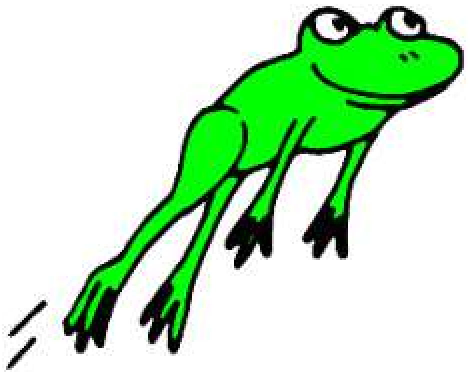
A gentle introduction to deep inference

6. Atomic flows



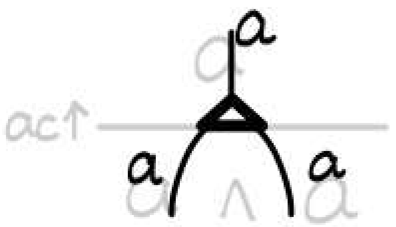
ESSLLI 2025

Victoria Barrett and Lutz Straßburger

From derivations to flows

$$
\mathsf{s}\;\dfrac{\mathsf{m}\;\dfrac{\boxed{\mathsf{ac}\!\uparrow\dfrac{a}{a \wedge a}} \;\vee\; \boxed{\mathsf{ac}\!\uparrow\dfrac{\bar{a}}{\bar{a} \wedge \bar{a}}}}{(a \vee \bar{a}) \wedge (a \vee \bar{a})}}{a \;\vee\; \boxed{\mathsf{s}\;\dfrac{\bar{a} \wedge (a \vee \bar{a})}{\boxed{\mathsf{ai}\!\uparrow\dfrac{\bar{a} \wedge a}{\bot}} \;\vee\; \dfrac{a}{\top}}}}
$$

From derivations to flows

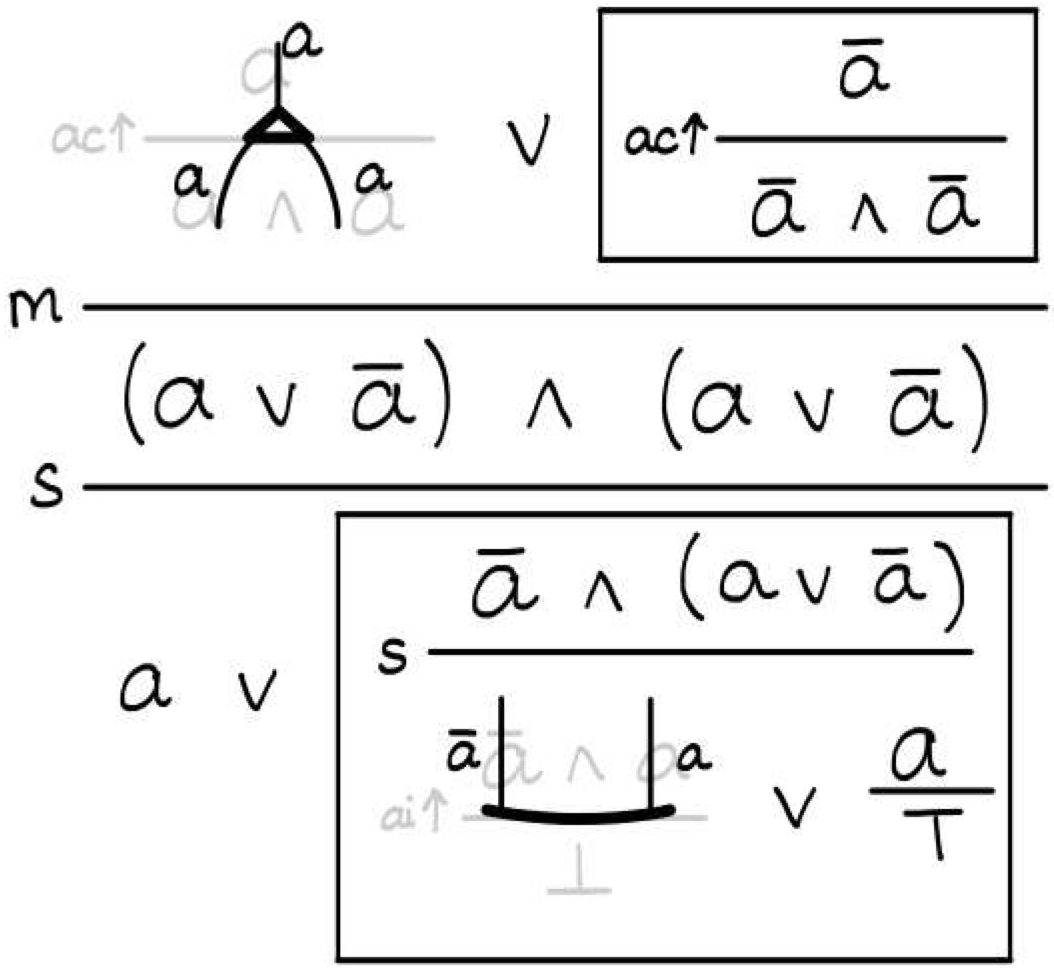


$$\vee \quad \boxed{\text{ac}\!\uparrow \frac{\bar{a}}{\bar{a} \wedge \bar{a}}}$$

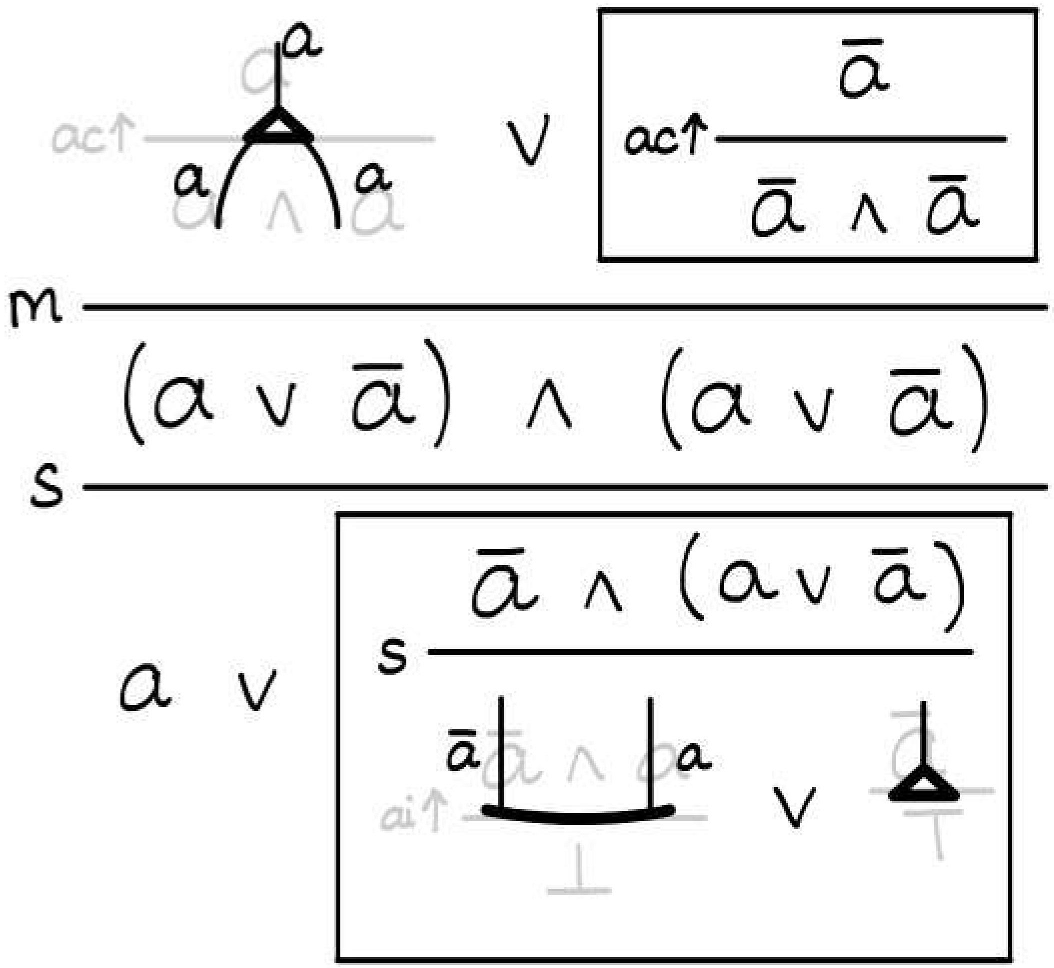
$$\text{m}\,\frac{\qquad\qquad}{(a \vee \bar{a}) \wedge (a \vee \bar{a})}$$

$$\text{s}\,\frac{\qquad\qquad}{a \vee \boxed{\text{s}\,\dfrac{\bar{a} \wedge (a \vee \bar{a})}{\boxed{\text{ai}\!\uparrow \dfrac{\bar{a} \wedge a}{\bot}} \vee \dfrac{a}{\top}}}}$$

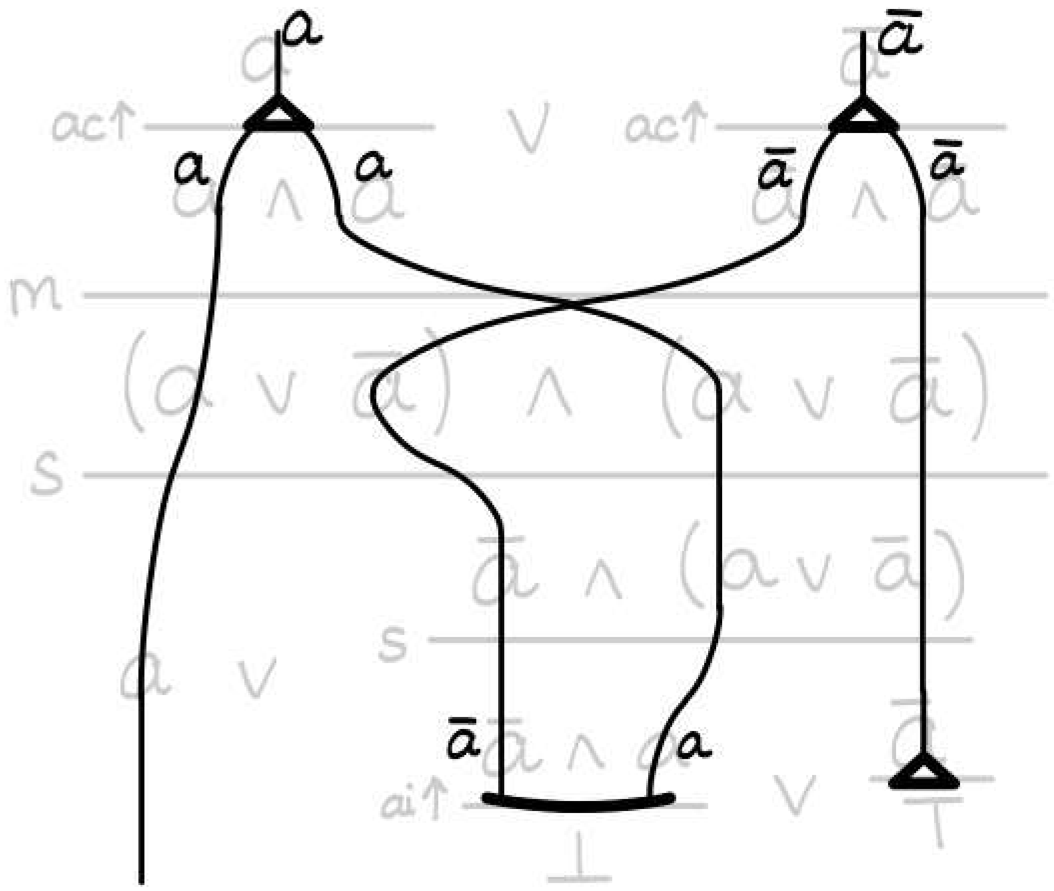
From derivations to flows



From derivations to flows

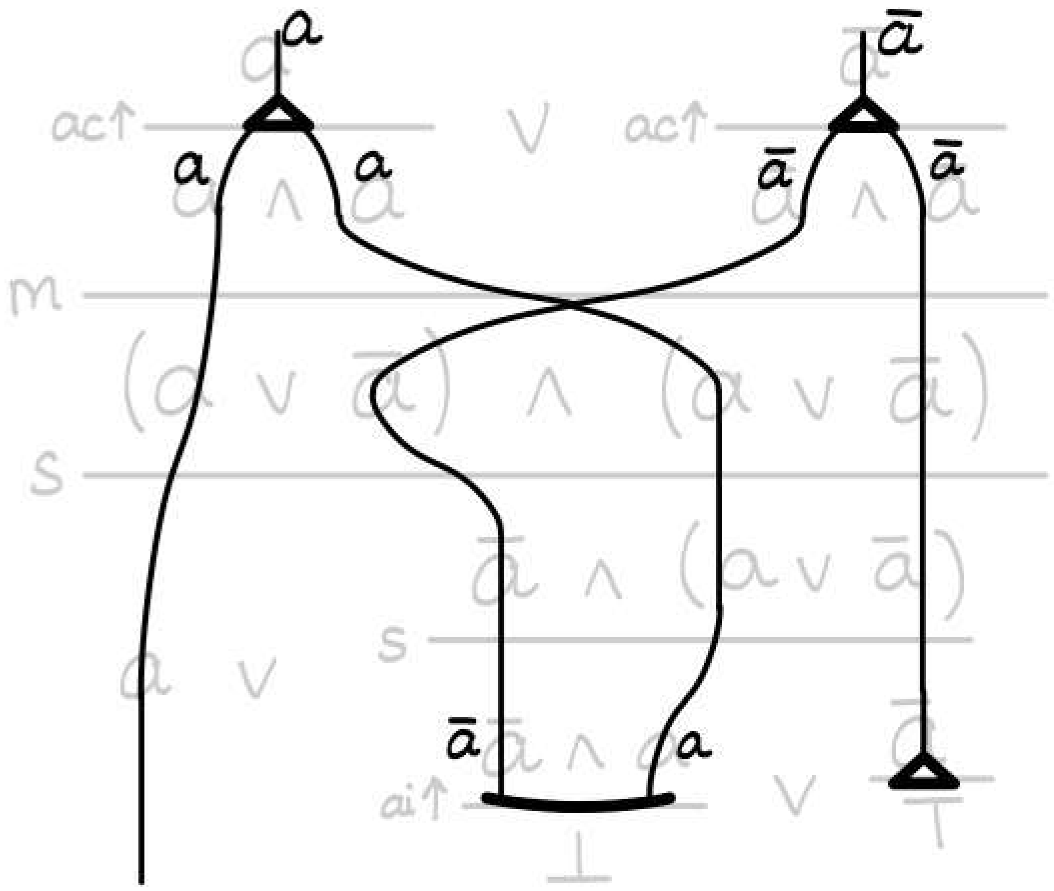


From derivations to flows

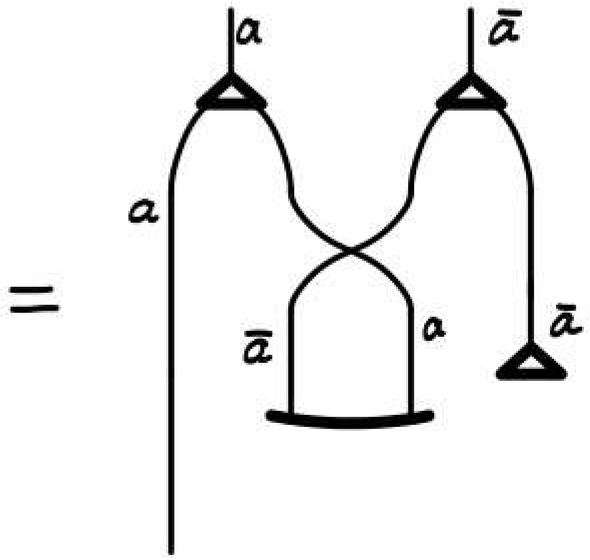


We retain only the structural information

From derivations to flows

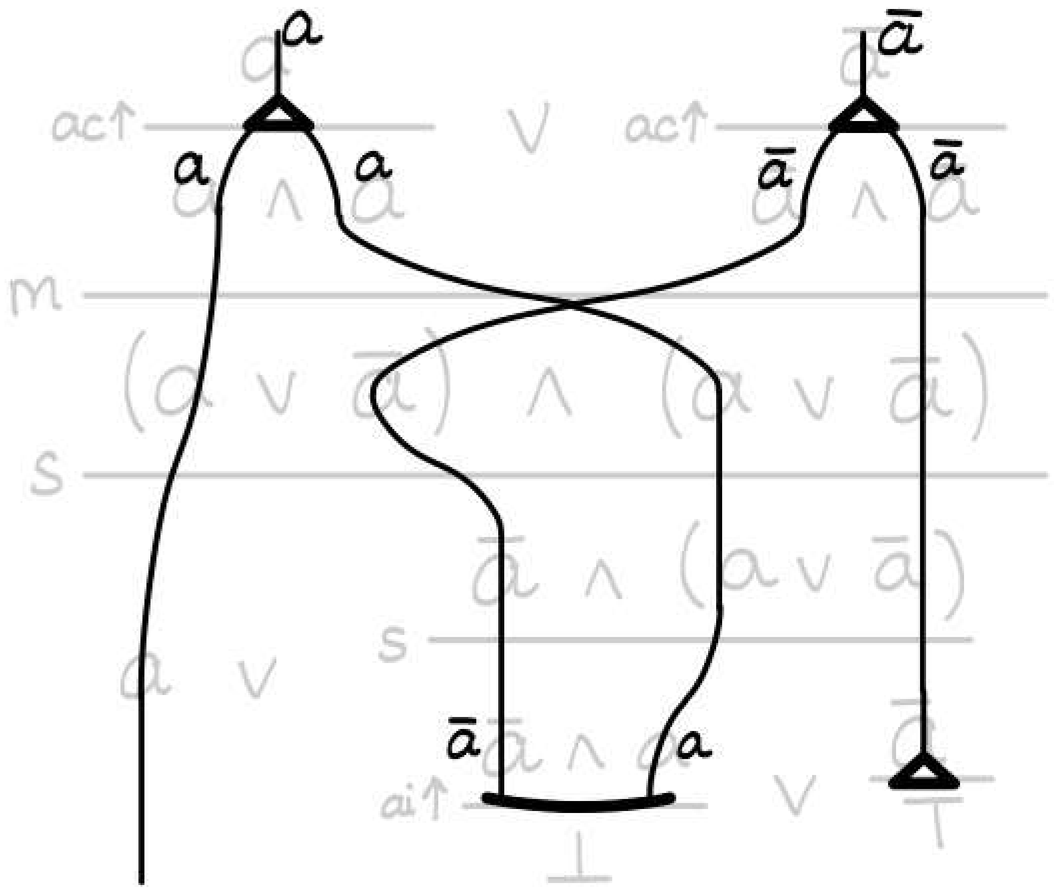


We retain only the structural information



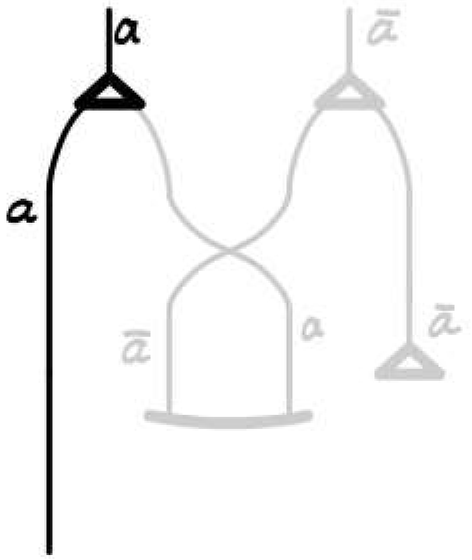
Flows are equal under continuous deformation

