flow diagram with a node labeled a . An incoming edge from the left is labeled a , and an outgoing edge to the right is also labeled a . A curved arrow labeled \wedge connects the two a s. Above the node, there is a small circle with a diagonal line through it, and above that, the word "act".
- To the right of the node is a vertical bar labeled \vee .
- Below the node is a rectangular box containing:
$$\frac{\text{act} \uparrow \bar{a}}{\bar{a} \wedge \bar{a}}$$
- The middle part of the derivation is enclosed in a horizontal line. It consists of two parts separated by a vertical bar:
$$(a \vee \bar{a}) \wedge (\bar{a} \vee a)$$
- The bottom part of the derivation is enclosed in a horizontal line. It consists of two parts separated by a vertical bar:
$$\bar{a} \wedge (a \vee \bar{a})$$
- The bottom-most part of the derivation is enclosed in a rectangular box:
$$\frac{s}{\frac{\bar{a} \vdash \bot}{\perp} \quad \frac{\bar{a} \vdash \top}{\top}}$$

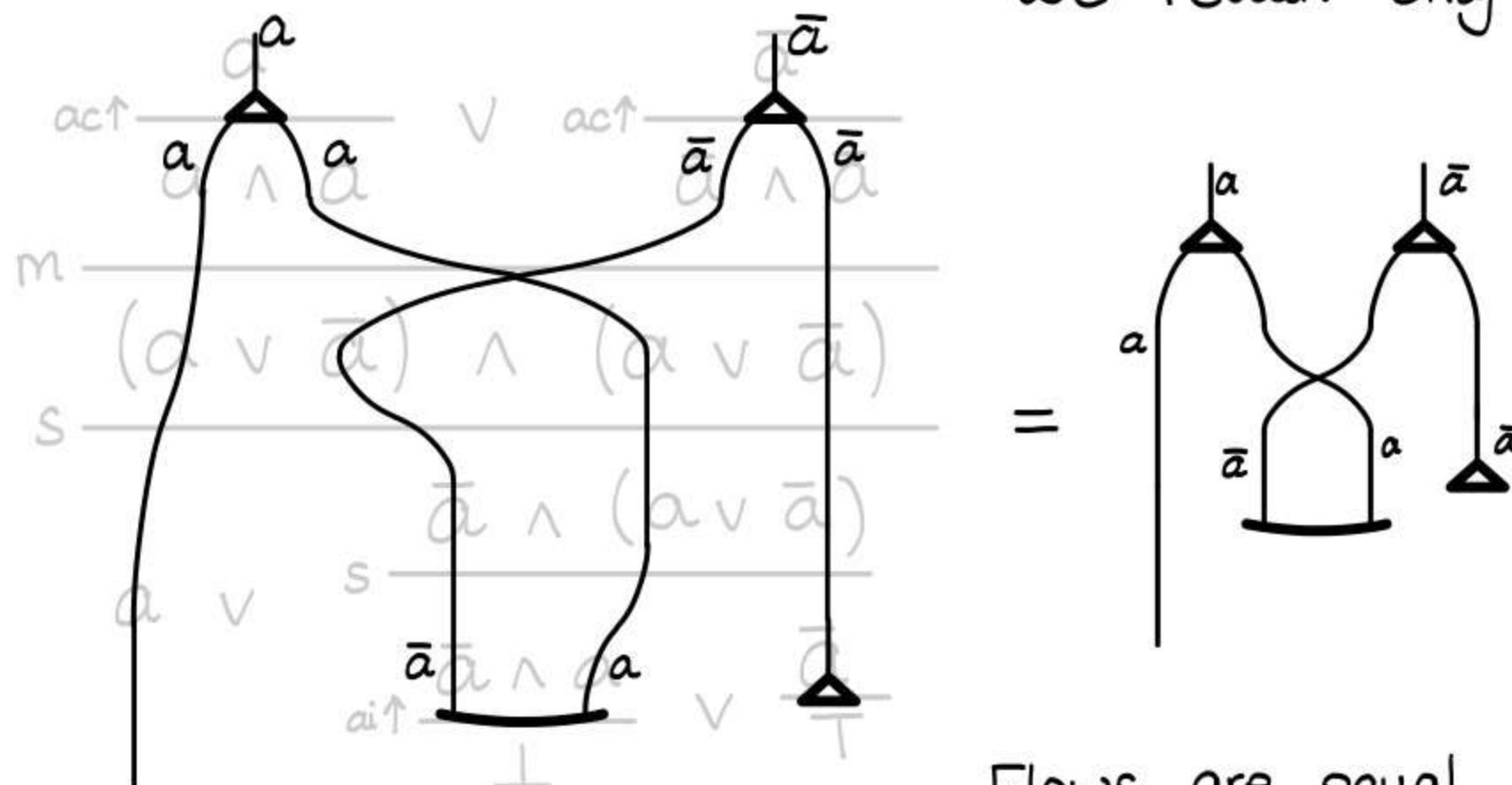
From derivations to flows

We retain only the structural information



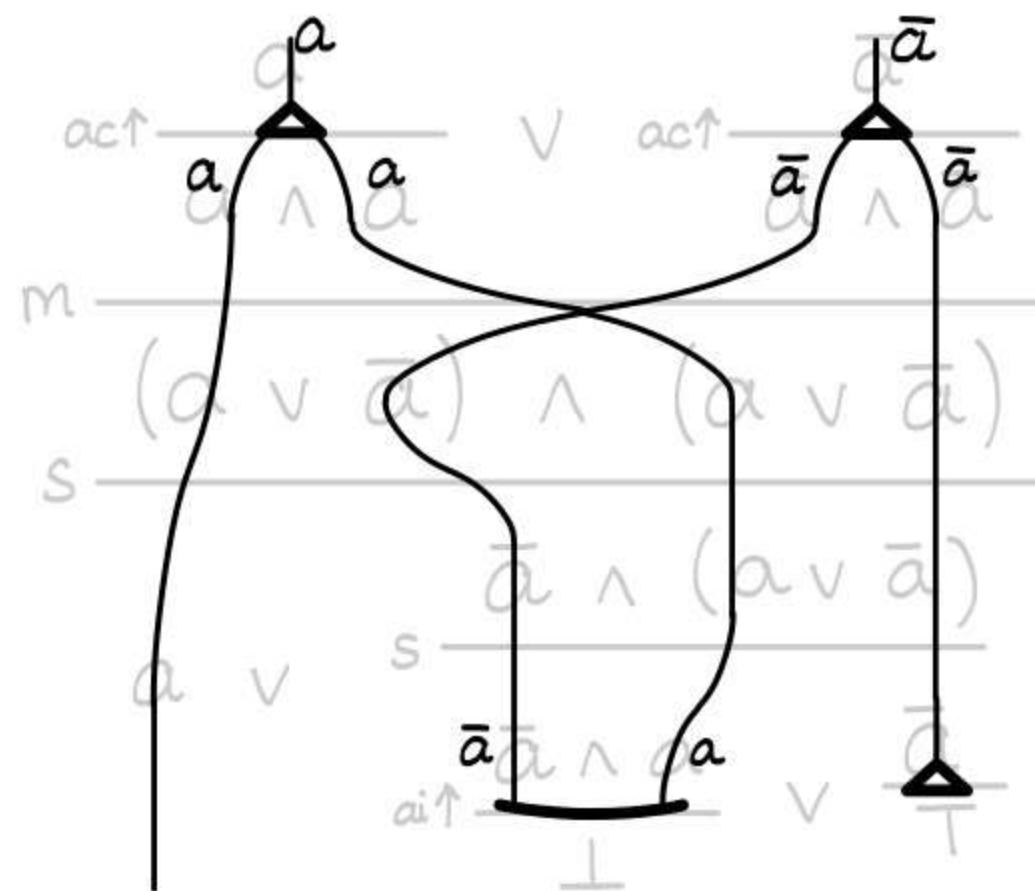
From derivations to flows

We retain only the structural information

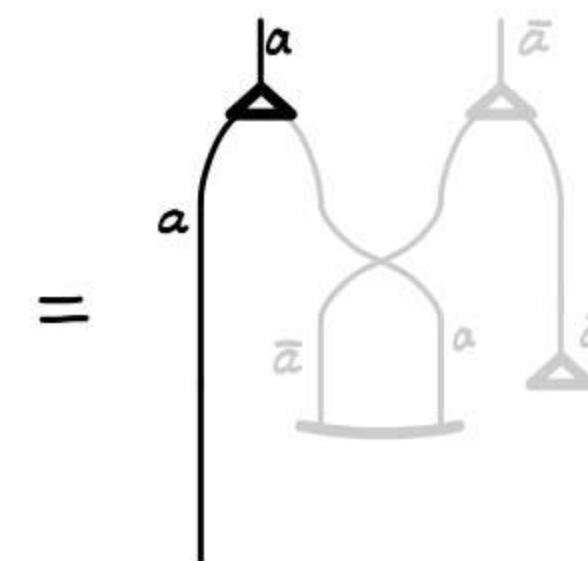


Flows are equal under continuous deformation

From derivations to flows

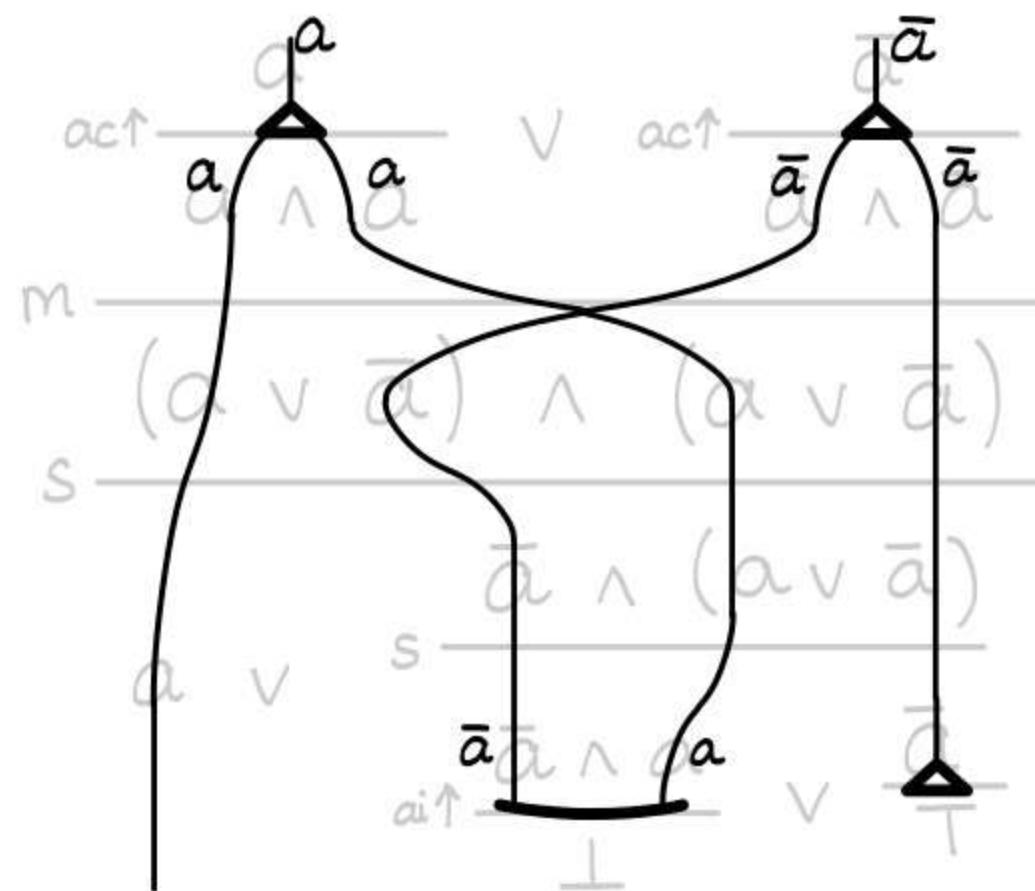


We retain only the structural information
and can trace the paths of atoms

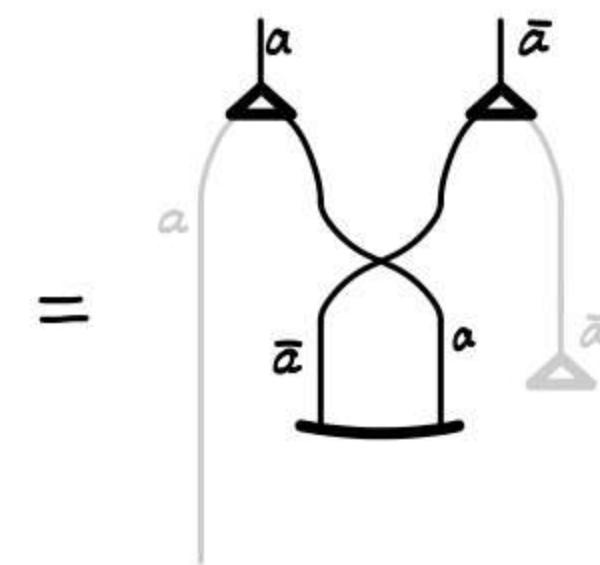


Flows are equal under continuous deformation

From derivations to flows

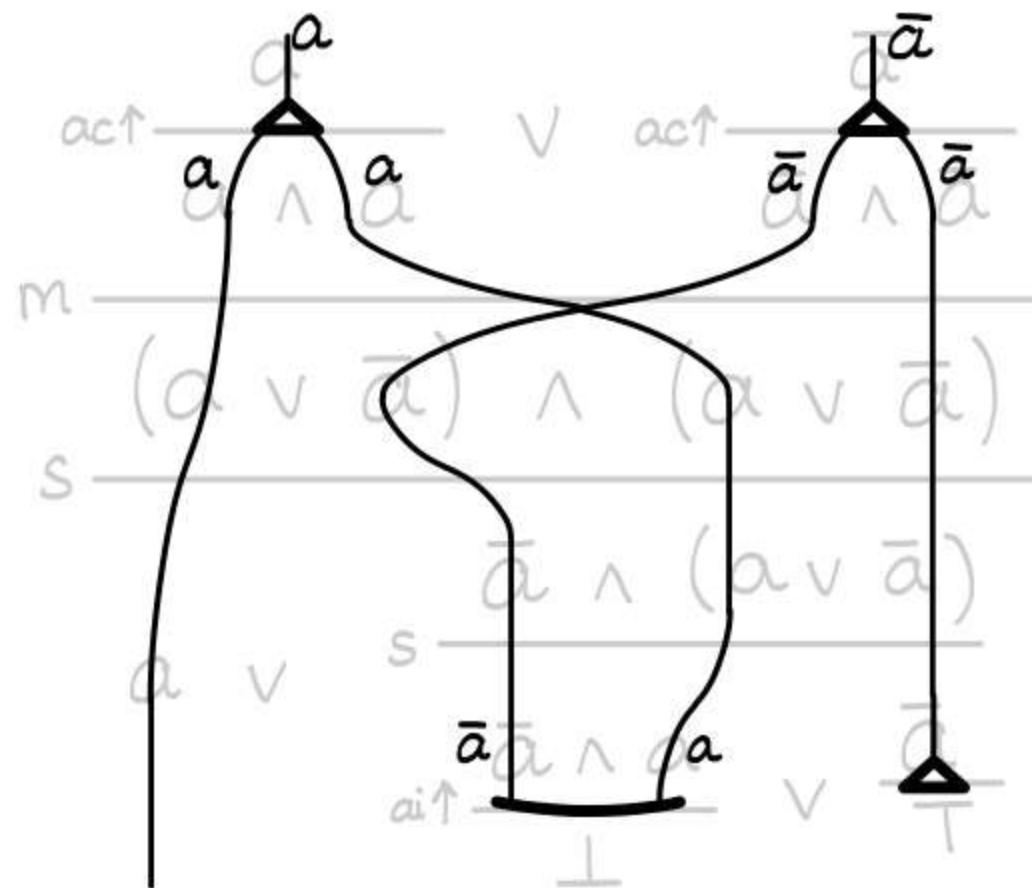


We retain only the structural information
and can trace the paths of atoms

