From Axioms to Rules:
The Factory of Modal Proof Systems

5. Lecture
Synthetic rules and bipoles

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Inversion phase

Negative formula on the right

\[ \Gamma_1 \vdash \Gamma_2 \vdash \Delta_1, A, B \vdash \Delta_2 \]

\[ \vdash_R \Gamma_1 \vdash \Gamma_2 \vdash \Delta_1, A \wedge B \vdash \Delta_2 \]

\[ \vdash_R \Gamma_1 \vdash \Gamma_2 \vdash \Delta_1, \neg \vdash \Delta_2 \]

Positive formula on the left

\[ \Gamma_1 \vdash \Gamma_2, A, B \vdash \Delta_1 \vdash \Delta_2 \]

\[ \vdash_L \Gamma_1 \vdash \Gamma_2, A \vdash B \vdash \Delta_1 \vdash \Delta_2 \]

\[ \vdash_L \Gamma_1 \vdash \Gamma_2, \bot \vdash \Delta_1 \vdash \Delta_2 \]

\[ \exists_L \Gamma_1 \vdash \Gamma_2, [y/x]A \vdash \Delta_1 \vdash \Delta_2 \]

\[ \exists_L \Gamma_1 \vdash \Gamma_2, \exists x A \vdash \Delta_1 \vdash \Delta_2 \]

Storage

\[ S_L \Gamma_1, A \vdash \Gamma_2 \vdash \Delta_1 \vdash \Delta_2 \]

\[ A = N \text{ or at} \]

\[ S_R \Gamma_1 \vdash \Gamma_2 \vdash A, \Delta_1 \vdash \Delta_2 \]

\[ A = P \text{ or at} \]
Inversion → Focus

\[ D_L \Gamma, N \downarrow N \vdash \Delta \quad D_R \Gamma \vdash P \downarrow P, \Delta \]

Focus phase

Positive formula on the right

\[ \vdash_R \Gamma \vdash A \downarrow \Delta \quad \vdash_R \Gamma \vdash B \downarrow \Delta \]

\[ \exists_R \Gamma \vdash [t/x]A \downarrow \Delta \quad t \text{ is a term} \]

Negative formula on the left

\[ \land_L \Gamma \downarrow A \vdash \Delta \quad \land_L \Gamma \downarrow B \vdash \Delta \]

\[ \land_L \Gamma \vdash A \lor B \vdash \Delta \quad \exists \Gamma \vdash \forall xA \downarrow \Delta \quad t \text{ is a term} \]

Identity

\[ \text{id}_L \Gamma \downarrow n \vdash n, \Delta \quad \text{id}_R \Gamma, p \vdash p \downarrow \Delta \]

Focus → Inversion

\[ R_L \Gamma \vdash P \vdash \Delta \quad R_R \Gamma \vdash N \vdash \Delta \]
Example

\[
\begin{align*}
\vdash a, b, \bar{a} & \quad \text{id} & \vdash a, b, b & \quad \text{id} \\
\vdash a, b, \bar{a} \land \bar{b} & \quad \land_R \\
\vdash a, b \lor \bar{a} \land \bar{b} & \quad \lor_{R1} \\
\vdash \exists x a, b \lor c, \bar{a} \land \bar{b} & \quad \exists_R \\
\vdash \exists x a, \exists y (b \lor c), \bar{a} \land \bar{b} & \quad \forall_R \\
\vdash \exists x a, \exists y (b \lor c), \forall z (\bar{a} \land \bar{B}) & \quad \forall_R
\end{align*}
\]
Motivation

Object

Reasoning
Motivation

Object

First order logic

Reasoning
Avantages of the sequent framework
(1) simple; (2) strong properties (*analyticity*); (3) easy implementation.
Motivation

Avantages of the sequent framework
(1) simple; (2) strong properties (*analyticity*); (3) easy implementation.

Add mathematical theories to first order logic
Motivation

Add mathematical theories to first order logic and reason about them using all the machinery already built for the sequent framework.