From derivations to flows



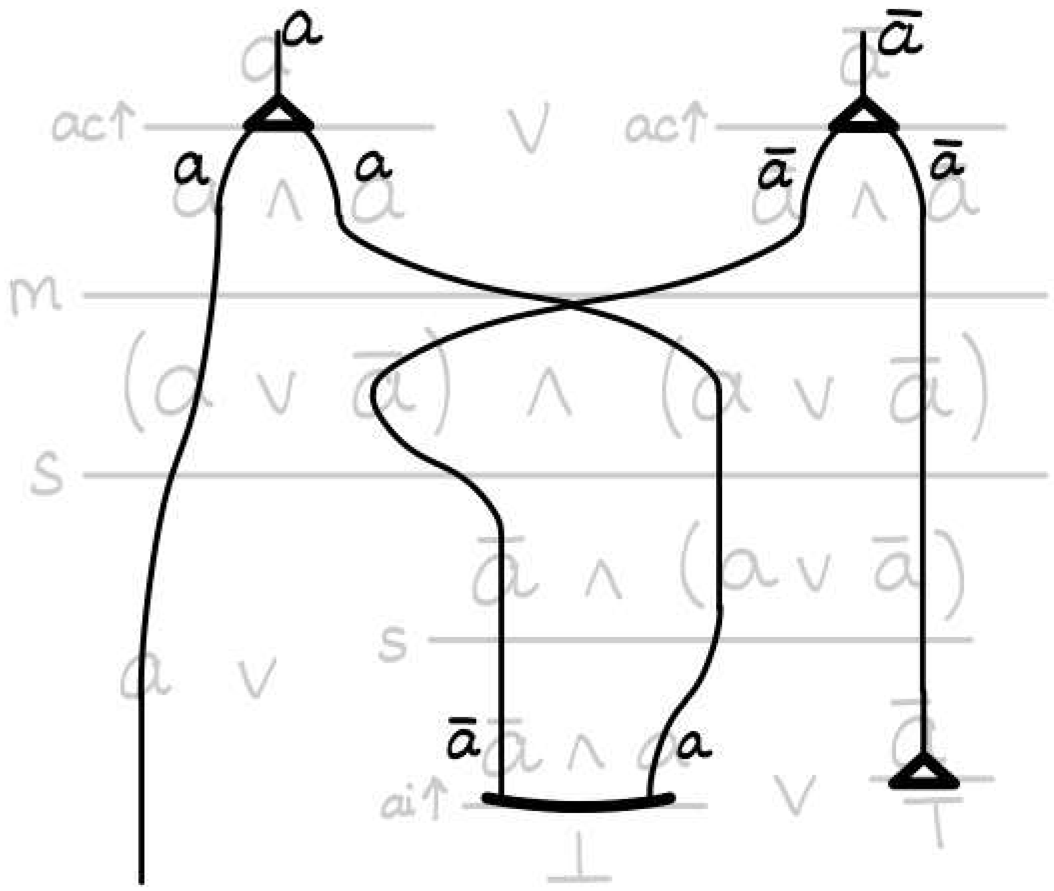
We retain only the structural information and can trace the paths of atoms

=



Flows are equal under continuous deformation

From derivations to flows



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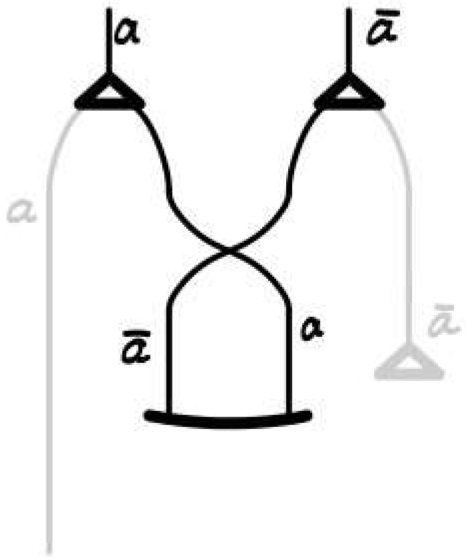
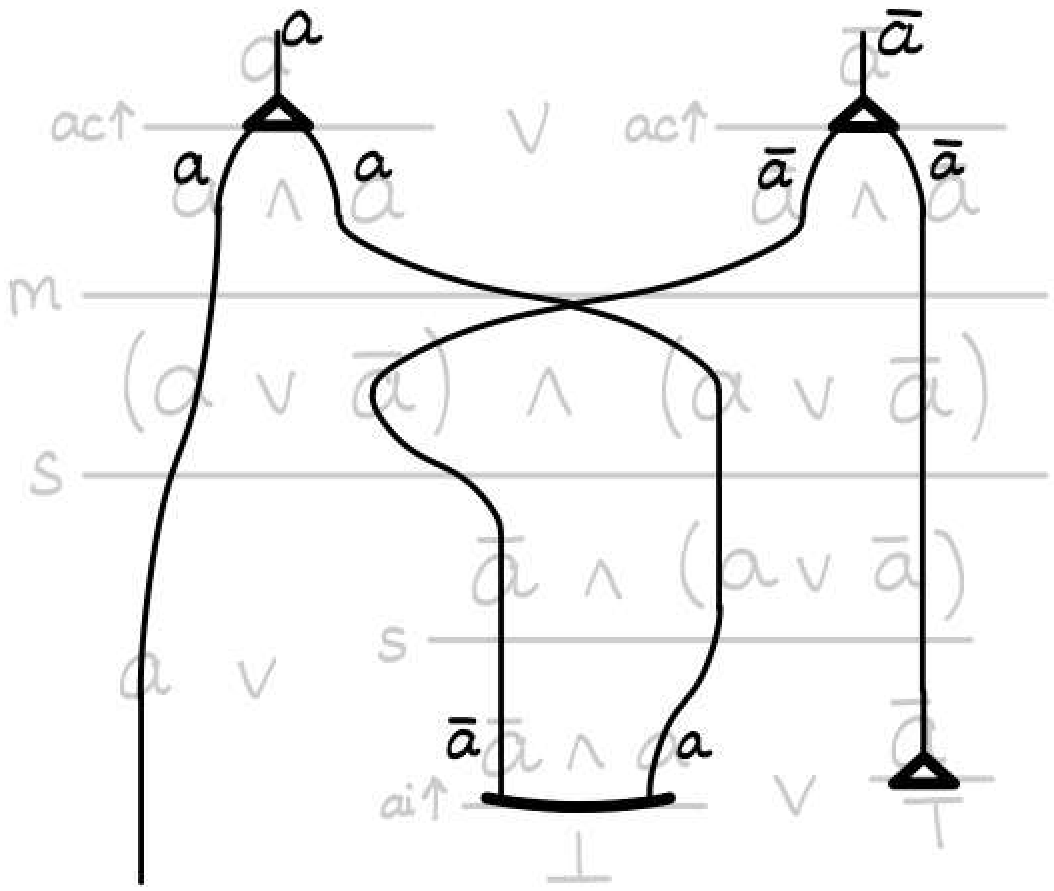
=



Flows are equal under continuous deformation

From derivations to flows



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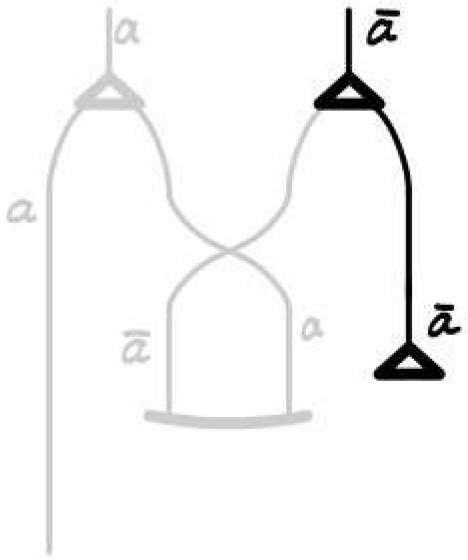
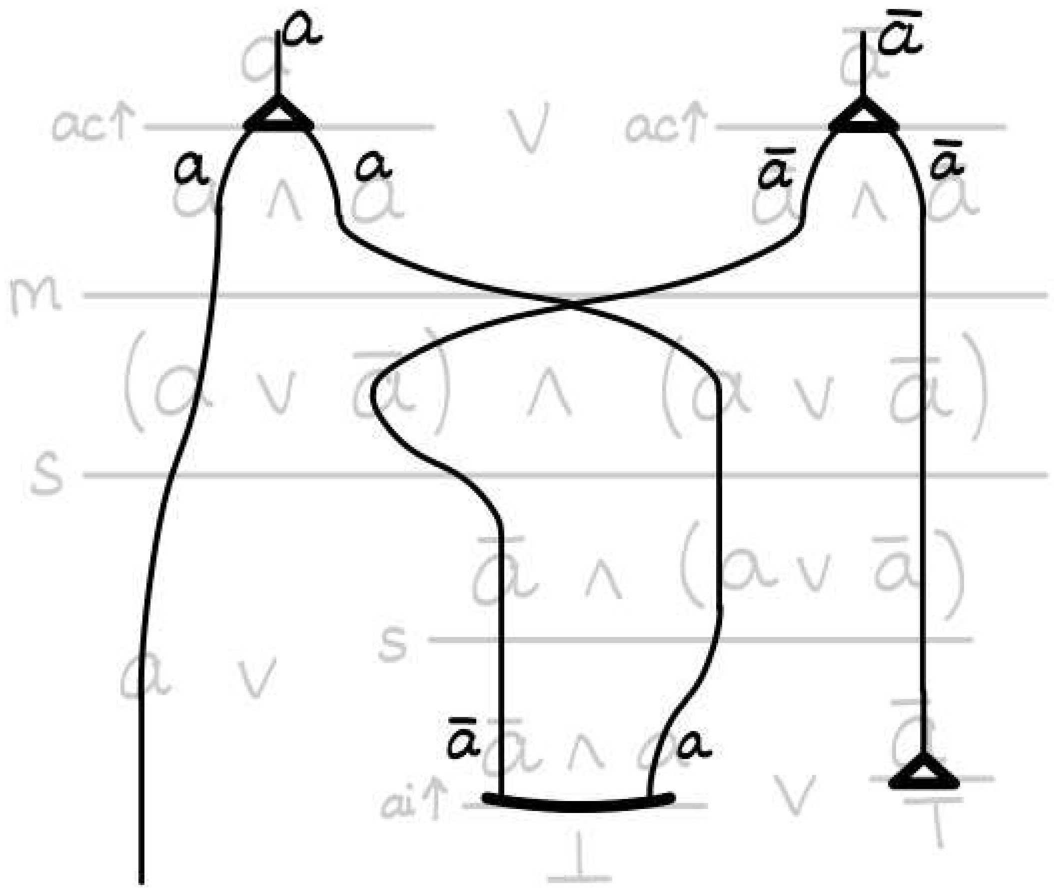


Flows are equal under continuous deformation

From derivations to flows

We retain only the structural information and can trace the paths of atoms



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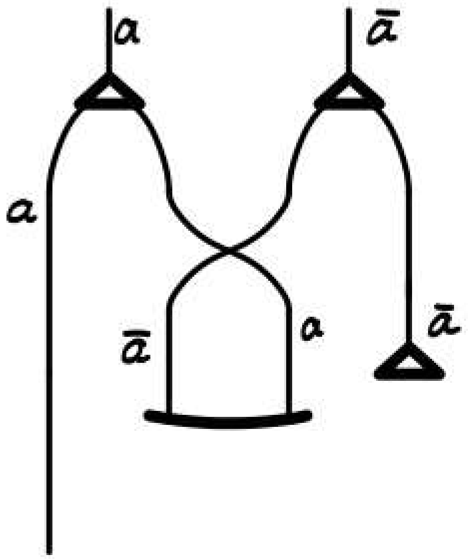
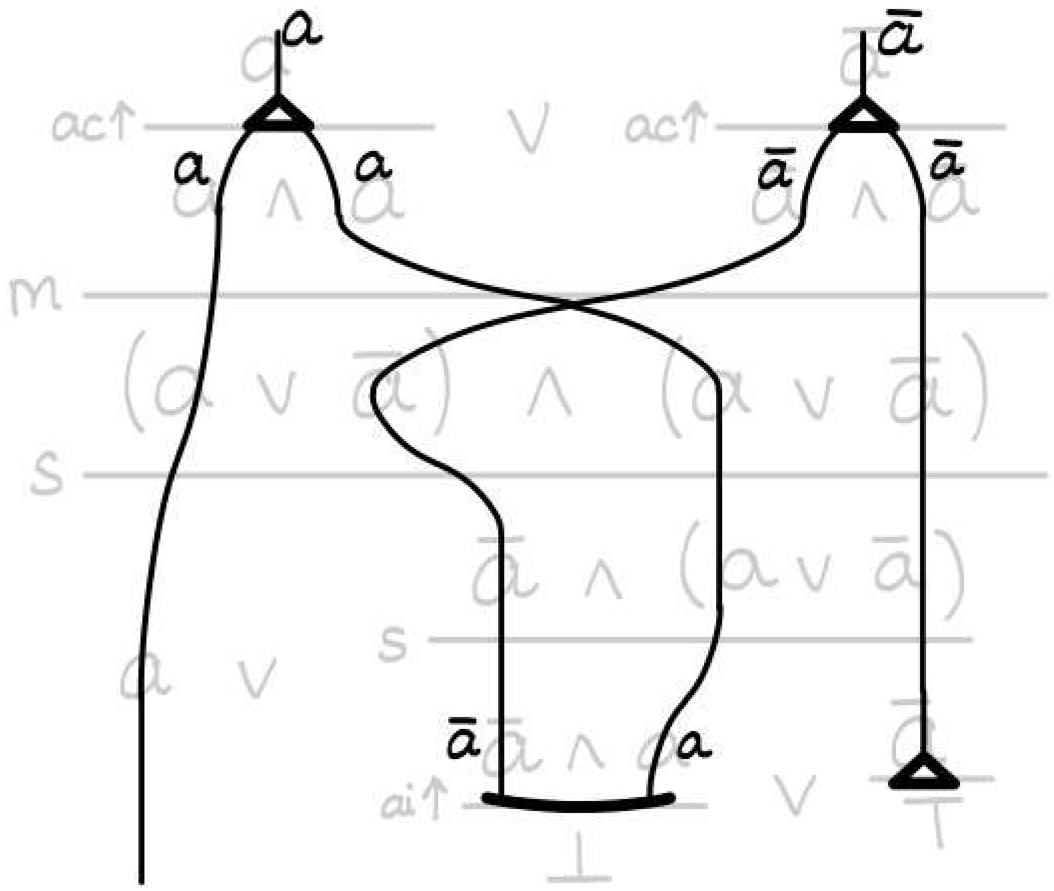


Flows are equal under continuous deformation

From derivations to flows



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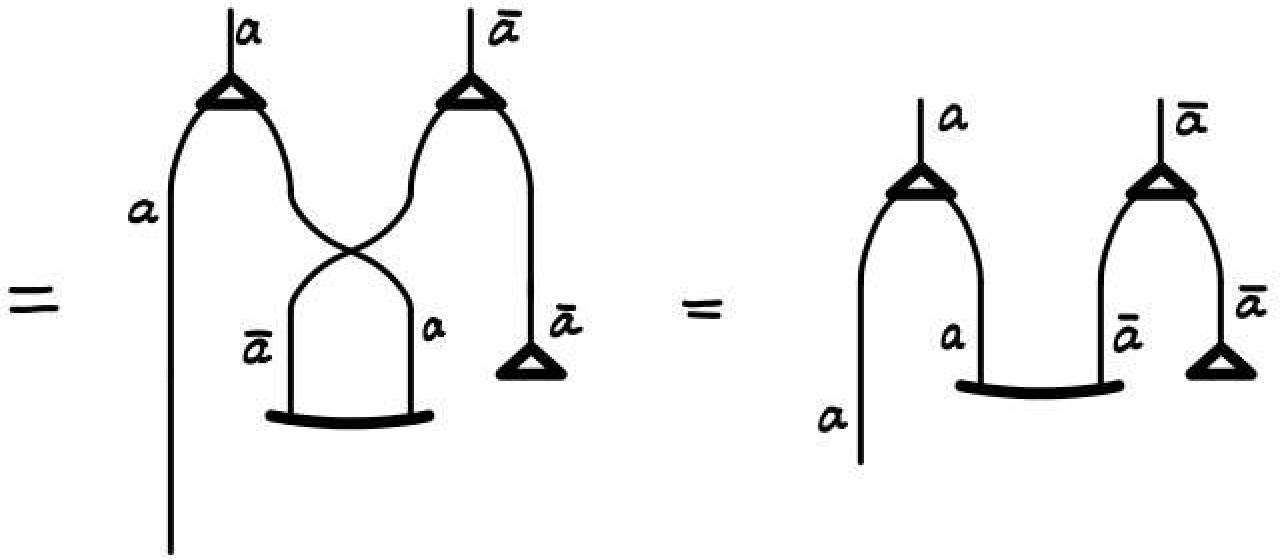
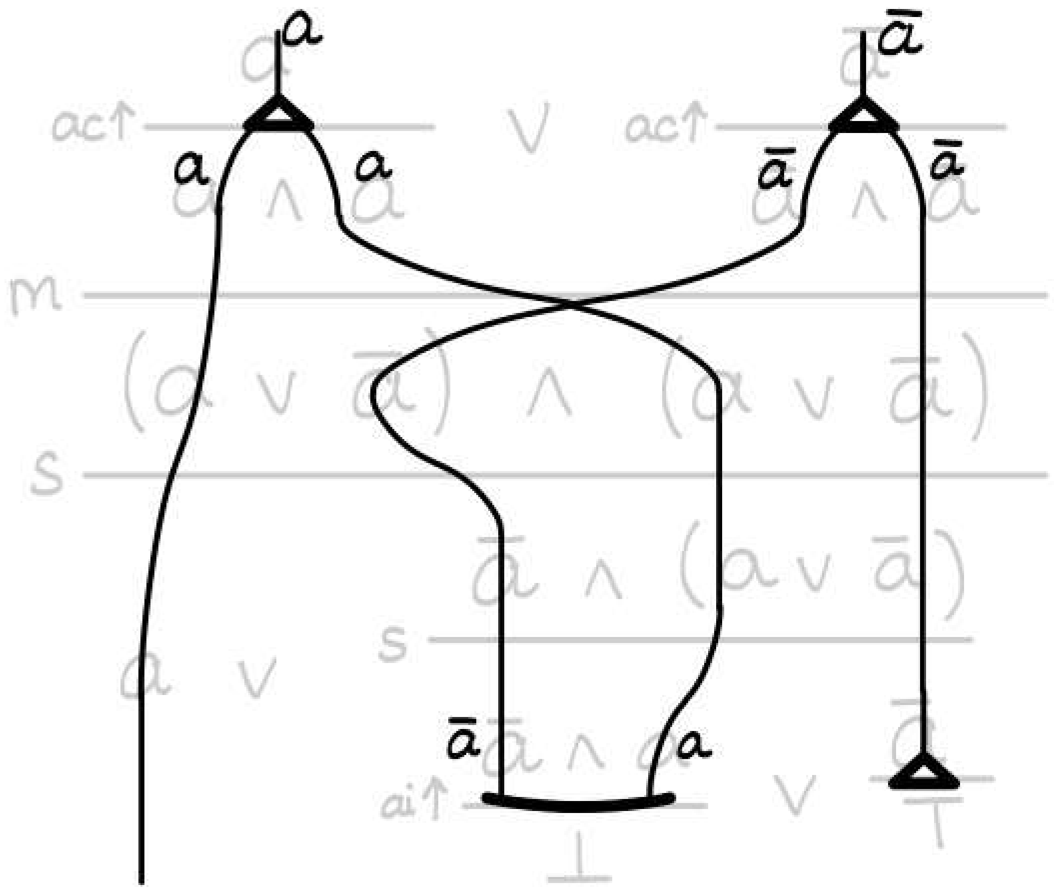


