From Axioms to Rules:

The Factory of Modal Proof Systems

# 9. Lecture

# Proof calculi for path logics

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# What are (Modal) Paths Logics?

**Modal logic K:** axioms for classical propositional logic

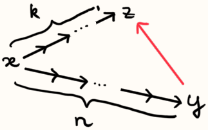
$+\ k:\ \Box(A \supset B) \supset (\Box A \supset \Box B)$

$+\ \text{nec}\ \dfrac{A}{\Box A} \qquad +\ \text{mp}\ \dfrac{A \qquad A \to B}{B}$

**Modal path axioms:** $\pi : \Diamond^k A \supset \Box^n \Diamond A \qquad k, n \in \mathbb{N}$

where $\heartsuit^0 A := A$ and $\heartsuit^{n+1} A := \heartsuit \heartsuit^n A$ is defined by induction.

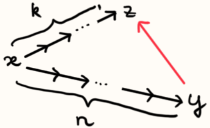
**Corresponding frame condition:** $C_\pi : \forall xyz ((R^n(x,y) \wedge R^k(x,z)) \supset R(y,z))$




$R^0(x,y)$ iff $x = y$

$R^{n+1}(x,y)$ iff there exists $u \in W$ s.t $R(x,u)$ & $R^n(u,y)$

$$\boxed{\pi : \Diamond^k A \supset \Box^n \Diamond A}$$




**Theorem** Let $\Pi$ be a set of path axioms. TFAE

1. Formula $A$ is a theorem of $K+\Pi$
2. Formula $A$ is valid in all frames that satisfy the conditions $C_\pi$ for each $\pi \in \Pi$.

**Examples:**

| | | | |
|---|---|---|---|
| $t$: $A \supset \Diamond A$ | $k=0$ | $n=0$ | $\forall x\, R(x,x)$ |
| $4$: $\Diamond\Diamond A \supset \Diamond A$ | $k=2$ | $n=0$ | $\forall xyz\, ((R(x,y) \land R(y,z)) \supset R(x,z))$ |
| $5$: $\Diamond A \supset \Box\Diamond A$ | $k=1$ | $n=1$ | $\forall xyz\, ((R(x,y) \land R(x,z)) \supset R(y,z))$ |
| $b$: $A \supset \Box\Diamond A$ | $k=0$ | $n=1$ | $\forall xy\, (R(x,y) \supset R(y,x))$ |

# Labelled rules for path axioms

The $C_\pi$ frame condition is a Horn formula :

$$\forall xyz\ (R^k(x,y) \wedge R^n(x,z)) \supset R(y,z))$$

① $\forall xyz\ (R^n(x,y) \wedge R^k(x,z)) \supset R(y,z))$

compatible with the $R^+$ version of $\Box_L / \Diamond_R$

$$\dfrac{\dfrac{\dfrac{\dfrac{R^n(x,y)\in\Gamma'}{\Gamma'\vdash R^n(x,y)\Downarrow\Delta'}\,id_R \qquad \dfrac{R^k(x,z)\in\Gamma'}{\Gamma'\vdash R^k(x,y)\Downarrow\Delta'}\,id_R}{\Gamma'\vdash R^n(x,y)\overset{+}{\wedge} R^k(x,y)\Downarrow\Delta'}\,\overset{+}{\wedge}_R \qquad \dfrac{\dfrac{\Gamma', R(y,z)\Uparrow\ \vdash\ \Uparrow\Delta'}{\Gamma'\Uparrow R(y,z)\vdash\ \Uparrow\Delta'}\,S_L}{\Gamma'\Downarrow R(y,z)\vdash\Delta'}\,R_L}{\Gamma'\Downarrow (R^n(x,y)\wedge R^k(x,z))\supset R(y,z)\vdash\Delta'}\,\supset_L}{\Gamma'\Downarrow\forall xyz\ (R^n(x,y)\wedge R^k(x,z))\supset R(y,z))\vdash\Delta'}\,\forall_L$$

$$\boxed{\dfrac{xRy,\ xR^n y,\ xR^k z,\ \Gamma\vdash\Delta}{xR^n y,\ xR^k z,\ \Gamma\vdash\Delta}\,\pi_1^s}$$

