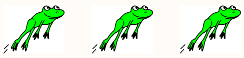
From Axioms to Rules:
The Factory of Modal Proof Systems

7. Lecture

Labelled Sequents for Modal Logics



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Recall: The Sequent Calculus for Modal Logics

k :	$\Box(A \supset B) \supset (\Box A \supset \Box B)$	$\leftrightarrow$	$\mathsf{k_R}\ \dfrac{\Gamma \vdash \Theta, A}{\Box\Gamma \vdash \Diamond\Theta, \Box A}$
t :	$A \supset \Diamond A$	$\leftrightarrow$	$\mathsf{t_R}\ \dfrac{\Gamma \vdash \Theta, A}{\Gamma \vdash \Theta, \Diamond A}$
4 :	$\Diamond\Diamond A \supset \Diamond A$	$\leftrightarrow$	$\mathsf{4_R}\ \dfrac{\Box\Gamma \vdash \Diamond\Theta, A}{\Box\Gamma \vdash \Diamond\Theta, \Box A}$
b :	$A \supset \Box\Diamond A$	$\leftrightarrow$	???

- **Exercise 7.1:** Show the correspondence between the axioms and the rules. I.e., show that the axioms are derivable when the corresponding rules are added to the system, and show that the rules are derivable when the axioms and cut are added.

From Forcing to Labelled Sequent Rules

$\vee$:	$x \Vdash A \vee B$ iff $x \Vdash A$ or $x \Vdash B$	$\rightsquigarrow$	$\vee_{\mathsf{R}}\ \dfrac{\Gamma \vdash \Theta, x{:}A, x{:}B}{\Gamma \vdash \Theta, x{:}A \vee B}$
$\Box$:	$x \Vdash \Box A$ iff $\forall y.\, xRy \supset y \Vdash A$	$\rightsquigarrow$	$\Box_{\mathsf{R}}\ \dfrac{xRy, \Gamma \vdash \Theta, y{:}A}{\Gamma \vdash \Theta, x{:}\Box A}\ y \text{ not in } \Gamma, \Theta$
t :	$\forall x.\, xRx$	$\rightsquigarrow$	$\mathsf{ref}\ \dfrac{xRx, \Gamma \vdash \Theta}{\Gamma \vdash \Theta}$
4 :	$\forall xyz.\, xRy \wedge yRz \supset xRz$	$\rightsquigarrow$	$\mathsf{trans}\ \dfrac{xRz, xRy, yRz, \Gamma \vdash \Theta}{xRy, yRz, \Gamma \vdash \Theta}$
b :	$\forall xy.\, xRy \supset yRx$	$\rightsquigarrow$	$\mathsf{sym}\ \dfrac{yRx, xRy, \Gamma \vdash \Theta}{xRy, \Gamma \vdash \Theta}$

What are Labelled Sequents?

- *labelled formula:* $x\!:\!A$

  relation atom: xRy

  where x, y are *labels* (from a countable set of label variables) and A is a (modal) formula.

- *labelled sequent:* $\mathcal{R}, \Gamma \vdash \Theta$

  where $\mathcal{R}$ is a multiset of relation atoms and Γ, Θ are multisets of labelled formulas (separated by comma)

- To simplify the presentation, we use multisets here instead of lists.
- Labelled systems for modal logics have first been studied in detail by Alex Simpson:
  - *Alex Simpson:* **"The Proof Theory and Semantics of Intuitionistic Modal Logic".** *PhD Thesis, University of Edinburgh, 1994*

  and Sara Negri:
  - *Sara Negri:* **"Proof Analysis in Modal Logics".** *Journal of Philosophical Logic 34, 2005*

Proof Rules for Labelled Sequents

Initial sequents / axioms:

$$\bot\ \frac{}{\mathcal{R}, \Gamma, x\!:\!\bot \vdash \Theta} \qquad \mathsf{id}\ \frac{}{\mathcal{R}, \Gamma, x\!:\!A \vdash x\!:\!A, \Theta} \qquad \top\ \frac{}{\mathcal{R}, \Gamma \vdash x\!:\!\top, \Theta}$$