Flows are equal under continuous deformation

From derivations to flows



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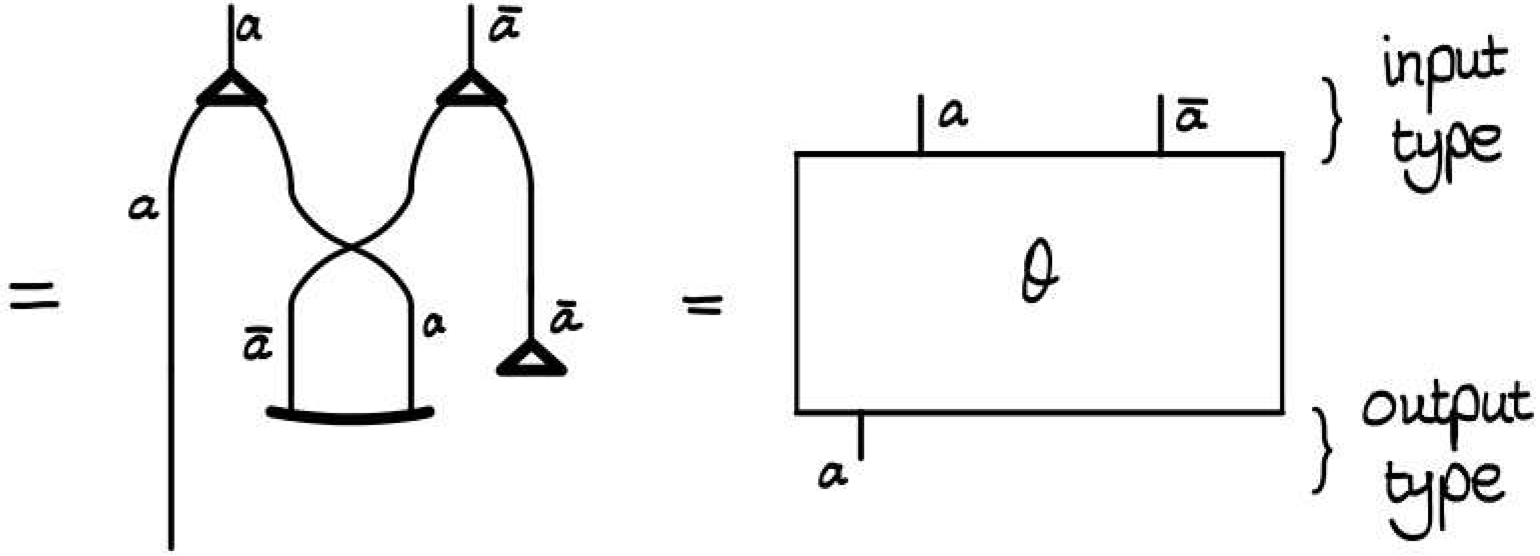
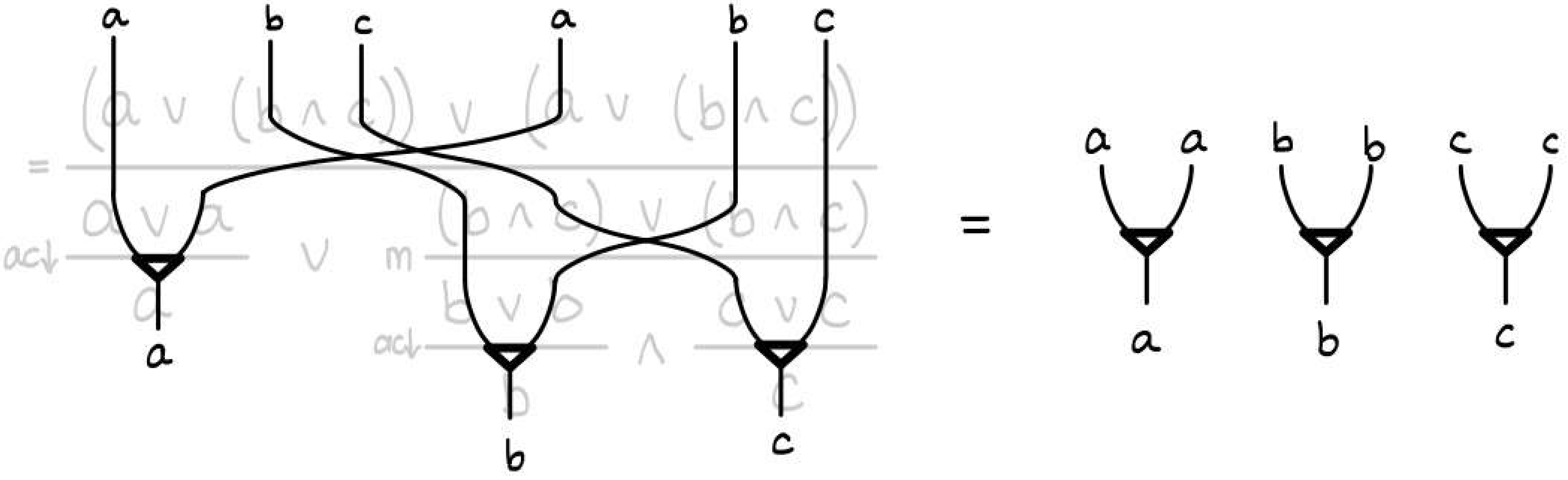


Flows are equal under continuous deformation

From derivations to flows

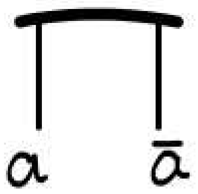
$$
{=}\,\frac{(a \vee (b \wedge c)) \vee (a \vee (b \wedge c))}{\mathsf{ac}{\downarrow}\,\dfrac{a \vee a}{a} \;\vee\; \mathsf{m}\,\dfrac{(b \wedge c) \vee (b \wedge c)}{\mathsf{ac}{\downarrow}\,\dfrac{b \vee b}{b} \;\wedge\; \dfrac{c \vee c}{c}}}
$$

From derivations to flows

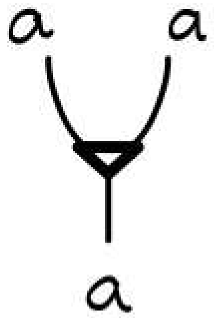


From derivations to flows

$$\mathsf{ai}\!\downarrow \frac{\top}{a \vee \bar{a}}$$

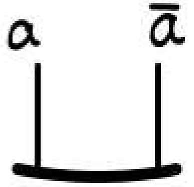


$$\mathsf{ac}\!\downarrow \frac{a \vee a}{a}$$

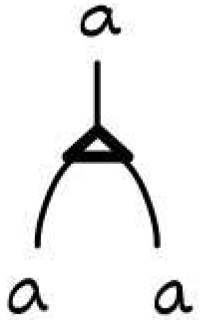


$$\mathsf{aw}\!\downarrow \frac{\bot}{a}$$

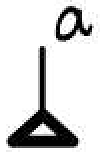
$$\mathsf{ai}\!\uparrow \frac{a \wedge \bar{a}}{\bot}$$



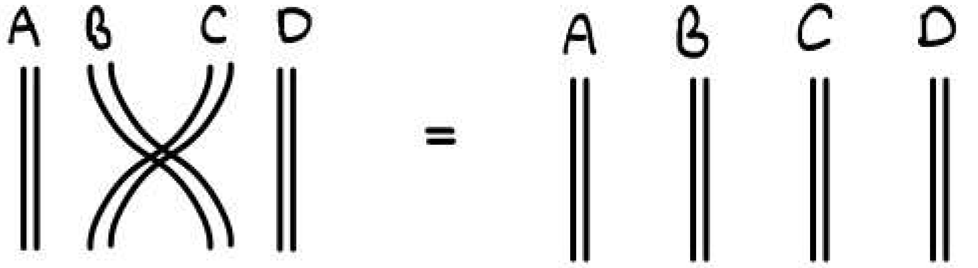
$$\mathsf{ac}\!\uparrow \frac{a}{a \wedge a}$$

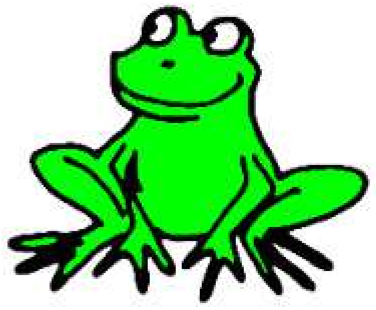


$$\mathsf{aw}\!\uparrow \frac{a}{\top}$$

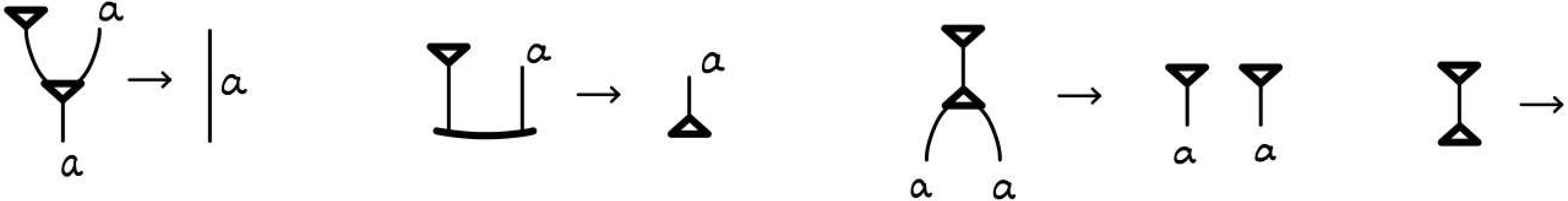


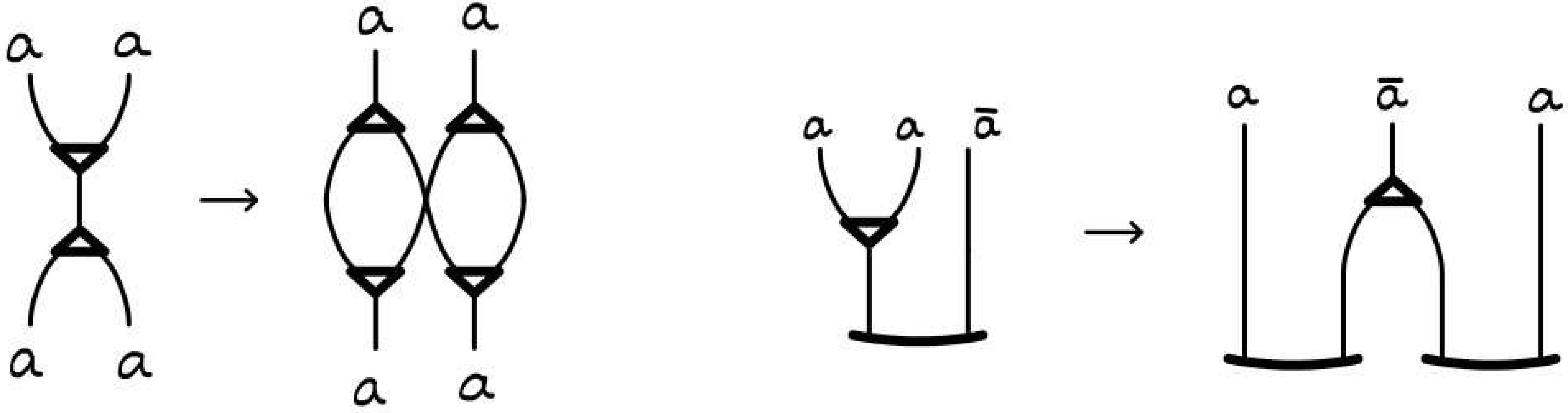
$$\mathsf{m} \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$



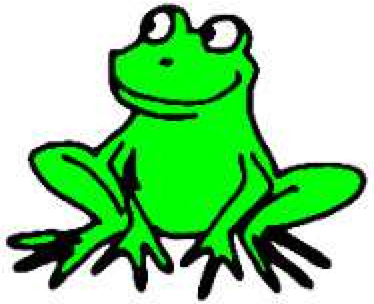


Rewriting flows

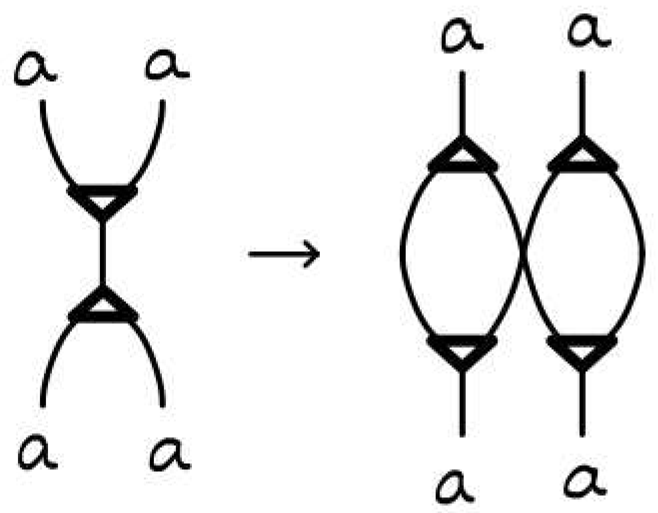




and duals



Rewriting flows lifts to derivations

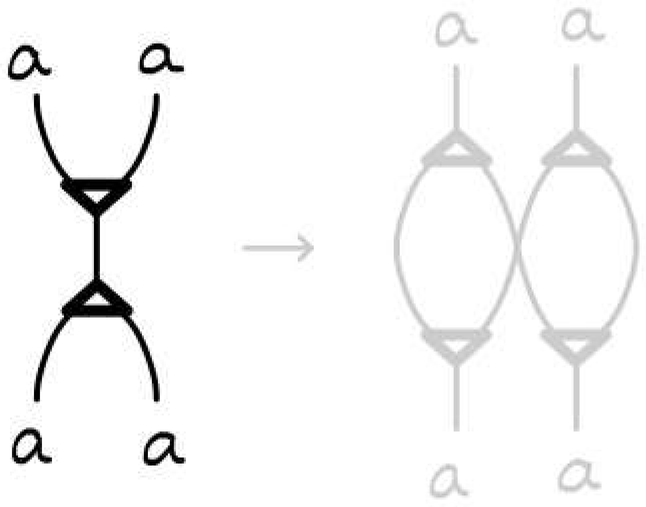


Rewriting flows lifts to derivations

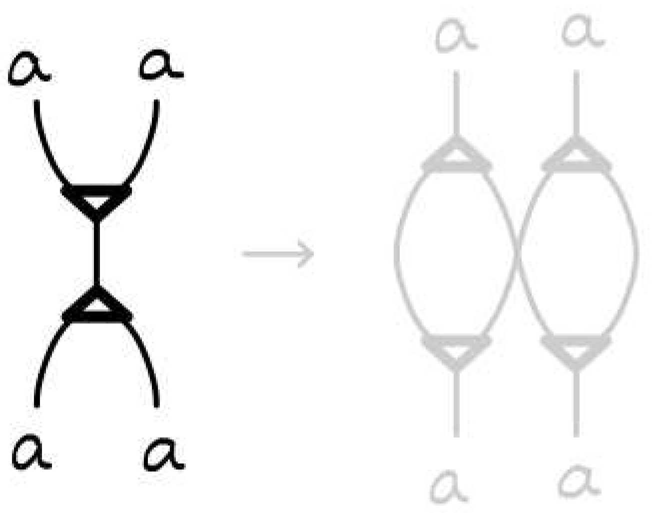
$$K\left\{ \mathsf{ac}{\downarrow}\frac{a \vee a}{a^{*}} \right\}$$

$$\Phi$$

$$H\left\{ \mathsf{ac}{\uparrow}\frac{a^{*}}{a \wedge a} \right\}$$

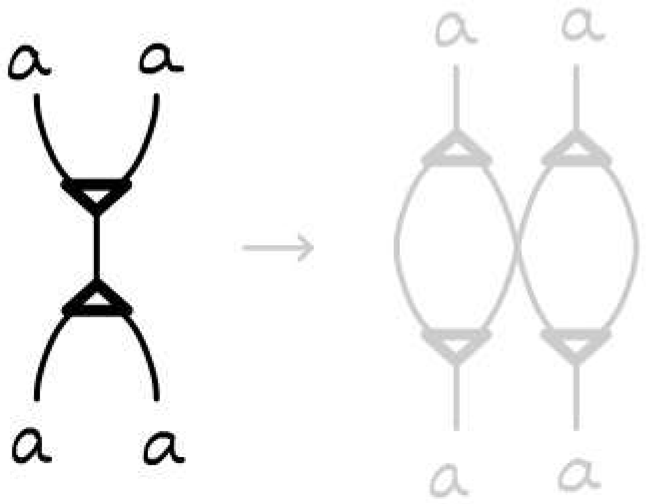


Rewriting flows lifts to derivations



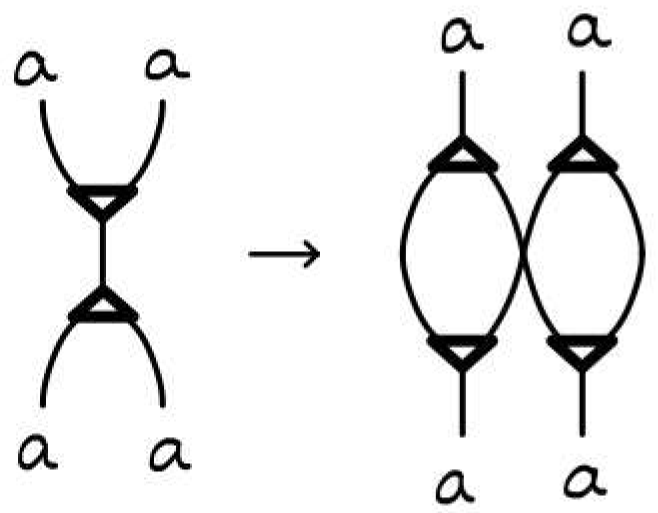
$$\begin{array}{c} K\left\{ \mathsf{ac}\!\downarrow \dfrac{a \vee a}{a^*} \right\} \\ \hline \Phi \\ \hline H\left\{ \mathsf{ac}\!\uparrow \dfrac{a^*}{a \wedge a} \right\} \end{array} \longrightarrow \begin{array}{c} K\{a \vee a\} \\ \hline [a \vee a / a^*]\,\Phi \end{array}$$

Rewriting flows lifts to derivations



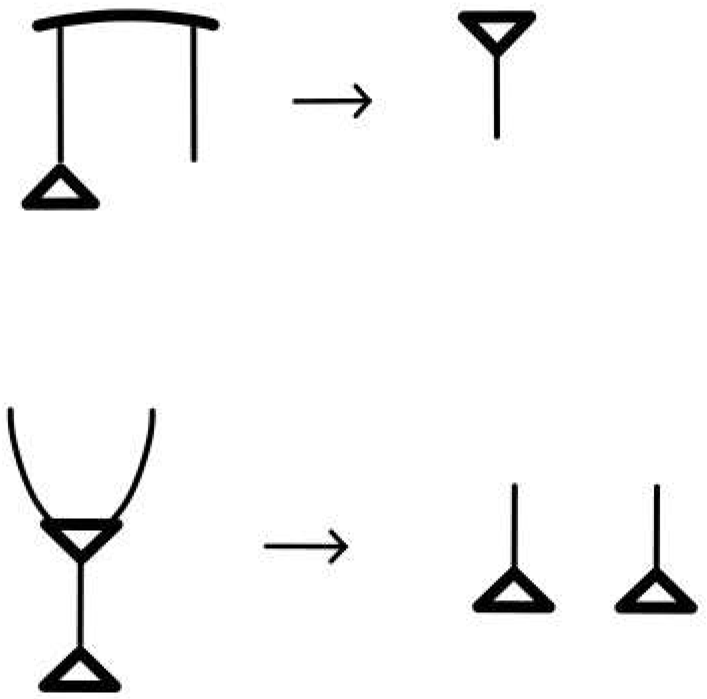
$$
\begin{array}{c}
K\left\{ \mathsf{ac}\!\downarrow \dfrac{a \vee a}{a^{*}} \right\} \\
\hdashline
\Phi \\
\hdashline
H\left\{ \mathsf{ac}\!\uparrow \dfrac{a^{*}}{a \wedge a} \right\}
\end{array}
\quad \longrightarrow \quad
\begin{array}{c}
K\{ a \vee a \} \\
\hdashline
[a \vee a / a^{*}]\, \Phi \\
\hdashline
H\left\{ \mathsf{m}\, \dfrac{\mathsf{ac}\!\uparrow \dfrac{a}{a \wedge a} \;\vee\; \mathsf{ac}\!\uparrow \dfrac{a}{a \wedge a}}{\mathsf{ac}\!\downarrow \dfrac{a \vee a}{a} \;\wedge\; \mathsf{ac}\!\downarrow \dfrac{a \vee a}{a}} \right\}
\end{array}
$$