Avantages of the sequent framework
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Motivation

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Control over choices in focused proof is known to improve proof search, but also allows for a compact synthetic representation.
Control over choices in focused proof is known to improve proof search, but also allows for a compact *synthetic* representation.

**Synthetic rules** result from looking only at border sequents:

\[ \Gamma \uparrow \cdot \vdash \cdot \uparrow \Delta \]
On the example

\[
\begin{align*}
\vdash a & \rightarrow [u/x]a \\ a & \rightarrow \exists x (b \land c) \\
\vdash a & \rightarrow \exists x, \exists y (b \land c) \\
\vdash \neg a & \rightarrow \exists x, \exists y (b \land c) \\
\vdash \forall z (\neg a \land \neg b) & \rightarrow \exists x, \exists y (b \land c) \\
\vdash \forall z (\neg a \land \neg b) & \rightarrow \exists x, \exists y (b \land c), \forall z (\neg a \land \neg b)
\end{align*}
\]
Hierarchical structure of positive and negative formulas.
(Inspired by [Ciabattoni et al.])

\( \mathcal{N}_0 \) and \( \mathcal{P}_0 \) consist of all atoms

\[
\begin{align*}
\mathcal{N}_{n+1} &::= \mathcal{P}_n \quad \mathcal{N}_{n+1} \vdash \mathcal{N}_{n+1} \quad \mathcal{N}_{n+1} \vdash \mathcal{N}_{n+1} \\
&\quad \vdash \quad \overline{\mathcal{I}} \\
&\quad \forall x \mathcal{N}_{n+1} \\
-\mathcal{P}_{n+1} &::= \mathcal{P}_{n+1} \quad \mathcal{P}_{n+1} \supset \mathcal{N}_{n+1} \\
\mathcal{P}_{n+1} &::= \mathcal{N}_n \quad \mathcal{P}_{n+1} \vdash \mathcal{P}_{n+1} \\
&\quad \vdash \quad \overline{\mathcal{I}} \\
&\quad \exists x \mathcal{P}_{n+1}
\end{align*}
\]
Polarity-based hierarchy

Hierarchy of negative and positive formulas.
(Inspired by [Ciabattoni et al.])

\( N_0 \) and \( P_0 \) consist of all atoms

\[
N_{n+1} ::= \begin{array}{c}
P_n \\
\land \ N_{n+1} \\
\land \ N_{n+1} \\
- P_{n+1} \\
\lor \ N_{n+1} \\
\lor \ N_{n+1} \\
\top \\
\bot \\
\forall x N_{n+1}
\end{array} \]

\[
P_{n+1} ::= \begin{array}{c}
N_n \\
\lor P_{n+1} \\
\lor P_{n+1} \\
\top \\
\bot \\
\exists x P_{n+1}
\end{array} \]

Bipolar formulas. Any formula in the class \( N_2 \) is a bipolar formula.
Let $B$ be a polarised negative formula.

A **bipole for $B$** is a synthetic rule obtained as a derivation in LKF.
Bipole

Let $B$ be a polarised negative formula.

A bipole for $B$ is a synthetic rule obtained as a derivation in LKF
1. starting with a left decide on $B$;
Bipole

Let $B$ be a polarised negative formula.

A bipole for $B$ is a synthetic rule obtained as a derivation in LKF

1. starting with a left decide on $B$;
2. no “focused” rule occurs above an “inversion” rule;
Bipole

Let $B$ be a polarised negative formula.

A bipole for $B$ is a synthetic rule obtained as a derivation in LKF

1. starting with a left decide on $B$;
2. no “focused” rule occurs above an “inversion” rule;
3. and only atomic formulas are stored.
Bipole

Let $B$ be a polarised negative formula.