(2) $\forall xyz\,(R^n(x,y) \bar{\wedge} R^k(x,z)) \supset R(y,z))$

( compatible with the $R^-$ version of $\Box_L / \Diamond_R$

$$\boxed{\dfrac{\Gamma \vdash xR^n y, \Delta \qquad \Gamma \vdash xR^k z, \Delta}{\Gamma \vdash yRz, \Delta}\,\pi_2^s}$$

Theorem Let $\Pi$ be a set of path axioms. TFAE

1. Formula $A$ is a theorem of $K+\Pi$
2. Formula $A$ is valid in all frames that satisfy the conditions for each $\pi \in \Pi$.
3. Sequent $\vdash x:A$ is provable in the labelled sequent calculus for $K$, either the $R^+$ or the $R^-$ version, extended with rule $\pi_1^s$ or $\pi_2^s$, resp.

The completeness proof for labelled sequents for K needs to be extended to the path axioms.

$$
\begin{array}{cl}
\hline & \text{id} \\
xR^n y_n, xR^k z_k, y_n R z_k, z_k : a \vdash z_k : a & \\
\hline & \Diamond_R \\
xR^n y_n, xR^k z_k, y_n R z_k, z_k : a \vdash y_n : \Diamond a & \\
\hline & \pi_1^s \\
xR^n y_n, xR^k z_k, z_k : a \vdash y_n : \Diamond a & \\
\hline\hline & \\
xR^n z_n, x : \Diamond^k a \vdash y_n : \Diamond a & \\
\hline\hline & \Box_R \\
x : \Diamond^k a \vdash x : \Box^n \Diamond a & \\
\hline & \supset_R \\
\vdash x : \Diamond^k a \supset \Box^n \Diamond a &
\end{array}
$$

This sort of labelled rules $\pi^s$ are called modal structural rules because they only affect relational atoms.

# Propagation rules

Let $\pi : \Diamond^k A \supset \Box^n \Diamond A$ be a modal path axiom.

$$
\begin{array}{lcl}
x R^n y_n, x R^k z_k, z_k : a & \vdash & z_k : a \\
\hdashline
x R^n y_n, x R^k z_k, y_n R z_k \;\; z_k : a & \vdash & z_k : a \\
\hline
x R^n y_n, x R^k z_k, y_n R z_k, z_k : a & \vdash & y_n : \Diamond a \\
\hline
x R^n y_n, x R^k z_k, z_k : a & \vdash & y_n : \Diamond a
\end{array}
\begin{array}{l}
w_L \\
\Diamond_R \\
\pi_1^s
\end{array}
$$

$$
\begin{array}{lcl}
x R^n y_n, x R^k z_k, z_k : a & \vdash & z_k : a \\
\hline
x R^n y_n, x R^k z_k, z_k : a & \vdash & y_n : \Diamond a
\end{array}
\; \pi^p
$$

The context is implicitly closed under the frame conditions

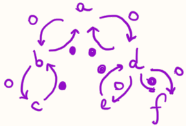
Let $S := \mathcal{G}, \Gamma \Rightarrow \Delta$ be a labelled sequent.

<u>Propagation graph</u> is a directed graph with $\{\circ, \bullet\}$-labelled edges. $PG(S) = (V, E)$

s.t. $V$ the set of labels occurring in $S$

and $E = \{(u, v, \circ), (v, u, \bullet) \mid uRv \in \mathcal{G}\}$

<u>Example</u>: $\mathcal{R} = aRb, bRc, aRd, dRe, dRf$




<u>Propagation path</u> from $w_1$ to $w_\ell$ in $PG(S)$ is an alternating sequence $\sigma = w_1 x_1 w_2 x_2 \dots x_{p-1} w_\ell$ s.t.

for $i \leq \ell$ $w_i \in V$ ; for $i < \ell$ $x_i \in \{\circ, \bullet\}$ ;

for $i < \ell$ $(w_i, w_{i+1}, x_i) \in E$.

