From Axioms to Rules:

The Factory of Modal Proof Systems



5. Lecture

Synthetic rules and bipoles



Sonia Marin and Lutz Straßburger

LKF

Inversion phase

Negative formula on the right

$$\bar{\vee}_R\ \frac{\Gamma_1 \Uparrow \Gamma_2 \vdash \Delta_1, A, B \Uparrow \Delta_2}{\Gamma_1 \Uparrow \Gamma_2 \vdash \Delta_1, A \mathbin{\bar{\vee}} B \Uparrow \Delta_2} \qquad \bar{\wedge}_R\ \frac{\Gamma_1 \Uparrow \Gamma_2 \vdash \Delta_1, A \Uparrow \Delta_2 \quad \Gamma_1 \Uparrow \Gamma_2 \vdash \Delta_1, B \Uparrow \Delta_2}{\Gamma_1 \Uparrow \Gamma_2 \vdash \Delta_1, A \mathbin{\bar{\wedge}} B \Uparrow \Delta_2}$$

$$\bar{\top}_R\ \frac{}{\Gamma_1 \Uparrow \Gamma_2 \vdash \Delta_1, \bar{\top} \Uparrow \Delta_2} \qquad \forall_R\ \frac{\Gamma_1 \Uparrow \Gamma_2 \vdash \Delta_1, [y/x]A \Uparrow \Delta_2}{\Gamma_1 \Uparrow \Gamma_2 \vdash \Delta_1, \forall x A \Uparrow \Delta_2}\ y \text{ is fresh}$$

$$\supset_R\ \frac{\Gamma_1 \Uparrow \Gamma_2, A \vdash \Delta_1, B \Uparrow \Delta_2}{\Gamma_1 \Uparrow \Gamma_2 \vdash \Delta_1, A \supset B \Uparrow \Delta_2} \qquad \neg_R\ \frac{\Gamma_1 \Uparrow \Gamma_2, A \vdash \Delta_1 \Uparrow \Delta_2}{\Gamma_1 \Uparrow \Gamma_2 \vdash \Delta_1, \neg A \Uparrow \Delta_2}$$

Positive formula on the left

$$\overset{+}{\wedge}_L\ \frac{\Gamma_1 \Uparrow \Gamma_2, A, B \vdash \Delta_1 \Uparrow \Delta_2}{\Gamma_1 \Uparrow \Gamma_2, A \mathbin{\overset{+}{\wedge}} B \vdash \Delta_1 \Uparrow \Delta_2} \qquad \overset{+}{\vee}_L\ \frac{\Gamma_1 \Uparrow \Gamma_2, A \vdash \Delta_1 \Uparrow \Delta_2 \quad \Gamma_1 \Uparrow \Gamma_2, B \vdash \Delta_1 \Uparrow \Delta_2}{\Gamma_1 \Uparrow \Gamma_2, A \mathbin{\overset{+}{\vee}} B \vdash \Delta_1 \Uparrow \Delta_2}$$

$$\overset{+}{\bot}_L\ \frac{}{\Gamma_1 \Uparrow \Gamma_2, \overset{+}{\bot} \vdash \Delta_1 \Uparrow \Delta_2} \qquad \exists_L\ \frac{\Gamma_1 \Uparrow \Gamma_2, [y/x]A \vdash \Delta_1 \Uparrow \Delta_2}{\Gamma_1 \Uparrow \Gamma_2, \exists x A \vdash \Delta_1 \Uparrow \Delta_2}\ y \text{ is fresh}$$

. . . .

