From Axioms to Rules:
The Factory of Modal Proof Systems

10. Lecture

Nested Sequents for Modal Logics

Sonia Marin and Lutz Straßburger

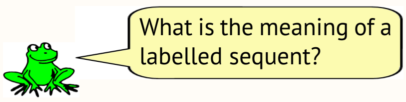
Recall: Sequents and Labelled Sequents

Sequents: $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$

Corresponding formula: $(A_1 \wedge \ldots \wedge A_n) \supset (B_1, \ldots, B_m)$

Labelled Sequents: $w_1Rz_1, \ldots, w_kRz_k, x_1 : A_1, \ldots, x_n : A_n \vdash y_1 : B_1, \ldots, y_m : B_m$

Corresponding formula: ???



From Sequents to Nested Sequents

Two-sided sequents: $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$

Corresponding formula: $(A_1 \wedge \cdots \wedge A_n) \supset (B_1 \vee \cdots \vee B_m)$

One-sided sequents: $\Gamma = B_1, \ldots, B_m$

Corresponding formula: $fm(\Gamma) = B_1 \vee \cdots \vee B_m$

Nested sequents: $\Gamma = B_1, \ldots, B_m, [\Gamma_1], \ldots, [\Gamma_k]$

Corresponding formula: $fm(\Gamma) = B_1 \vee \cdots \vee B_m \vee \Box fm(\Gamma_1) \vee \cdots \vee \Box fm(\Gamma_k)$

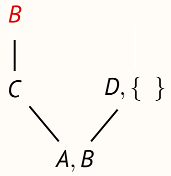
$$\begin{array}{ccc} \Gamma_1 & \cdots & \Gamma_n \\ & A_1, \ldots, A_m & \end{array}$$

- Every two-sided sequent system (for a logic with De Morgan duality) can be transformed into a one-sided system:
  - consider only formulas in *negation normal form* (negation is primitive only on atoms, and is defined inductively for compound formulas)
  - the two sided sequent
    $$A_1, \ldots, A_n \vdash B_1, \ldots, B_m$$
    is transformed into
    $$\bar{A}_1, \ldots, \bar{A}_n, B_1, \ldots, B_m$$
  - only half as many rules are needed
- **Exercise 10.1:** Take the derivations from previous exercises and write them as on-sided.

Nested Sequents for Modal Logic **K**

Formulas: $A ::= a \mid \bar{a} \mid A \wedge A \mid A \vee A \mid \Box A \mid \Diamond A$

Sequent context: $\Gamma\{B\}\{A,[C]\} = A, B, [C,[B]], [D, A, [C]]$



Rules:

$$\mathsf{id}\ \frac{}{\Gamma\{a,\bar{a}\}} \qquad \vee\ \frac{\Gamma\{A,B\}}{\Gamma\{A\vee B\}} \qquad \wedge\ \frac{\Gamma\{A\}\quad \Gamma\{B\}}{\Gamma\{A\wedge B\}}$$

$$\Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \Diamond\ \frac{\Gamma\{\Diamond A,[A,\Delta]\}}{\Gamma\{\Diamond A,[\Delta]\}}$$

- Nested sequents have been independently indroduced by Kashima, Brünnler, and Poggiolesi:
  - *Ryo Kashima:* **"Cut-free sequent calculi for some tense logics".** *Studia Logica 53(1), 1994, pp 119–136*
  - *Kai Brünnler:* **"Deep Sequent Systems for Modal Logic".** *Archive for Mathematical Logic 48(6), 2009, pp. 551–577*
  - *Francesca Poggiolesi:* **"The Method of Tree-Hypersequents for Modal Propositional Logic".** *Towards Mathematical Philosophy 28, 2009, pp 31–51, Trends in Logic, Springer*
- The systems presented here are based on Brünnler's work.

Nested Sequents vs Labelled Sequents

$$\mathsf{id}\ \frac{}{\Gamma\{a,\bar{a}\}} \iff \mathsf{id}\ \frac{}{\mathcal{R} \vdash x{:}a, x{:}\bar{a}, \Gamma}$$

$$\vee\ \frac{\Gamma\{A,B\}}{\Gamma\{A\vee B\}} \iff \vee\ \frac{\mathcal{R} \vdash x{:}A, x{:}B, \Gamma}{\mathcal{R} \vdash x{:}A\vee B, \Gamma}$$

$$\wedge\ \frac{\Gamma\{A\}\quad \Gamma\{B\}}{\Gamma\{A\wedge B\}} \iff \wedge\ \frac{\mathcal{R} \vdash x{:}A, \Gamma \qquad \mathcal{R}, \Gamma \vdash x{:}B, \Gamma}{\mathcal{R} \vdash x{:}A\wedge B, \Gamma}$$

$$\Diamond\ \frac{\Gamma\{\Diamond A,[A,\Delta]\}}{\Gamma\{\Diamond A,[\Delta]\}} \iff \Diamond\ \frac{xRy, \mathcal{R} \vdash y{:}A, x{:}\Diamond A, \Gamma}{xRy, \mathcal{R} \vdash x{:}\Diamond A, \Gamma}$$

$$\Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \iff \Box_{\mathsf{R}}\ \frac{xRy, \mathcal{R} \vdash y{:}A, \Gamma}{\mathcal{R} \vdash x{:}\Box A, \Gamma}\ x \text{ not in } \mathcal{R}, \Gamma$$

- If $\mathcal{R}$ describes a tree, then the labelled and the nested sequent rules are only notational variants of each other.
- **Exercise 10.2:** Transform some of the labelled sequent derivations from previous exercises into nested sequent derivations.