$\longrightarrow$ $s(\sigma) = x_1 \dots x_{p-1} \in \{\circ, \bullet\}^{p-1}$ is the <u>string</u> of the path $\sigma$

Technically:

We can use words on the $\{\bullet, \circ\}$ alphabet as a symbolic representation of propagation behaviour.

Let $s = (x_1 \dots x_m, x_0)$ and $t = (y_1 \dots y_\ell, y_0)$ s.t $x_i, y_j \in \{\circ, \bullet\}$

IF $x_0 = y_i$, the composition of $s$ with $t$ at position $i$ is the pair

$$s \triangleright^i t := (y_1 \dots y_{i-1} x_1 \dots x_\ell y_{i+1} \dots y_m, y_0)$$

For a set of path axioms $\Pi$, the completion $\Pi^*$ of $\Pi$ is the smallest set such that

(i) it contains reflections $\pi^\circ$ and $\pi^\bullet$ for each $\pi \in \Pi$ & identities $(\circ, \circ)$ and $(\bullet, \bullet)$

$$\pi^\circ = (\bullet^k \circ^n, \circ) \quad \text{and} \quad \pi^\bullet = (\circ^k \bullet^n, \bullet)$$

(ii) it is closed under composition (at any position)

## Propagation rule corresponding to axiom $\pi$

$$\frac{\mathcal{G}, \Gamma \Rightarrow \Delta, v : A}{\mathcal{G}, \Gamma \Rightarrow \Delta, u : \Diamond A} \; \pi^P \, (*)$$

(*) There is a path $\sigma$ from $u$ to $v$ in $PG(\mathcal{G}, \Gamma \Rightarrow \Delta, u : \Diamond A)$ such that $s(\sigma)$ appears in the completion of $\pi$

**Example** $\quad t : \; A \supset \Diamond A$

$$t^* = \{ (\cdot, \circ), (\cdot, \bullet), (\circ, \circ), (\bullet, \bullet) \}$$

$$\left\{ \frac{\mathcal{G}, \Gamma \vdash \Delta, x : A}{\mathcal{G}, \Gamma \vdash \Delta, x : \Diamond A} \right\}$$

only one rule $\neq \Diamond_R$

$4 : \; \Diamond\Diamond A \supset \Diamond A$

$$4^* = \{ (\overset{m}{\circ}, \circ), (\overset{m}{\bullet}, \bullet) \mid m \geqslant 1 \}$$

$$\left\{ \frac{x_0 R x_1, \ldots, x_{m-1} R x_m, \mathcal{G}, \Gamma \vdash \Delta, x_m : A}{x R x_1, \ldots x_{m-1} R x_m, \mathcal{G}, \Gamma \vdash \Delta, x_0 : \Diamond A} \;\middle|\; m \geqslant 1 \right\}$$

<u>Theorem</u> Let $\Pi$ be a set of path axioms. TFAE

1. Formula $A$ is a theorem of $K + \Pi$
2. Formula $A$ is valid in all frames that satisfy the conditions for each $\pi \in \Pi$.
3. Sequent $\vdash x : A$ is provable in the labelled sequent calculus for $K$, either the $R^+$ or the $R^-$ version, extended with rule $\pi_1^S$ or $\pi_2^S$, resp.
4. Sequent $\vdash x : A$ is provable in the labelled sequent calculus for $K$ (in the $R^+$ version) extended with rule $\pi^P$

[Goré, Postniece & Tiu 2011]

Propagation rules might seem more complicated than structural ones at first.

But in reality they simplify the structure described by the relational atoms

$\longrightarrow$ it remains a TREE throughout the proof

A tree-labelled sequent is a labelled sequent $\mathcal{G}, \Gamma \vdash \Delta$ such that $\mathcal{G}$ forms a tree.

$\downarrow$

in this specific case we can recover a formula interpretation of sequents! [Goré & Ramanayake 2014]


Questions?