Storage

$$S_L\ \frac{\Gamma_1, A \Uparrow \Gamma_2 \vdash \Delta_1 \Uparrow \Delta_2}{\Gamma_1 \Uparrow \Gamma_2, A \vdash \Delta_1 \Uparrow \Delta_2}\ A = N \text{ or } at \qquad S_R\ \frac{\Gamma_1 \Uparrow \Gamma_2 \vdash \Delta_1 \Uparrow A, \Delta_2}{\Gamma_1 \Uparrow \Gamma_2 \vdash A, \Delta_1 \Uparrow \Delta_2}\ A = P \text{ or } at$$

Inversion $\rightarrow$ Focus

$$\mathsf{D_L}\ \frac{\Gamma, N \Downarrow N \vdash \Delta}{\Gamma, N \Uparrow \vdash \Uparrow \Delta} \qquad \mathsf{D_R}\ \frac{\Gamma \vdash P \Downarrow P, \Delta}{\Gamma \Uparrow \vdash \Uparrow P, \Delta}$$

. . . .

Focus phase

Positive formula on the right

$$\overset{+}{\vee}_{R1}\ \frac{\Gamma \vdash A \Downarrow \Delta}{\Gamma \vdash A \overset{+}{\vee} B \Downarrow \Delta} \quad \overset{+}{\vee}_{R2}\ \frac{\Gamma \vdash B \Downarrow \Delta}{\Gamma \vdash A \overset{+}{\vee} B \Downarrow \Delta} \quad \overset{+}{\wedge}_{R}\ \frac{\Gamma \vdash A \Downarrow \Delta \quad \Gamma \vdash B \Downarrow \Delta}{\Gamma \vdash A \overset{+}{\wedge} B \Downarrow \Delta}$$

$$\overset{+}{\top}_{R}\ \frac{}{\Gamma \vdash \overset{+}{\top} \Downarrow \Delta} \quad \exists_R\ \frac{\Gamma \vdash [t/x]A \Downarrow \Delta}{\Gamma \vdash \exists x A \Downarrow \Delta}\ t \text{ is a term}$$

Negative formula on the left

$$\overset{-}{\wedge}_{L1}\ \frac{\Gamma \Downarrow A \vdash \Delta}{\Gamma \Downarrow A \overset{-}{\wedge} B \vdash \Delta} \quad \overset{-}{\wedge}_{L2}\ \frac{\Gamma \Downarrow B \vdash \Delta}{\Gamma \Downarrow A \overset{-}{\wedge} B \vdash \Delta} \quad \overset{-}{\vee}_{L}\ \frac{\Gamma \Downarrow A \vdash \Delta \quad \Gamma \Downarrow B \vdash \Delta}{\Gamma \Downarrow A \overset{-}{\vee} B \vdash \Delta} \quad \overset{-}{\bot}_{L}\ \frac{}{\Gamma \Downarrow \overset{-}{\bot} \vdash \Delta}$$

$$\supset_L\ \frac{\Gamma \vdash A \Downarrow \Delta \quad \Gamma \Downarrow B \vdash \Delta}{\Gamma \Downarrow A \supset B \vdash \Delta} \quad \neg_L\ \frac{\Gamma \vdash A \Downarrow \Delta}{\Gamma \Downarrow \neg A \vdash \Delta} \quad \forall_L\ \frac{\Gamma \vdash [t/x]A \Downarrow \Delta}{\Gamma \vdash \forall x A \Downarrow \Delta}\ t \text{ is a term}$$

. . . .

Identity

$$\mathsf{id_L}\ \frac{}{\Gamma \Downarrow n \vdash n, \Delta} \quad \mathsf{id_R}\ \frac{}{\Gamma, p \vdash p \Downarrow \Delta}$$

Focus $\rightarrow$ Inversion

$$\mathsf{R_L}\ \frac{\Gamma \Uparrow P \vdash \Uparrow \Delta}{\Gamma \Downarrow P \vdash \Delta} \quad \mathsf{R_R}\ \frac{\Gamma \vdash N \Uparrow \Delta}{\Gamma \vdash N \Downarrow \Delta}$$