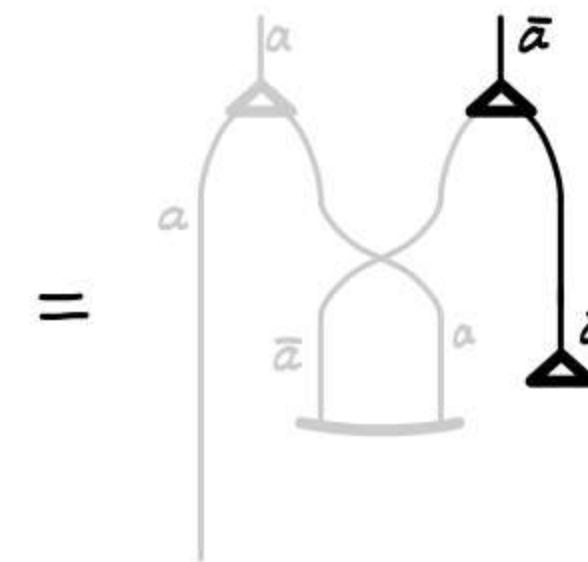


Flows are equal under continuous deformation

From derivations to flows

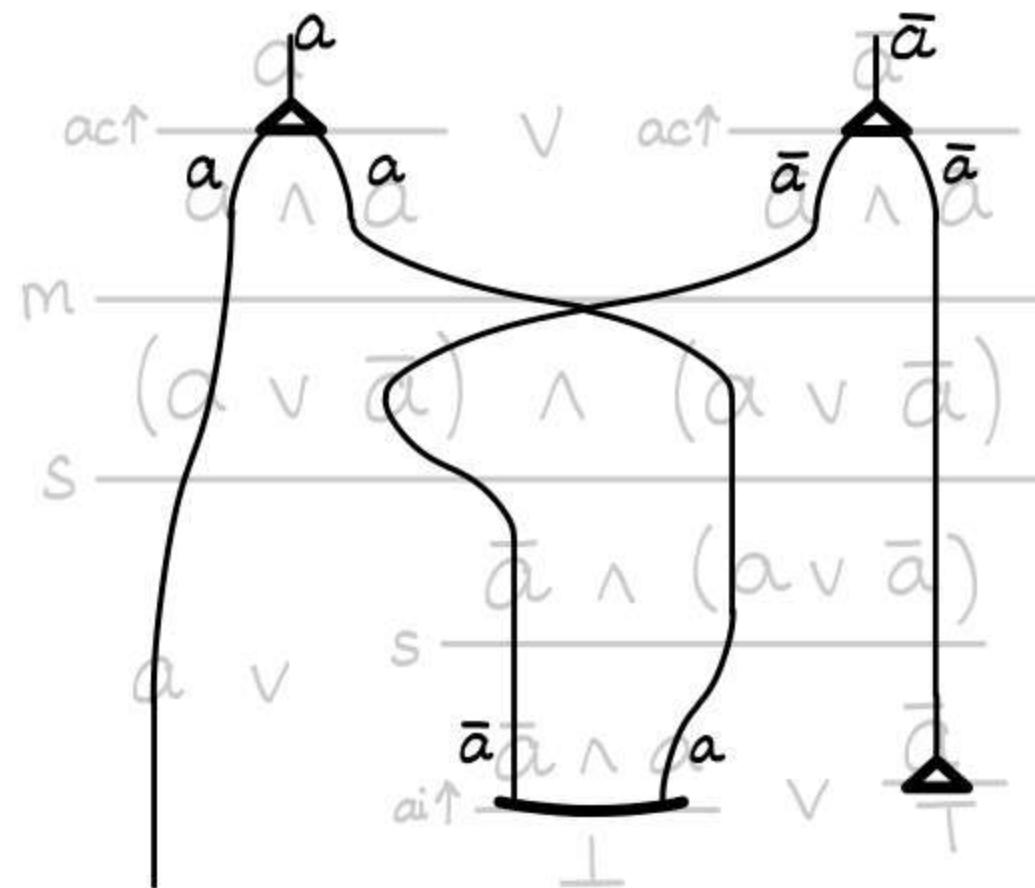


We retain only the structural information
and can trace the paths of atoms

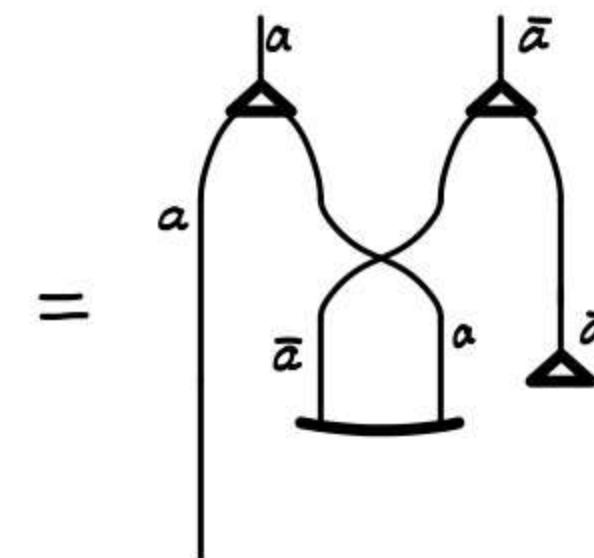


Flows are equal under continuous deformation

From derivations to flows

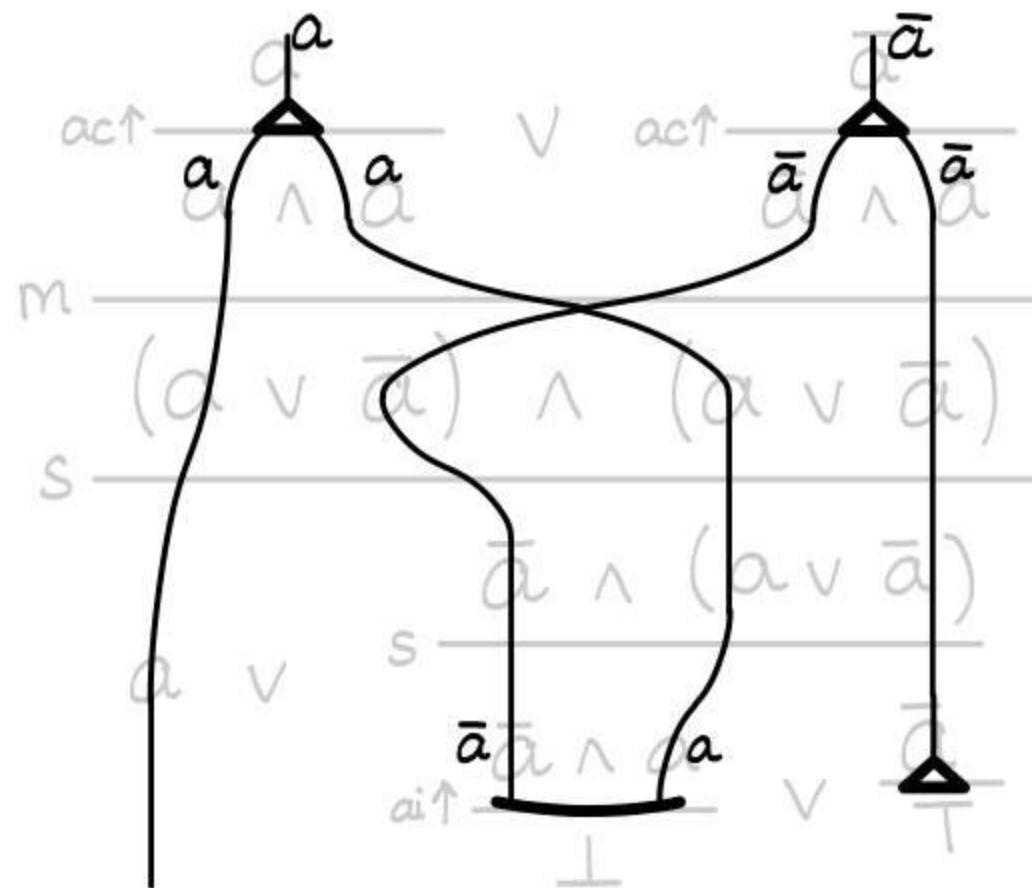


We retain only the structural information
and can trace the paths of atoms

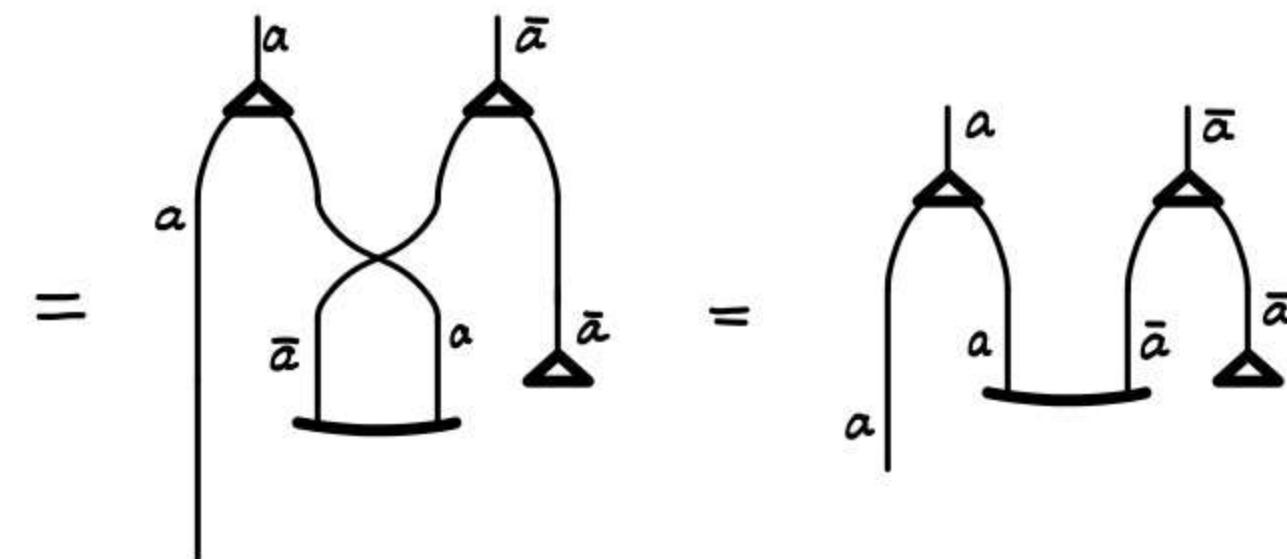


Flows are equal under continuous deformation

From derivations to flows

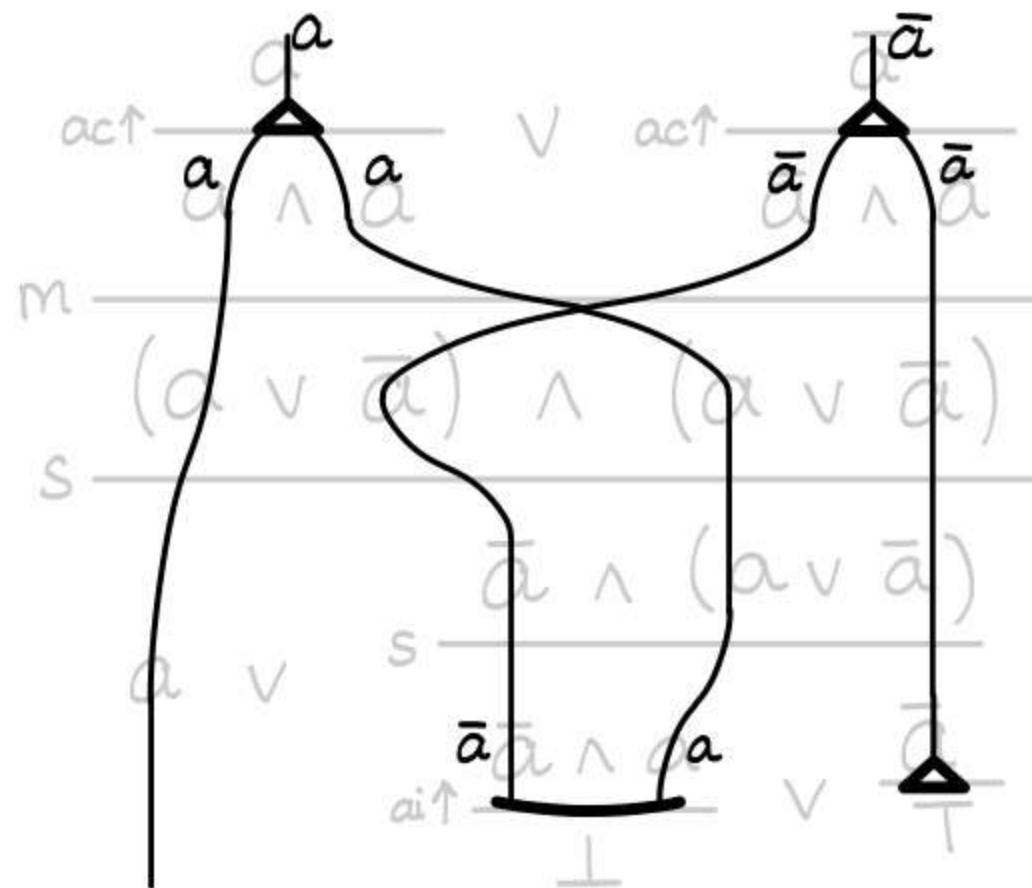


We retain only the structural information
and can trace the paths of atoms

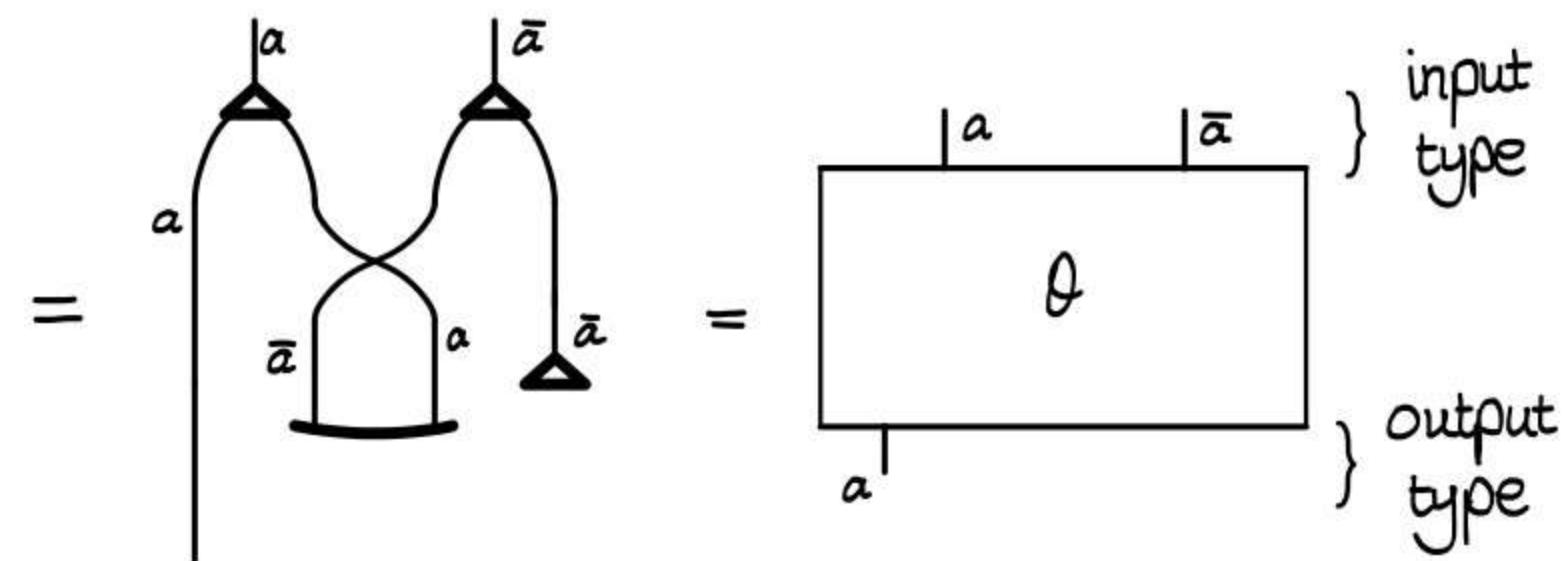


Flows are equal under continuous deformation

From derivations to flows



We retain only the structural information
and can trace the paths of atoms

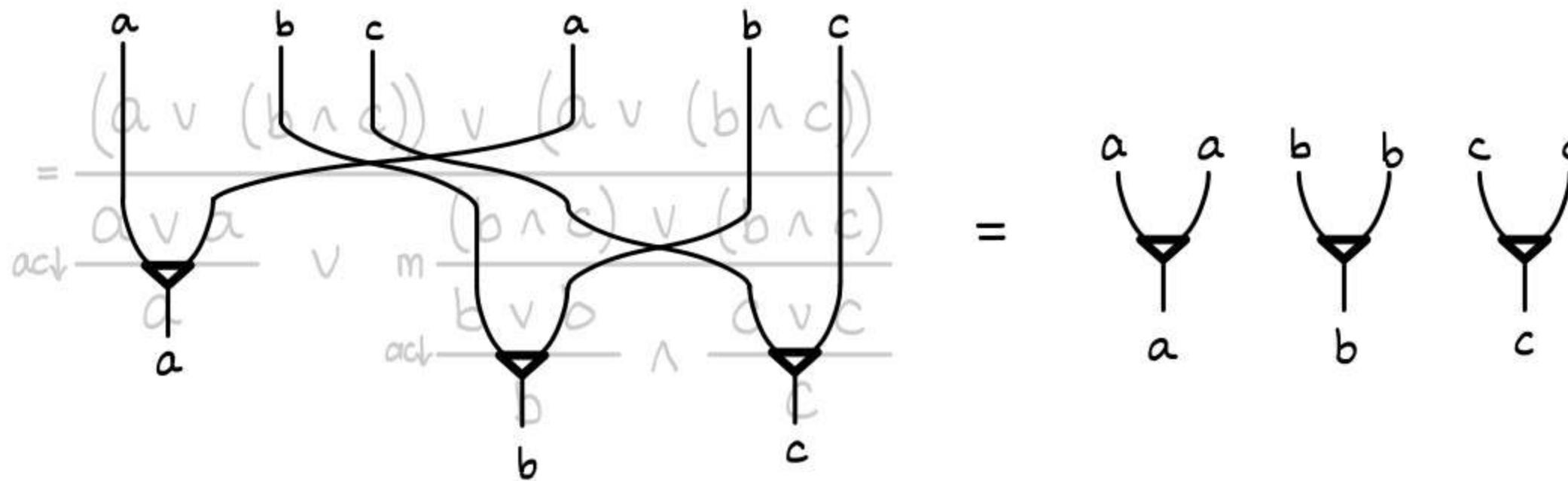


Flows are equal under continuous deformation

From derivations to flows

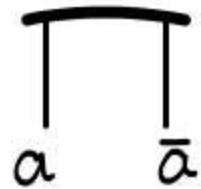
$$= \frac{(a \vee (b \wedge c)) \vee (a \vee (b \wedge c))}{\text{act} \downarrow \frac{a \vee a}{a} \vee \text{m} \frac{(b \wedge c) \vee (b \wedge c)}{\text{act} \downarrow \frac{b \vee b}{b} \wedge \frac{c \vee c}{c}}}$$

From derivations to flows



From derivations to flows

$$ai \downarrow \frac{T}{a \vee \bar{a}}$$



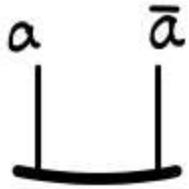
$$act \downarrow \frac{a \vee a}{a}$$



$$aw \downarrow \frac{\perp}{a}$$



$$ai \uparrow \frac{a \wedge \bar{a}}{\perp}$$



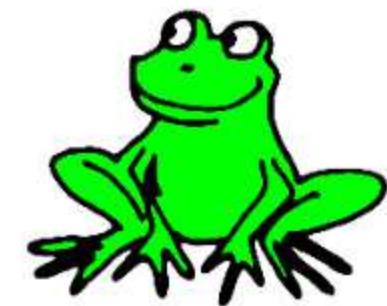
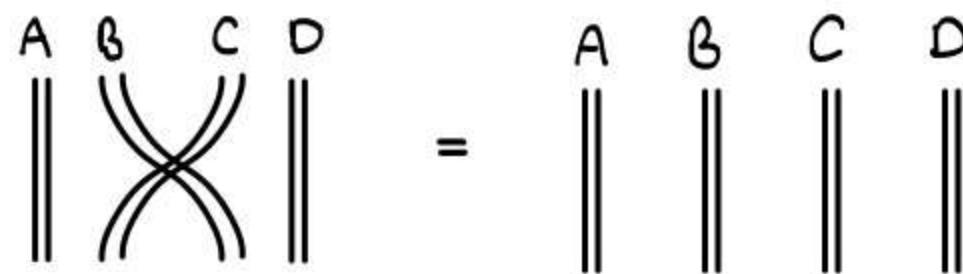
$$act \uparrow \frac{a}{a \wedge a}$$