Rewriting flows lifts to derivations



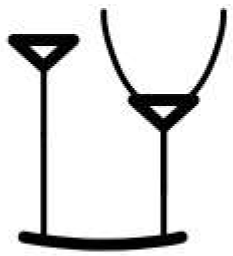
$$\begin{array}{c} K\left\{ \mathsf{ac}{\downarrow}\dfrac{a \vee a}{a^{*}} \right\} \\ \hdashline \Phi \\ \hdashline H\left\{ \mathsf{ac}{\uparrow}\dfrac{a^{*}}{a \wedge a} \right\} \end{array} \quad \longrightarrow \quad \begin{array}{c} K\{a \vee a\} \\ \hdashline [a \vee a / a^{*}]\,\Phi \\ \hdashline H\left\{ \mathsf{m}\dfrac{\mathsf{ac}{\uparrow}\dfrac{a}{a \wedge a} \vee \mathsf{ac}{\uparrow}\dfrac{a}{a \wedge a}}{\mathsf{ac}{\downarrow}\dfrac{a \vee a}{a} \wedge \mathsf{ac}{\downarrow}\dfrac{a \vee a}{a}} \right\} \end{array}$$

Rewriting flows lifts to derivations

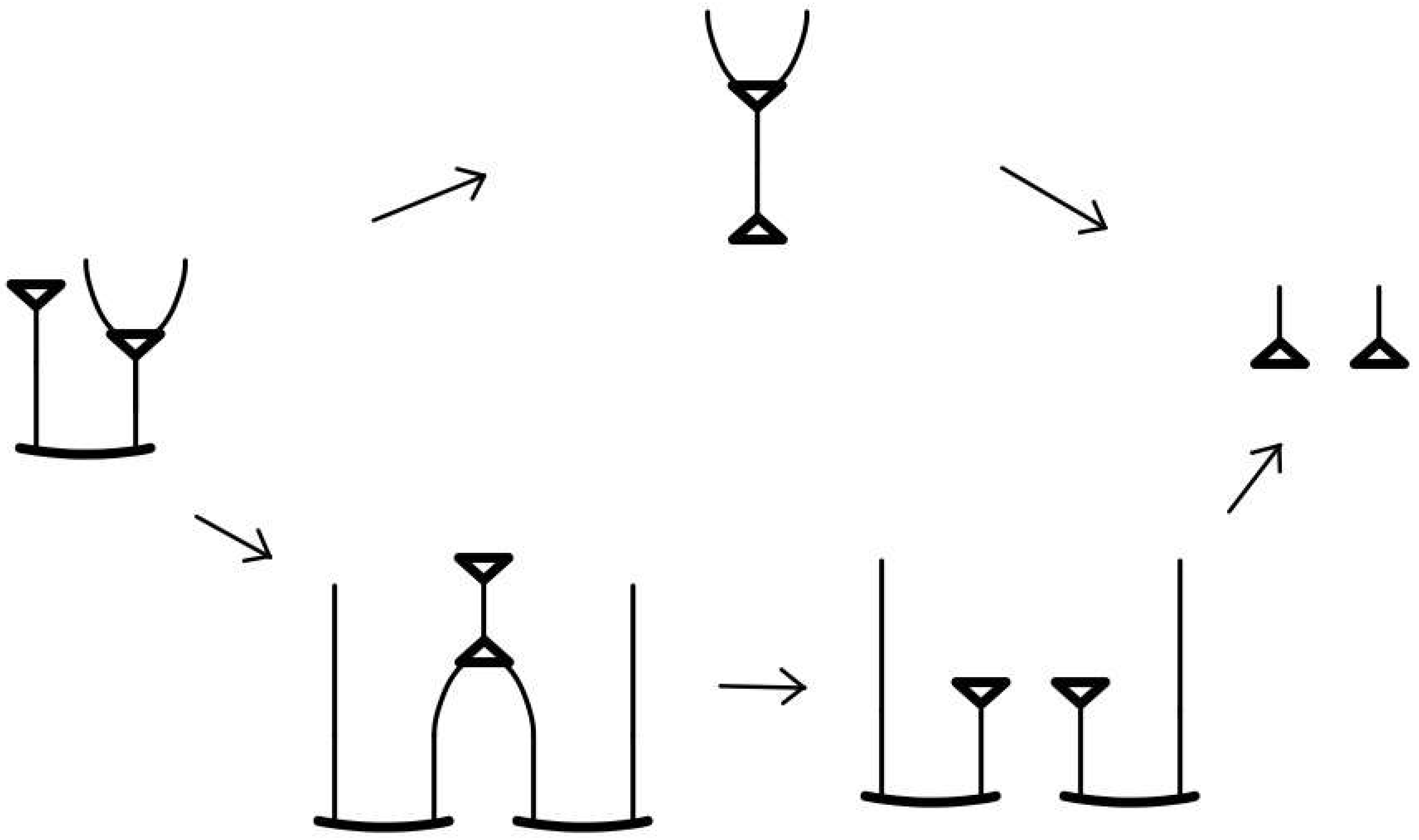


We've seen these rewrite steps before in the cut elimination procedure for SKS!

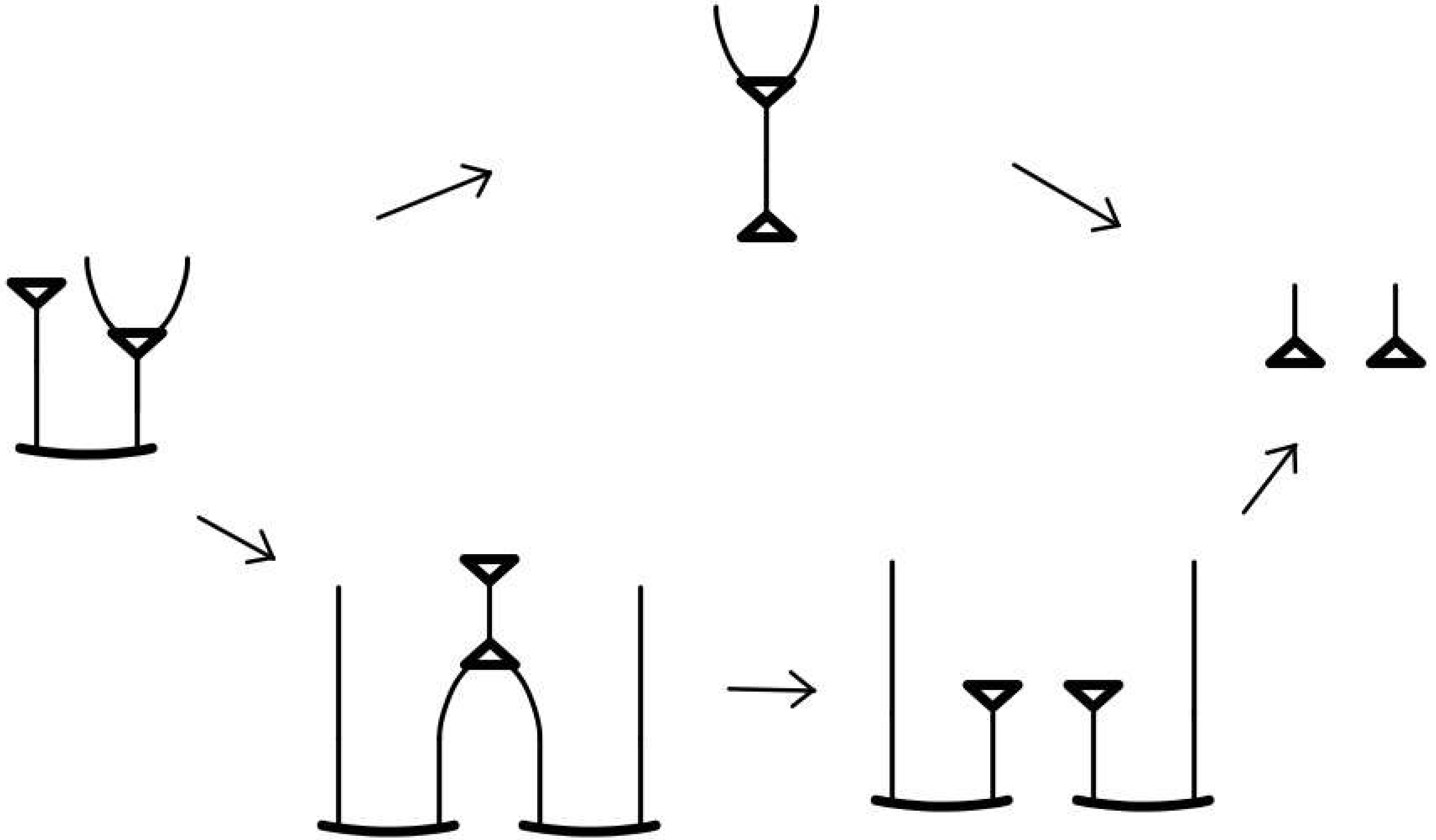
Rewriting flows is locally confluent



Rewriting flows is locally confluent

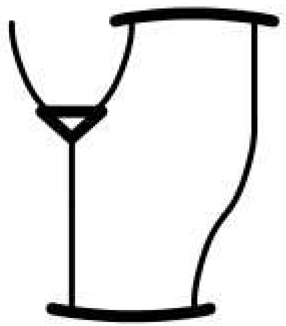


Rewriting flows is locally confluent

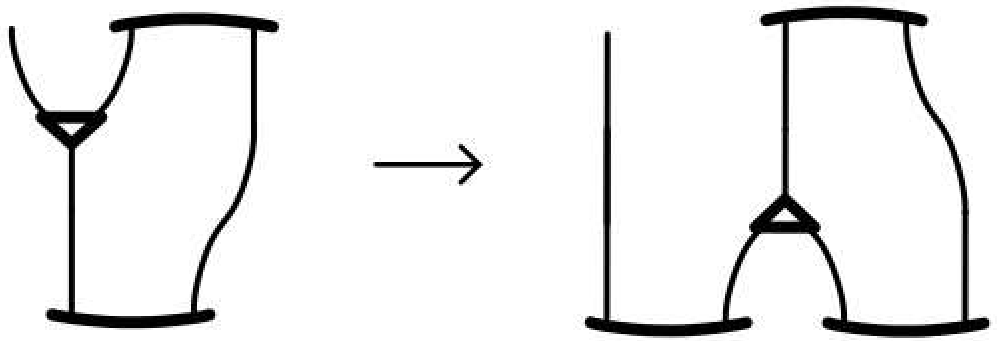


sometimes rewriting eliminates cuts, sometimes it duplicates them

Rewriting flows does not always terminate!



Rewriting flows does not always terminate!



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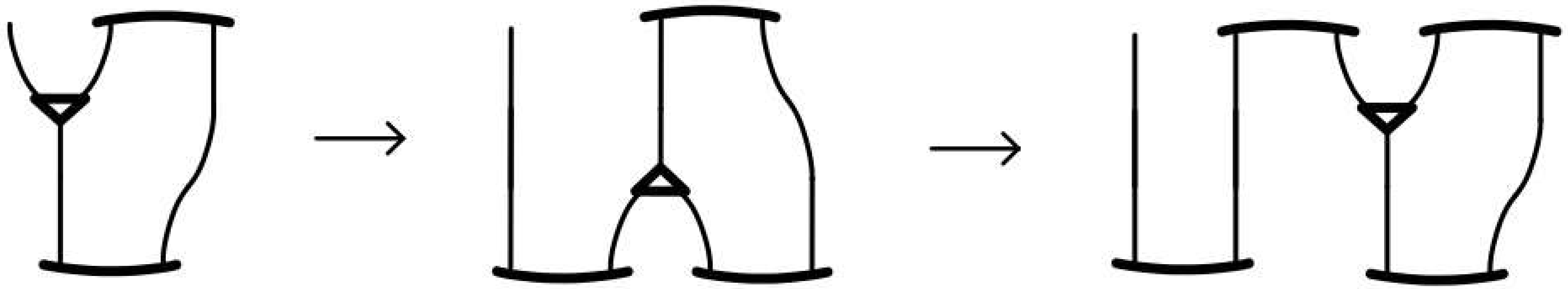

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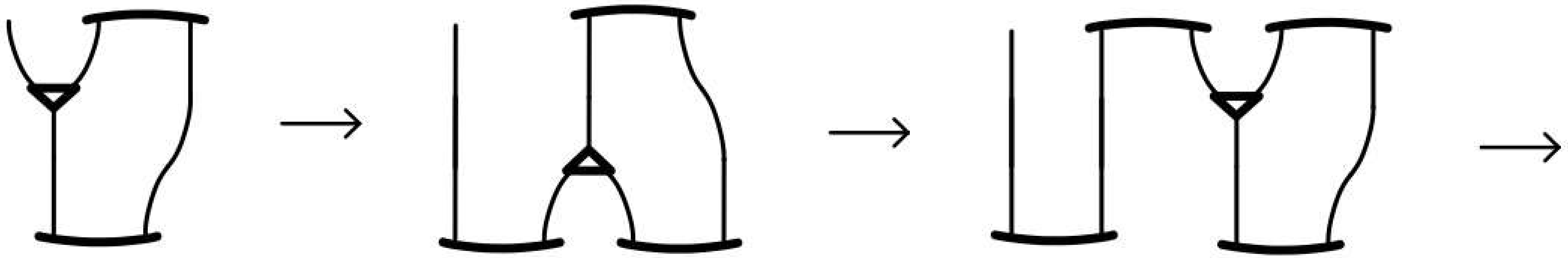
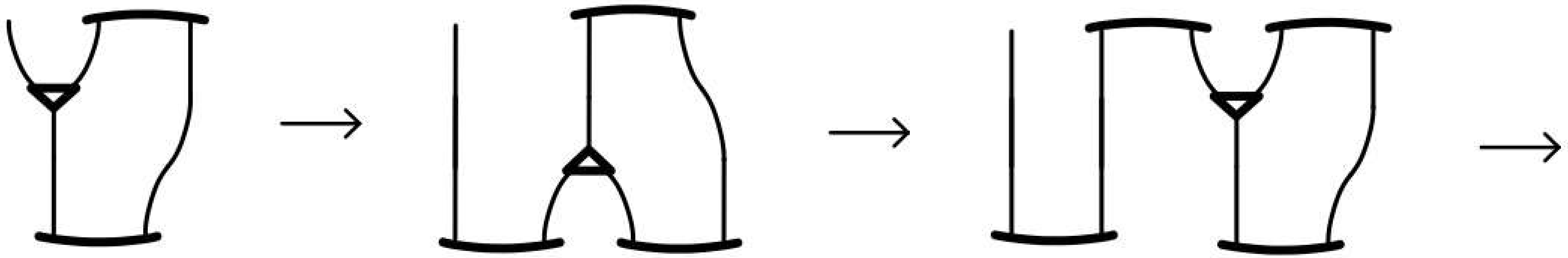
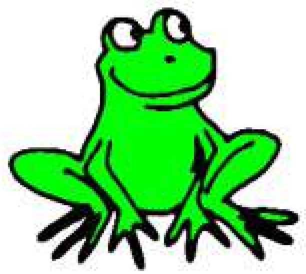


the problem arises when a (co)contraction gets into a path from an identity to a cut

Rewriting flows does not always terminate!



the problem arises when a (co)contraction gets into a path from an identity to a cut



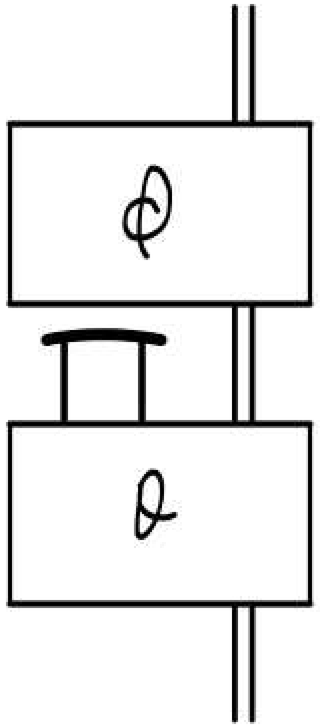
break the paths between cuts and identities

Identities can be pulled up derivations

$$\begin{array}{c} B \\ \Phi \Big\| \\ K\left\{ \text{ai}\!\downarrow \dfrac{\top}{a \vee \bar{a}} \right\} \\ \Theta \Big\| \\ C \end{array}$$

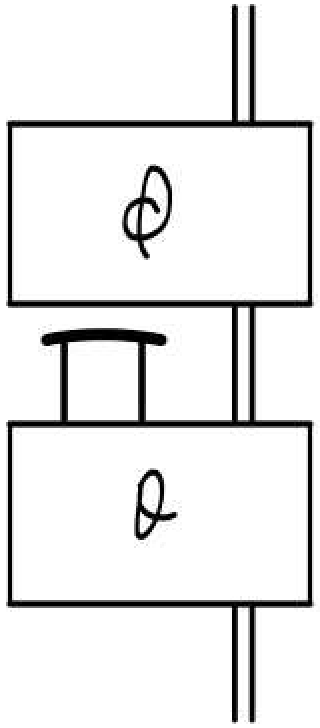
Identities can be pulled up derivations

$$\begin{array}{c} B \\ \Phi \Big\| \\ K\left\{ \text{ai}\!\downarrow \dfrac{\top}{a \vee \bar{a}} \right\} \\ \Theta \Big\| \\ C \end{array}$$



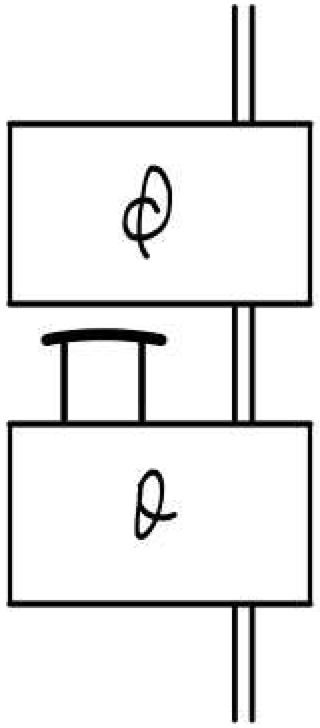
Identities can be pulled up derivations

$$
\begin{array}{c}
B \\
\Phi \, \Big\| \\
K\left\{ \mathsf{ai}{\downarrow}\,\dfrac{T}{a \vee \bar{a}} \right\} \\
\Theta \, \Big\| \\
C
\end{array}
\quad \longrightarrow \quad
\begin{array}{c}
B \\
\Phi \, \Big\| \\
K\{T\}
\end{array}
$$



Identities can be pulled up derivations

$$\begin{array}{c} B \\ \Phi \Big\| \\ K\left\{ \text{ai}\!\downarrow \dfrac{\top}{a \vee \bar{a}} \right\} \\ \Theta \Big\| \\ C \end{array} \quad \rightarrow \quad \boxed{\begin{array}{c} B \\ \Phi \Big\| \\ K\{\top\} \end{array} \quad \wedge \quad \text{ai}\!\downarrow \dfrac{\top}{a \vee \bar{a}}}$$