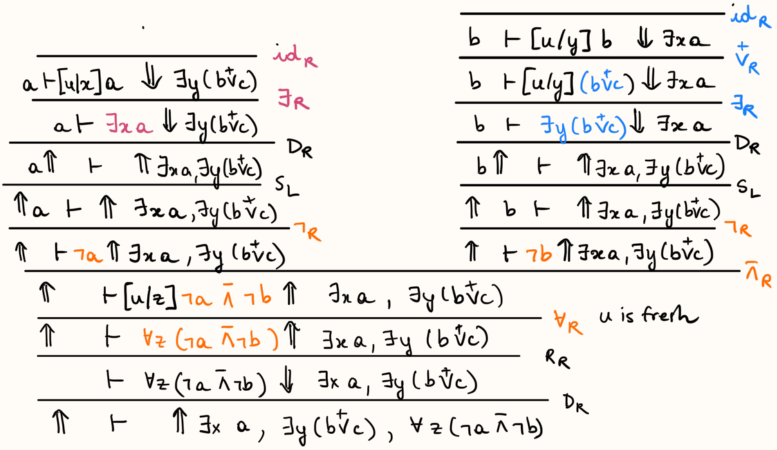
Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash a, b, \bar{a}}\;\text{id} \qquad \dfrac{}{\vdash a, b, \bar{b}}\;\text{id}}{\vdash a, b, \bar{a} \wedge \bar{b}}\;\wedge_R}{\vdash a, b \vee c, \bar{a} \wedge \bar{b}}\;\vee_{R1}}{\vdash \exists x a, b \vee c, \bar{a} \wedge \bar{b}}\;\exists_R}{\vdash \exists x a, \exists y(b \vee c), \bar{a} \wedge \bar{b}}\;\exists_R}{\vdash \exists x a, \exists y(b \vee c), \forall z(\bar{a} \wedge \bar{B})}\;\forall_R
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash a, \exists y(b \vee c), \bar{a}}\;\text{id}}{\vdash \exists x a, \exists y(b \vee c), \bar{a}}\;\exists_R \qquad \dfrac{\dfrac{\dfrac{}{\vdash \exists x a, b, \bar{b}}\;\text{id}}{\vdash \exists x a, b \vee c, \bar{b}}\;\vee_{R1}}{\vdash \exists x a, \exists y(b \vee c), \bar{b}}\;\exists_R}{\vdash \exists x a, \exists y(b \vee c), \bar{a} \wedge \bar{b}}\;\vee_R}{\vdash \exists x a, \exists y(b \vee c), \forall z(\bar{a} \wedge \bar{b})}\;\forall_R
$$

Unfocused → Focused

Example



a ⊢ [u/x]a ⇓ ∃y(b ∨̇ c)
id_R
∃_R
a ⊢ ∃x a ⇓ ∃y(b ∨̇ c)
D_R
a ⇑ ⊢ ⇑ ∃x a, ∃y(b ∨̇ c)
S_L
⇑ a ⊢ ⇑ ∃x a, ∃y(b ∨̇ c)
¬_R
⇑ ⊢ ¬a ⇑ ∃x a, ∃y (b ∨̇ c)
b ⊢ [u/y] b ⇓ ∃x a
id_R
∨̇_R
b ⊢ [u/y] (b ∨̇ c) ⇓ ∃x a
∃_R
b ⊢ ∃y (b ∨̇ c) ⇓ ∃x a
D_R
b ⇑ ⊢ ⇑ ∃x a, ∃y (b ∨̇ c)
S_L
⇑ b ⊢ ⇑ ∃x a, ∃y (b ∨̇ c)
¬_R
⇑ ⊢ ¬b ⇑ ∃x a, ∃y (b ∨̇ c)
∧̄_R
⇑ ⊢ [u/z] ¬a ∧̄ ¬b ⇑ ∃x a, ∃y (b ∨̇ c)
∀_R
u is fresh
⇑ ⊢ ∀z (¬a ∧̄ ¬b) ⇑ ∃x a, ∃y (b ∨̇ c)
R_R
⊢ ∀z (¬a ∧̄ ¬b) ⇓ ∃x a, ∃y (b ∨̇ c)
D_R
⇑ ⊢ ⇑ ∃x a, ∃y (b ∨̇ c), ∀z (¬a ∧̄ ¬b)