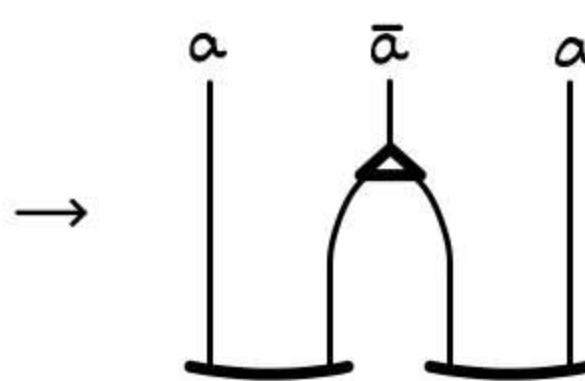
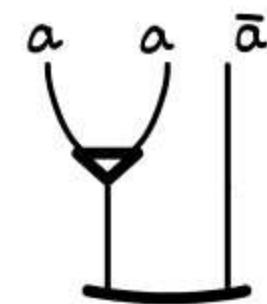
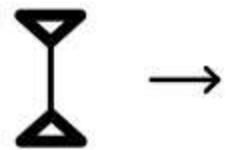
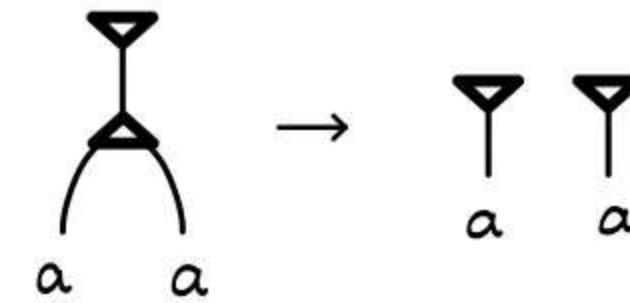
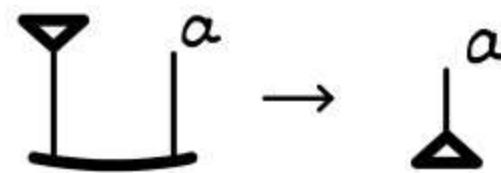
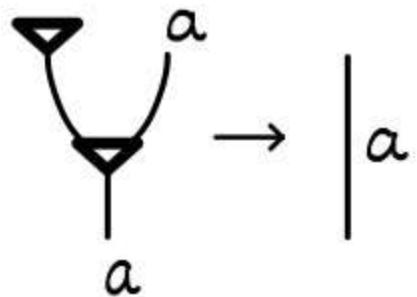
$$aw \uparrow \frac{a}{T}$$



$$m \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$



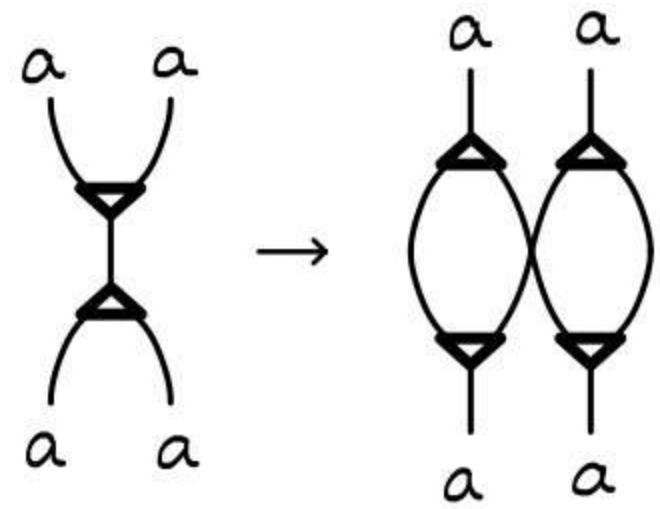
Rewriting flows



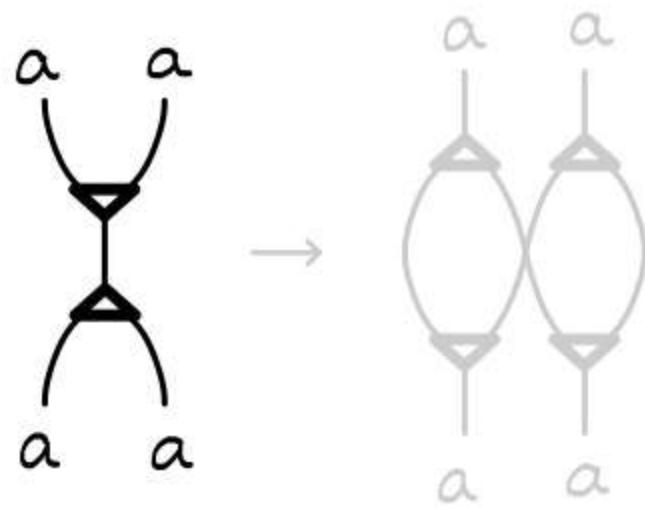
and duals



Rewriting flows lifts to derivations



Rewriting flows lifts to derivations

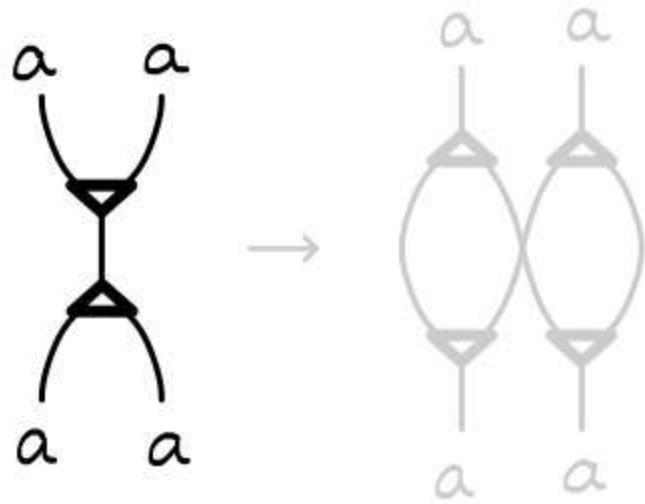


$$K \left\{ \text{act} \frac{a \vee a}{a^*} \right\}$$

$$\Phi$$

$$H \left\{ \text{act} \frac{a^*}{a \wedge a} \right\}$$

Rewriting flows lifts to derivations



$$K \left\{ \frac{\text{act} \downarrow a \vee a}{a^*} \right\}$$

$$\frac{K \{ a \vee a \}}{[a \vee a / a^*] \Phi}$$

Φ

\rightarrow

$$H \left\{ \frac{\text{act} \downarrow a^*}{a \wedge a} \right\}$$

Rewriting flows lifts to derivations



→



$$K \left\{ \frac{\text{act} \downarrow a \vee a}{a^*} \right\}$$

Φ

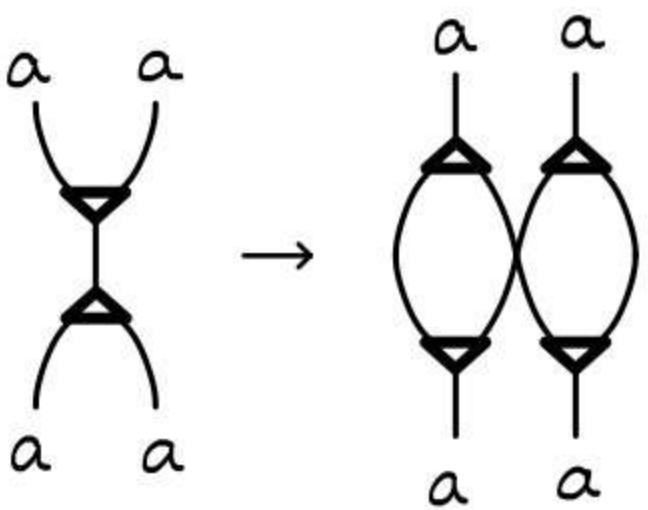
$$H \left\{ \frac{\text{act} \downarrow a^*}{a \wedge a} \right\}$$

→

$$H \left\{ \frac{\begin{array}{c} \text{act} \frac{a}{a \wedge a} \vee \text{act} \frac{a}{a \wedge a} \\ m \frac{a \vee a}{a} \end{array}}{\begin{array}{c} a \\ \wedge \text{act} \frac{a \vee a}{a} \end{array}} \right\}$$

$K \{ a \vee a \}$
[$a \vee a / a^*$] Φ

Rewriting flows lifts to derivations



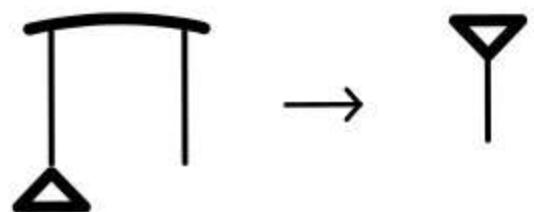
$$K \left\{ \frac{\text{act} \downarrow a \vee a}{a^*} \right\}$$

$$\Phi$$

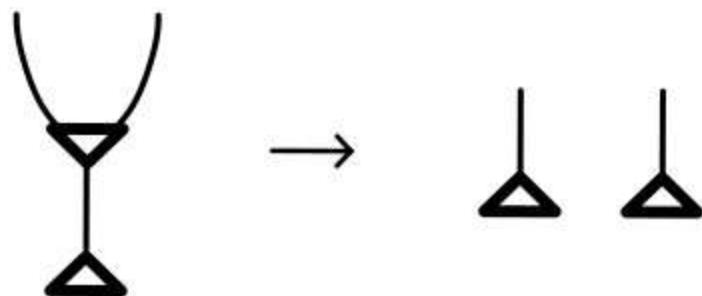
$$H \left\{ \frac{\text{act} \downarrow a^*}{a \wedge a} \right\}$$

$$\begin{array}{c}
 K \{ a \vee a \} \\
 \hline
 [a \vee a / a^*] \Phi \\
 \hline
 H \left\{ \frac{\begin{array}{c} \text{act} \downarrow a \\ \vee \text{act} \downarrow a \end{array}}{\begin{array}{c} a \wedge a \\ a \end{array}} \right\}
 \end{array}$$

Rewriting flows lifts to derivations



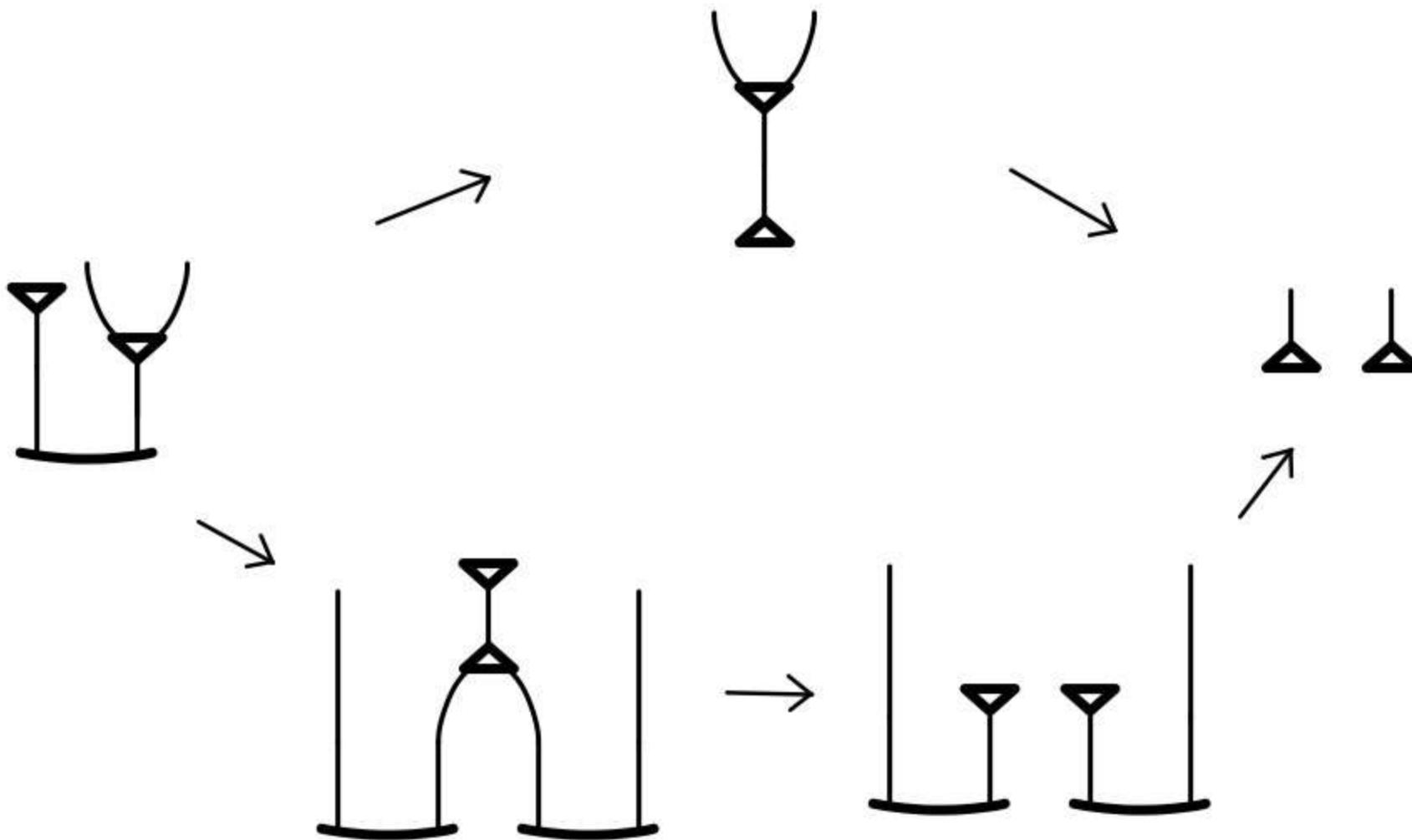
We've seen these rewrite steps
before in the cut elimination
procedure for SKS!



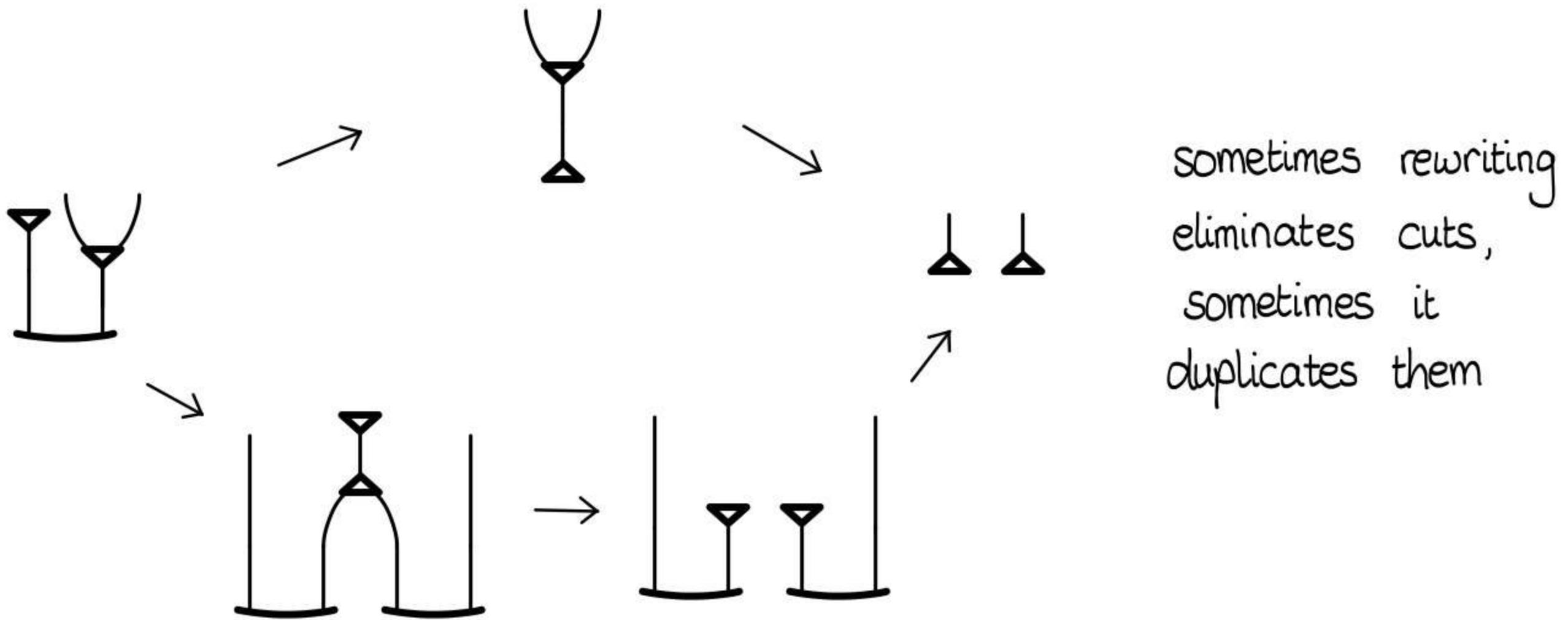
Rewriting flows is locally confluent



Rewriting flows is locally confluent



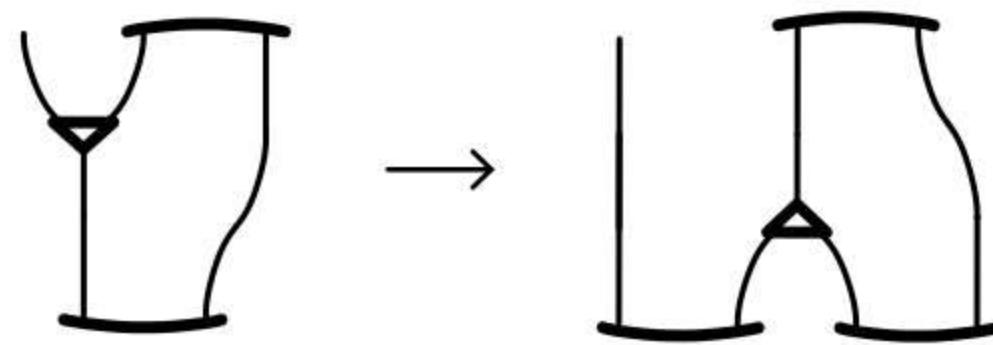
Rewriting flows is locally confluent



Rewriting flows does not always terminate!