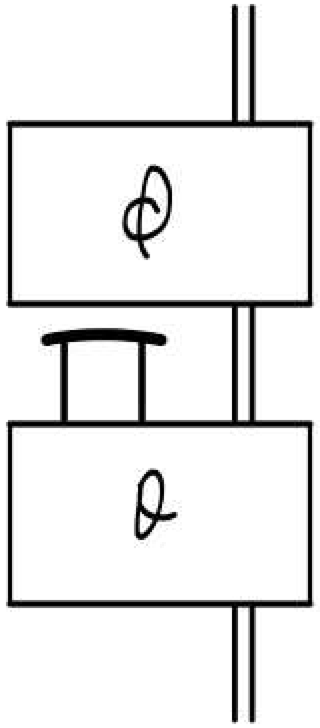
Identities can be pulled up derivations

$$
\begin{array}{c}
B \\
\Phi \, \Big\| \\
K\left\{ \text{ai}\!\downarrow \dfrac{\top}{a \vee \bar{a}} \right\} \\
\Theta \, \Big\| \\
C
\end{array}
\quad \longrightarrow \quad
\boxed{
\begin{array}{c}
B \\
\Phi \, \Big\| \\
K\{\top\}
\end{array}
\quad \wedge \quad
\text{ai}\!\downarrow \dfrac{\top}{a \vee \bar{a}}
}
$$

induction on $K\{\}$

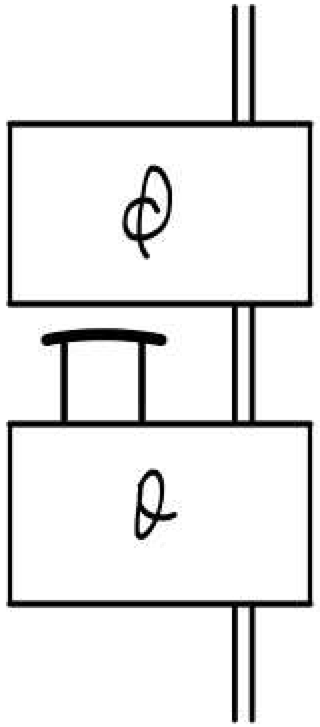
$$
\Big\| \; s, =
$$

$$
K\{a \vee \bar{a}\}
$$



Identities can be pulled up derivations

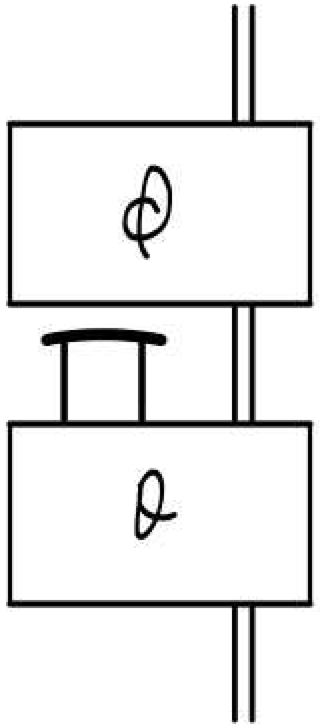
$$\begin{array}{c} B \\ \Phi \Big\| \\ K\left\{ \mathsf{ai}\!\downarrow \dfrac{\top}{a \vee \bar{a}} \right\} \\ \Theta \Big\| \\ C \end{array} \quad \rightarrow \quad \begin{array}{c} \boxed{\begin{array}{c} B \\ \Phi \Big\| \\ K\{\top\} \end{array} \quad \wedge \quad \mathsf{ai}\!\downarrow \dfrac{\top}{a \vee \bar{a}}} \\ \Big\| \; \mathsf{s}, = \\ K\{a \vee \bar{a}\} \\ \Theta \Big\| \\ C \end{array}$$



induction on K{}

Identities can be pulled up derivations

$$\begin{array}{c} B \\ \Phi \Big\| \\ K\left\{ \text{ai}\!\downarrow \dfrac{\top}{a \vee \bar{a}} \right\} \\ \Theta \Big\| \\ C \end{array}$$

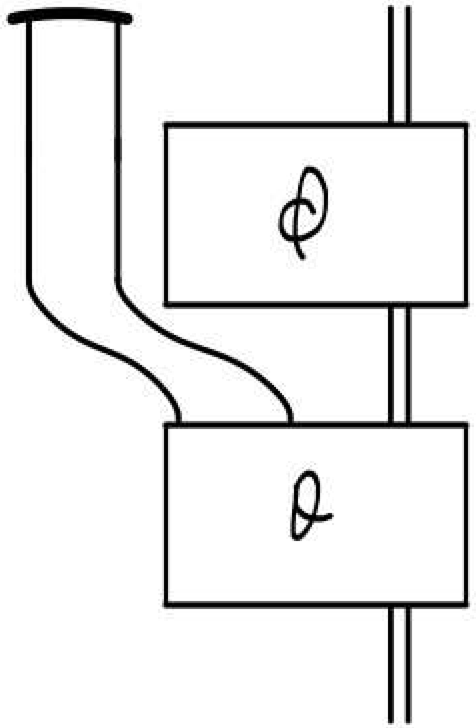


$\longrightarrow$

induction on $K\{\,\}$

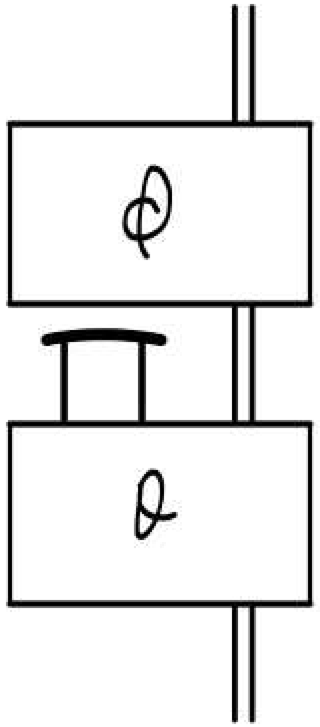
$$\boxed{\begin{array}{c} B \\ \Phi \Big\| \\ K\{\top\} \end{array} \quad \wedge \quad \text{ai}\!\downarrow \dfrac{\top}{a \vee \bar{a}}}$$

$$\begin{array}{c} \Big\| \; s, = \\ K\{a \vee \bar{a}\} \\ \Theta \Big\| \\ C \end{array}$$



Identities can be pulled up derivations

$$
\begin{array}{c}
B \\
\Phi \Big\| \\
K\left\{ \mathsf{ai}\!\downarrow \dfrac{\top}{a \vee \bar{a}} \right\} \\
\Theta \Big\| \\
C
\end{array}
$$

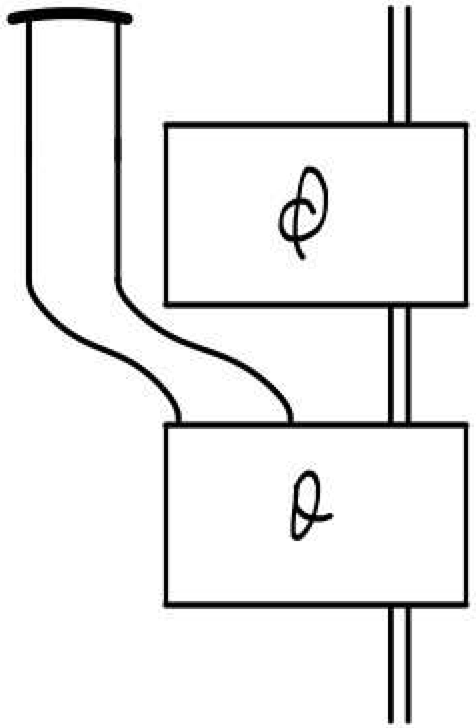


$\rightarrow$

$$
\boxed{
\begin{array}{c}
B \\
\Phi \Big\| \\
K\{\top\}
\end{array}
\quad \wedge \quad
\mathsf{ai}\!\downarrow \dfrac{\top}{a \vee \bar{a}}
}
$$

induction on $K\{\}$

$$
\begin{array}{c}
\Big\| \; \mathsf{s}, = \\
K\{a \vee \bar{a}\} \\
\Theta \Big\| \\
C
\end{array}
$$



this justifies identifying their flows

For some atom a, pull every identity up and every cut down to get this:

$$\begin{array}{c} B \wedge \overbrace{\dfrac{\top}{a \vee \bar{a}} \wedge \cdots \wedge \dfrac{\top}{a \vee \bar{a}}}^{m \geqslant 0} \\ \Xi \, \Big\| \\ C \wedge \underbrace{\dfrac{a \wedge \bar{a}}{\bot} \vee \cdots \vee \dfrac{a \wedge \bar{a}}{\bot}}_{n \geqslant 0} \end{array}$$

For some atom a, pull every identity up and every cut down to get this:

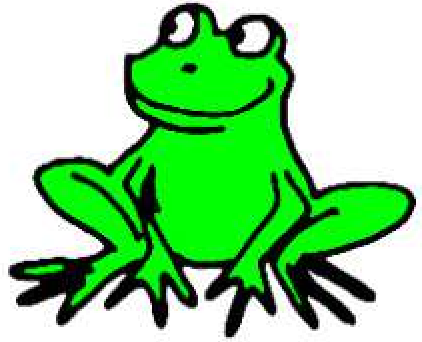
$$
\begin{array}{c}
B \wedge \overbrace{\dfrac{\top}{a \vee \bar{a}} \wedge \cdots \wedge \dfrac{\top}{a \vee \bar{a}}}^{m \geqslant 0} \\
\Xi \, \Vert \\
C \wedge \underbrace{\dfrac{a \wedge \bar{a}}{\bot} \vee \cdots \vee \dfrac{a \wedge \bar{a}}{\bot}}_{n \geqslant 0}
\end{array}
$$

then transform it like this to get a single cut and identity on a

$\rightarrow$

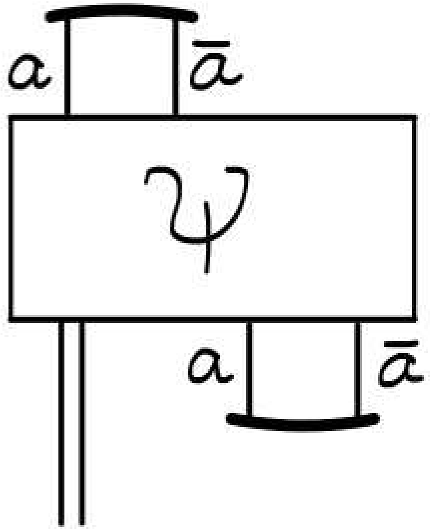
$$
\boxed{
\begin{array}{c}
B \wedge \boxed{\begin{array}{c} \dfrac{\top}{a \vee \bar{a}} \\ \Vert \; ac\uparrow, m, = \\ (a \vee \bar{a}) \wedge \cdots \wedge (a \vee \bar{a}) \end{array}} \\
\Xi \, \Vert \\
C \vee \boxed{\begin{array}{c} (a \wedge \bar{a}) \vee \cdots \vee (a \wedge \bar{a}) \\ \Vert \; ac\uparrow, m, = \\ \dfrac{a \wedge \bar{a}}{\bot} \end{array}}
\end{array}
}
$$

Every proof Φ that contains a cut $\dfrac{a \wedge \bar{a}}{\bot}$ can be transformed into a proof



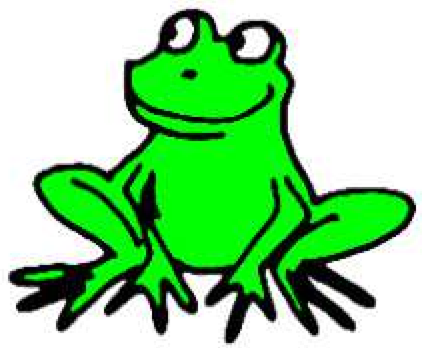
$$\begin{array}{c} \dfrac{\top}{a \vee \bar{a}} \\ \Psi \, \Vert \\ \boxed{B \vee \dfrac{a \wedge \bar{a}}{\bot}} \end{array}$$

with flow



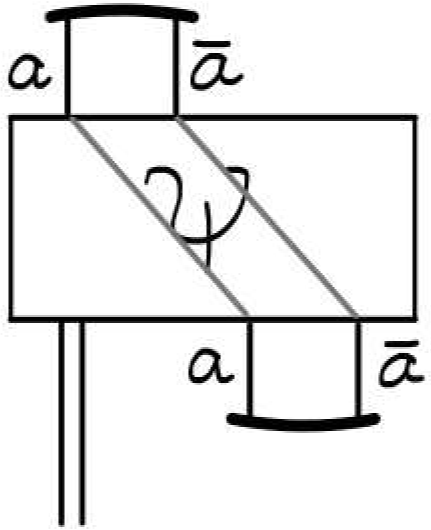
where Ψ contains no cuts or identities on a.

Every proof Φ that contains a cut $\dfrac{a \wedge \bar{a}}{\bot}$ can be transformed into a proof



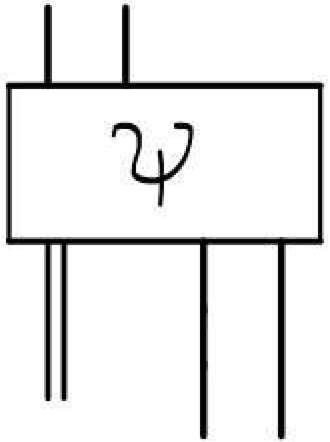
$$\begin{array}{c} \dfrac{\top}{a \vee \bar{a}} \\ \Psi \, \Big\Vert \\ \boxed{B \vee \dfrac{a \wedge \bar{a}}{\bot}} \end{array}$$

with flow

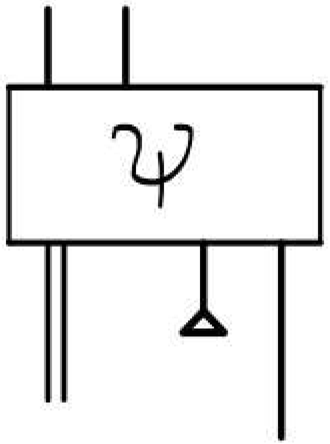


where Ψ contains no cuts or identities on a.

Ψ contains paths from the identity to the cut and these are what we want to break



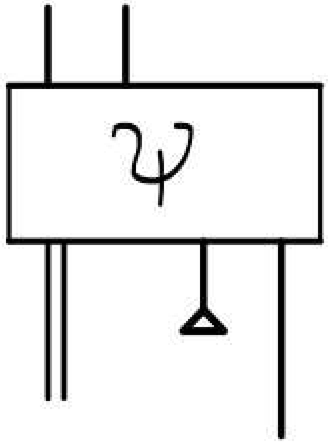
$$
\begin{array}{c}
a \vee \bar{a} \\
\Psi \Big\Vert \\
B \vee (a \wedge \bar{a})
\end{array}
$$



$$a \vee \bar{a}$$

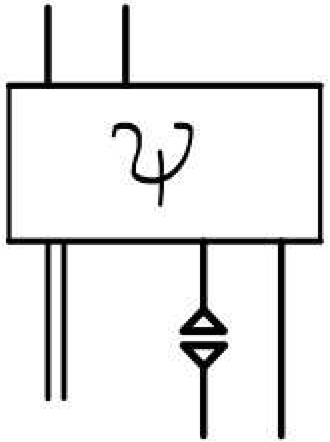
$$\Psi \,\|$$

$$B \vee (\underset{\top}{\underline{a}} \wedge \bar{a})$$



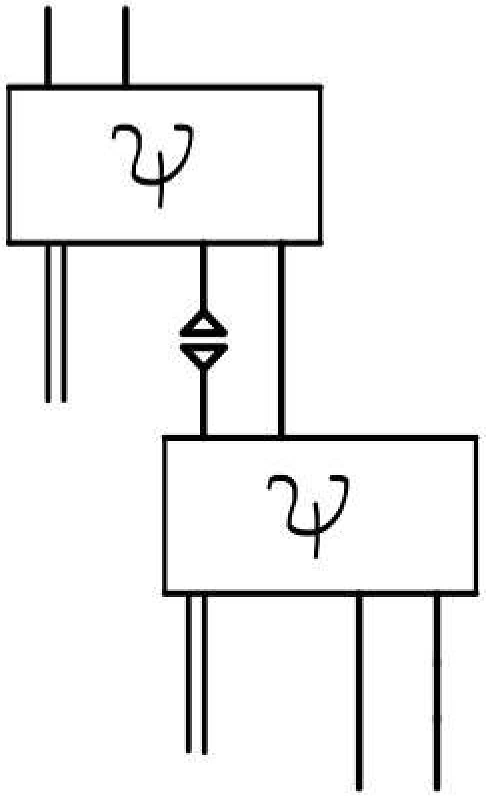
$$\begin{array}{c} a \vee \bar{a} \\ \Psi \Big\| \\ B \vee \left(\dfrac{a}{\top} \wedge \bar{a} \right) \end{array}$$

$$\Pi_a = \boxed{\begin{array}{rc} \text{aw}\uparrow & \dfrac{a}{\top} \wedge \bar{a} \\ = & \hline & \bar{a} \\ = & \hline \text{aw}\downarrow & \dfrac{\bot}{a} \vee \bar{a} \end{array}}$$



$$\begin{array}{c} a \vee \bar{a} \\ \Psi \,\Big\| \\ B \vee \left(\dfrac{a}{\top} \wedge \bar{a}\right) \end{array}$$

$$\Pi_a = \boxed{\begin{array}{l} \text{aw}\!\uparrow \dfrac{a}{\top} \wedge \bar{a} \\ = \overline{} \\ \qquad \bar{a} \\ = \overline{} \\ \text{aw}\!\downarrow \dfrac{\bot}{a} \vee \bar{a} \end{array}}$$

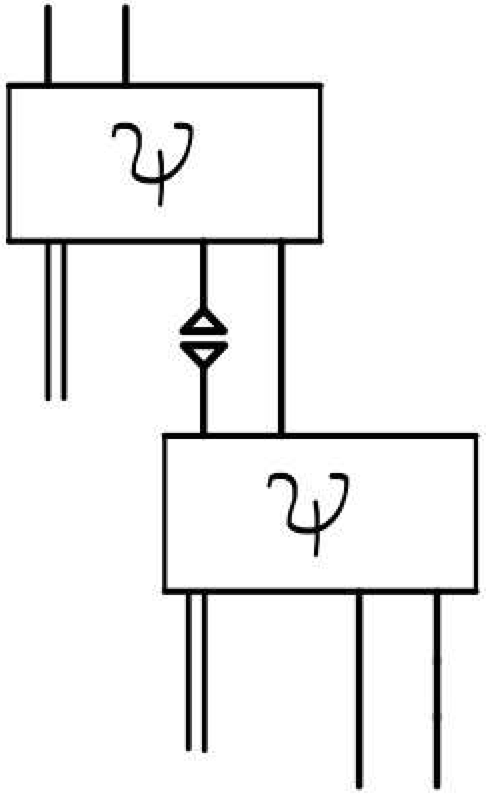


$$a \vee \bar{a}$$

$$\Psi \,\Vert$$

$$B \vee \left(\frac{a}{\top} \wedge \bar{a} \right)$$

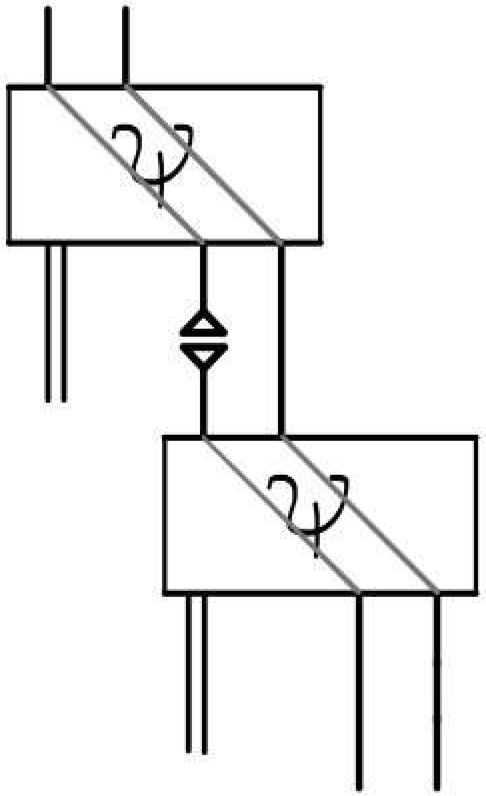
$$\Pi_a = \boxed{\begin{array}{c} {}_{aw\uparrow}\dfrac{a}{\top} \wedge \bar{a} \\ = \overline{} \\ \bar{a} \\ = \overline{} \\ {}_{aw\downarrow}\dfrac{\bot}{a} \vee \bar{a} \end{array}}$$



$$\begin{array}{c} a \vee \bar{a} \\ \Psi \Downarrow \\ \beta \vee (a \wedge \bar{a}) \end{array}$$

$$\begin{array}{c} a \vee \bar{a} \\ \Psi \Downarrow \\ \beta \vee \boxed{\begin{array}{c} (a \wedge \bar{a}) \\ \Pi_a \Downarrow \\ a \vee \bar{a} \\ \Psi \Downarrow \\ \beta \vee (a \wedge \bar{a}) \end{array}} \end{array}$$

$$\Pi_a = \boxed{\begin{array}{l} \text{aw}\uparrow \dfrac{a}{\top} \wedge \bar{a} \\ = \dfrac{}{\bar{a}} \\ = \dfrac{}{\text{aw}\downarrow \dfrac{\bot}{a} \vee \bar{a}} \end{array}}$$