Logical rules:

$$\vee_{\mathsf{L}}\ \frac{\mathcal{R}, \Gamma, x\!:\!A \vdash \Theta \qquad \mathcal{R}, \Gamma, x\!:\!B \vdash \Theta}{\mathcal{R}, \Gamma, x\!:\!A \vee B \vdash \Theta} \qquad \vee_{\mathsf{R}}\ \frac{\mathcal{R}, \Gamma \vdash x\!:\!A, x\!:\!B, \Theta}{\mathcal{R}, \Gamma \vdash x\!:\!A \vee B, \Theta}$$

$$\wedge_{\mathsf{L}}\ \frac{\mathcal{R}, \Gamma, x\!:\!A, x\!:\!B \vdash \Theta}{\mathcal{R}, \Gamma, x\!:\!A \wedge B \vdash \Theta} \qquad \wedge_{\mathsf{R}}\ \frac{\mathcal{R}, \Gamma \vdash x\!:\!A, \Theta \qquad \mathcal{R}, \Gamma \vdash x\!:\!B, \Theta}{\mathcal{R}, \Gamma \vdash x\!:\!A \wedge B, \Theta}$$

$$\supset_{\mathsf{L}}\ \frac{\mathcal{R}, \Gamma \vdash x\!:\!A, \Theta \qquad \mathcal{R}, \Gamma, x\!:\!B \vdash \Theta}{\mathcal{R}, \Gamma, x\!:\!A \supset B \vdash \Theta} \qquad \supset_{\mathsf{R}}\ \frac{\mathcal{R}, \Gamma, x\!:\!A \vdash x\!:\!B, \Theta}{\mathcal{R}, \Gamma \vdash x\!:\!A \supset B, \Theta}$$

$$\neg_{\mathsf{L}}\ \frac{\mathcal{R}, \Gamma \vdash x\!:\!A, \Theta}{\mathcal{R}, \Gamma, x\!:\!\neg A \vdash \Theta} \qquad \neg_{\mathsf{R}}\ \frac{\mathcal{R}, \Gamma, x\!:\!A \vdash \Theta}{\mathcal{R}, \Gamma \vdash x\!:\!\neg A, \Theta}$$

$$\Box_{\mathsf{L}}\ \frac{xRy, \mathcal{R}, \Gamma, x\!:\!\Box A, y\!:\!A \vdash \Theta}{xRy, \mathcal{R}, \Gamma, x\!:\!\Box A \vdash \Theta} \qquad \Box_{\mathsf{R}}\ \frac{xRy, \mathcal{R}, \Gamma \vdash y\!:\!A, \Theta}{\mathcal{R}, \Gamma \vdash x\!:\!\Box A, \Theta}\ y \text{ not in } \mathcal{R}, \Gamma, \Theta$$

$$\Diamond_{\mathsf{L}}\ \frac{xRy, \mathcal{R}, \Gamma, y\!:\!A \vdash \Theta}{\mathcal{R}, \Gamma, x\!:\!\Diamond A \vdash \Theta}\ y \text{ not in } \mathcal{R}, \Gamma, \Theta \qquad \Diamond_{\mathsf{R}}\ \frac{xRy,\ \mathcal{R}, \Gamma \vdash y\!:\!A, x\!:\!\Diamond A, \Theta}{xRy, \mathcal{R}, \Gamma \vdash x\!:\!\Diamond A, \Theta}$$

Proof Rules for Labelled Sequents (cont.)

cut rule:

$$\mathsf{cut}\ \frac{\mathcal{R}, \Gamma \vdash x\!:\!A, \Theta \qquad \mathcal{S}, \Delta, x\!:\!A \vdash \Lambda}{\mathcal{R}, \mathcal{S}, \Gamma, \Delta \vdash \Theta, \Lambda}$$

frame rules:

$$\mathsf{d}: \quad \forall x.\, \exists y.\, xRy \quad \to \quad \mathsf{ser}\ \frac{xRy, \mathcal{R}, \Gamma \vdash \Theta}{\mathcal{R}, \Gamma \vdash \Theta}\ y \text{ not in } \mathcal{R}, \Gamma, \Theta$$

$$\mathsf{t}: \quad \forall x.\, xRx \quad \to \quad \mathsf{ref}\ \frac{xRx, \mathcal{R}, \Gamma \vdash \Theta}{\mathcal{R}, \Gamma \vdash \Theta}$$

$$\mathsf{b}: \quad \forall xy.\, xRy \supset yRx \quad \to \quad \mathsf{sym}\ \frac{yRx, xRy, \mathcal{R}, \Gamma \vdash \Theta}{xRy, \mathcal{R}, \Gamma \vdash \Theta}$$