Motivation

Object

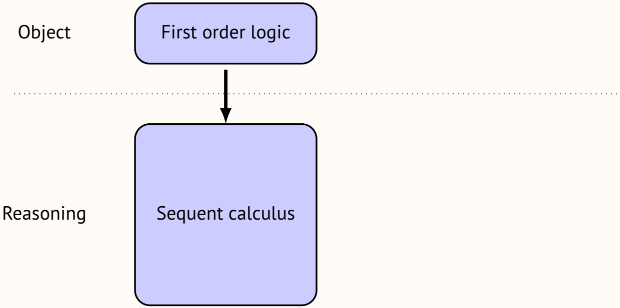
Reasoning

Motivation

Object First order logic

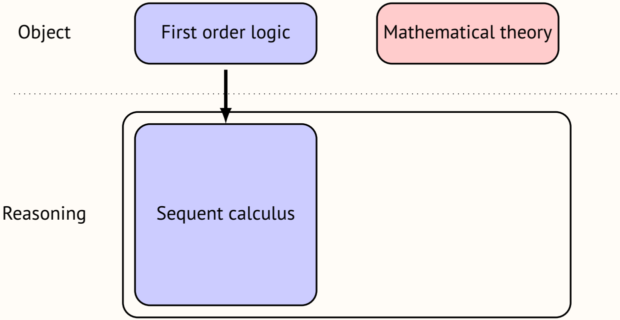
Reasoning

Motivation



Avantages of the sequent framework
(1) simple; (2) strong properties (*analyticity*); (3) easy implementation.

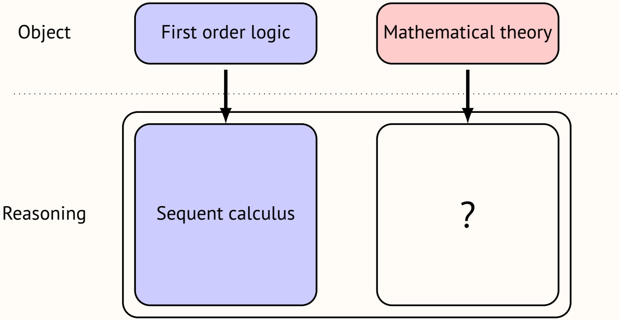
Motivation



Avantages of the sequent framework
(1) simple; (2) strong properties (*analyticity*); (3) easy implementation.

Add mathematical theories to first order logic

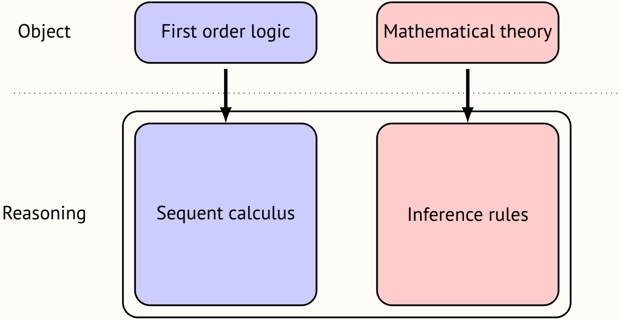
Motivation



Avantages of the sequent framework
(1) simple; (2) strong properties (*analyticity*); (3) easy implementation.

Add mathematical theories to first order logic and reason about them using all the machinery already built for the sequent framework.

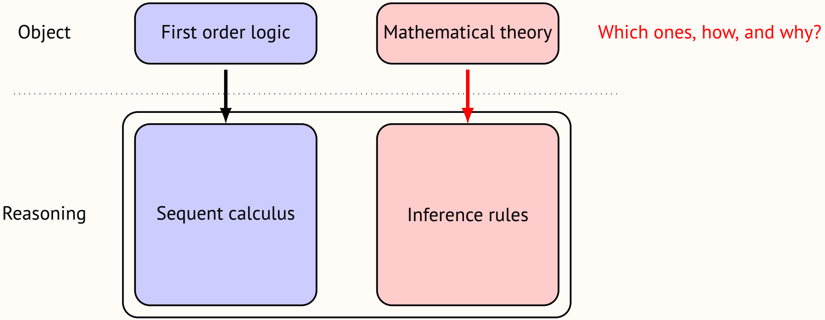
Motivation



Avantages of the sequent framework
(1) simple; (2) strong properties (*analyticity*); (3) easy implementation.

