Recall: Sequents and Labelled Sequents

**Sequents:**
\[ A_1, \ldots, A_n \vdash B_1, \ldots, B_m \]

Corresponding formula:
\[ (A_1 \land \ldots \land A_n) \supset (B_1, \ldots, B_m) \]

**Labelled Sequents:**
\[ w_1 R z_1, \ldots, w_k R z_k, x_1 : A_1, \ldots, x_n : A_n \vdash y_1 : B_1, \ldots, y_m : B_m \]

Corresponding formula:

What is the meaning of a labelled sequent?

From Sequents to Nested Sequents

**Two-sided sequents:**
\[ A_1, \ldots, A_n \vdash B_1, \ldots, B_m \]

Corresponding formula:
\[ (A_1 \land \ldots \land A_n) \supset (B_1 \lor \ldots \lor B_m) \]

**One-sided sequents:**
\[ \Gamma = B_1, \ldots, B_m \]

Corresponding formula:
\[ \text{fm}(\Gamma) = B_1 \lor \ldots \lor B_m \]

**Nested sequents:**
\[ \Gamma = B_1, \ldots, B_m, [\Gamma_1], \ldots, [\Gamma_k] \]

Corresponding formula:
\[ \text{fm}(\Gamma) = B_1 \lor \ldots \lor B_m \lor \Box \text{fm}(\Gamma_1) \lor \ldots \lor \Box \text{fm}(\Gamma_k) \]

- Every two-sided sequent system (for a logic with De Morgan duality) can be transformed into a one-sided system:
  - consider only formulas in **negation normal form** (negation is primitive only on atoms, and is defined inductively for compound formulas)
  - the two sided sequent
    \[ A_1, \ldots, A_n \vdash B_1, \ldots, B_m \]
  - is transformed into
    \[ \bar{A}_1, \ldots, \bar{A}_n, B_1, \ldots, B_m \]
  - only half as many rules are needed

- **Exercise 10.1:** Take the derivations from previous exercises and write them as one-sided.
Nested Sequents for Modal Logic K

Formulas: 
\[ A ::= a \mid \bar{a} \mid A \land A \mid A \lor A \mid \Box A \mid \Diamond A \]

Sequent context: 
\[ \Gamma (B) \{ A, [C] \} = A, B, [C, [B]], [D, A, [C]] \]

Rules:

\[
\begin{align*}
\text{id} & : \frac{}{\Gamma \{ a, \bar{a} \}} \\
\lor & : \frac{\Gamma \{ A, B \}, \Gamma \{ A \lor B \}}{\Gamma \{ A \} \lor \Gamma \{ B \}} \\
\land & : \frac{\Gamma \{ A \}, \Gamma \{ B \}}{\Gamma \{ A \land B \}} \\
\Diamond & : \frac{\Gamma \{ \Box A, [A, \Delta] \}}{\Gamma \{ \Diamond A, [\Delta] \}} \\
\Box & : \frac{\Gamma \{ A \}}{\Gamma \{ \Box A \}}
\end{align*}
\]

Nested Sequents vs Labelled Sequents

\[
\begin{align*}
\text{id} & : \frac{}{\Gamma \{ a, \bar{a} \}} \\
\lor & : \frac{\Gamma \{ A \}, \Gamma \{ A \lor B \}}{\Gamma \{ A \} \lor \Gamma \{ B \}} \\
\land & : \frac{\Gamma \{ A \}, \Gamma \{ B \}}{\Gamma \{ A \land B \}} \\
\Diamond & : \frac{\Gamma \{ \Diamond A, [A, \Delta] \}}{\Gamma \{ \Box A, [\Delta] \}} \\
\Box & : \frac{\Gamma \{ A \}}{\Gamma \{ \Box A \}}
\end{align*}
\]

Structural Rules in Nested Sequents

\[
\begin{align*}
\text{ser} & : \frac{xRz, \overline{R}, \overline{\Gamma} \vdash \Theta}{\overline{R}, \overline{\Gamma} \vdash \Theta}, \text{y fresh} \\
\text{ref} & : \frac{xRz, \overline{R}, \overline{\Gamma} \vdash \Theta}{\overline{R}, \overline{\Gamma} \vdash \Theta} \\
\text{sym} & : \frac{xRy, \overline{xRy}, \overline{R}, \overline{\Gamma} \vdash \Theta}{\overline{xRy}, \overline{R}, \overline{\Gamma} \vdash \Theta} \\
\text{trans} & : \frac{xRy, \overline{xRy}, \overline{R}, \overline{\Gamma} \vdash \Theta}{\overline{xRy}, \overline{R}, \overline{\Gamma} \vdash \Theta} \\
\text{exc} & : \frac{xRy, \overline{xRy}, \overline{R}, \overline{\Gamma} \vdash \Theta}{\overline{xRy}, \overline{R}, \overline{\Gamma} \vdash \Theta}
\end{align*}
\]

Adding structural rules does not always lead to complete nested sequent systems!
Propagation Rules in Nested Sequents (General Form)

**Path axiom:**
\[ \emptyset \Box A \supset \Box \emptyset \Box A \]

**Corresponding propagation rule:**
\[ p^\Box \frac {\Box \{ \Box A \}} {\Box \{ \Box A \} \emptyset \{ \Box A \}} \quad \text{there is a propagation path from node } u \text{ to node } v \]

**Example:**
4: \[ \Box \Box A \supset \Box A \supseteq \exists \Box \{ \Box A, \emptyset \} \]
5: \[ \Box \Box A \supset \Box A \supseteq \exists \Box \{ \Box A, \emptyset \} \]

Nested sequents simplify the presentation of propagation rules.

Propagation Rules in Nested Sequents (Modal Cube)

**id**
\[ \frac {\Gamma \{ A, \emptyset \} \land \Gamma \{ A \}} {\Gamma \{ A \} \land \Gamma \{ A \}} \]
**d\Box**
\[ \frac {\Box \{ A \} \land \Box \{ A \}} {\Box \{ A \} \land \Box \{ A \}} \]
**t\Box**
\[ \frac {\Box \{ \Box A, A \} \land \Box \{ \Box A, A \}} {\Box \{ \Box A, A \} \land \Box \{ \Box A, A \}} \]
**b\Box**
\[ \frac {\Box \{ \Box A, \Box A, A \} \land \Box \{ \Box A, \Box A \}} {\Box \{ \Box A, \Box A, A \} \land \Box \{ \Box A, \Box A \}} \]

With the propagation rules we can have cut elimination in nested sequent systems.

- These rules have been given in
  - Rajeev Goré and Linda Postniece and Alwen Tiu: "On the Correspondence between Display Postulates and Deep Inference in Nested Sequent Calculi for Tense Logics", Logical Methods in Computer Science 7(2), 2011

- These propagation rules for the modal cube have been introduced by Brünnler.
In this paper there is also a simple completeness proof via a terminating proof search for nested sequents.