$$\mathsf{4}: \quad \forall xyz.\, xRy \wedge yRz \supset xRz \quad \to \quad \mathsf{trans}\ \frac{xRz, xRy, yRz, \mathcal{R}, \Gamma \vdash \Theta}{xRy, yRz, \mathcal{R}, \Gamma \vdash \Theta}$$

$$\mathsf{5}: \quad \forall xyz.\, xRy \wedge xRz \supset yRz \quad \to \quad \mathsf{euc}\ \frac{yRz, xRy, xRz, \mathcal{R}, \Gamma \vdash \Theta}{xRy, xRz, \mathcal{R}, \Gamma \vdash \Theta}$$

Soundness and Completeness for Labelled Sequents

Definition: Let **KX** be a logic in the modal cube. A formula A is *provable* in a labelled sequent calculus for **KX** iff the sequent $\vdash x:A$ is derivable from the initial sequents via

- the logical rules,
- the cut rule, and
- the frame rules corresponding to the axioms of **KX**.

Definition: The formula A is *cut-free provable* iff the sequent $\vdash x:A$ is derivable without the cut rule.

Theorem: TFAE:

1. A is a theorem of **KX**.
2. A is valid for **KX**.
3. A is provable in the labelled sequent calculus for **KX**.
4. A is cut-free provable in the labelled sequent calculus for **KX**.

- Soundness, i.e., any of the implications $1 \implies 2$ or $3 \implies 2$ or $4 \implies 2$ can be proved by an induction on the structure of the proof of A.
- **Exercise 7.2:** Prove one of those three implications. (Hint: Prove the contrapositive by induction on the size of the proof. For this, show for each inference rule, if there is a countermodel for the conclusion, then there is a countermodel for one of the premises.)
- Completeness (i.e., the other direction) is now much easier with the help of the labelled sequents. (See Exercise 7.8)

Where are the structural rules?

structural rules:

$$\mathsf{weak}_\mathsf{L}\ \frac{\mathcal{R},\Gamma\vdash\Theta}{xRy,\mathcal{R},\Gamma\vdash\Theta}\qquad \mathsf{weak}_\mathsf{L}\ \frac{\mathcal{R},\Gamma\vdash\Theta}{\mathcal{R},\Gamma,x:A\vdash\Theta}\qquad \mathsf{weak}_\mathsf{R}\ \frac{\mathcal{R},\Gamma\vdash\Theta}{\mathcal{R},\Gamma\vdash x:A,\Theta}$$

$$\mathsf{con}_\mathsf{L}\ \frac{xRy,xRy,\mathcal{R},\Gamma\vdash\Theta}{xRy,\mathcal{R},\Gamma\vdash\Theta}\qquad \mathsf{con}_\mathsf{L}\ \frac{\mathcal{R},\Gamma,x:A,x:A\vdash\Theta}{\mathcal{R},\Gamma,x:A\vdash\Theta}\qquad \mathsf{con}_\mathsf{R}\ \frac{\mathcal{R},\Gamma\vdash x:A,x:A,\Theta}{\mathcal{R},\Gamma\vdash x:A,\Theta}$$

Theorem: For every **KX** in the modal cube, the structural rules are *height-preserving admissible* for the labelled sequent systems for **KX**.

Theorem: For every **KX** in the modal cube, the logical rules and the frame rules are *height-preserving invertible* in the labelled sequent systems for **KX**.

- The exchange rules are not needed because we consider multisets instead of lists.
- **Definition:** A rule

$$\mathsf{r}\ \frac{\mathcal{R}_1,\Gamma_1\vdash\Theta_1\quad\dots\quad\mathcal{R}_n,\Gamma_n\vdash\Theta_n}{\mathcal{R},\Gamma\vdash\Theta}$$

is *height-preserving invertible* in a system **S** if for every proof of $\mathcal{R},\Gamma\vdash\Theta$ in **S**, there are proofs of $\mathcal{R}_1,\Gamma_1\vdash\Theta_1$, ..., $\mathcal{R}_n,\Gamma_n\vdash\Theta_n$ of at most the same height.

- **Definition:** A rule r is *admissible* in a system **S** if every sequent that is provable in $\mathbf{S}+\mathsf{r}$ is also provable in **S**. It is *height-preserving admissible* if the new proof in has at most the same height as the original proof in $+\,\mathsf{r}$.
- **Exercise 7.3:** Prove these two theorems. (Hint: proceed by induction on the height of the proof. Show first admissibility of weakening, then invertability of of the logical rules and the frame rules, and finally admisibility of contraction.)