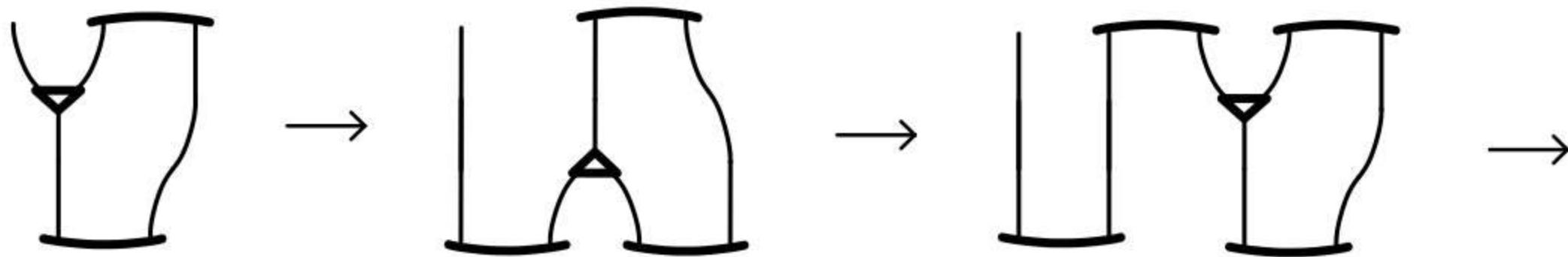
Rewriting flows does not always terminate!



Rewriting flows does not always terminate!

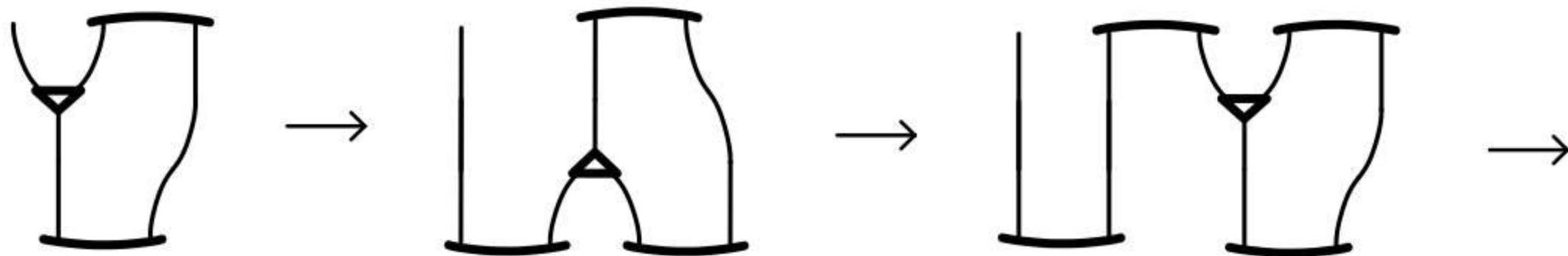


Rewriting flows does not always terminate!



the problem arises when a (co)contraction gets into a path
from an identity to a cut

Rewriting flows does not always terminate!



the problem arises when a (co)contraction gets into a path
from an identity to a cut



break the paths between
cuts and identities

Identities can be pulled up derivations

B

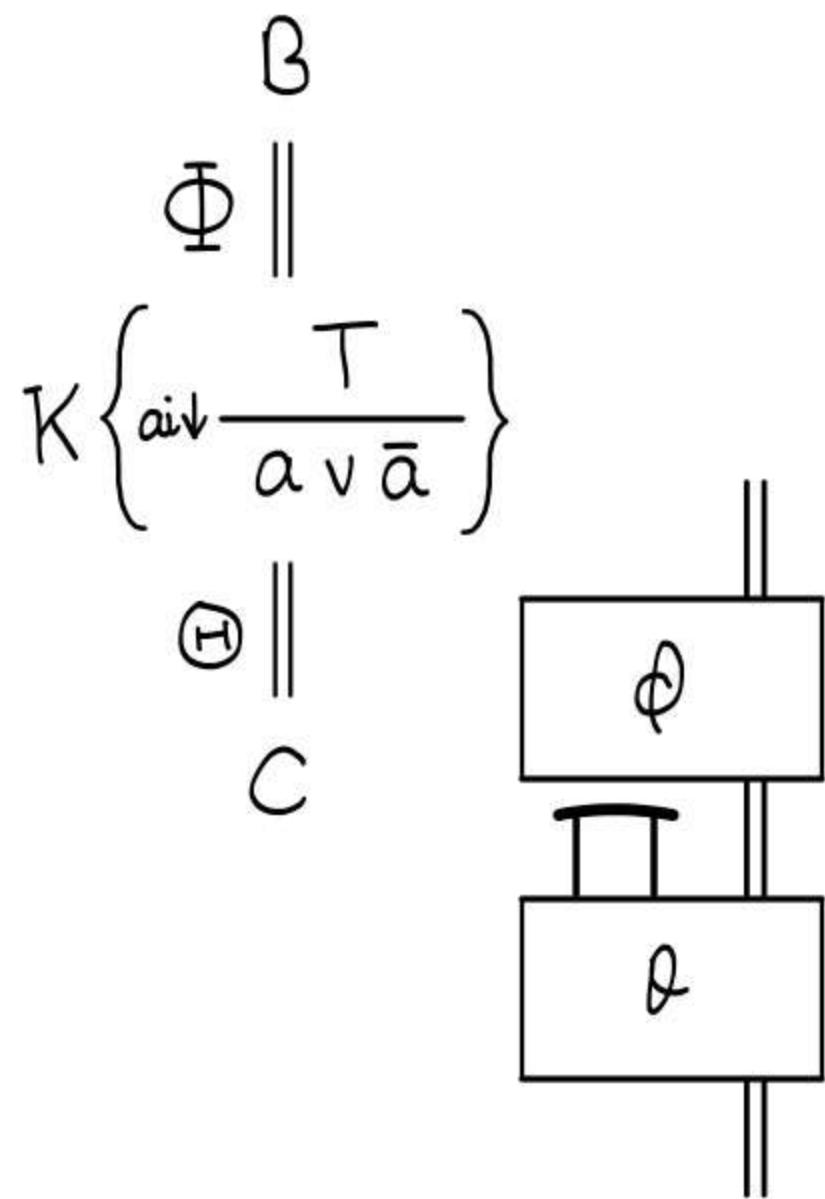
$\Phi \parallel$

$$K \left\{ \frac{\alpha \downarrow T}{\alpha \vee \bar{\alpha}} \right\}$$

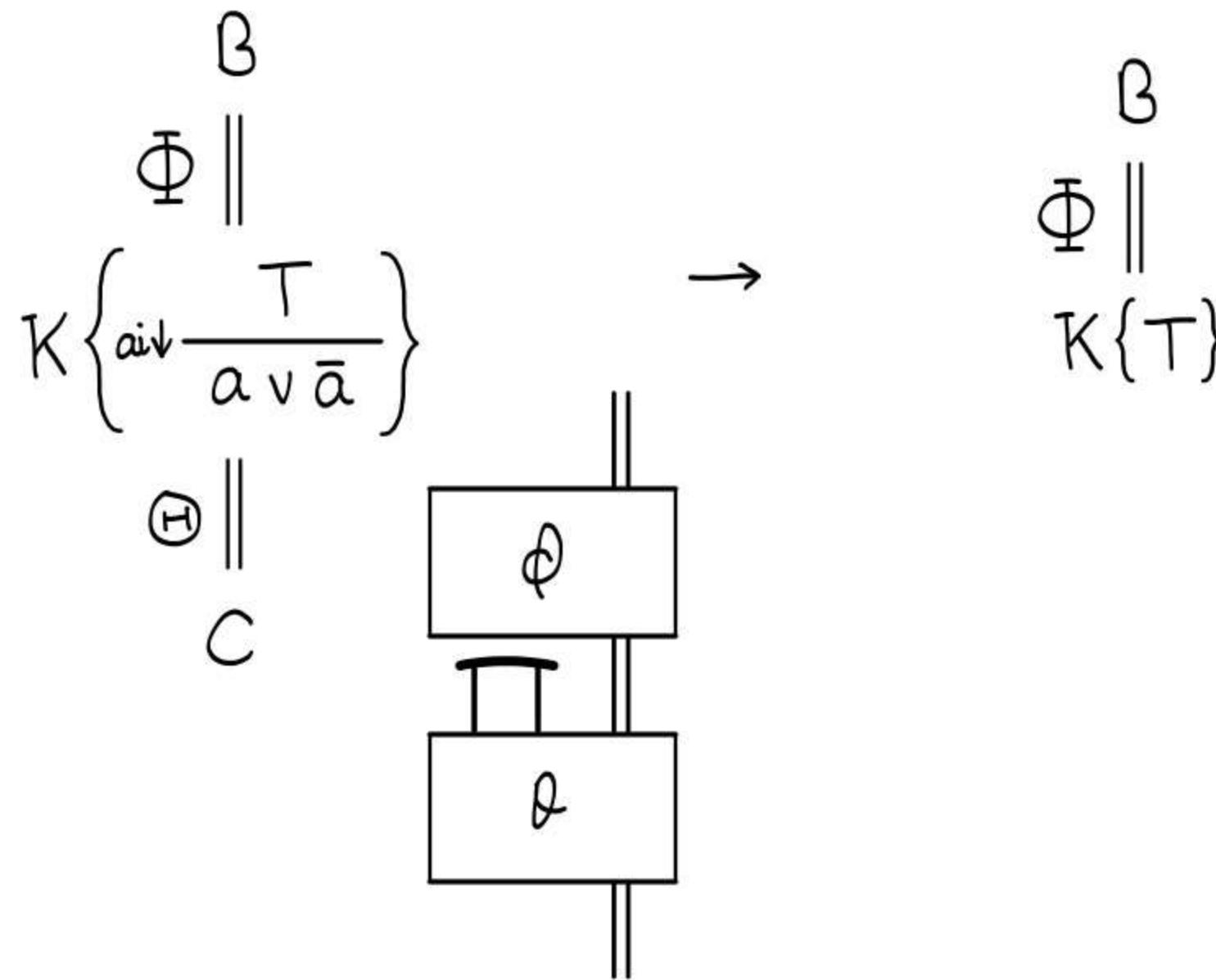
$\Theta \parallel$

C

Identities can be pulled up derivations



Identities can be pulled up derivations

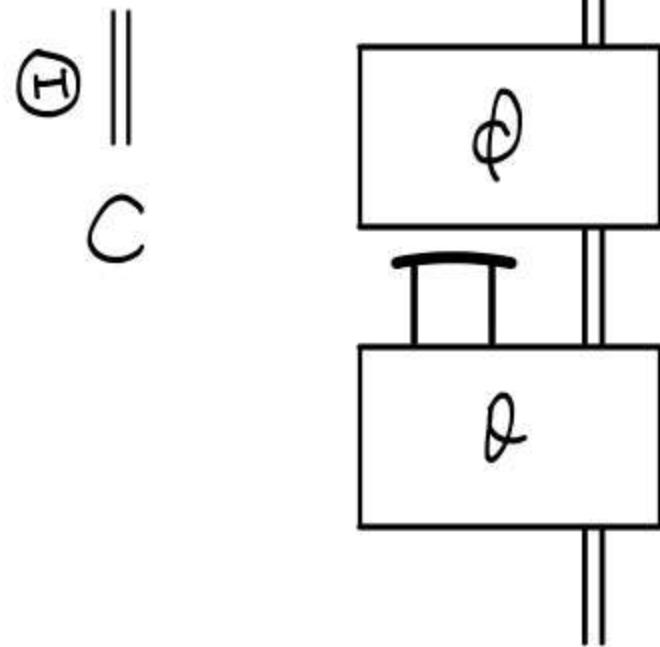


Identities can be pulled up derivations

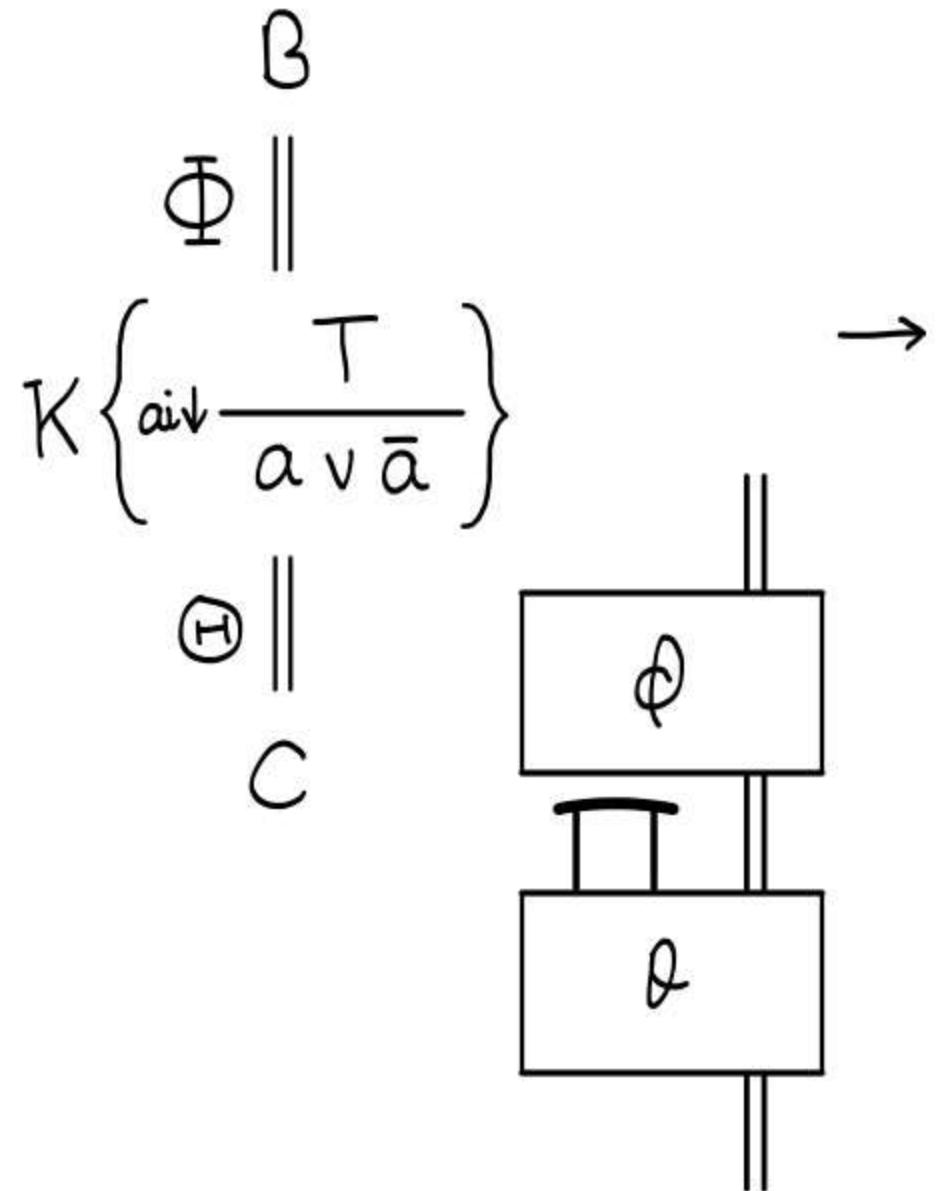
$$\frac{\Phi \parallel \beta}{K\left\{ \frac{a \downarrow T}{a \vee \bar{a}} \right\}}$$



$$\boxed{\frac{\Phi \parallel \beta \wedge \frac{T}{a \downarrow \frac{T}{a \vee \bar{a}}}}{K\{T\}}}$$



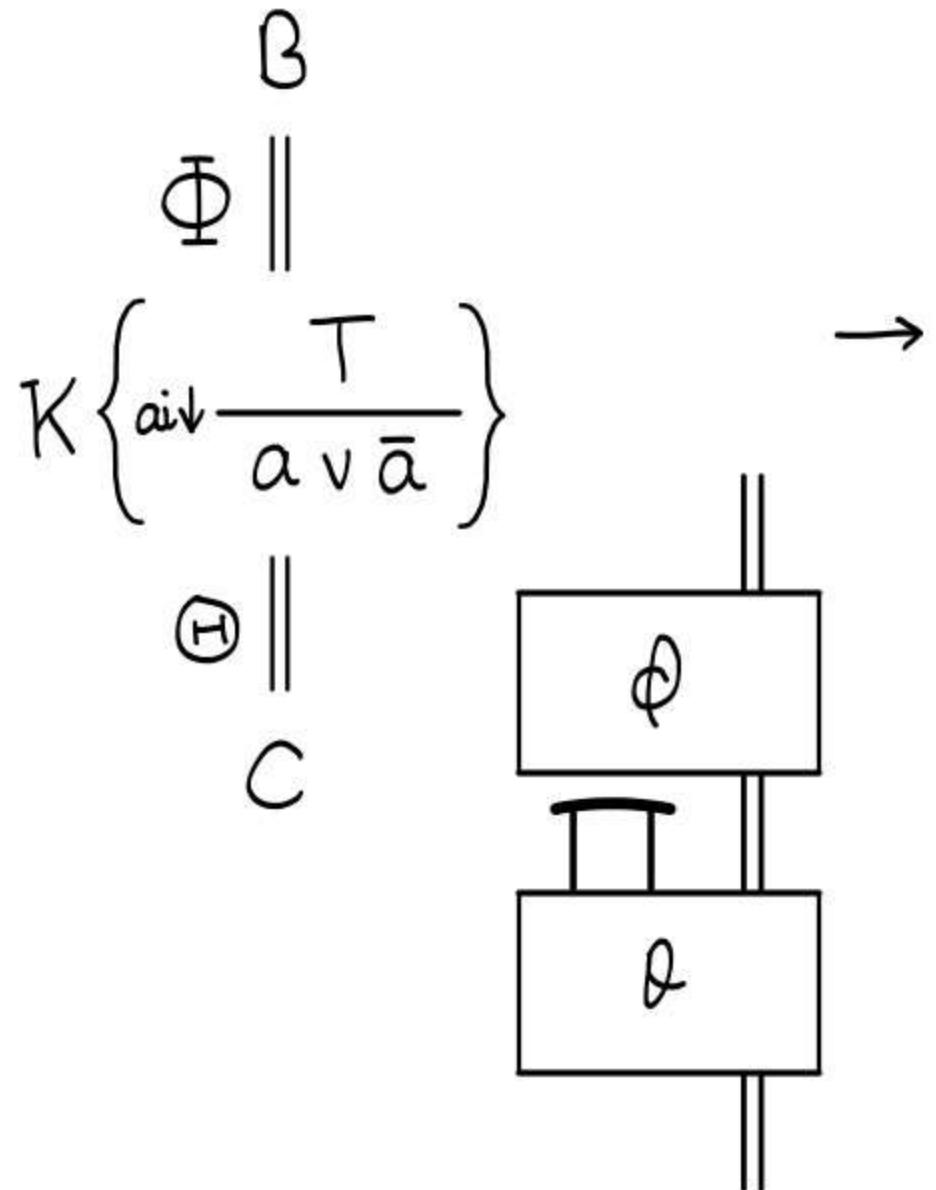
Identities can be pulled up derivations



$$\boxed{\begin{array}{c} \beta \\ \Phi \parallel \\ K\{T\} \\ \wedge \quad \frac{T}{a \vee \bar{a}} \end{array}}$$