A bipole for $B$ is a synthetic rule obtained as a derivation in LKF

1. starting with a left decide on $B$;
2. no “focused” rule occurs above an “inversion” rule;
3. and only atomic formulas are stored.

\[ \Gamma_1 \uparrow \cdot \vdash \cdot \uparrow \Delta_1 \ldots \Gamma_n \uparrow \cdot \vdash \cdot \uparrow \Delta_n \]

\[ \Gamma \vdash A \downarrow \Delta \quad \Gamma \vdash B \downarrow \Delta \]

\[ \Gamma, B \uparrow \cdot \vdash \cdot \uparrow \Delta \]

**Atomic storage**

Atoms are stored

\[ C, \Gamma \uparrow \Theta \vdash \Omega \uparrow \Delta \]

\[ \Gamma \uparrow \vdash \Theta \uparrow \Omega \uparrow \Delta \]

\[ \Gamma \uparrow \vdash \Omega \uparrow D, \Delta \]

\[ \Gamma \uparrow \vdash \Omega \uparrow D, \uparrow \Delta \]

**Asynchronous phase**

Invertible rules are applied eagerly

\[ \Gamma \uparrow \vdash A, \Omega \uparrow \Delta \quad \Gamma \uparrow \vdash B, \Omega \uparrow \Delta \]

\[ \Gamma \uparrow \vdash A \land^- B, \Omega \uparrow \Delta \]

\[ \Gamma \uparrow \vdash A \land^+ B \downarrow \Delta \]

**Synchronous phase**

Focusing persists

\[ \Gamma \vdash A \downarrow \Delta \quad \Gamma \vdash B \downarrow \Delta \]

\[ \Gamma \vdash A \land^+ B \downarrow \Delta \]
Bipole

Let $B$ be a polarised negative formula.

A bipole for $B$ is a synthetic rule obtained as a derivation in LKF

1. starting with a left decide on $B$;
2. no “focused” rule occurs above an “inversion” rule;
3. and only atomic formulas are stored.

\[
\Gamma_1 \downarrow \vdash \cdots \uparrow \Delta_1 \quad \ldots \quad \Gamma_n \downarrow \vdash \cdots \uparrow \Delta_n
\]

\[\Gamma, B \downarrow \vdash \Delta \quad D_l\]

Corresponding synthetic rule in LK

\[
\frac{\Gamma_1 \vdash \Delta_1 \quad \ldots \quad \Gamma_n \vdash \Delta_n}{\Gamma \vdash \Delta}
\]
Let $B$ be a polarised negative formula.

**Theorem:**
- If $B$ is bipolar, then any synthetic rule for $B$ is a *bipole*.
- If every synthetic rule for $B$ is a bipole, then $B$ is *bipolar*.

This delineates precisely the scope of the relationship between axioms and rules!

▷ And provides the answer to *Which ones?*
Rules from axioms

How?
Rules from axioms

Unpolarised Axiom

\[ \forall x((P_1(x) \supset P_2(x)) \land Q(x)) \supset \exists y R(x, y) \]
Rules from axioms

Unpolarised Axiom

- Polarising

✓ Bipole

Derivation in LKF

Synthesizing
Rules from axioms

Unpolarised Axiom

Polarised Axiom

Polarised Axiom

\[ \forall x((P_1(x) \supset P_2(x)) \land Q(x)) \supset \exists y R(x, y) \]

\[ \forall x((P_1(x) \supset P_2(x)) \land Q(x)) \supset \exists y R(x, y) \]

\[ \forall x((P_1(x) \supset P_2(x)) \land Q(x)) \supset \exists y R(x, y) \]

Is it bipolar?

✓ ×

Derivation in LKF

Bipole rule for LK

Synthesizing
Rules from axioms

Unpolarised Axiom

- $\forall x (((P_1(x) \supset P_2(x)) \land Q(x)) \supset \exists y R(x, y))$
- $\forall x (((P_1(x) \supset P_2(x)) \landbar Q(x)) \supset \exists y R(x, y))$
- $\forall x (((P_1(x) \supset P_2(x)) \landbar Q(x)) \supset \exists y R(x, y))$
- $\forall x (((P_1(x) \supset P_2(x)) \landbar Q(x)) \supset \exists y R(x, y))$

Polarised Axiom

Polarising

Is it bipolar?