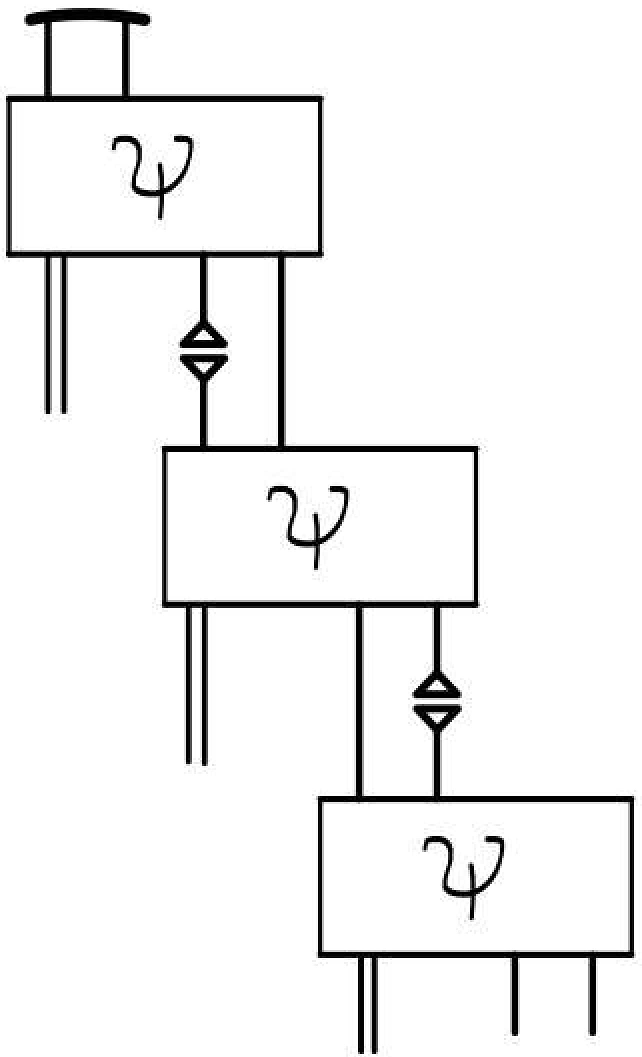


$$\begin{array}{c} a \vee \bar{a} \\ \Psi \,\| \\ \beta \vee (a \wedge \bar{a}) \end{array}$$

$$\begin{array}{c} a \vee \bar{a} \\ \Psi \,\| \end{array}$$

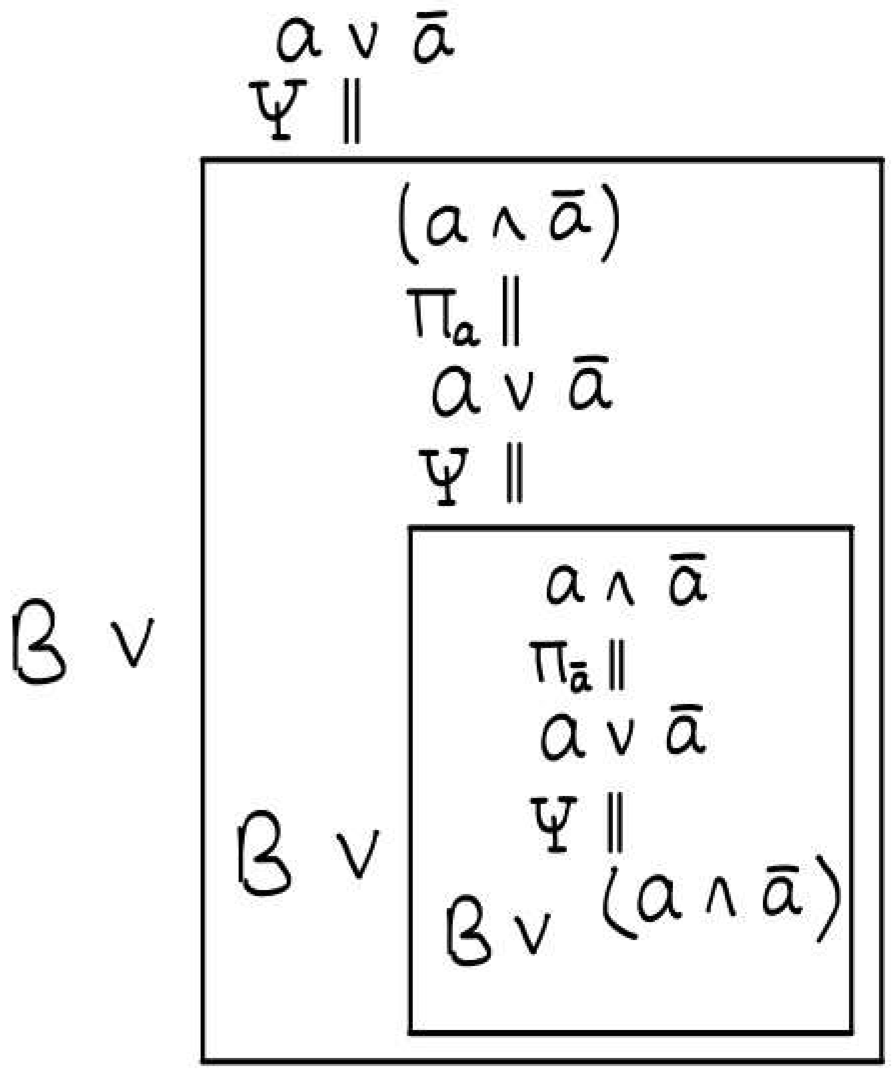
$$\beta \vee \boxed{\begin{array}{c} (a \wedge \bar{a}) \\ \Pi_a \,\| \\ a \vee \bar{a} \\ \Psi \,\| \\ \beta \vee (a \wedge \bar{a}) \end{array}}$$

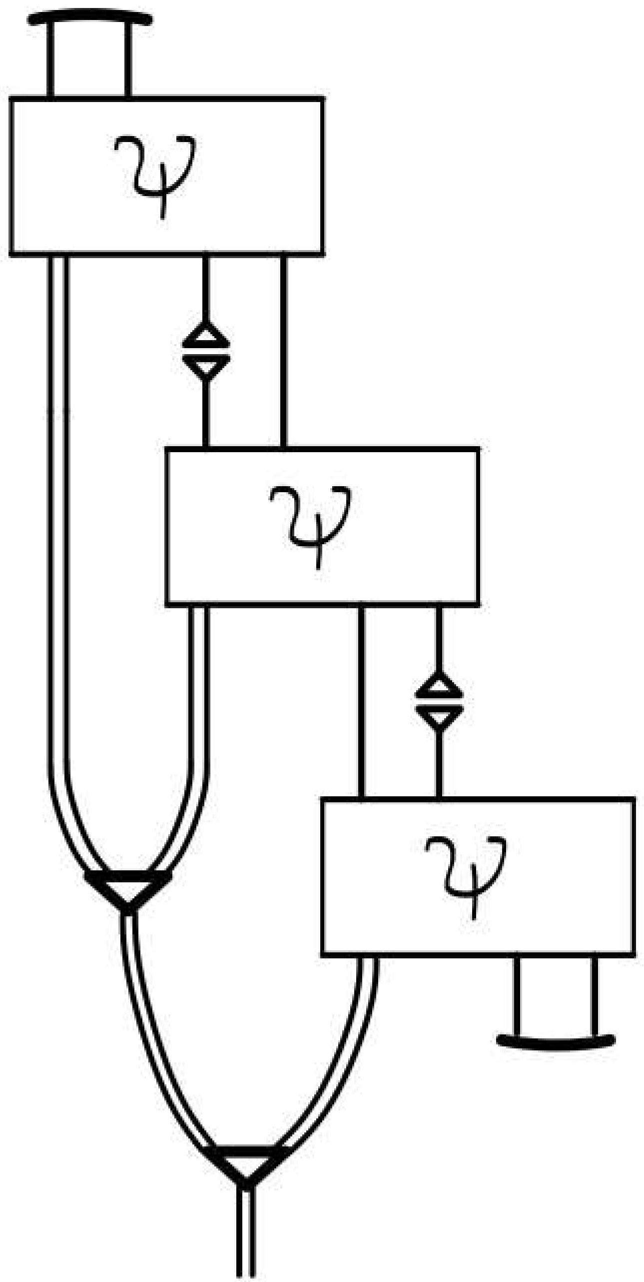
$$\Pi_a = \boxed{\begin{array}{c} \text{aw}\uparrow \dfrac{a}{\top} \wedge \bar{a} \\ = \overline{} \\ \bar{a} \\ = \overline{} \\ \text{aw}\downarrow \dfrac{\bot}{a} \vee \bar{a} \end{array}}$$



$$\begin{array}{c} a \vee \bar{a} \\ \Psi \Vert \\ B \vee (a \wedge \bar{a}) \end{array}$$

$$\Pi_a = \boxed{\begin{array}{c} \text{aw}\uparrow\dfrac{a}{\top} \wedge \bar{a} \\ = \overline{} \\ \bar{a} \\ = \overline{} \\ \text{aw}\downarrow\dfrac{\bot}{a} \vee \bar{a} \end{array}}$$





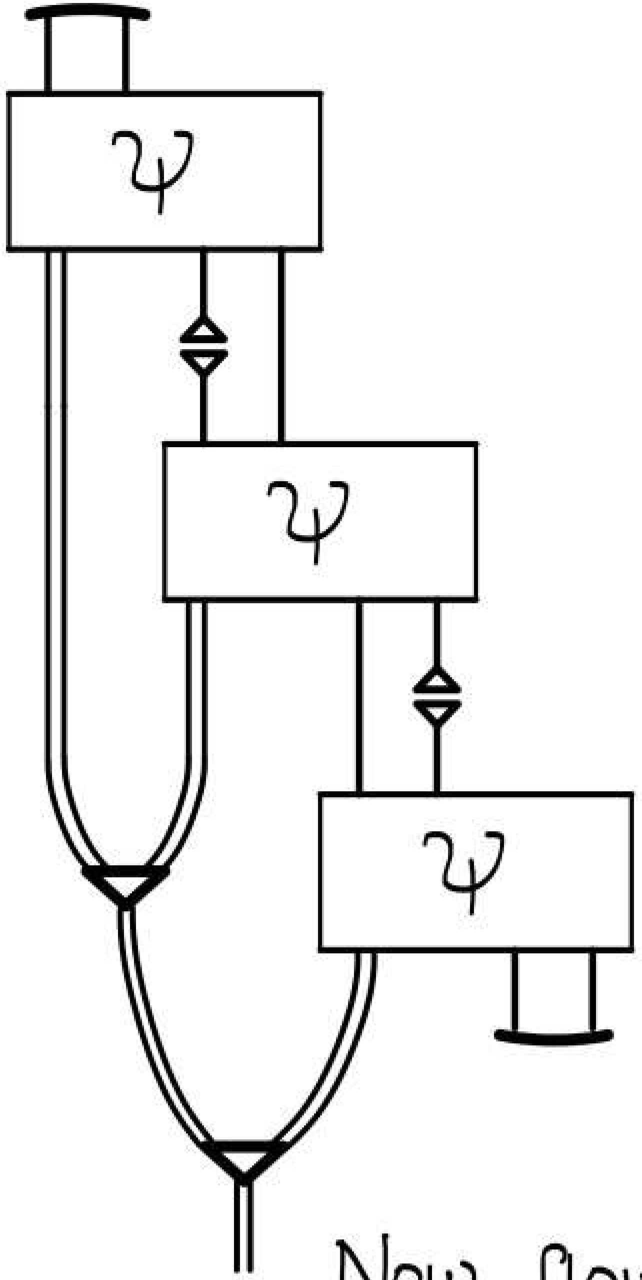
$$\begin{array}{c} a \vee \bar{a} \\ \Psi \Vert \\ \beta \vee (a \wedge \bar{a}) \end{array}$$

$$\Pi_a = \boxed{\begin{array}{c} \text{aw}{\uparrow}\,\dfrac{a}{\top} \wedge \bar{a} \\ = \overline{\qquad\qquad\qquad} \\ \bar{a} \\ = \overline{\qquad\qquad\qquad} \\ \text{aw}{\downarrow}\,\dfrac{\bot}{a} \vee \bar{a} \end{array}}$$

$$\boxed{\begin{array}{c} \dfrac{\top}{a \vee \bar{a}} \\ \Psi \Vert \\ \beta \vee \boxed{\begin{array}{c} (a \wedge \bar{a}) \\ \Pi_a \Vert \\ a \vee \bar{a} \\ \Psi \Vert \\ \beta \vee \boxed{\begin{array}{c} a \wedge \bar{a} \\ \Pi_{\bar{a}} \Vert \\ a \vee \bar{a} \\ \Psi \Vert \\ \beta \vee \text{ai}{\uparrow}\,\dfrac{a \wedge \bar{a}}{\bot} \\ = \overline{\qquad\qquad\qquad} \\ \beta \end{array}} \end{array}} \end{array}}$$

$$\Vert$$

$$\beta$$



$$\begin{array}{c} a \vee \bar{a} \\ \Psi \, \Big\| \\ B \vee (a \wedge \bar{a}) \end{array}$$

$$\Pi_a = \boxed{\begin{array}{l} \mathsf{aw}{\uparrow}\dfrac{a}{\top} \wedge \bar{a} \\ = \dfrac{\qquad\qquad}{\bar{a}} \\ = \dfrac{\qquad\qquad}{\mathsf{aw}{\downarrow}\dfrac{\bot}{a} \vee \bar{a}} \end{array}}$$

$$\boxed{\begin{array}{c} \dfrac{\top}{a \vee \bar{a}} \\ \Psi \, \Big\| \\ B \vee \boxed{\begin{array}{c} (a \wedge \bar{a}) \\ \Pi_a \, \Big\| \\ a \vee \bar{a} \\ \Psi \, \Big\| \\ B \vee \boxed{\begin{array}{c} a \wedge \bar{a} \\ \Pi_{\bar{a}} \, \Big\| \\ a \vee \bar{a} \\ \Psi \, \Big\| \\ B \vee \mathsf{ai}{\uparrow}\dfrac{a \wedge \bar{a}}{\bot} \\ = \dfrac{\qquad\qquad}{B} \end{array}} \end{array}} \end{array}}$$

$$\| \\ B$$

Now flow rewriting will terminate! (for a)

Atomic flows do not form a proof system: essential information is lost and there is no polynomial correctness criterion* (Das 2013)

*unless integer factoring is in P/poly

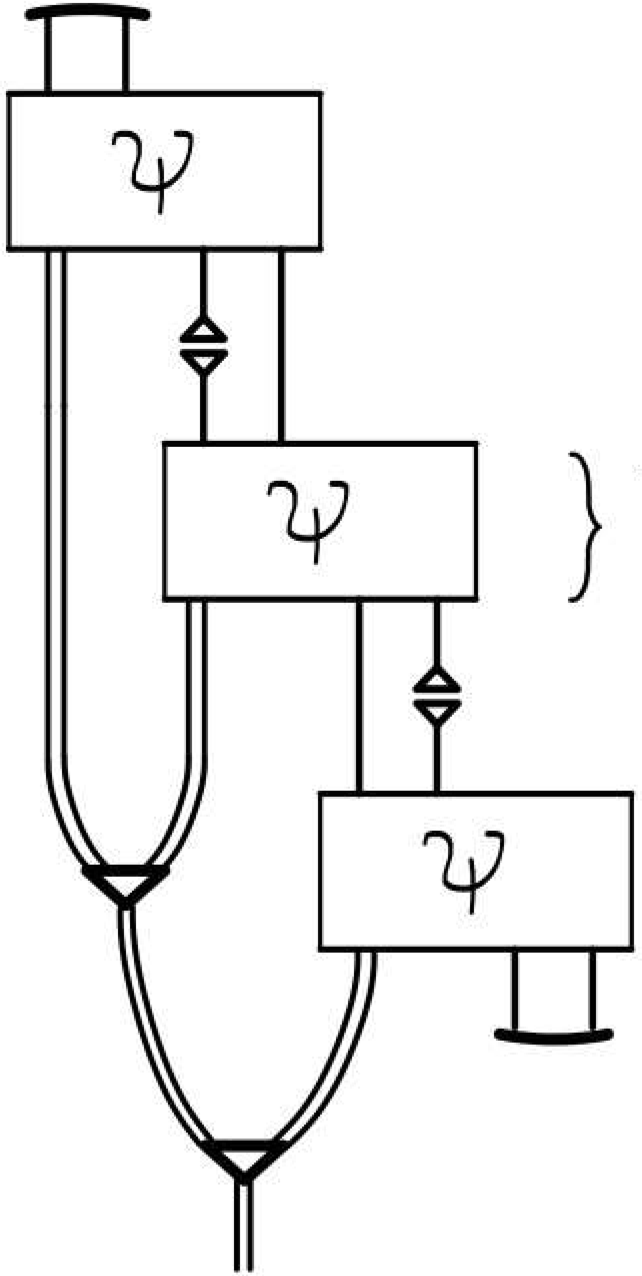
Atomic flows do not form a proof system: essential information is lost and there is no polynomial correctness criterion* (Das 2013)

*unless integer factoring is in P/poly

but they give us some idea for how the substitution of proofs could work

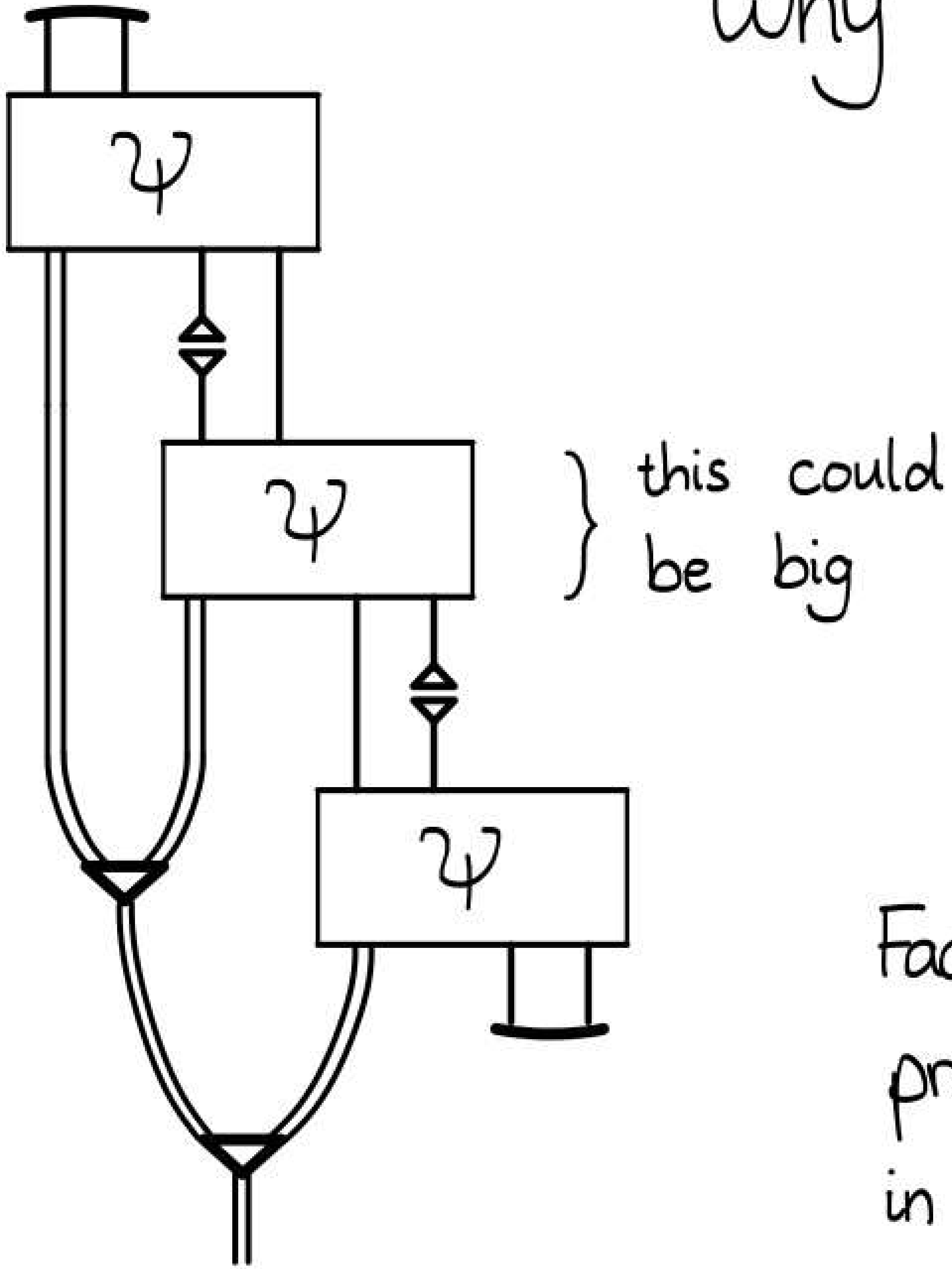
Why substitution of proofs?

