From Axioms to Rules:
The Factory of Modal Proof Systems

8. Lecture

Synthetic rules in labelled sequents

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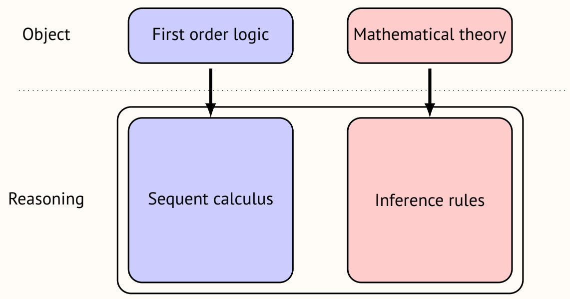
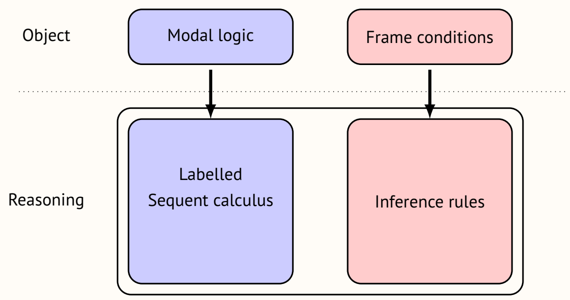
Remember?



What if...



The standard translation

Modal language $\Rightarrow$ **FO language** with:
- a binary predicate R
- a unary predicate a for each atom a

$$
\begin{aligned}
ST_x(a) &= a(x)\\
ST_x(A \wedge B) &= ST_x(A) \wedge ST_x(B)\\
ST_x(\Box A) &= \forall y(R(x,y) \supset ST_y(A))\\
ST_x(\Diamond A) &= \exists y(R(x,y) \wedge ST_y(A))
\end{aligned}
$$

where x is a free variable.

Equivalence
For any modal formula A, any model $\mathcal{M}$ and any world w

$$\mathcal{M}, w \Vdash A \quad \text{iff} \quad \mathcal{M} \vDash [x/w]ST_x(A)$$

Delays

Delay operators ∂^+/∂^- force a formula to be positive / negative.

$$\frac{\Gamma \vdash B \Downarrow \Delta}{\Gamma \vdash \partial^+ B \Downarrow \Delta}\ \partial^+_R \qquad \frac{\Gamma \Uparrow \Theta, B \vdash \Omega \Uparrow \Delta}{\Gamma \Uparrow \Theta, \partial^+ B \vdash \Omega \Uparrow \Delta}\ \partial^+_L$$

$$\frac{\Gamma \Uparrow \Theta \vdash B, \Omega \Uparrow \Delta}{\Gamma \Uparrow \Theta \vdash \partial^- B, \Omega \Uparrow \Delta}\ \partial^-_R \qquad \frac{\Gamma \Downarrow B \vdash \Delta}{\Gamma \Downarrow \partial^- B \vdash \Delta}\ \partial^-_L$$

Polarised translation

We define a **translation for labelled sequents**

$$[\mathcal{G}_1, \Gamma \vdash \mathcal{G}_2, \Delta] = [\mathcal{G}_1, \Gamma]_l \vdash [\mathcal{G}_2, \Delta]_r$$

$$
\begin{aligned}
[x : a]_l = [x : a]_r &= a(x)\\
[xRy]_l = [xRy]_r &= R(x,y)\\
[x : A \wedge B]_l &= \partial^+ [x : A]_l \mathbin{\bar{\wedge}} \partial^+ [x : B]_l\\
[x : A \wedge B]_r &= \partial^+ ([x : A]_r \mathbin{\bar{\wedge}} [x : B]_r)\\
[x : \Box A]_l &= \forall y(R(x,y) \supset \partial^+ [y : A]_l)\\
[x : \Box A]_r &= \partial^+ \forall y(R(x,y) \supset [y : A]_r)\\
[x : \Diamond A]_l &= \partial^- \exists y(R(x,y) \mathbin{\overset{+}{\wedge}} [y : A]_l)\\
[x : \Diamond A]_r &= \exists y(R(x,y) \mathbin{\overset{+}{\wedge}} \partial^- [y : A]_r)
\end{aligned}
$$

Labelled rules are synthesised!

An **inference rule** in the labelled modal proof system corresponds to a **synthetic rule** in the focused proof system LKF.

$$\Box\ \frac{xRy, \mathcal{G}, \Gamma \vdash \Delta, y : A}{\mathcal{G}, \Gamma \vdash \Delta, x : \Box A}$$