Add mathematical theories to first order logic and reason about them using all the machinery already built for the sequent framework.

Motivation



Avantages of the sequent framework

(1) simple; (2) strong properties (*analyticity*); (3) easy implementation.

Add mathematical theories to first order logic and reason about them using all the machinery already built for the sequent framework.

Big steps reasoning

Control over choices in focused proof is known to improve proof search, but also allows for a compact synthetic representation.

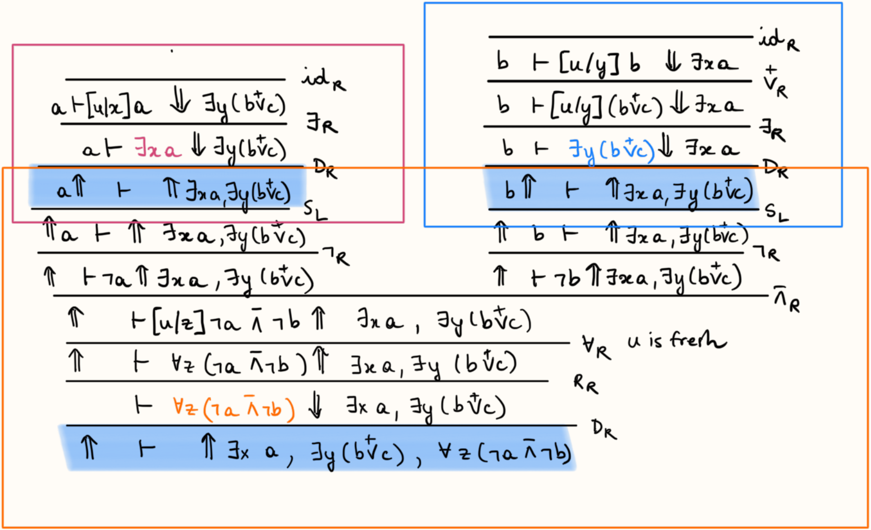
Big steps reasoning

Control over choices in focused proof is known to improve proof search, but also allows for a compact synthetic representation.

Synthetic rules result from looking only at border sequents:

$$\Gamma \Uparrow \cdot \vdash \cdot \Uparrow \Delta$$

On the example



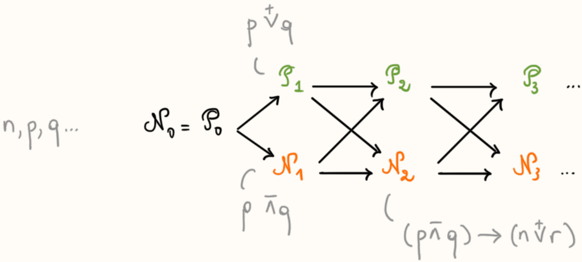
Polarity-based hierarchy

Hierarchy of negative and positive formulas.
(Inspired by [Ciabattoni et al.])

$\mathcal{N}_0$ and $\mathcal{P}_0$ consist of all atoms

$$\mathcal{N}_{n+1} \ ::= \ \mathcal{P}_n \mid \mathcal{N}_{n+1} \mathbin{\bar{\wedge}} \mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} \mathbin{\bar{\vee}} \mathcal{N}_{n+1} \mid \bar{\top} \mid \bar{\bot} \mid \forall x \mathcal{N}_{n+1} \mid \neg \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \supset \mathcal{N}_{n+1} \mid$$

$$\mathcal{P}_{n+1} \ ::= \ \mathcal{N}_n \mid \mathcal{P}_{n+1} \mathbin{\overset{+}{\wedge}} \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \mathbin{\overset{+}{\vee}} \mathcal{P}_{n+1} \mid \overset{+}{\top} \mid \overset{+}{\bot} \mid \exists x \mathcal{P}_{n+1}$$



Polarity-based hierarchy

Hierarchy of negative and positive formulas.
(Inspired by [Ciabattoni et al.])