Structural Rules in Nested Sequents

$$\mathsf{ser}\ \frac{xRy, \mathcal{R}, \Gamma \vdash \Theta}{\mathcal{R}, \Gamma \vdash \Theta}\ y \text{ fresh} \iff \mathsf{d}^{[]}\ \frac{\Gamma\{[]\}}{\Gamma\{\emptyset\}}$$

$$\mathsf{ref}\ \frac{xRx, \mathcal{R}, \Gamma \vdash \Theta}{\mathcal{R}, \Gamma \vdash \Theta} \iff \mathsf{t}^{[]}\ \frac{\Gamma\{\Delta,[\Delta]\}}{\Gamma\{\Delta\}}$$

$$\mathsf{sym}\ \frac{yRx, xRy, \mathcal{R}, \Gamma \vdash \Theta}{xRy, \mathcal{R}, \Gamma \vdash \Theta} \iff \mathsf{b}^{[]}\ \frac{\Gamma\{\Sigma,[\Delta,[\Sigma]]\}}{\Gamma\{[\Delta,[\Sigma]]\}}$$

$$\mathsf{trans}\ \frac{xRz, xRy, yRz, \mathcal{R}, \Gamma \vdash \Theta}{xRy, yRz, \mathcal{R}, \Gamma \vdash \Theta} \iff \mathsf{4}^{[]}\ \frac{\Gamma\{[\Sigma],[\Delta,[\Sigma]]\}}{\Gamma\{[\Delta,[\Sigma]]\}}$$

$$\mathsf{euc}\ \frac{yRz, xRy, xRz, \mathcal{R}, \Gamma \vdash \Theta}{xRy, xRz, \mathcal{R}, \Gamma \vdash \Theta} \iff \mathsf{5}^{[]}\ \frac{\Gamma\{[\Sigma,[\Delta]]\}\{[[\Delta],\Lambda]\}}{\Gamma\{[\Sigma]\},\{[[\Delta],\Lambda]\}}$$

Adding structural rules does not always lead to complete nested sequent systems!

- These structural rules have first been introduced by Brünnler as auxiliary step in the cut elimination argument. However, the cut elimination proof usually fails if only the structural rules are present. More details can be found here:
  - *Sonia Marin and Lutz Straßburger:* **"Label-free Modular Systems for Classical and Intuitionistic Modal Logics".** *Advances in Modal Logic 10, 2014*

Propagation Rules in Nested Sequents (General Form)

Path axiom:

$$\Diamond^h A \supset \Box^k \Diamond A$$

Corresponding propagation rule:

$$\mathsf{p}^{\Diamond}\ \frac{\Gamma\{\Diamond A\}_u\{A\}_v}{\Gamma\{\Diamond A\}_u\{\emptyset\}_v}\quad \text{there is a propagation path from node } u \text{ to node } v$$

Example:

$$4\colon \Diamond\Diamond A \supset \Diamond A \quad \rightsquigarrow \quad 4^*\ \frac{\Gamma\{\Delta\{A\}\}}{\Gamma\{\Diamond A, \Delta\{\emptyset\}\}}$$

$$\left\{ \frac{x_0 R x_1, \ldots, x_{m-1} R x_m, \mathcal{R}, \Gamma \vdash \Theta, x_m : A}{x_0 R x_1, \ldots, x_{m-1} R x_m, \mathcal{R}, \Gamma \vdash \Theta, x_0 : \Diamond A} \;\middle|\; m \geq 1 \right\}$$

Nested sequents simplify the presentation of propagation rules.

- These rules have been given in
  - *Rajeev Goré and Linda Postniece and Alwen Tiu:* **"On the Correspondence between Display Postulates and Deep Inference in Nested Sequent Calculi for Tense Logics".** *Logical Methods in Computer Science 7(2), 2011*

Propagation Rules in Nested Sequents (Modal Cube)

$$\mathsf{id}\ \frac{}{\Gamma\{a, \bar{a}\}} \qquad \vee\ \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}} \qquad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \qquad \Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \Diamond\ \frac{\Gamma\{\Diamond A, [A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$$

$$\mathsf{d}^{\Diamond}\ \frac{\Gamma\{\Diamond A, [A]\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{t}^{\Diamond}\ \frac{\Gamma\{\Diamond A, A\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{b}^{\Diamond}\ \frac{\Gamma\{[\Delta, \Diamond A], A\}}{\Gamma\{[\Delta, \Diamond A]\}}$$

$$4^{\Diamond}\ \frac{\Gamma\{\Diamond A, [\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \qquad 5^{\Diamond}\ \frac{\Gamma\{\Diamond A\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}}\ \mathrm{depth}(\Gamma\{\,\}\{[\Delta]\}) \geq 1$$

With the propagation rules we can have cut elimination in nested sequent systems.

- These propagation rules for the modal cube have been introduced by Brünnler.
  - *Kai Brünnler:* **"Deep Sequent Systems for Modal Logic".** *Archive for Mathematical Logic 48(6), 2009, pp. 551–577*

  In this paper there is also a simple completeness proof via a terminating proof search for nested sequents.