$$\begin{array}{rc}
 & \mathcal{G}, \Gamma' {\Uparrow} \vdash {\Uparrow} \Delta', \partial^+[\Box A]_x, \partial^+[A]_y \\
\mathsf{S_R} & \overline{\mathcal{G}, \Gamma', R(x,y) {\Uparrow} \vdash \partial^+[A]_y \Uparrow \Delta', \partial^+[\Box A]_x} \\
\mathsf{S_L} & \overline{\mathcal{G}, \Gamma' \Uparrow R(x,y) \vdash \partial^+[A]_y \Uparrow \Delta', \partial^+[\Box A]_x} \\
\supset_{\mathsf{R}} & \overline{\mathcal{G}, \Gamma' {\Uparrow} \vdash R(x,y) \supset \partial^+[A]_y \Uparrow \Delta', \partial^+[\Box A]_x} \\
\forall_{\mathsf{R}} & \overline{\mathcal{G}, \Gamma' {\Uparrow} \vdash \forall y(R(x,y) \supset \partial^+[A]_y) \Uparrow \Delta', \partial^+[\Box A]_x} \\
\mathsf{R_R} & \overline{\mathcal{G}, \Gamma' \vdash \forall y(R(x,y) \supset \partial^+[A]_y) \Downarrow \Delta', \partial^+[\Box A]_x} \\
\partial^+_{\mathsf{R}} & \overline{\mathcal{G}, \Gamma' \vdash \partial^+ \forall y(R(x,y) \supset \partial^+[A]_y) \Downarrow \Delta', \partial^+[\Box A]_x} \\
\mathsf{D_R} & \overline{\mathcal{G}, \Gamma' {\Uparrow} \vdash {\Uparrow} \Delta', \partial^+[\Box A]_x}
\end{array}$$

Note: in this case, the polarity assigned to $R(x,y)$ does not matter, it can always be stored.

Alternative synthetic rules

$R(x,y)$ **positive:**

$$\frac{y : A, xRy, \Gamma \vdash \Delta}{x : \Box A, xRy, \Gamma \vdash \Delta}\ \Box_{L1} \qquad \frac{xRy, \Gamma \vdash \Delta, y : A}{xRy, \Gamma \vdash \Delta, x : \Diamond A}\ \Diamond_{R1}$$

$R(x,y)$ **negative:**

$$\frac{\Gamma \vdash \Delta, xRy \quad y : A, \Gamma \vdash \Delta}{x : \Box A, \Gamma \vdash \Delta}\ \Box_{L2} \qquad \frac{\Gamma \vdash \Delta, xRy \quad \Gamma \vdash \Delta, y : A}{\Gamma \vdash \Delta, x : \Diamond A}\ \Diamond_{R2}$$

Pre-existing extensions

Horn clauses as bipoles.

$$\forall \bar{z}(P_1 \wedge \ldots \wedge P_m \supset Q)$$

$$\forall \bar{z}(P_1^{\pm} \wedge \ldots \wedge P_m^{\pm} \supset Q^{\pm})$$

$$\forall \bar{z}(P_1^{+} \overset{+}{\wedge} \ldots \overset{+}{\wedge} P_m^{+} \supset Q^{+}) \qquad \forall \bar{z}(P_1^{-} \overset{-}{\wedge} \ldots \overset{-}{\wedge} P_m^{-} \supset Q^{-})$$

$$\frac{Q, \Gamma \vdash C}{P_1, \ldots, P_m, \Gamma \vdash C}\ (H1) \qquad \frac{\Gamma \vdash P_1, \quad \ldots \quad \Gamma \vdash P_m}{\Gamma \vdash Q}\ (H2)$$

Geometric axioms as bipoles.

$$\forall \bar{z}(P_1 \wedge \ldots \wedge P_m \supset \exists \bar{x}_1(\wedge Q_1) \vee \ldots \vee \exists \bar{x}_n(\wedge Q_n))$$

$$\forall \bar{z}(P_1^{\pm} \wedge \ldots \wedge P_m^{\pm} \supset \exists \bar{x}_1(\overset{+}{\wedge} Q_1^{\pm}) \vee \ldots \vee \exists \bar{x}_n(\overset{+}{\wedge} Q_n^{\pm}))$$

$$\forall \bar{z}(P_1^{+} \overset{+}{\wedge} \ldots \overset{+}{\wedge} P_m^{+} \supset \exists \bar{x}_1(\overset{+}{\wedge} Q_1^{\pm}) \vee \ldots \vee \exists \bar{x}_n(\overset{+}{\wedge} Q_n^{\pm}))$$

$$\frac{\overline{Q}_1, \Gamma \vdash C \quad \ldots \quad \overline{Q}_n, \Gamma \vdash C}{P_1, \ldots, P_m, \Gamma' \vdash C}\ (G)$$

In the modal cube

Axiom	Rule scheme 1	Rule scheme 2
t	$\dfrac{R(x,x), \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\ \mathrm{t}_1$	$\dfrac{}{\Gamma \vdash \Delta, R(x,x)}\ \mathrm{t}_2$
4	$\dfrac{R(x,z), \Gamma \vdash \Delta}{R(x,y), R(y,z), \Gamma \vdash \Delta}\ 4_1$	$\dfrac{\Gamma \vdash \Delta, R(x,y) \quad \Gamma \vdash \Delta, R(y,z)}{\Gamma \vdash \Delta, R(x,z)}\ 4_2$
5	$\dfrac{R(y,z), \Gamma \vdash \Delta}{R(x,y), R(x,z), \Gamma \vdash \Delta}\ 5_1$	$\dfrac{\Gamma \vdash \Delta, R(x,y) \quad \Gamma \vdash \Delta, R(x,z)}{\Gamma \vdash \Delta, R(y,z)}\ 5_2$
b	$\dfrac{R(y,x), \Gamma \vdash \Delta}{R(x,y), \Gamma \vdash \Delta}\ \mathrm{b}_1$	$\dfrac{\Gamma \vdash \Delta, R(x,y)}{\Gamma \vdash \Delta, R(y,x)}\ \mathrm{b}_2$