- $\forall x (((P_1(x) \supset P_2(x)) \land Q(x)) \supset \exists y R(x, y))$
- $\forall x (((P_1(x) \supset P_2(x)) \landbar Q(x)) \supset \exists y R(x, y))$
- $\forall x (((P_1(x) \supset P_2(x)) \landbar Q(x)) \supset \exists y R(x, y))$
- $\forall x (((P_1(x) \supset P_2(x)) \landbar Q(x)) \supset \exists y R(x, y))$
Rules from axioms

Unpolarised Axiom

Polarising

∀x((P_1(x) ⊃ P_2(x)) ∧ Q(x)) ⊃ ∃yR(x, y)

Is it bipolar?

✓

Polarised Axiom

Bipole rule for LK

Synthesizing

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Rules from axioms

Unpolarised Axiom

Polarised Axiom

Polarising

{Is it bipolar?}

Derivation in LKF

Γ, \( P_1(t) \uparrow \vdash \uparrow P_2(t), \Delta \)

\[ \frac{\Gamma, P_1(t) \uparrow \vdash \uparrow P_2(t), \Delta}{\Gamma \vdash (P_1(t) \supset P_2(t)) \uparrow Q(t) \downarrow \Delta} \]

\[ \frac{\Gamma \vdash (P_1(t) \supset P_2(t)) \uparrow Q(t) \downarrow \Delta}{\Gamma \vdash (P_1(t) \supset P_2(t)) \uparrow \vdash \uparrow \Delta} \]

\[ \frac{\Gamma \vdash (P_1(t) \supset P_2(t)) \uparrow \vdash \uparrow \Delta}{\Gamma \vdash \forall x((P_1(x) \supset P_2(x)) \uparrow Q(x)) \supset \exists y R(x, y) \vdash \Delta} \]
Rules from axioms

Unpolarised Axiom

Polarised Axiom

Polarised Axiom

Is it bipolar?

Synthesising

Γ, P_1(t) ⊩ ⬆ ⬆ P_2(t), Δ
Γ ⊩ P_1(t) ⊩ P_2(t) ⊬ Δ
Γ ⊩ P_1(t) ⊩ P_2(t) ⬆ Δ
Γ ⊩ P_1(t) ⊩ P_2(t) ⬇ Δ
Γ ⊩ (P_1(t) ⪰ P_2(t)) ⌑ Q(t) ⬇ Δ
Γ ⊩ ((P_1(t) ⪰ P_2(t)) ⌑ Q(t)) ⪰ ⬇ Δ
Γ ⊩ ∀x(((P_1(x) ⪰ P_2(x)) ⌑ Q(x)) ⪰ ⬇ Δ

Γ = Γ', Q(t) ⬇ Δ

Γ ⊩ R(t, z) ⊩ ⬆ ⬆ Δ
Γ ⊩ Q(t) ⬇ Δ
Γ ⊩ Q(t) ⬇ Δ
Γ ⊩ R(t, z) ⊩ ⬆ Δ
Γ ⊩ ∀yR(t, y) ⊩ ⬆ Δ
Γ = Γ', Q(t) ⬇ Δ

Γ ⊩ R(t, z) ⊩ ⬆ ⬆ Δ

Γ ⊩ Q(t) ⬇ Δ
Γ ⊩ Q(t) ⬇ Δ
Γ ⊩ R(t, z) ⊩ ⬆ Δ
Γ ⊩ ∀yR(t, y) ⊩ Δ

Γ ⊩ R(t, z) ⊩ ⬆ ⬆ Δ

Γ ⊩ Q(t) ⬇ Δ
Γ ⊩ Q(t) ⬇ Δ
Rules from axioms

Polarised Axiom

Unpolarised Axiom

Derivation in LKF

Polarised Axiom

Bipole rule for LK

Γ, P_1(t) ⊬ P_2(t), Δ
Γ, R(t, z) ⊬ Δ
Γ = Γ', Q(t) ⊬ Δ

Γ ⊬ P_1(t) ⊢ P_2(t), Δ
Γ ⊬ Q(t) ⊬ Δ

Γ ⊬ (P_1(t) ⊢ P_2(t)) ⊢ Q(t) ⊬ Δ

Γ ⊬ ∀x(((P_1(x) ⊢ P_2(x)) ⊢ Q(x)) ⊢ ∃y R(x, y)) ⊬ Δ
Rules from axioms

Is it bipolar?