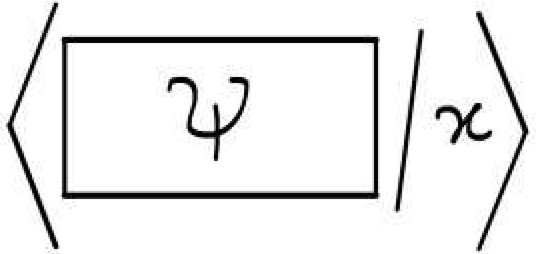
Why substitution of proofs?

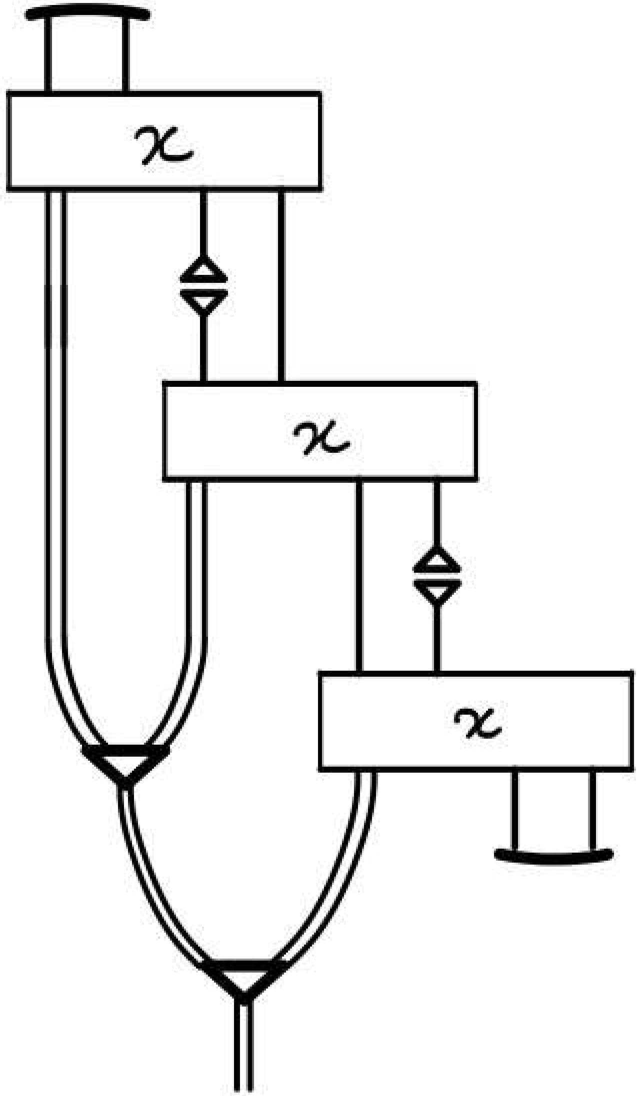


this could be big

Why substitution of proofs?



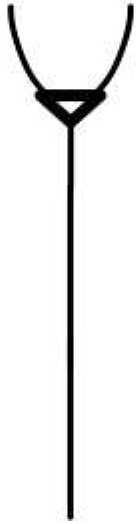




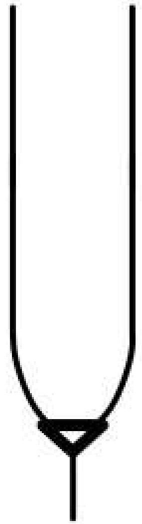
Factorisation would lead to proof compression, especially in normalisation procedures

Why substitution of proofs?

Why substitution of proofs?



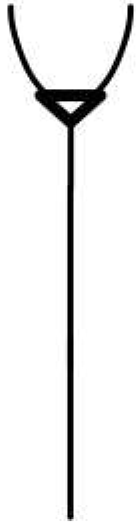
$$\begin{array}{c} K\left\{ \mathsf{ac}\!\downarrow \dfrac{a \vee a}{a} \right\} \\ \hdashline [a/x]\Phi \\ \hdashline H\{a\} \end{array}$$



$$\begin{array}{c} K\{a \vee a\} \\ \hdashline [a \vee a/x]\,\Phi \\ \hdashline H\left\{ \mathsf{ac}\!\downarrow \dfrac{a \vee a}{a} \right\} \end{array}$$

Why substitution of proofs?

$$\begin{array}{c} K\{a \vee a\} \\ \hline \left\langle \mathsf{ac}\!\downarrow \dfrac{a \vee a}{a} \middle/ x \right\rangle \Phi \\ \hline H\{a\} \end{array}$$



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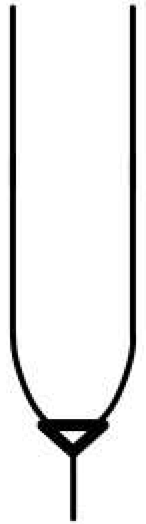

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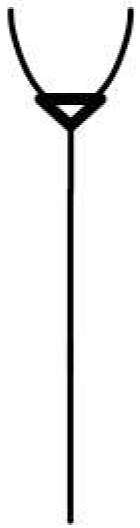
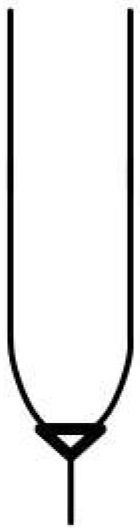
$$\begin{array}{c} K\{a \vee a\} \\ \hline \left\langle \mathsf{ac}{\downarrow}\,\dfrac{a \vee a}{a} \Big/ x \right\rangle \Phi \\ \hline H\{a\} \end{array}$$



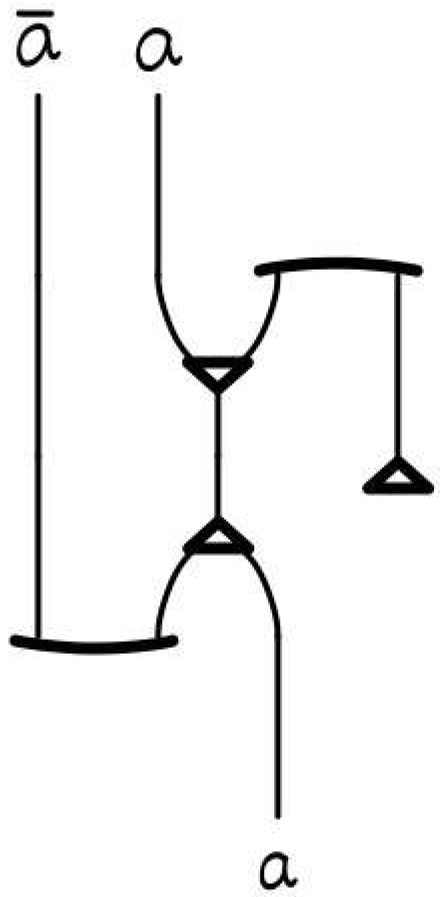
$$\begin{array}{c} K\left\{ \mathsf{ac}{\downarrow}\,\dfrac{a \vee a}{a} \right\} \\ \hdashline [a / x]\Phi \\ \hdashline H\{a\} \end{array}$$

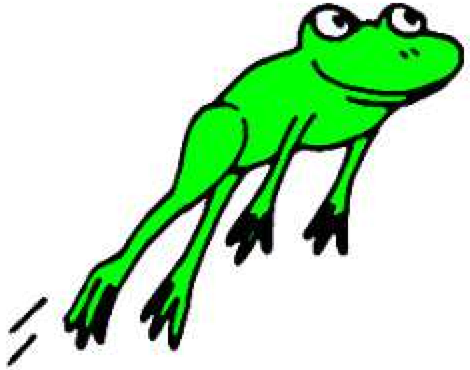
Factorisation would reduce bureaucracy by identifying proofs whose flows are continuous deformations



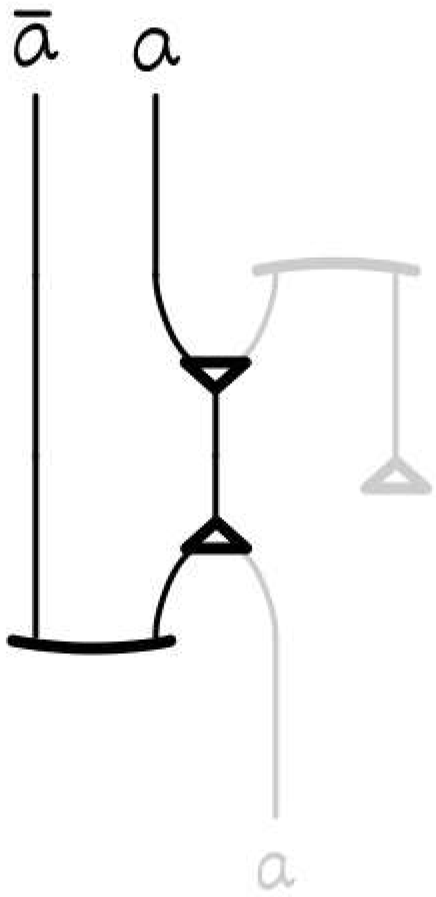
$$\begin{array}{c} K\{a \vee a\} \\ \hdashline [a \vee a / x]\,\Phi \\ \hdashline H\left\{ \mathsf{ac}{\downarrow}\,\dfrac{a \vee a}{a} \right\} \end{array}$$

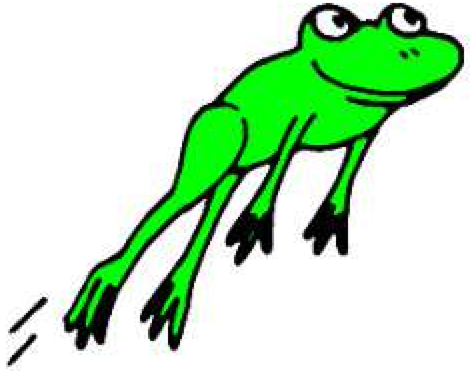
$$\left\langle \begin{matrix} B \\ \Phi \| \\ C \end{matrix} \middle/ a \right\rangle$$

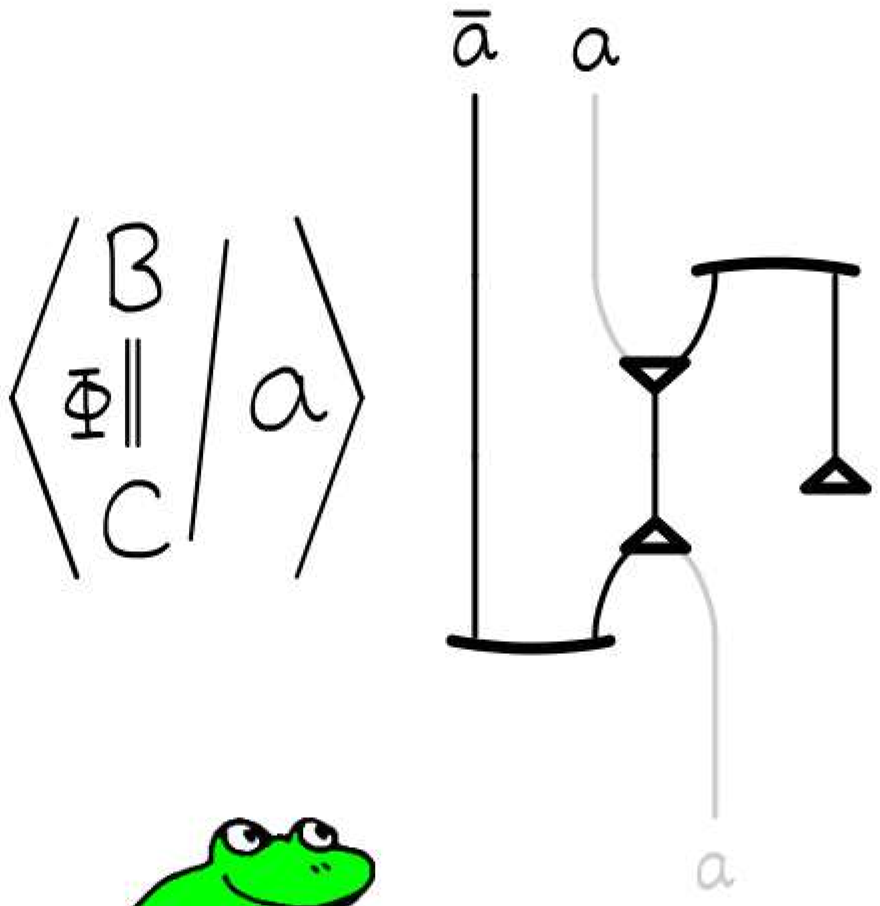




$$\left\langle \begin{matrix} B \\ \Phi \| \\ C \end{matrix} \middle/ a \right\rangle$$



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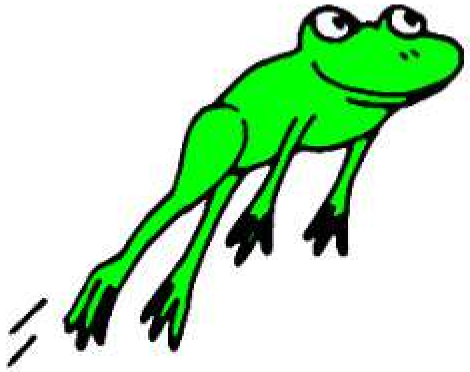
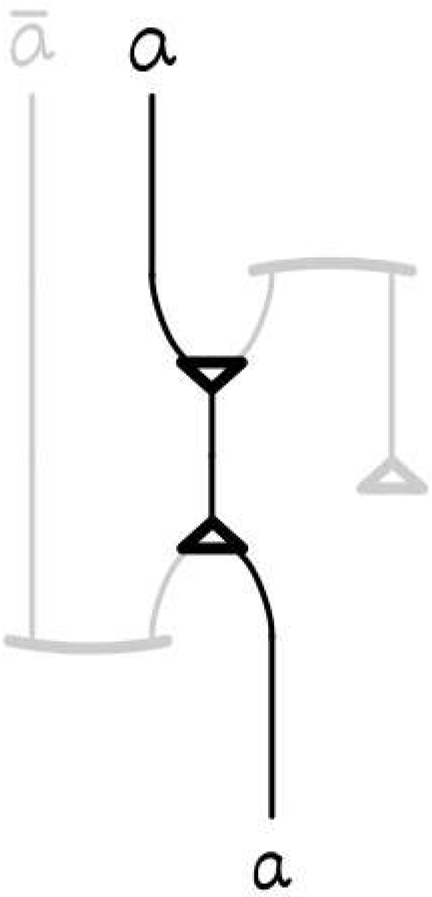
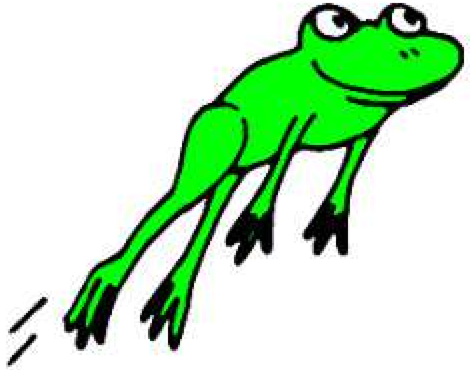
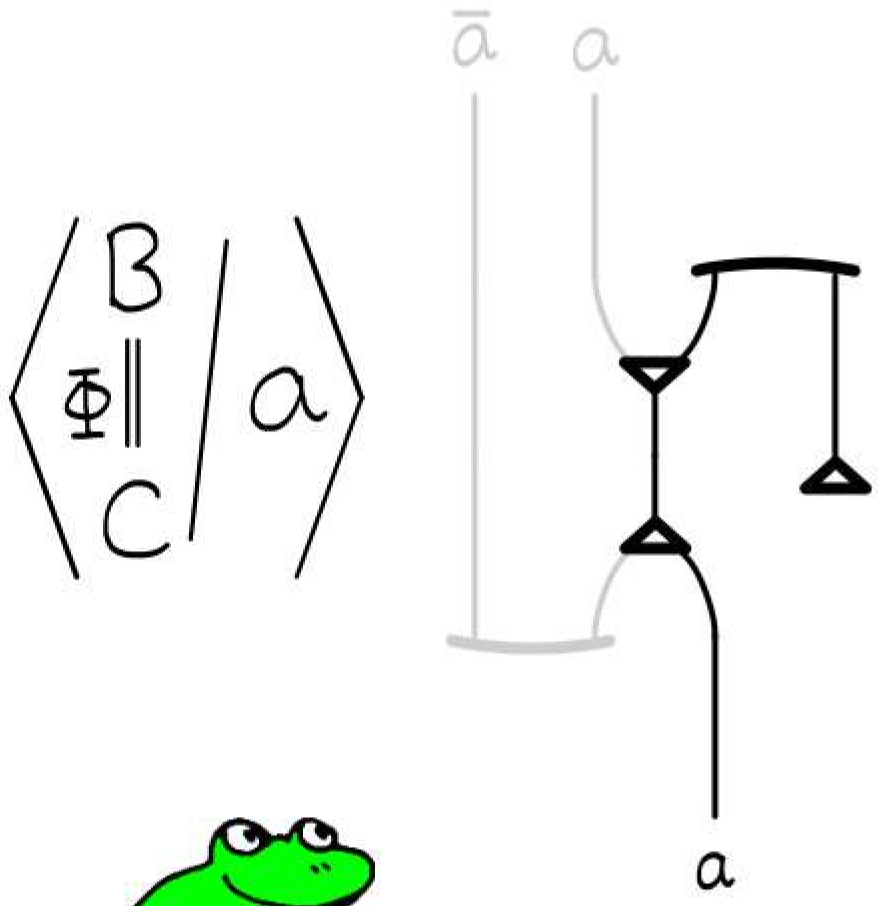


$$\left\langle \begin{matrix} B \\ \Phi \| \\ C \end{matrix} \middle/ a \right\rangle$$