induction
on $K\{\}$ $\parallel s, =$
 $K\{a \vee \bar{a}\}$

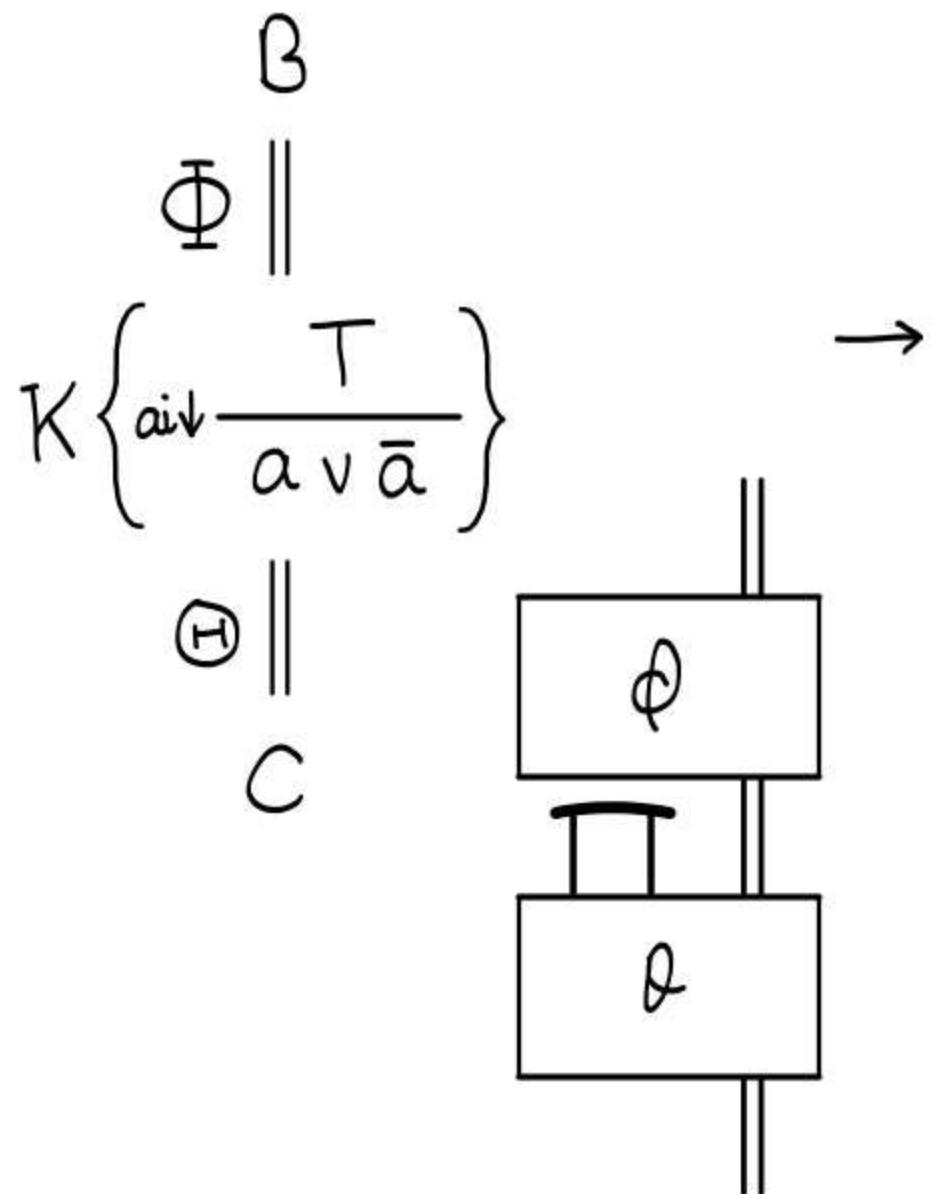
Identities can be pulled up derivations



$$\boxed{\frac{\Phi \parallel \text{B} \quad K\{T\} \quad \text{ai}\downarrow \frac{T}{a \vee \bar{a}}}{\Phi \parallel \text{B} \wedge K\{T\}}}$$

$$\frac{\parallel s, =}{K\{a \vee \bar{a}\}}$$
$$\frac{\Theta \parallel C}{\Theta \parallel C}$$

Identities can be pulled up derivations



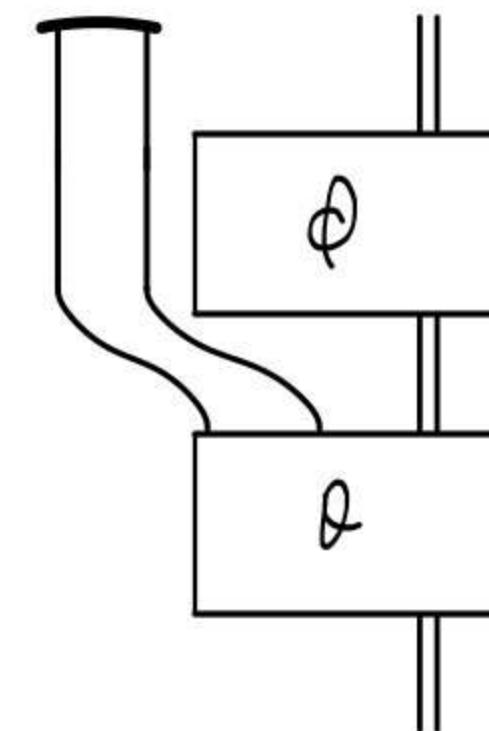
$$\frac{\Phi \parallel \text{B} \quad K\{T\}}{\Phi \parallel \text{B} \wedge \frac{T}{a \downarrow \frac{T}{a \vee \bar{a}}}}$$

induction
on $K\{\}$

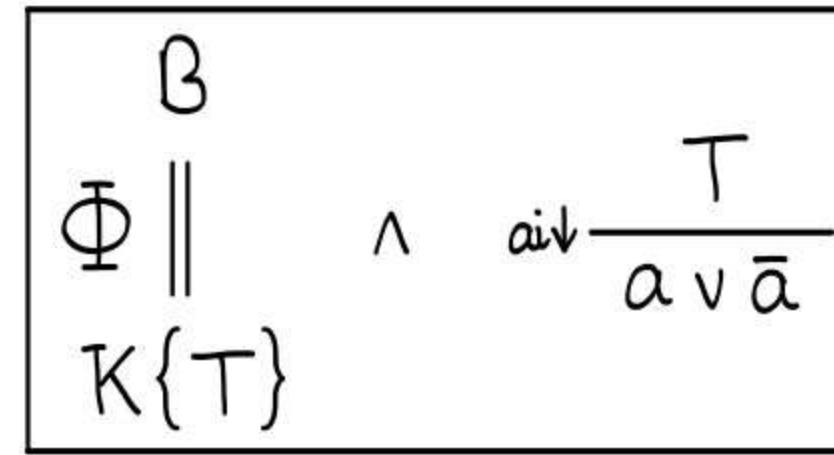
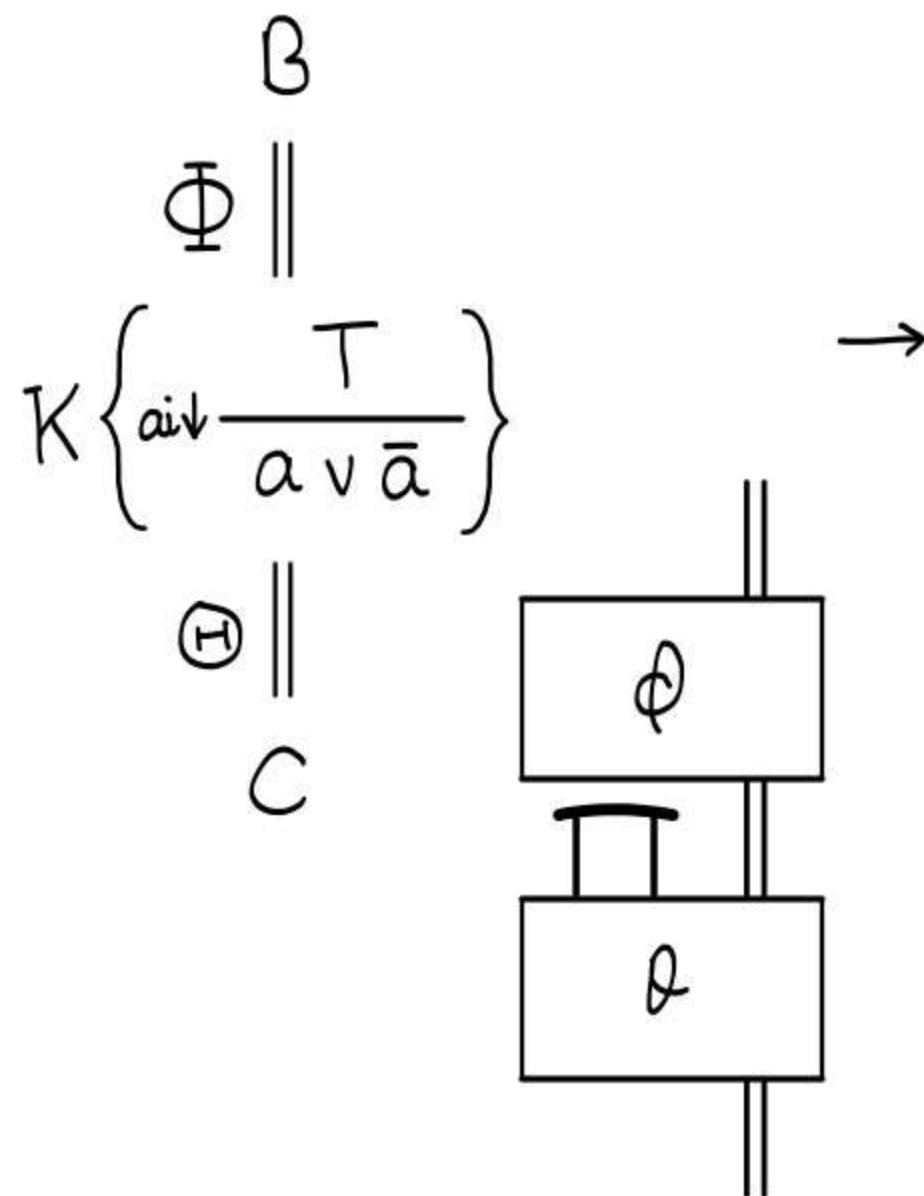
$\parallel s, =$

$$K\{a \vee \bar{a}\}$$

$$\frac{\Theta \parallel \text{C}}{\Phi \parallel \text{B}}$$



Identities can be pulled up derivations

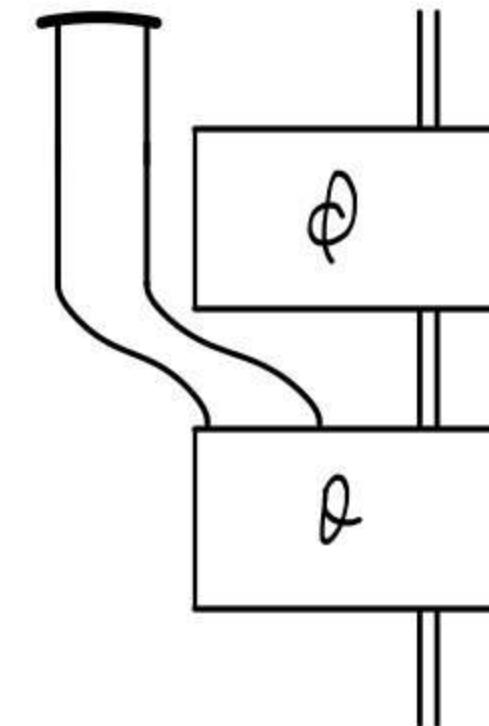


induction
on $K\{\}$

$\parallel s, =$
 $K\{a \vee \bar{a}\}$

$\Theta \parallel$

C
this justifies identifying
their flows



For some atom a , pull every identity up and every cut down to get this :

$$\frac{\overbrace{B \wedge \frac{T}{a \vee \bar{a}} \wedge \dots \wedge \frac{T}{a \vee \bar{a}}}^{m \geq 0}}{E \parallel} \\ C \wedge \frac{a \wedge \bar{a}}{\perp} \vee \dots \vee \frac{a \wedge \bar{a}}{\perp} \underbrace{\quad}_{n \geq 0}$$

For some atom a , pull every identity up and every cut down to get this :

$$B \wedge \frac{T}{a \vee \bar{a}} \wedge \dots \wedge \frac{T}{a \vee \bar{a}}$$

$\overbrace{\quad\quad\quad}^{m \geq 0}$

$$C \wedge \frac{a \wedge \bar{a}}{\perp} \vee \dots \vee \frac{a \wedge \bar{a}}{\perp}$$

$\underbrace{\quad\quad\quad}_{n \geq 0}$

$E \parallel$

then transform it like this to get a single cut and identity on a

\rightarrow

$$B \wedge$$

$$\frac{T}{a \vee \bar{a}} \parallel ac^{\uparrow}, m, =$$

$$(a \vee \bar{a}) \wedge \dots \wedge (a \vee \bar{a})$$

$$E \parallel$$

$$C \vee$$

$$(a \wedge \bar{a}) \vee \dots \vee (a \wedge \bar{a}) \parallel ac^{\uparrow}, m, =$$

$$\frac{a \wedge \bar{a}}{\perp}$$

Every proof Φ that contains a cut $\frac{a \wedge \bar{a}}{\perp}$

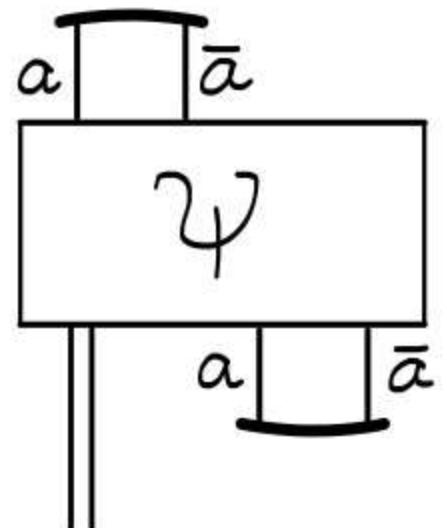
can be transformed into a proof

$$\frac{T}{a \vee \bar{a}}$$

$$\Psi \parallel$$

$$B \vee \frac{a \wedge \bar{a}}{\perp}$$

with flow



where Ψ contains no cuts or identities on a .



Every proof Φ that contains a cut $\frac{a \wedge \bar{a}}{\perp}$

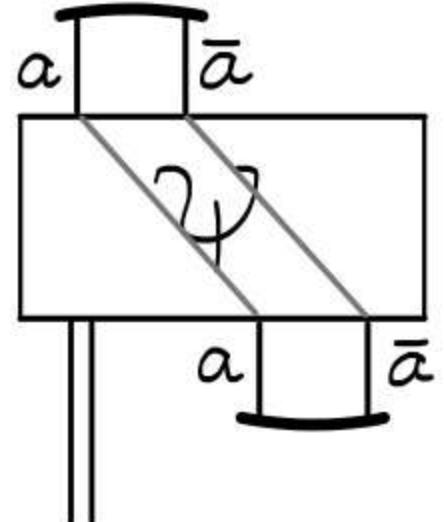
can be transformed into a proof

$$\frac{T}{a \vee \bar{a}}$$

$$\Psi \parallel$$

$$B \vee \frac{a \wedge \bar{a}}{\perp}$$

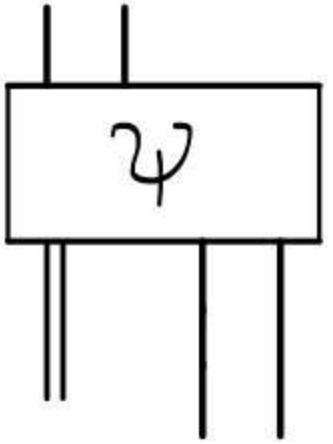
with flow



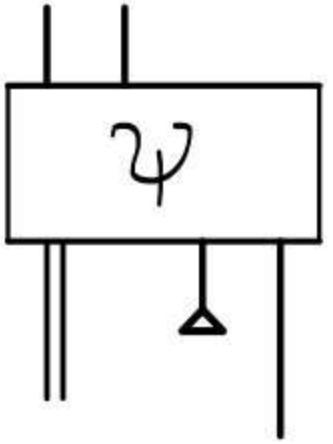
where Ψ contains no cuts or identities on a .