Synthesizing

Polarised Axiom

Derivation in LKF

Bipole rule for LK

\[ \Gamma, P_1(t) \vdash P_2(t), \Delta \quad \Gamma, R(t, z) \vdash \Delta \]

\[ \Gamma = \Gamma', Q(t) \vdash \Delta \]

Unpolarised Axiom

Polarising

Polarised Axiom

Derivation in LKF
Rules from axioms

Unpolarised Axiom

- \( \forall x (((P_1(x) \supset P_2(x)) \land Q(x)) \supset \exists y R(x, y)) \)
- \( \forall x (((P_1(x) \supset P_2(x)) \land Q(x)) \supset \exists y R(x, y)) \)
- \( \forall x (((P_1(x) \supset P_2(x)) \land Q(x)) \supset \exists y R(x, y)) \)
- \( \forall x (((P_1(x) \supset P_2(x)) \land Q(x)) \supset \exists y R(x, y)) \)

Polarised Axiom

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Is it bipolar?
Rules from axioms

Unpolarised Axiom

Polarised Axiom

Is it bipolar?

Polarising

∀x(((P₁(x) ⊃ P₂(x)) ∧ Q(x)) ⊃ ∃yR(x, y))

Derivation in LKF
Bipole rule for LK
Synthesizing
Rules from axioms

Unpolarised Axiom

Polarised Axiom

Derivation in LK

\[ \Gamma, P_1(t) \vdash \perp \vdash P_2(t), \Delta \]
\[ \Gamma, P_1(t) \vdash \perp \vdash P_2(t) \vdash \Delta \]
\[ \Gamma \vdash P_1(t) \vdash P_2(t) \vdash \Delta \]
\[ \Gamma \vdash P_1(t) \vdash P_2(t) \vdash \Delta \]
\[ \Gamma \vdash Q(t), \Delta \]
\[ \Gamma \vdash Q(t) \vdash \Delta \]
\[ \Gamma \vdash (P_1(t) \supset P_2(t)) \supset Q(t) \vdash \Delta \]
\[ \Gamma \vdash (P_1(t) \supset P_2(t)) \supset Q(t) \vdash \Delta \]
\[ \Gamma \vdash \forall x((P_1(x) \supset P_2(x)) \supset Q(x)) \vdash \exists y R(x, y) \vdash \Delta \]

Is it bipolar?

\[ \text{Bipole rule for LK} \]

Synthesizing
Rules from axioms

Is it bipolar?

Polarised Axiom → Derivation in LKF → Bipole rule for LK

Γ, P₁(t) ⊬⁺ · ⊬⁺ P₂(t), Δ

Γ, P₁(t) ⊬⁺ · ⊬⁺ P₂(t) ⊬⁺ Δ

Γ ⊬⁺ P₁(t) ⊬⁺ P₂(t) ⊬⁺ Δ

Γ ⊬⁺ · ⊬⁺ P₁(t) ⊬⁺ P₂(t) ⊬⁺ Δ

Γ, (P₁(t) ⊬⁺ P₂(t)) ⊬⁺ Q(t) ⊬⁺ Δ

Γ ⊬⁺ (P₁(t) ⊬⁺ P₂(t)) ⊬⁺ Q(t) ⊬⁺ Δ

Γ ⊬⁺ ∀x((P₁(x) ⊬⁺ P₂(x)) ⊬⁺ Q(x)) ⊬⁺ ∀yR(x, y) ⊬⁺ Δ
Rules from axioms

Is it bipolar?