Cut Elimination

With height-preserving admissibility of structural rules, cut elimination is simpler. No multi-cut is needed.

- *commutative cases:*

$$\mathsf{cut}\ \frac{\mathsf{r}\ \dfrac{\mathcal{R}',\Gamma'\vdash x:A,\Theta'}{\mathcal{R},\Gamma\vdash x:A,\Theta}\qquad \mathcal{S},\Delta,x:A\vdash\Lambda}{\mathcal{R},\mathcal{S},\Gamma,\Delta\vdash\Theta,\Lambda}\quad\rightsquigarrow\quad \mathsf{r}\ \frac{\mathsf{cut}\ \dfrac{\mathcal{R}',\Gamma'\vdash x:A,\Theta'\qquad \mathcal{S},\Delta,x:A\vdash\Lambda}{\mathcal{R}',\mathcal{S},\Gamma',\Delta\vdash\Theta',\Lambda}}{\mathcal{R},\mathcal{S},\Gamma,\Delta\vdash\Theta,\Lambda}$$

- *key cases:*

$$\mathsf{cut}\ \frac{\Box_\mathsf{R}\ \dfrac{xRz,\mathcal{R},\Gamma\vdash z:A,\Theta}{\mathcal{R},\Gamma\vdash x:\Box A,\Theta}\qquad \Box_\mathsf{L}\ \dfrac{xRy,\mathcal{S},\Delta,x:\Box A,y:A\vdash\Lambda}{xRy,\mathcal{S},\Delta,x:\Box A\vdash\Lambda}}{xRy,\mathcal{R},\mathcal{S},\Gamma,\Delta\vdash\Theta,\Lambda}\quad\rightsquigarrow$$

$$\mathsf{con}_\mathsf{L},\mathsf{con}_\mathsf{R}\ \frac{\mathsf{cut}\ \dfrac{xRy,\mathcal{R},\Gamma\vdash y:A,\Theta\qquad \mathsf{cut}\ \dfrac{\Box_\mathsf{R}\ \dfrac{xRz,\mathcal{R},\Gamma\vdash z:A,\Theta}{\mathcal{R},\Gamma\vdash x:\Box A,\Theta}\qquad xRy,\mathcal{S},\Delta,x:\Box A,y:A\vdash\Lambda}{xRy,\mathcal{R},\mathcal{S},\Gamma,\Delta,y:A\vdash\Theta,\Lambda}}{xRy,xRy,\mathcal{R},\mathcal{R},\mathcal{S},\Gamma,\Gamma,\Delta\vdash\Theta,\Theta,\Lambda}}{xRy,\mathcal{R},\mathcal{S},\Gamma,\Delta\vdash\Theta,\Lambda}$$

- Cut elimination is the implication $3 \implies 4$ from above.
- **Exercise 7.4:** Why is the duplication of the cut not a problem here (but was a problem before)?
- **Exercise 7.5:** Complete the proof, i.e., list all the other cases, and argue why this process of step-wise cut reduction is terminating.

Proof Search and Decidability

Basic idea:
If the proof system is complete and proof search is terminating, then we have a decision procedure.

- all rules are (height-preserving) invertible
  $\Longrightarrow$ *no backtracking is needed!*
- contraction is (height-preserving) admissible
  $\Longrightarrow$ *we do not need to apply rules that add a formula that is already present in the conclusion!*
- all rules have the subformula property
  $\Longrightarrow$ *the number of formulas that can occur in the proof is bounded!*

$\Longrightarrow$ for termination, it remains to restrict the number of labels created during the proof search

- In general, proof search produces infinite trees.
- Sometimes we can impose restrictions to make proof search finite.
- Then we have to show that the proof search is still complete, i.e., if we don't find a proof then there is indeed no proof.
- Usually, this is done by constructing a countermodel out of a failed proof search.