Ψ contains paths from the identity to the cut and these are what we want to break

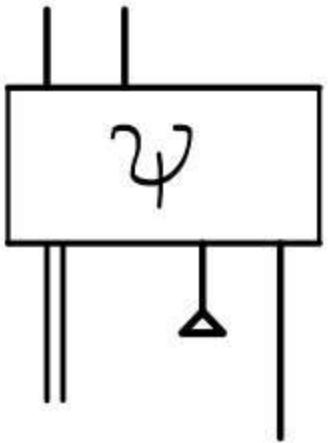




$$\begin{array}{c} \alpha \vee \bar{\alpha} \\ \neg \psi \parallel \\ \beta \vee (\alpha \wedge \bar{\alpha}) \end{array}$$

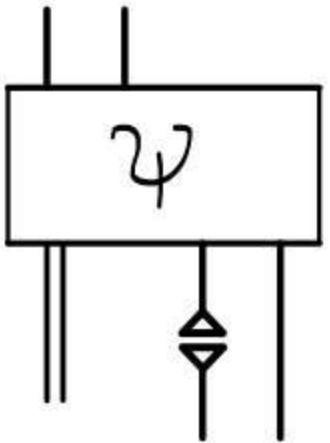


$$\frac{\alpha \vee \bar{\alpha}}{\Psi \parallel} \\ \beta \vee \frac{\alpha \wedge \bar{\alpha}}{\top}$$



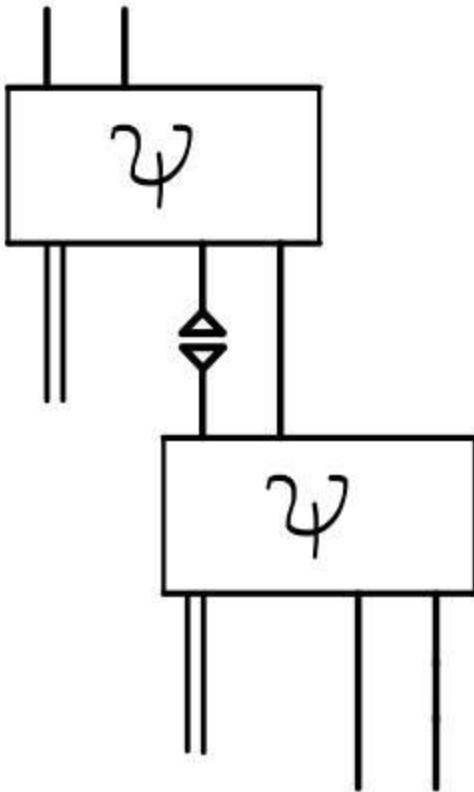
$$\begin{array}{c}
 a \vee \bar{a} \\
 \Psi \parallel \\
 B \vee (\frac{a \wedge \bar{a}}{\top})
 \end{array}$$

$$\Pi_a = \boxed{
 \begin{array}{c}
 \text{awt} \frac{a}{\top} \wedge \bar{a} \\
 = \frac{}{\bar{a}} \\
 = \frac{\perp}{a} \\
 \text{awt} \frac{\perp}{a} \vee \bar{a}
 \end{array}
 }$$



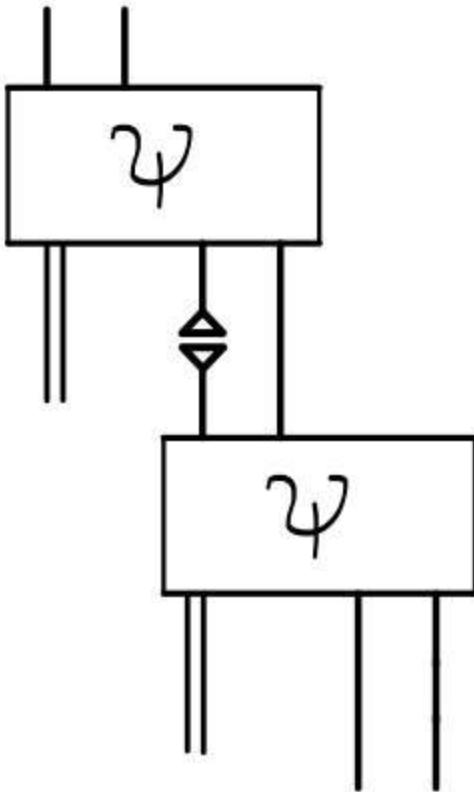
$$\begin{array}{c}
 a \vee \bar{a} \\
 \vdash \Psi \parallel \\
 B \vee (\frac{a}{\top} \wedge \bar{a})
 \end{array}$$

$$\Pi_a = \boxed{
 \begin{array}{c}
 \text{awt} \frac{a}{\top} \wedge \bar{a} \\
 = \frac{}{\bar{a}} \\
 = \frac{}{\perp} \\
 \text{awt} \frac{\perp}{a} \vee \bar{a}
 \end{array}
 }$$



$$\begin{array}{c}
 a \vee \bar{a} \\
 \vdash \Psi \parallel \\
 B \vee \frac{a \wedge \bar{a}}{\top}
 \end{array}$$

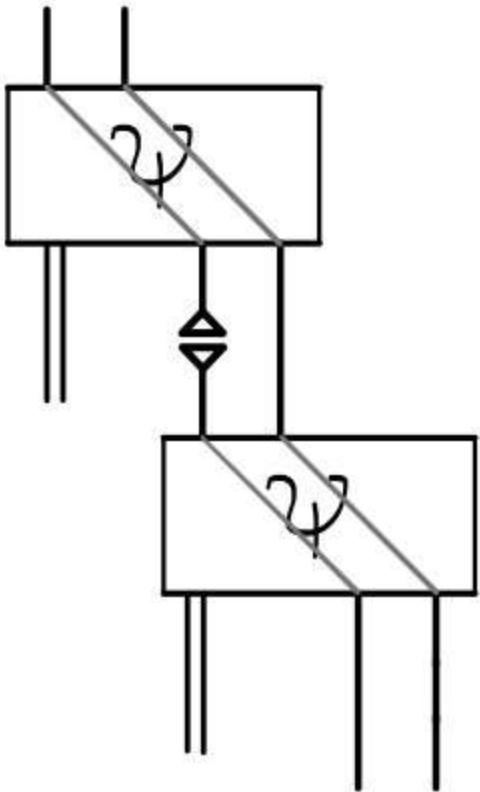
$$\Pi_a = \boxed{
 \begin{array}{c}
 \text{awt} \frac{a}{\top} \wedge \bar{a} \\
 = \frac{}{\bar{a}} \\
 = \frac{}{\perp} \\
 \text{awt} \frac{\perp}{a} \vee \bar{a}
 \end{array}
 }$$



$$\begin{array}{c}
 a \vee \bar{a} \\
 \Psi \parallel \\
 B \vee (a \wedge \bar{a})
 \end{array}$$

$$\Pi_a = \boxed{
 \begin{array}{c}
 \text{awt} \frac{a}{\top} \wedge \bar{a} \\
 = \frac{}{\bar{a}} \\
 = \frac{\perp}{\text{awt} \frac{\perp}{a} \vee \bar{a}}
 \end{array}
 }$$

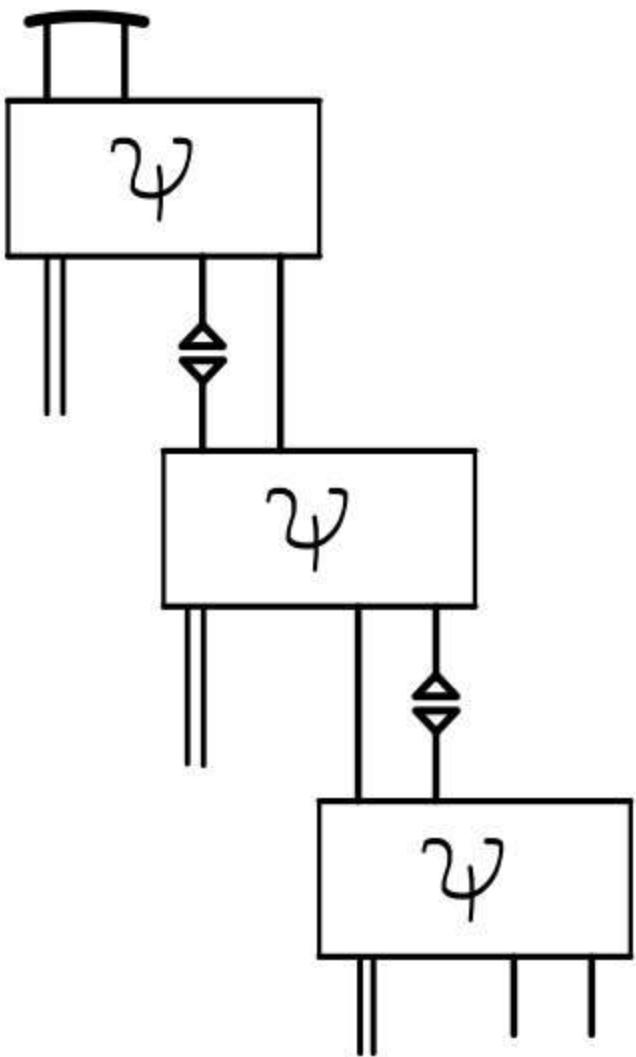
$$\begin{array}{c}
 a \vee \bar{a} \\
 \Psi \parallel \\
 (a \wedge \bar{a}) \\
 \Pi_a \parallel \\
 a \vee \bar{a} \\
 \Psi \parallel \\
 B \vee (a \wedge \bar{a})
 \end{array}$$



$$\begin{array}{c}
 a \vee \bar{a} \\
 \Psi \parallel \\
 B \vee (a \wedge \bar{a})
 \end{array}$$

$$\Pi_a = \boxed{
 \begin{array}{c}
 \text{awt} \frac{a}{\top} \wedge \bar{a} \\
 = \frac{}{\bar{a}} \\
 = \frac{\perp}{\text{awt} \frac{\perp}{a} \vee \bar{a}}
 \end{array}
 }$$

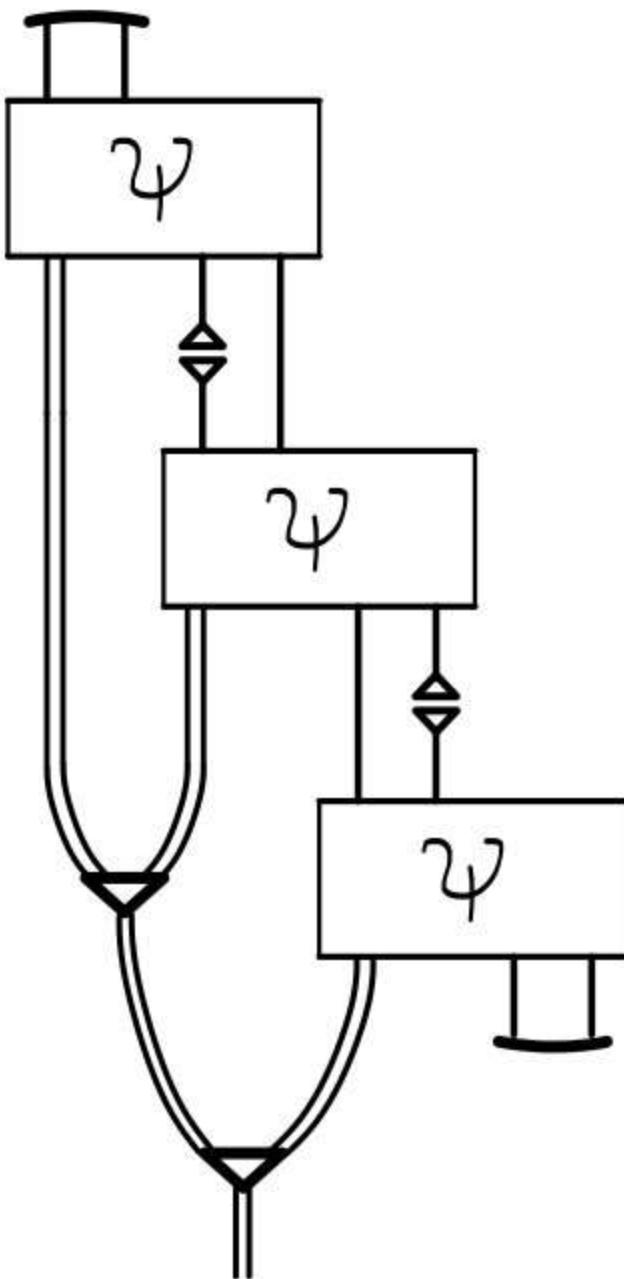
$$\begin{array}{c}
 a \vee \bar{a} \\
 \Psi \parallel \\
 (a \wedge \bar{a}) \\
 \Pi_a \parallel \\
 a \vee \bar{a} \\
 \Psi \parallel \\
 B \vee (a \wedge \bar{a})
 \end{array}$$



$$\begin{array}{c}
 a \vee \bar{a} \\
 \vdash \Psi \parallel \\
 B \vee (a \wedge \bar{a})
 \end{array}$$

$$\Pi_a = \boxed{
 \begin{array}{c}
 \text{awt} \frac{a}{\top} \wedge \bar{a} \\
 = \frac{\top}{\bar{a}} \\
 = \frac{\perp}{a} \\
 \text{awt} \frac{\perp}{a} \vee \bar{a}
 \end{array}
 }$$

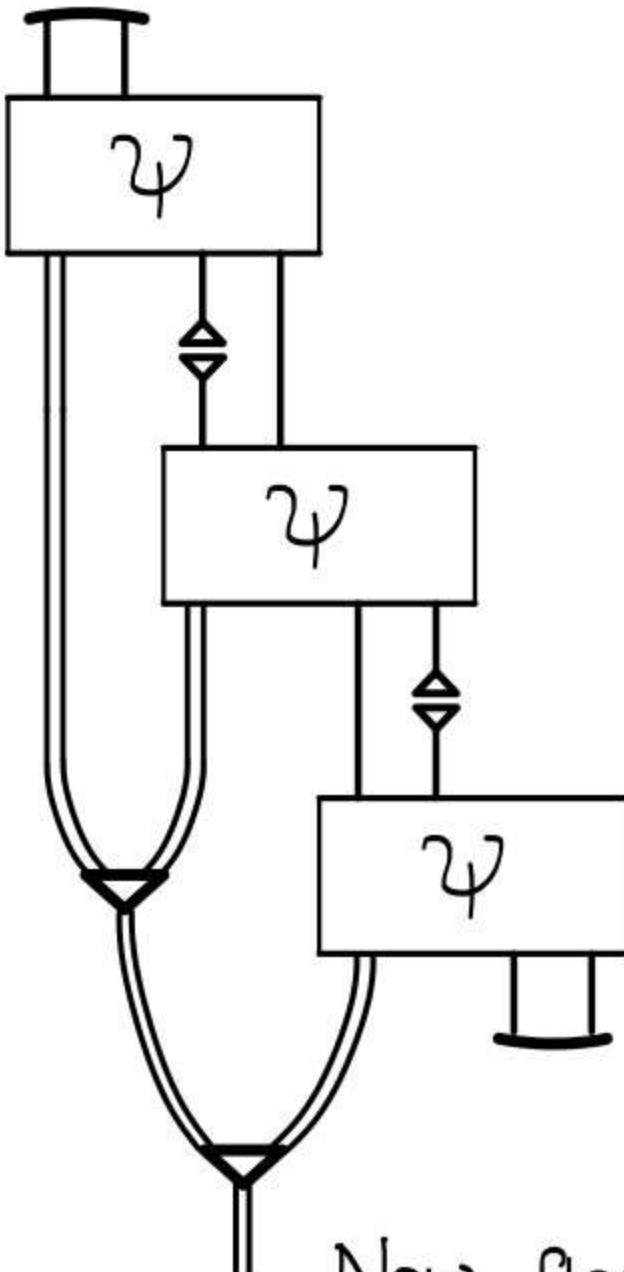
$$\begin{array}{c}
 a \vee \bar{a} \\
 \vdash \Psi \parallel \\
 (a \wedge \bar{a}) \\
 \Pi_a \parallel \\
 \bar{a} \vee \bar{a} \\
 \vdash \Psi \parallel \\
 B \vee \\
 B \vee \\
 a \wedge \bar{a} \\
 \Pi_{\bar{a}} \parallel \\
 \bar{a} \vee \bar{a} \\
 \vdash \Psi \parallel \\
 B \vee (a \wedge \bar{a})
 \end{array}$$



$$\begin{array}{c}
 a \vee \bar{a} \\
 \vdash \Psi \parallel \\
 B \vee (a \wedge \bar{a})
 \end{array}$$

$$\Pi_a = \boxed{
 \begin{array}{c}
 \text{awt} \frac{a}{\top} \wedge \bar{a} \\
 = \frac{\top}{\bar{a}} \\
 = \frac{\perp}{a} \\
 \text{awt} \frac{\perp}{a} \vee \bar{a}
 \end{array}
 }$$

$$\begin{array}{c}
 \frac{\top}{a \vee \bar{a}} \\
 \vdash \Psi \parallel \\
 (a \wedge \bar{a}) \\
 \Pi_a \parallel \\
 \frac{a \vee \bar{a}}{\Psi \parallel} \\
 B \vee \\
 B \vee \\
 \frac{B \vee \text{awt} \frac{a \wedge \bar{a}}{\perp}}{B} \\
 \parallel
 \end{array}$$



$$\begin{array}{c}
 a \vee \bar{a} \\
 \Psi \parallel \\
 B \vee (a \wedge \bar{a})
 \end{array}$$

$$\Pi_a = \boxed{
 \begin{array}{c}
 \text{awt} \frac{a}{T} \wedge \bar{a} \\
 = \frac{}{\bar{a}} \\
 = \frac{\perp}{a} \\
 \text{awt} \frac{\perp}{a} \vee \bar{a}
 \end{array}
 }$$

Now flow rewriting will terminate! (for a)

$$\begin{array}{c}
 T \\
 \frac{a \vee \bar{a}}{\Psi \parallel} \\
 (a \wedge \bar{a}) \\
 \Pi_a \parallel \\
 \frac{a \vee \bar{a}}{\Psi \parallel} \\
 B \vee \\
 B \vee \\
 \frac{B \vee \text{awt} \frac{a \wedge \bar{a}}{\perp}}{B}
 \end{array}$$

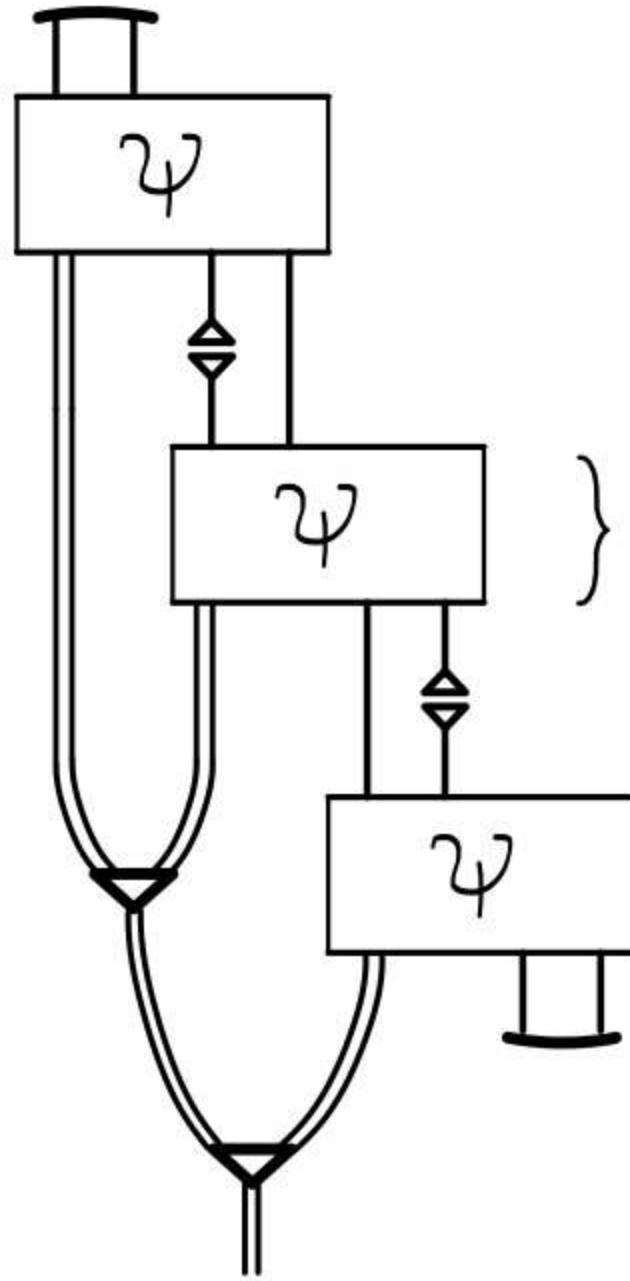
Atomic flows do not form a proof system : essential information is lost and there is no polynomial correctness criterion* (Das 2013)