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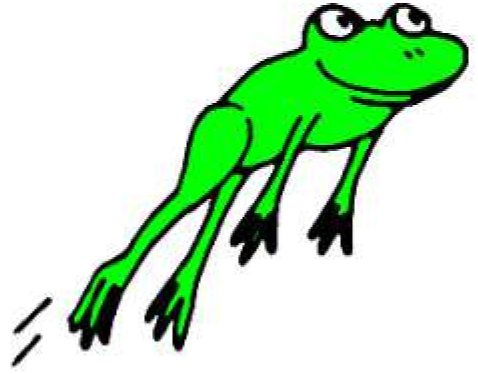
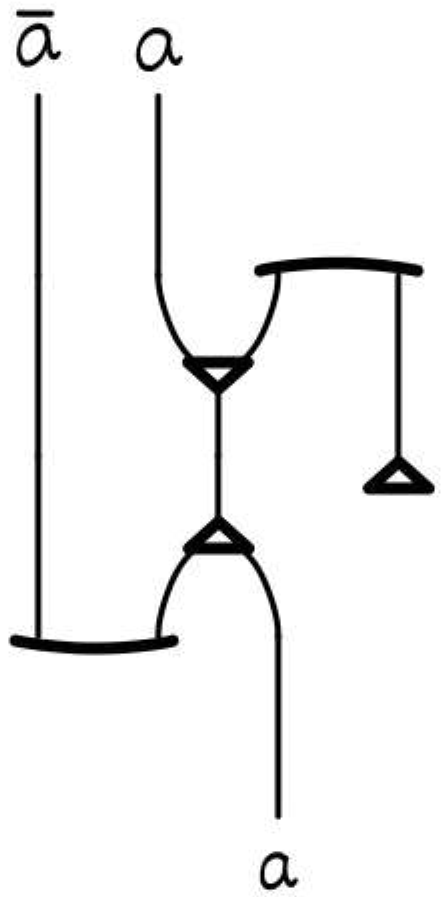
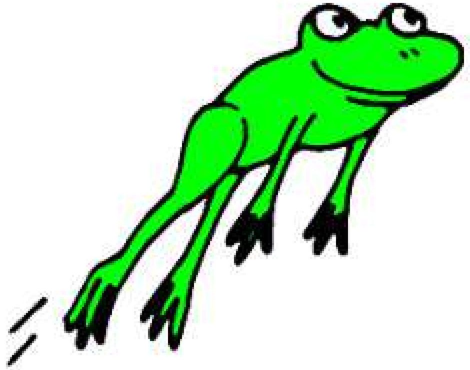
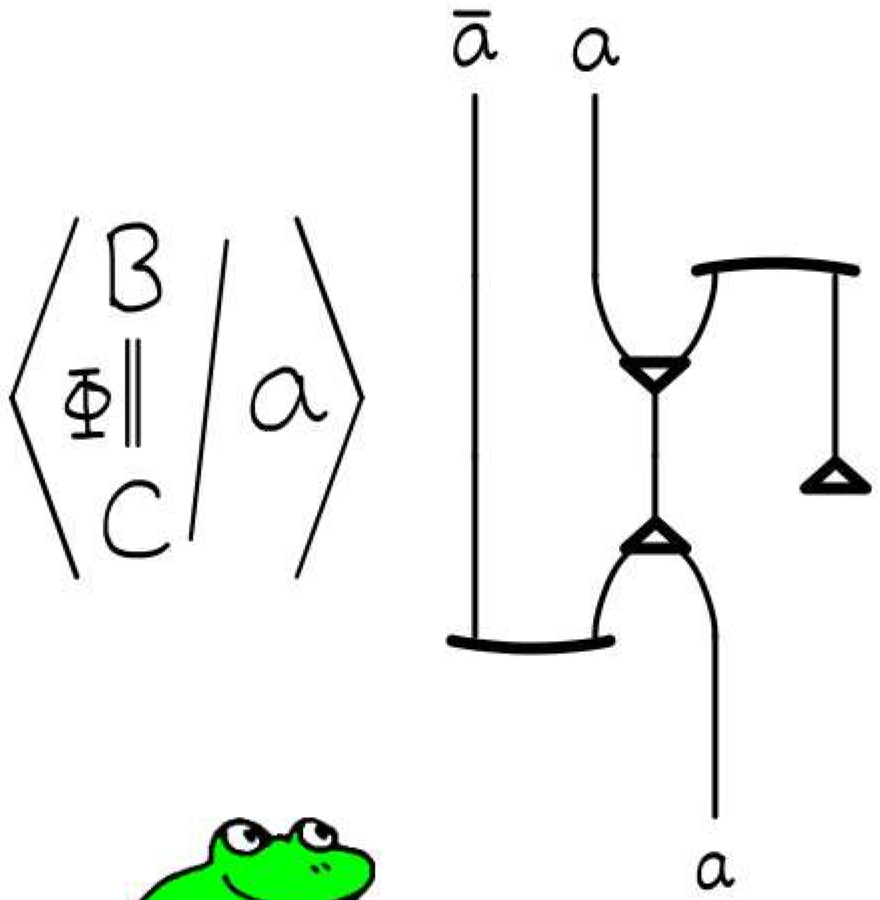


$$\left\langle \begin{matrix} B \\ \Phi \| \\ C \end{matrix} \middle/ a \right\rangle$$



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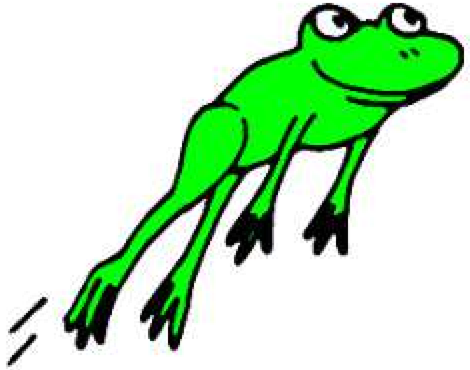

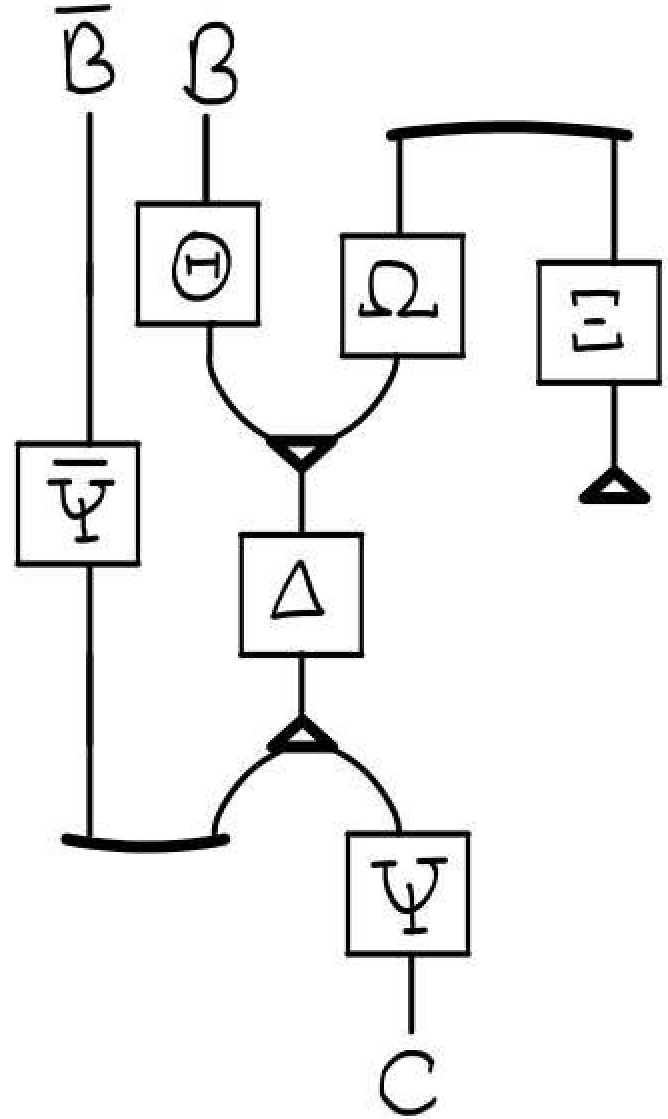
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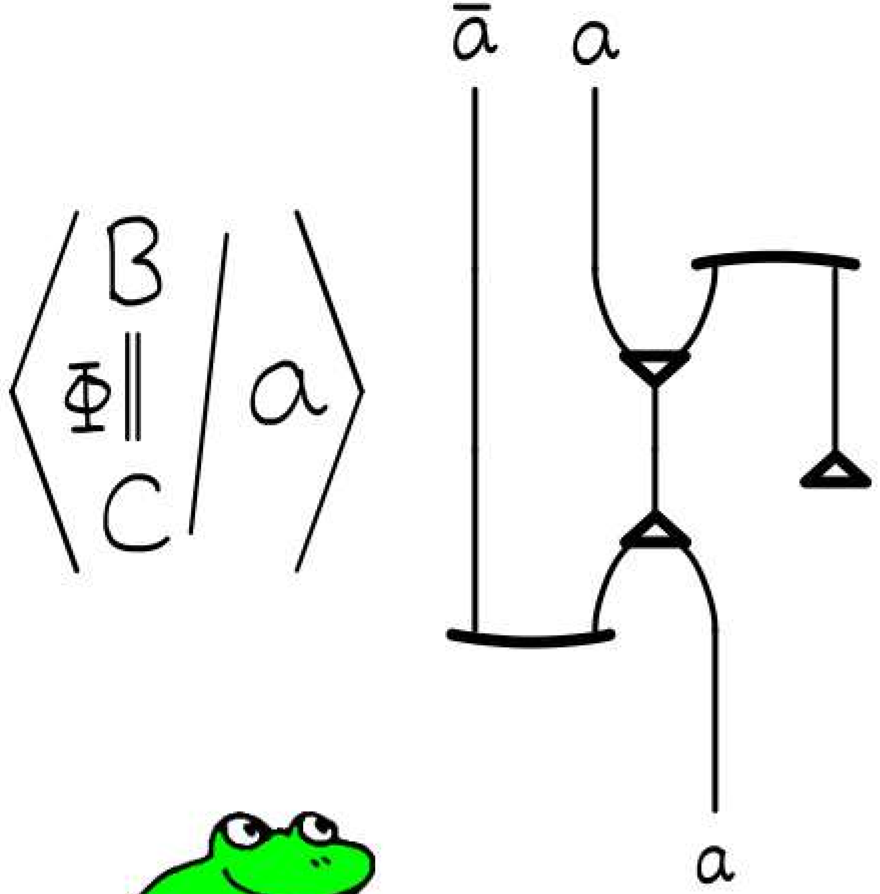
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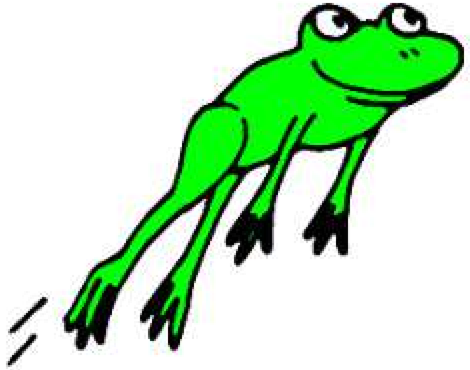

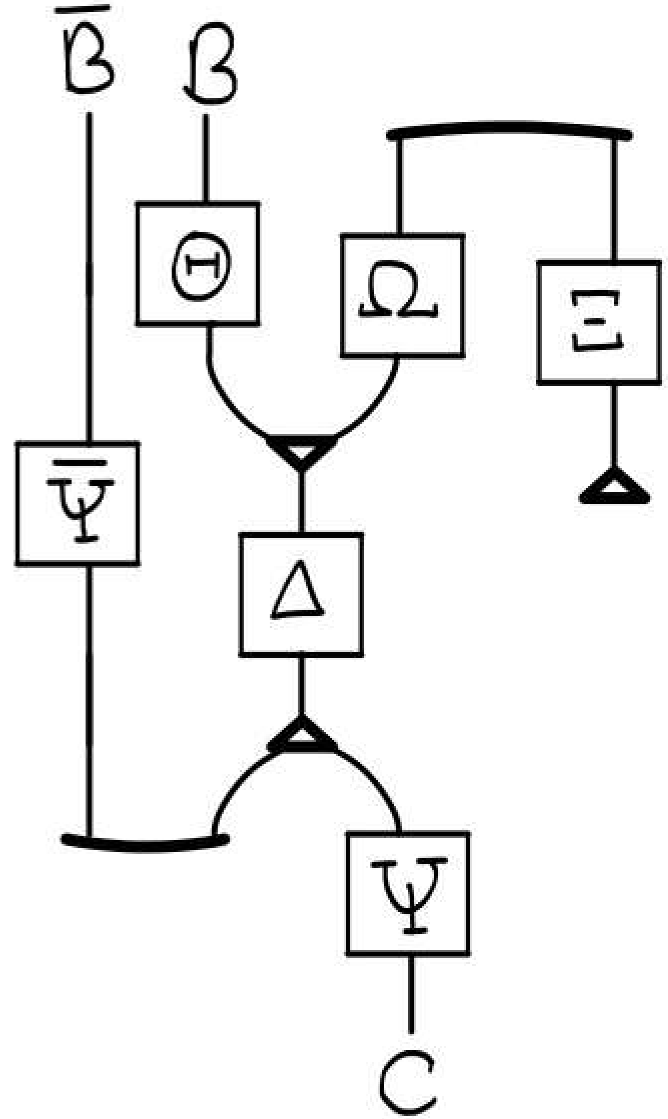
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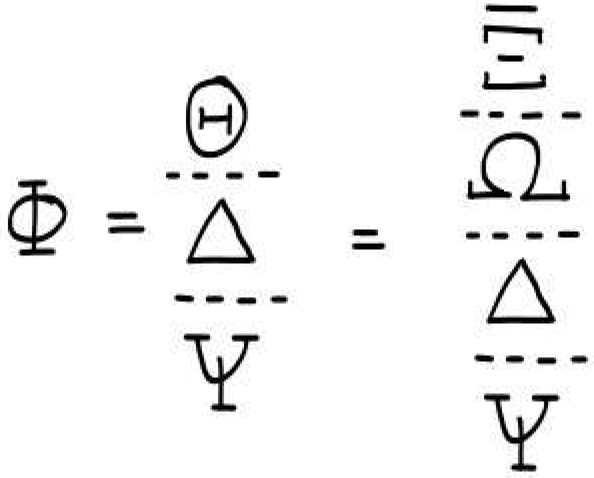
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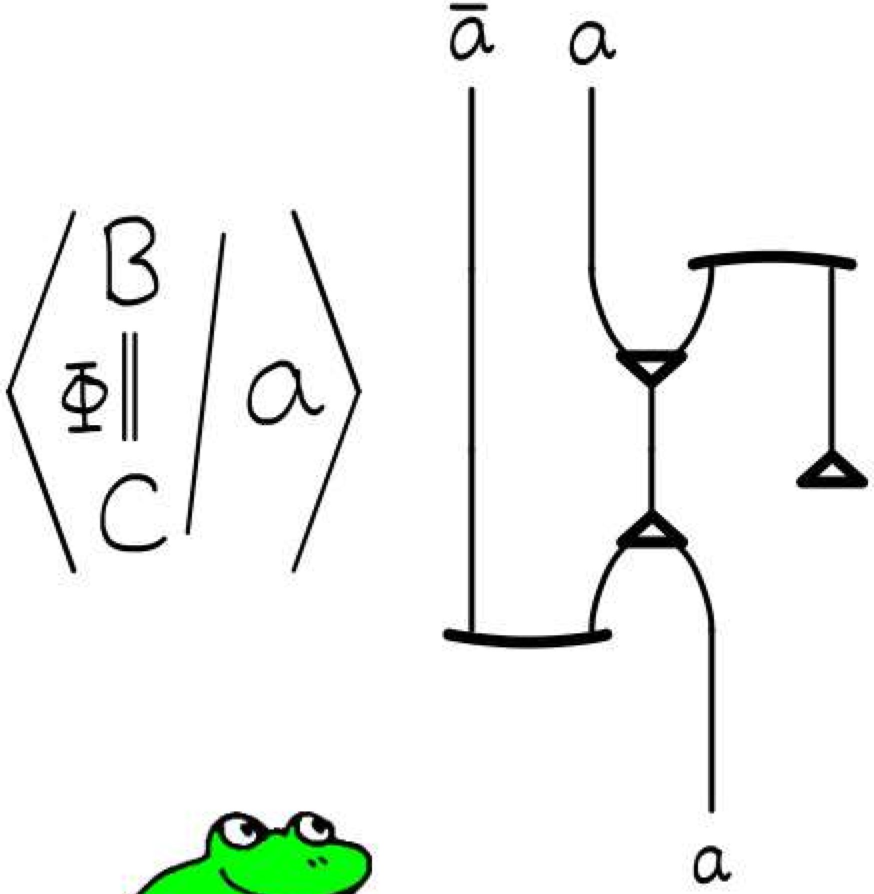

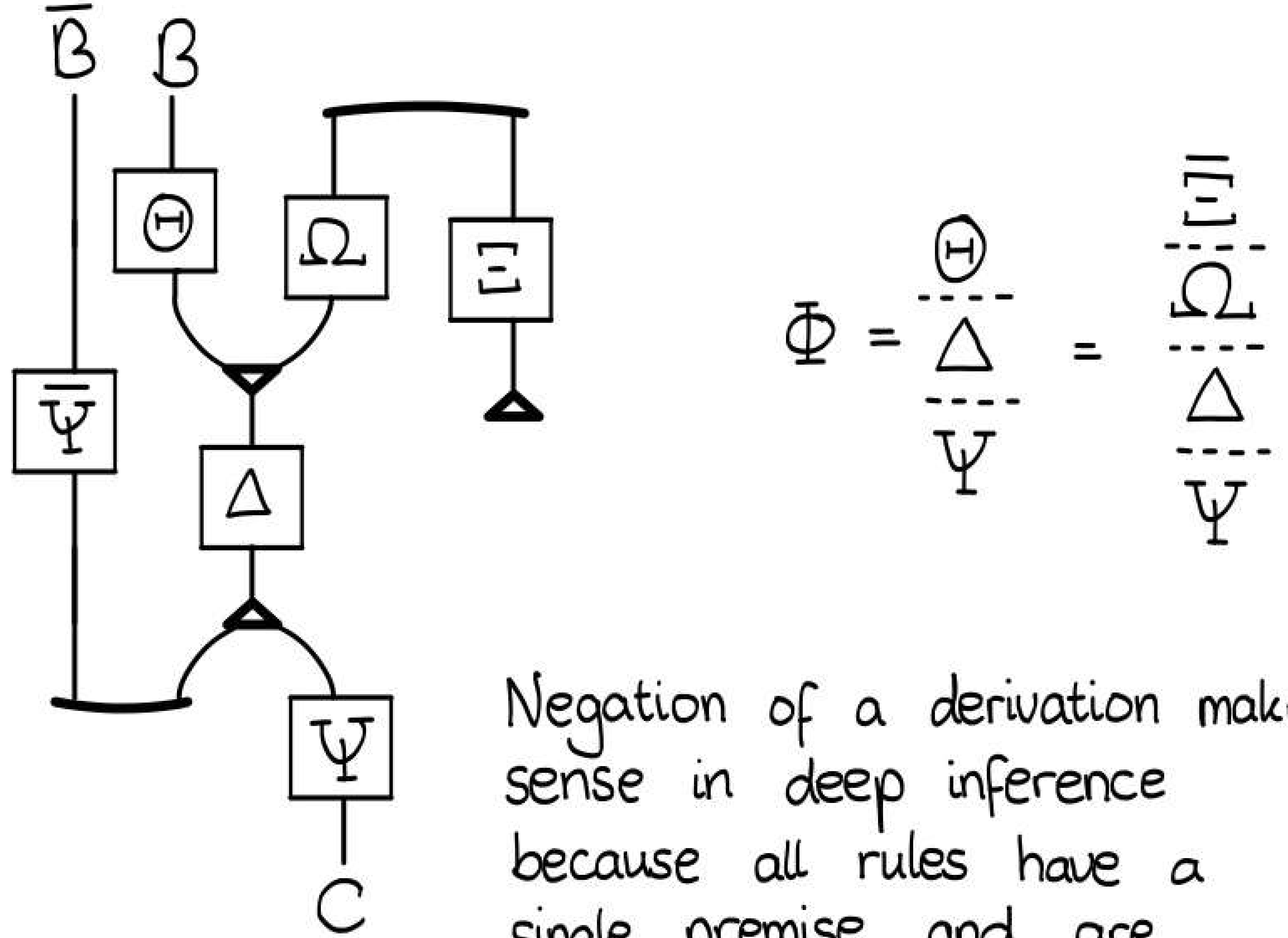

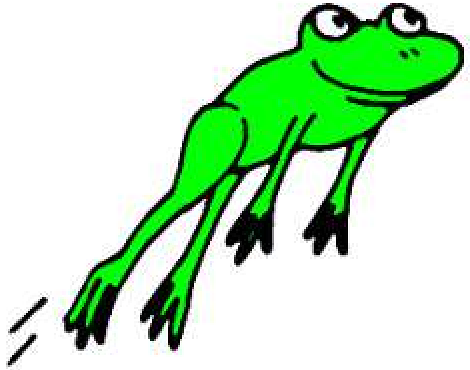

Negation of a derivation makes sense in deep inference because all rules have a single premise and are dualisable