Decidability — Part 1: Axioms k, d, t, b

$$\mathsf{ref}\ \frac{xRx, \mathcal{R}, \Gamma \vdash \Theta}{\mathcal{R}, \Gamma \vdash \Theta} \qquad \mathsf{sym}\ \frac{yRx, xRy, \mathcal{R}, \Gamma \vdash \Theta}{xRy, \mathcal{R}, \Gamma \vdash \Theta} \qquad \mathsf{ser}\ \frac{xRy, \mathcal{R}, \Gamma \vdash \Theta}{\mathcal{R}, \Gamma \vdash \Theta}\ y \text{ fresh}$$

$$\Box_{\mathsf{L}}\ \frac{xRy, \mathcal{R}, \Gamma, x:\Box A, y:A \vdash \Theta}{xRy, \mathcal{R}, \Gamma, x:\Box A \vdash \Theta} \qquad \Box_{\mathsf{R}}\ \frac{xRy, \mathcal{R}, \Gamma \vdash y:A, \Theta}{\mathcal{R}, \Gamma \vdash x:\Box A, \Theta}\ y \text{ fresh}$$

$$\Diamond_{\mathsf{L}}\ \frac{xRy, \mathcal{R}, \Gamma, y:A \vdash \Theta}{\mathcal{R}, \Gamma, x:\Diamond A \vdash \Theta}\ y \text{ fresh} \qquad \Diamond_{\mathsf{R}}\ \frac{xRy,\ \mathcal{R}, \Gamma \vdash y:A, x:\Diamond A, \Theta}{xRy, \mathcal{R}, \Gamma \vdash x:\Diamond A, \Theta}$$

- we need only labels that occur in the conclusion or are introduced as *eigenvariables* (in ser, $\Box_{\mathsf{R}}$, $\Diamond_{\mathsf{L}}$)
  $\Longrightarrow$ *limits the possible applications of* ref
- the rules $\Box_{\mathsf{R}}$, $\Diamond_{\mathsf{L}}$, and ser only need to be applied once (for each $x:\Box A$, each $x:\Diamond A$, each x, respectively)
  $\Longrightarrow$ *limits the number fresh eigenvariables in the proof*

$\Longrightarrow$ We have a terminating proof search procedure.

- The proof search is terminating because there is an upper bound for the size of each sequent in the proof (if the number of labels is bound and the number of formulas is bound, the number of possible labelled formulas that can occur in the sequent is bound).
- More details on this termination argument can be found in
  - *Sara Negri:* **"Proof Analysis in Modal Logics".** *Journal of Philosophical Logic 34, 2005*
- If (in our case) proof search fails, there is a sequent to which no further rule can be applied. This sequent defines a countermodel for the endsequent of the proof.
- **Exercise 7.6:** Construct this countermodel.

Decidability — Part 2: Axioms 4, 5

$$\mathsf{trans}\ \frac{xRz, xRy, yRz, \mathcal{R}, \Gamma \vdash \Theta}{xRy, yRz, \mathcal{R}, \Gamma \vdash \Theta} \qquad \mathsf{euc}\ \frac{yRz, xRy, xRz, \mathcal{R}, \Gamma \vdash \Theta}{xRy, xRz, \mathcal{R}, \Gamma \vdash \Theta}$$

$$\Box_{\mathsf{L}}\ \frac{xRy, \mathcal{R}, \Gamma, x:\Box A, y:A \vdash \Theta}{xRy, \mathcal{R}, \Gamma, x:\Box A \vdash \Theta} \qquad \Box_{\mathsf{R}}\ \frac{xRy, \mathcal{R}, \Gamma \vdash y:A, \Theta}{\mathcal{R}, \Gamma \vdash x:\Box A, \Theta}\ y \text{ fresh}$$

$$\Diamond_{\mathsf{L}}\ \frac{xRy, \mathcal{R}, \Gamma, y:A \vdash \Theta}{\mathcal{R}, \Gamma, x:\Diamond A \vdash \Theta}\ y \text{ fresh} \qquad \Diamond_{\mathsf{R}}\ \frac{xRy,\ \mathcal{R}, \Gamma \vdash y:A, x:\Diamond A, \Theta}{xRy, \mathcal{R}, \Gamma \vdash x:\Diamond A, \Theta}$$