*unless integer factoring is in P/poly

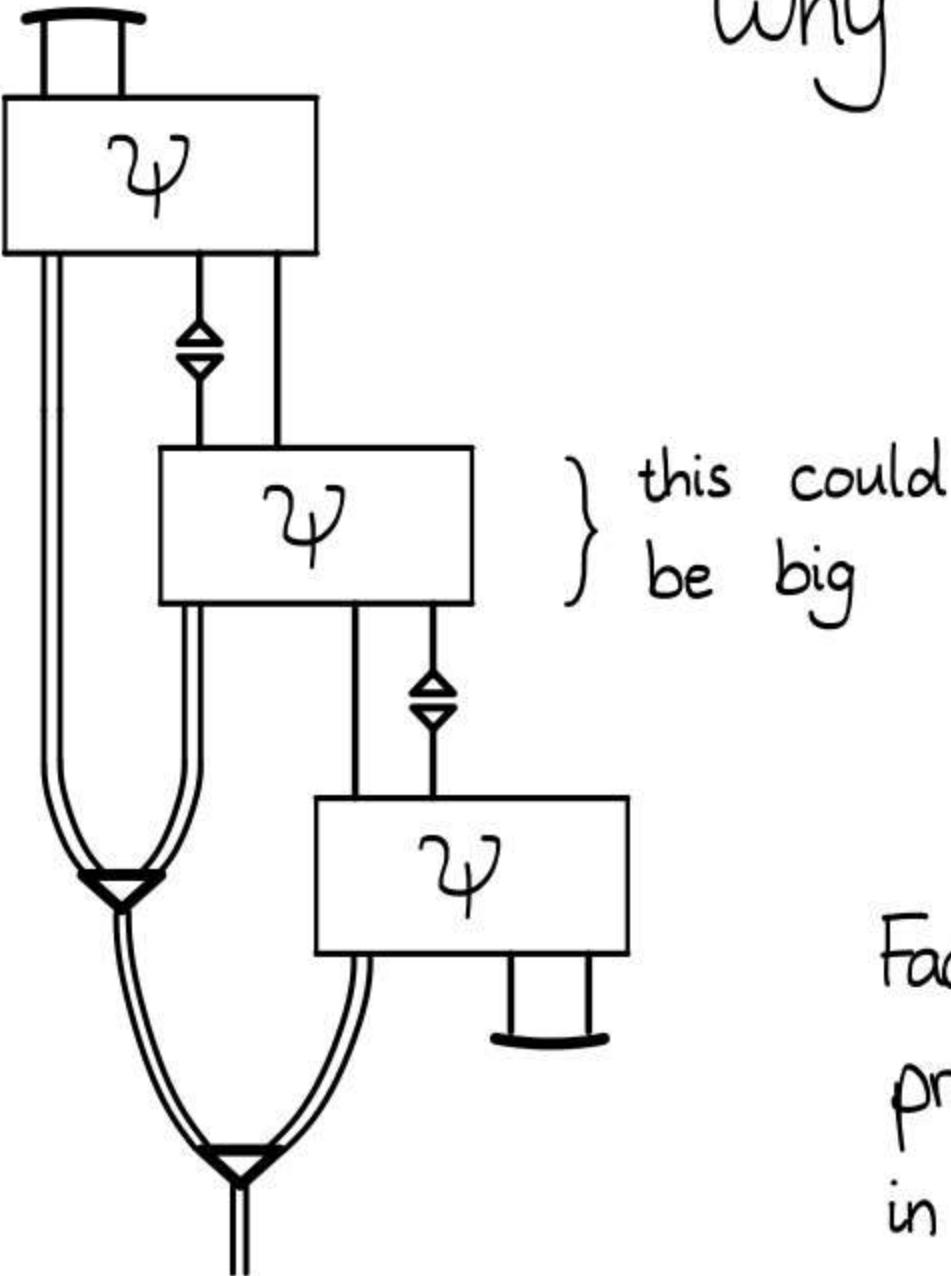
Atomic flows do not form a proof system : essential information is lost and there is no polynomial correctness criterion* (Das 2013) *unless integer factoring is in P/poly

but they give us some idea for how the substitution of proofs could work

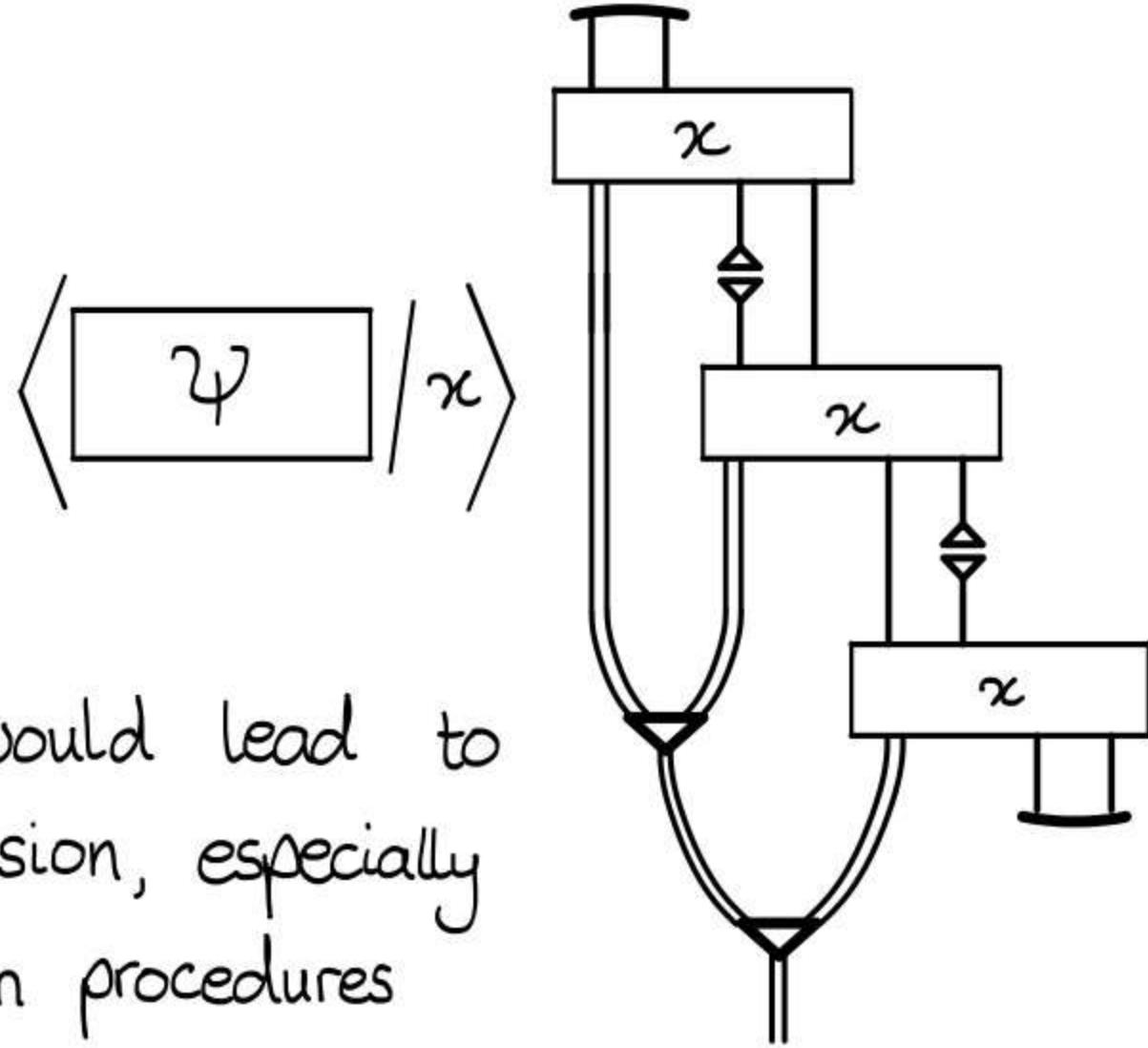
Why substitution of proofs?



Why substitution of proofs?

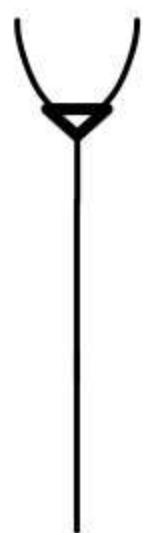


Factorisation would lead to proof compression, especially in normalisation procedures

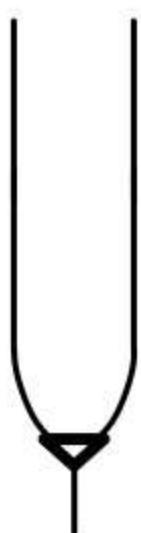


Why substitution of proofs?

Why substitution of proofs?



$$\frac{K \left\{ \frac{a \vee a}{a} \right\}}{[a/x]\Phi}$$
$$\frac{[a/x]\Phi}{H\{a\}}$$



$$\frac{K \left\{ a \vee a \right\}}{\frac{[a \vee a/x]\Phi}{H \left\{ \frac{a \vee a}{a} \right\}}}$$

Why substitution of proofs?

$$\frac{K\{a \vee a\}}{\frac{\left\langle \text{act} \frac{a \vee a}{a} / \chi \right\rangle \Phi}{H\{a\}}}$$


$$\frac{K\left\{ \text{act} \frac{a \vee a}{a} \right\}}{\frac{[a / \chi] \Phi}{H\{a\}}}$$


$$\frac{K\{a \vee a\}}{\frac{[a \vee a / \chi] \Phi}{\frac{H\left\{ \text{act} \frac{a \vee a}{a} \right\}}{}}}$$

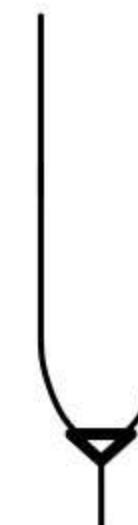
Why substitution of proofs?

$$\frac{K\{a \vee a\}}{\left\langle \text{act} \frac{a \vee a}{a} / \chi \right\rangle \Phi}$$

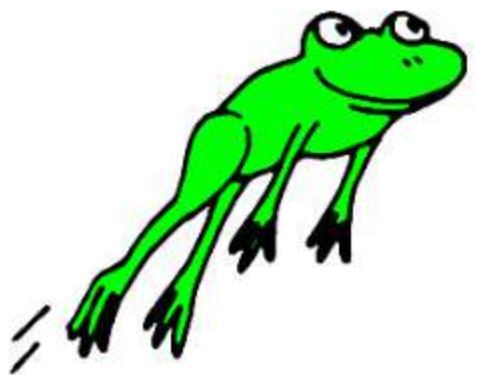
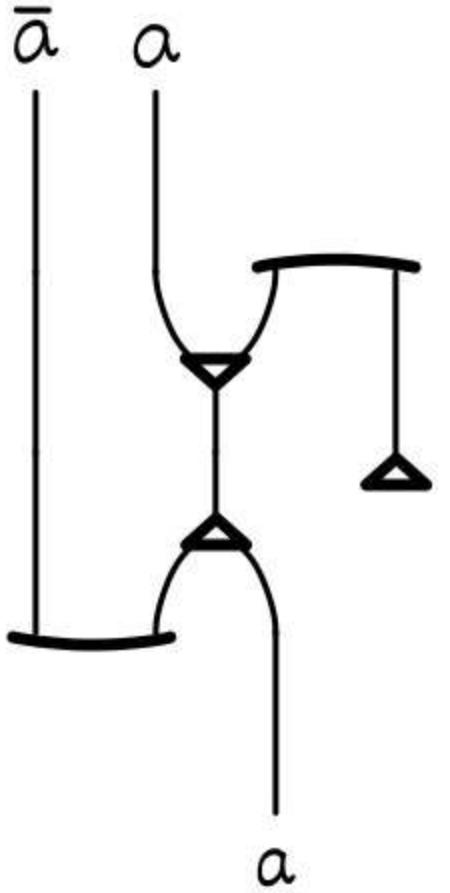


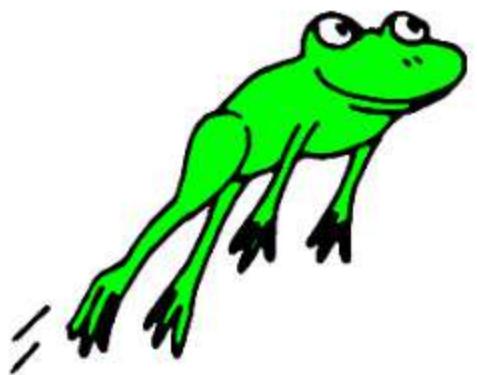
$$\frac{K\left\{ \text{act} \frac{a \vee a}{a} \right\}}{\begin{array}{c} \cdots \\ [a/\chi]\Phi \\ \cdots \\ H\{a\} \end{array}}$$

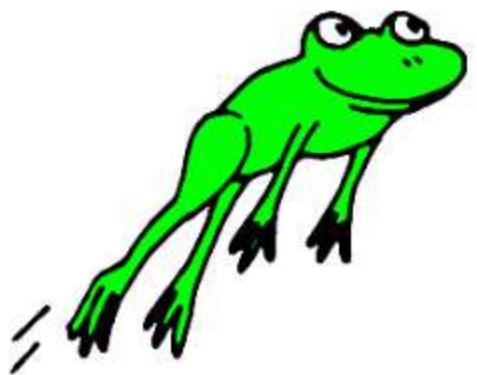
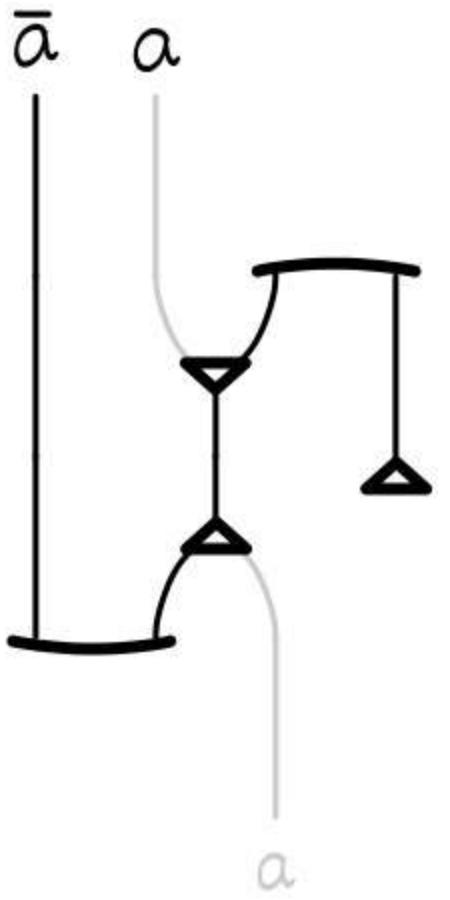
Factorisation would reduce bureaucracy by identifying proofs whose flows are continuous deformations

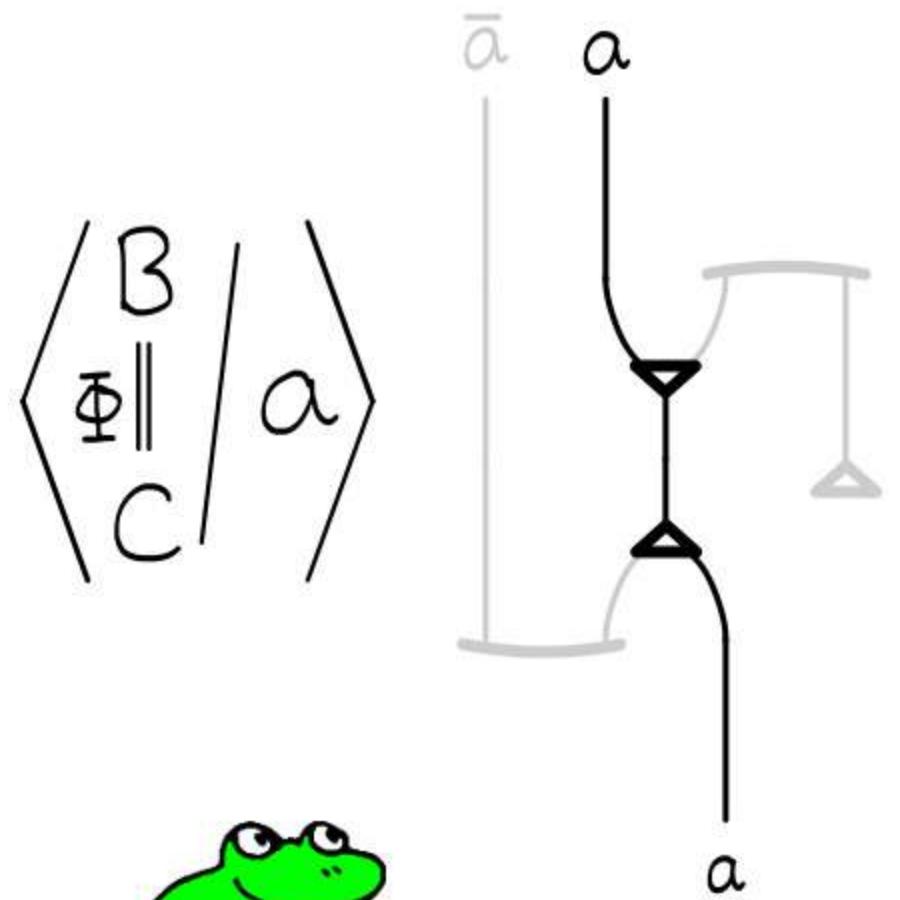
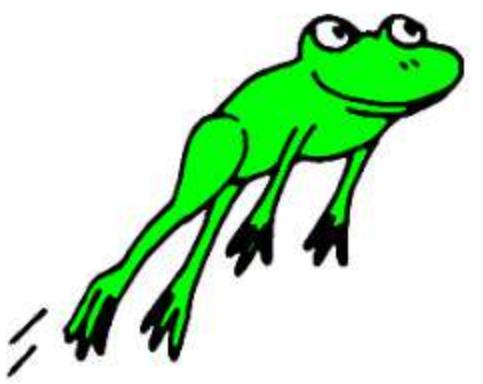