Unpolarised Axiom

Polarised Axiom

Synthesizing

Derivation in LKF

Bipole rule for LK

\[
\Gamma, P_1(t) \vdash P_2(t), \Delta \quad \Gamma \vdash Q(t), \Delta \quad \Gamma, R(t, z) \vdash \Delta
\]

\[
\therefore \Gamma \vdash \Delta
\]
Rules from axioms

Unpolarised Axiom

Polarised Axiom

\[ \forall x((P_1(x) \supset P_2(x)) \land Q(x)) \supset \exists y R(x, y) \]

\[ \forall x((P_1(x) \supset P_2(x)) \land Q(x)) \supset \exists y R(x, y) \]

\[ \forall x((P_1(x) \supset P_2(x)) \land Q(x)) \supset \exists y R(x, y) \]

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Is it bipolar?
Rules from axioms

Is it bipolar?

Unpolarised Axiom

Polarised Axiom

\[ \forall x ((P_1(x) \supset P_2(x)) \land Q(x)) \supset \exists y R(x, y) \]
Rules from axioms

- Polarised Axiom
  - Polarising

Unpolarised Axiom

Polarised Axiom

Is it bipolar? Yes

Derivation in LKF

Bipole rule for LK

Synthesizing
Rules from axioms

\[ \forall x ((P_1(x) \supset P_2(x)) \land Q(x)) \supset \exists y R(x, y) \]

Unpolarised Axiom

Polarised Axiom

Is it bipolar?

Synthesizing

Polarising

Derivation in LKF

Bipole rule for LK

Γ, P_1(t) \vdash P_2(t), \Delta \quad Γ, R(t, z) \vdash \Delta

Γ = \Gamma', Q(t) \vdash \Delta

Γ, P_1(t) \vdash P_2(t), \Delta \quad Γ \vdash Q(t), \Delta \quad Γ, R(t, z) \vdash \Delta

Γ \vdash \Delta
Rules from axioms

- **Polarised Axiom**
  - Derivation in LKF
  - Bipole rule for LK
- **Unpolarised Axiom**
  - Polarising
  - Is it bipolar?
  - Synthesizing
Cut admissibility

Let $\mathcal{T}$ be a set of bipolar formulas. 

$LK\mathcal{T}$ denotes the extension of LK with the synthetic inference rules corresponding to a bipole for each $B \in \mathcal{T}$.

**Theorem:** The cut rule is admissible for the proof system $LK\mathcal{T}$. 

Note: the proof is simple! It is a direct consequence of cut admissibility in $LKF$. 

This is why bipoles live in harmony within the sequent framework.
Cut admissibility

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$LK\mathcal{T}$ denotes the extension of LK with the synthetic inference rules corresponding to a bipole for each $B \in \mathcal{T}$.

**Theorem:** The cut rule is admissible for the proof system $LK\mathcal{T}$. Note: the proof is simple!

It is a direct consequence of cut admissibility in LKF.

\[
\frac{\Gamma \uparrow \cdot \vdash B \uparrow \Delta \quad \Gamma \uparrow B \vdash \cdot \uparrow \Delta}{\Gamma \uparrow \cdot \vdash \cdot \uparrow \Delta \quad \text{Cut}}
\]

This is why bipoles live in harmony within the sequent framework.
Let $\mathcal{T}$ be a set of bipolar formulas. $\text{LK}\mathcal{T}$ denotes the extension of LK with the synthetic inference rules corresponding to a bipole for each $B \in \mathcal{T}$.

**Theorem:** The cut rule is admissible for the proof system $\text{LK}\mathcal{T}$. Note: the proof is *simple*!

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$$
\Gamma \uparrow \vdash \vdash B \uparrow \Delta \quad \Gamma \uparrow B \vdash \cdot \uparrow \Delta \\
\Gamma \uparrow \cdot \vdash \cdot \uparrow \Delta \quad \text{Cut}
$$

This is why bipoles live in harmony within the sequent framework.
Questions?