Decidability — Part 2: Axioms 4, 5

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{xRy, yRz, xRz, zRw, xRw, wRv, x:\Box\Diamond A, z:A, w:A, v:A \vdash y:B}\text{trans}}{xRy, yRz, xRz, zRw, xRw, x:\Box\Diamond A, z:A, w:A, w:\Diamond A \vdash y:B}\Diamond_L}{xRy, yRz, xRz, zRw, xRw, x:\Box\Diamond A, z:A, w:A \vdash y:B}\Box_L}{xRy, yRz, xRz, zRw, x:\Box\Diamond A, z:A, w:A \vdash y:B}\text{trans}}{xRy, yRz, xRz, x:\Box\Diamond A, z:A, z:\Diamond A \vdash y:B}\Diamond_L}{xRy, yRz, xRz, x:\Box\Diamond A, z:A \vdash y:B}\Box_L}{xRy, yRz, x:\Box\Diamond A, z:A \vdash y:B}\text{trans}}{xRy, x:\Box\Diamond A, y:\Diamond A \vdash y:B}\Diamond_L}{xRy, x:\Box\Diamond A \vdash y:B}\Box_L}{x:\Box\Diamond A \vdash x:\Box B}\Box_R}{\vdash x:\Box\Diamond A \supset \Box B}\supset_R
$$

- Naive proof search would not terminate.

Decidability — Part 2: Axioms 4, 5

$$
\text{trans}\ \frac{xRz, xRy, yRz, \mathcal{R}, \Gamma \vdash \Theta}{xRy, yRz, \mathcal{R}, \Gamma \vdash \Theta}
\qquad
\text{euc}\ \frac{yRz, xRy, xRz, \mathcal{R}, \Gamma \vdash \Theta}{xRy, xRz, \mathcal{R}, \Gamma \vdash \Theta}
$$

$$
\Box_L\ \frac{xRy, \mathcal{R}, \Gamma, x:\Box A, y:A \vdash \Theta}{xRy, \mathcal{R}, \Gamma, x:\Box A \vdash \Theta}
\qquad
\Box_R\ \frac{xRy, \mathcal{R}, \Gamma \vdash y:A, \Theta}{\mathcal{R}, \Gamma \vdash x:\Box A, \Theta}\ y \text{ fresh}
$$

$$
\Diamond_L\ \frac{xRy, \mathcal{R}, \Gamma, y:A \vdash \Theta}{\mathcal{R}, \Gamma, x:\Diamond A \vdash \Theta}\ y \text{ fresh}
\qquad
\Diamond_R\ \frac{xRy,\ \mathcal{R}, \Gamma \vdash y:A, x:\Diamond A, \Theta}{xRy, \mathcal{R}, \Gamma \vdash x:\Diamond A, \Theta}
$$

- the rules trans and euc create new relational atoms xRy to which the rules $\Box_L$ and $\Diamond_R$ have to be applied
  $\Longrightarrow$ *possibly creating infinite chains*
- when we have zRw for some labels z and w, and z and w label the same set of formulas (on the left and on the right of the turnstile), then we can stop
  $\Longrightarrow$ *this terminates*

- As the number of formulas that can occur in a sequent is bounded (by the number of subformulas of the endsequent), there cannot be an infinte chain x_1Rx_2, x_2Rx_3, x_3Rx_4, ..., such that the formula set for each label x_i is different.
- Similary as above, such a sequent determines a countermodel. But here we have to identify the two labels z and w (i.e., they define the same world in the model).
- **Exercise 7.7:** Construct this countermodel and show that it is indeed a countermodel.

Proof of Completeness (finally)

Theorem: If a formula A is valid, then it is provable.

Proof:

- We show the contrapositive.
- Assume the formula A is not provable.
- By our terminating proof search, we obtain a *failed sequent*.
- From that we construct a countermodel:
  - let our endsequent be $\vdash x:A$,
  - let failed sequent be $\mathcal{R}, \Gamma \vdash \Theta$,
  - let W be the set of labels in $\mathcal{R}, \Gamma \vdash \Theta$, and
  - let V such that $w \Vdash a$ for all $w:a$ in Θ,
  - then $\langle W, \mathcal{R}, V\rangle$ is a model with $x \in W$ and $x \nVdash A$
- Hence, A is not valid. Contradiction. $\Box$

- With *failed sequent* we mean a sequent which is not axiomatic, but the proof search procedure described above has stopped.
- **Exercise 7.8:** Work out the details of the completeness proof, i.e., show that $x \nVdash A$. (Hint: for all sequents $\mathcal{R}', \Gamma' \vdash \Theta'$ in the proof on the path from the root (the endsequent) to the failed sequent, one can show that for all $z:B \in \Gamma'$ we have $z \Vdash B$ and for all $y:C \in \Theta'$, we have that $y \nVdash C$.)