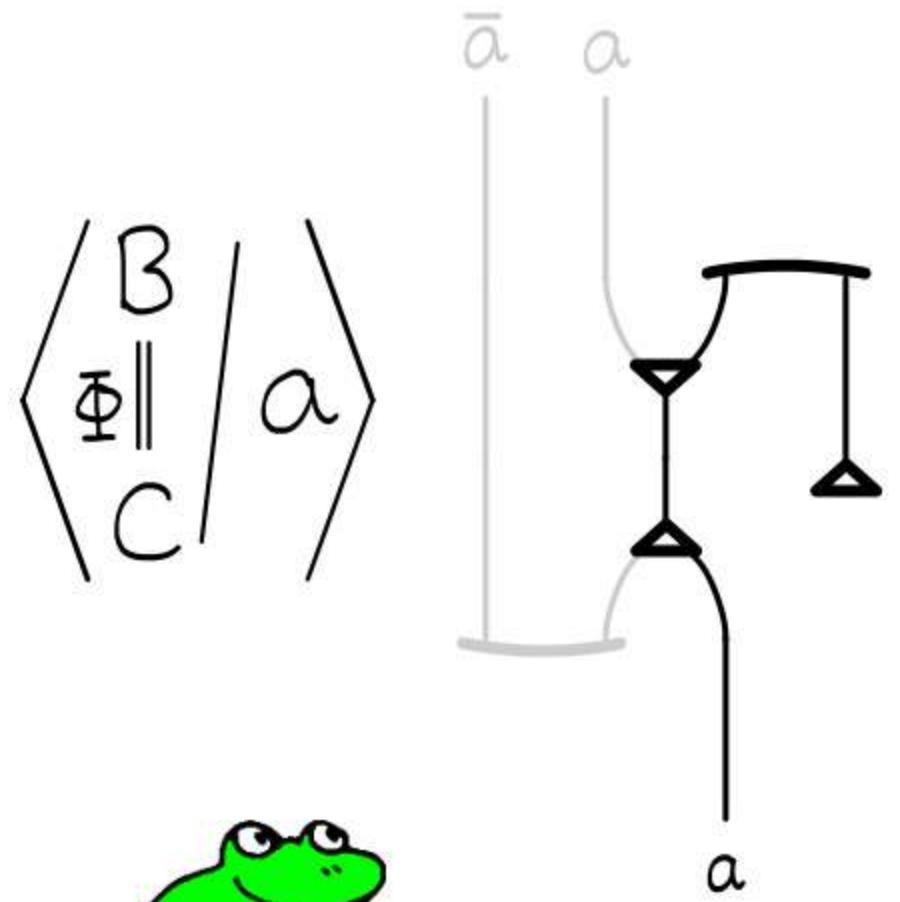
$$\frac{K\{a \vee a\}}{\begin{array}{c} \cdots \\ [a \vee a / \chi] \Phi \\ \cdots \\ H\left\{ \text{act} \frac{a \vee a}{a} \right\} \end{array}}$$

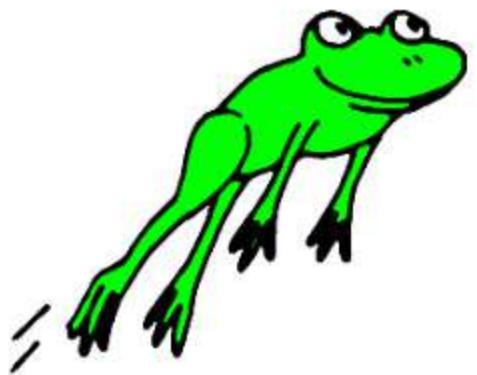
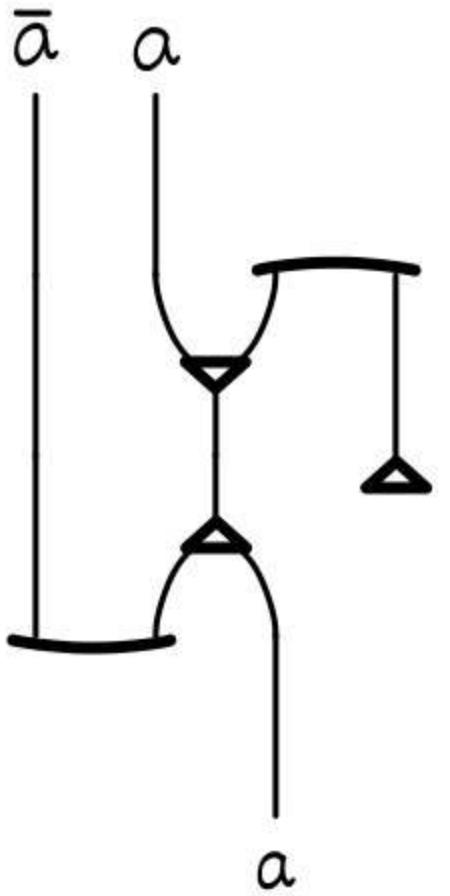
$$\langle \Phi | \begin{matrix} B \\ \parallel \\ C \end{matrix} | a \rangle$$


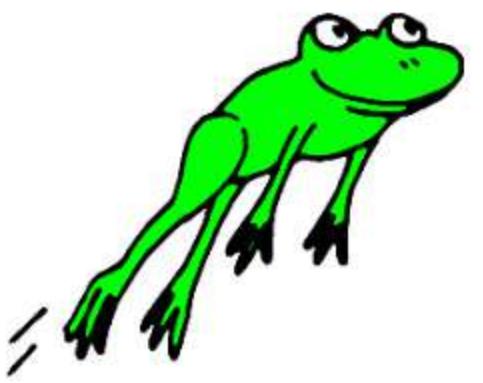
$$\langle \Phi | \begin{matrix} B \\ \parallel \\ C \end{matrix} | a \rangle$$


$$\langle \Phi | \begin{matrix} B \\ \parallel \\ C \end{matrix} | a \rangle$$


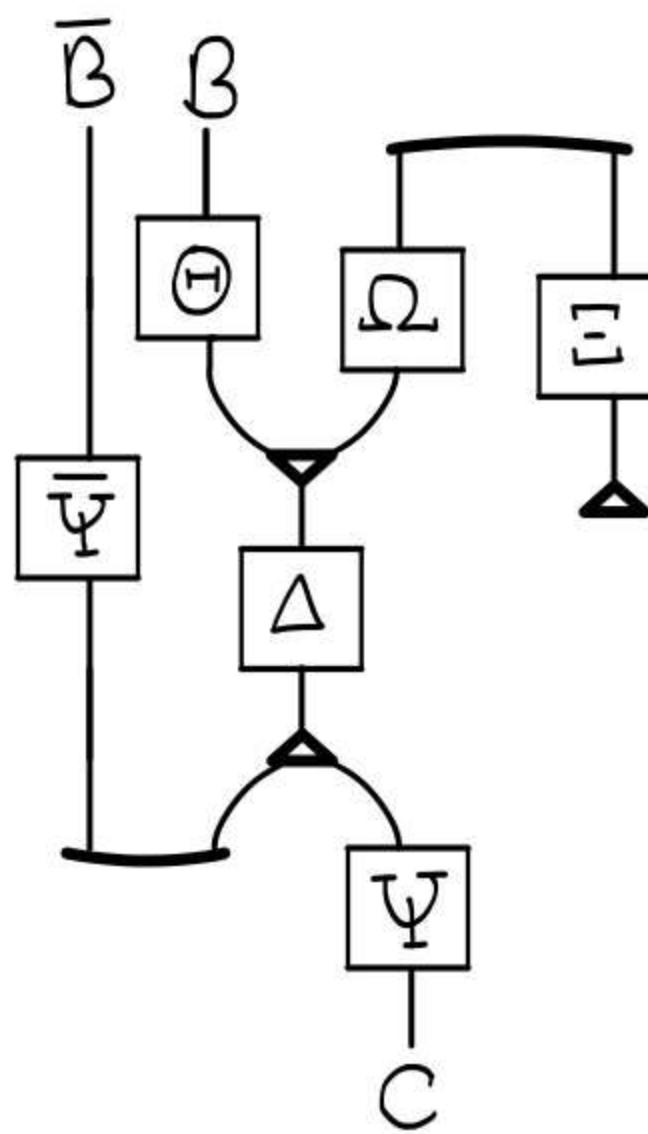
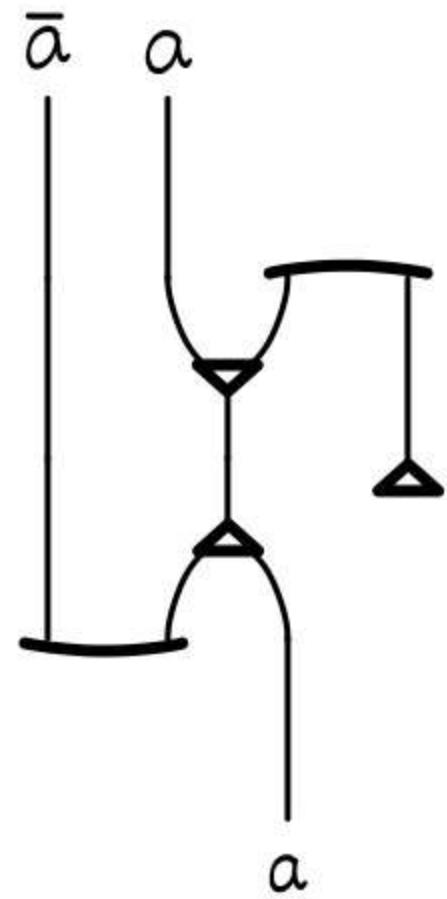


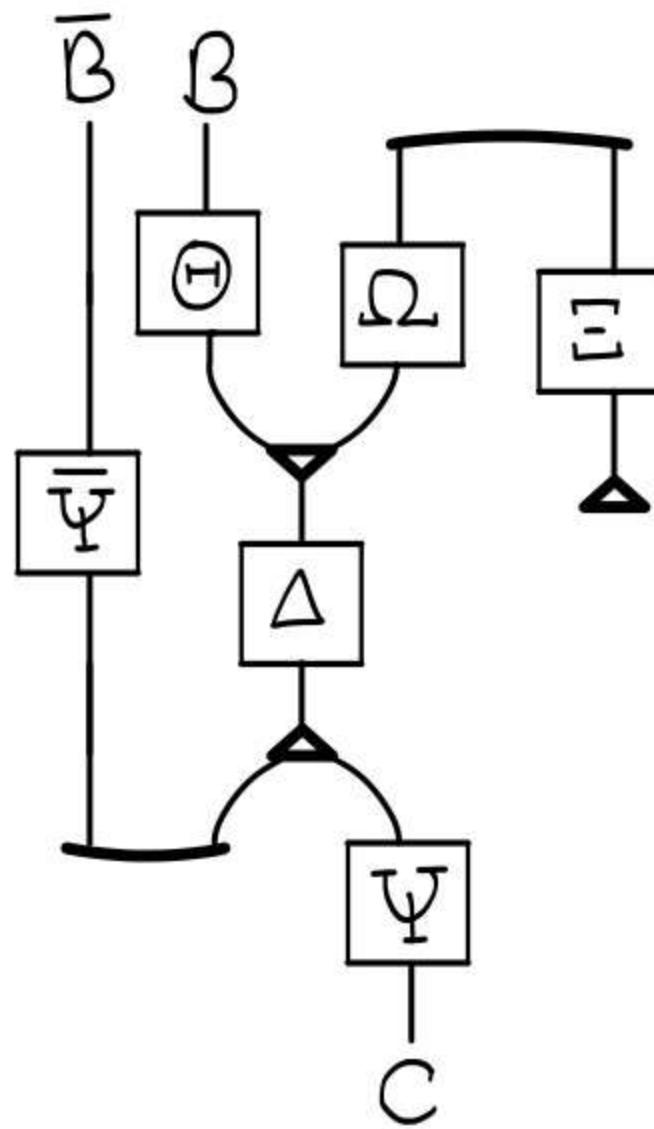
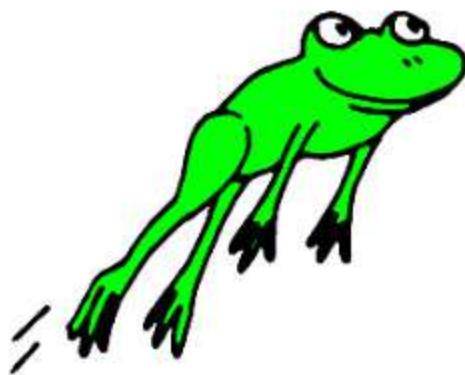
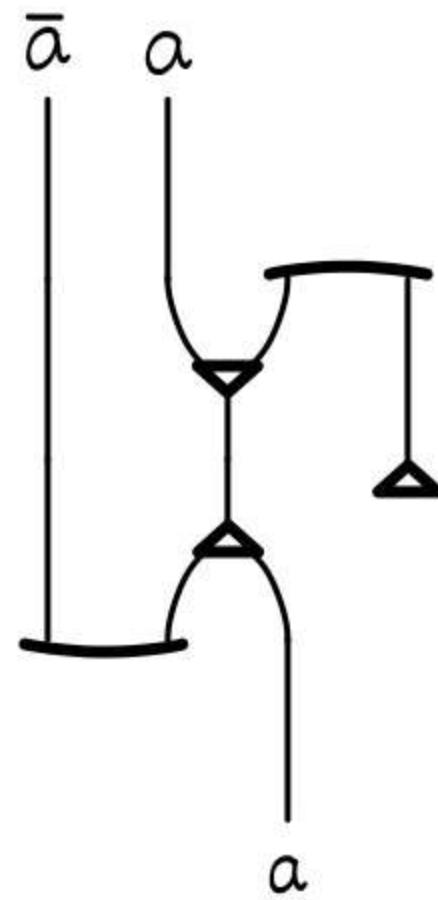


$$\langle \Phi | \begin{matrix} B \\ \parallel \\ C \end{matrix} | a \rangle$$




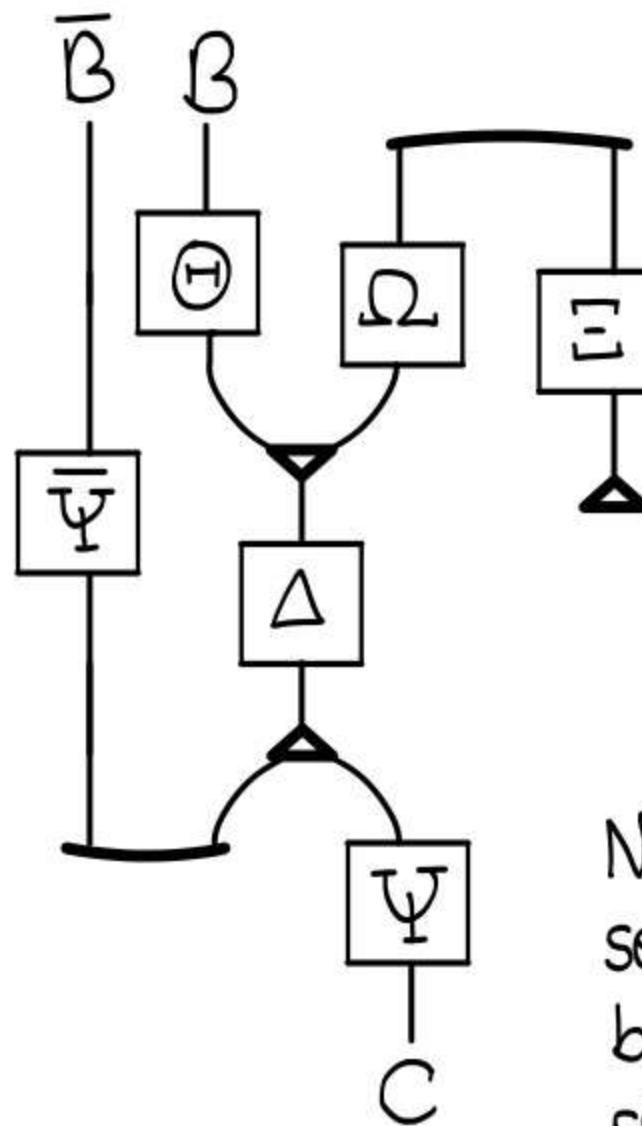
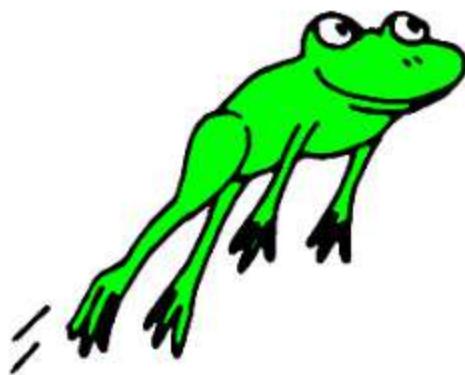
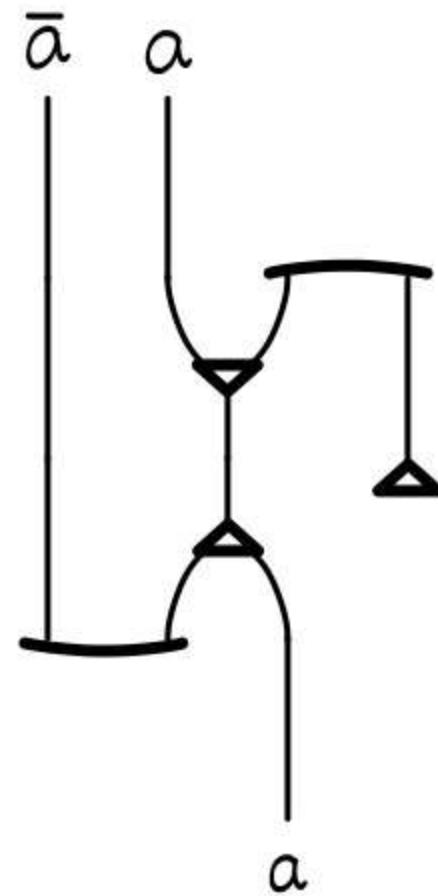
$$\langle \bar{\Phi} \mid \begin{matrix} B \\ \oplus \\ C \end{matrix} \mid a \rangle$$



$\langle \Phi | B_C | a \rangle$ 

$$\Phi = \frac{H}{\Delta} = \frac{\Psi}{\bar{\Psi}}$$

$$\frac{\bar{\Omega}}{\Omega} = \frac{\bar{\Psi}}{\Psi}$$

$\langle \Phi \models_C B \rangle$ 

$$\Phi = \frac{\text{H}}{\Delta} = \frac{\text{H}}{\Psi}$$

$$\frac{\text{E}}{\Omega} = \frac{\text{E}}{\Delta} = \frac{\text{E}}{\Psi}$$

Negation of a derivation makes sense in deep inference because all rules have a single premise and are dualisable