$\mathcal{N}_0$ and $\mathcal{P}_0$ consist of all atoms

$$\mathcal{N}_{n+1} ::= \mathcal{P}_n \mid \mathcal{N}_{n+1} \bar{\wedge} \mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} \bar{\vee} \mathcal{N}_{n+1} \mid \bar{\top} \mid \bar{\bot} \mid \forall x \mathcal{N}_{n+1}$$
$$\neg \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \supset \mathcal{N}_{n+1} \mid$$

$$\mathcal{P}_{n+1} ::= \mathcal{N}_n \mid \mathcal{P}_{n+1} \overset{+}{\wedge} \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \overset{+}{\vee} \mathcal{P}_{n+1} \mid \overset{+}{\top} \mid \overset{+}{\bot} \mid \exists x \mathcal{P}_{n+1}$$

Bipolar formulas. Any formula in the class $\mathcal{N}_2$ is a bipolar formula.

Bipole

Let B be a polarised negative formula.

A bipole for B is a synthetic rule obtained as a derivation in LKF

Bipole

Let B be a polarised negative formula.

A bipole for B is a synthetic rule obtained as a derivation in LKF

1. starting with a left decide on B;

Bipole

Let B be a polarised negative formula.

A bipole for B is a synthetic rule obtained as a derivation in LKF

1. starting with a left decide on B;
2. no "focused" rule occurs above an "inversion" rule;

Bipole

Let B be a polarised negative formula.

A bipole for B is a synthetic rule obtained as a derivation in LKF

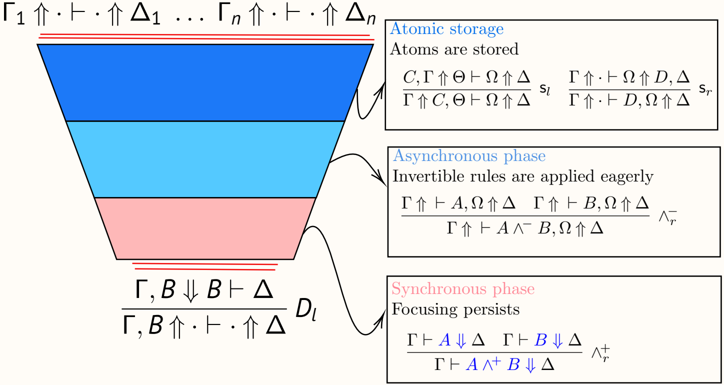
1. starting with a left decide on B;
2. no “focused” rule occurs above an “inversion” rule;
3. and only atomic formulas are stored.

Bipole

Let B be a polarised negative formula.

A bipole for B is a synthetic rule obtained as a derivation in LKF

1. starting with a left decide on B;
2. no "focused" rule occurs above an "inversion" rule;
3. and only atomic formulas are stored.



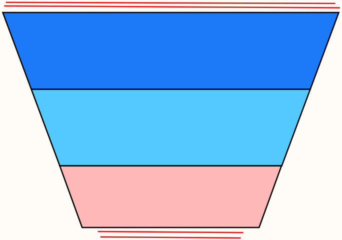
Bipole

Let B be a polarised negative formula.

A bipole for B is a synthetic rule obtained as a derivation in LKF

1. starting with a left decide on B;
2. no “focused” rule occurs above an “inversion” rule;
3. and only atomic formulas are stored.

$$\Gamma_1 \Uparrow \cdot \vdash \cdot \Uparrow \Delta_1 \quad \ldots \quad \Gamma_n \Uparrow \cdot \vdash \cdot \Uparrow \Delta_n$$



$$\frac{\Gamma, B \Downarrow B \vdash \Delta}{\Gamma, B \Uparrow \cdot \vdash \cdot \Uparrow \Delta}\ D_l$$

Corresponding synthetic rule

in LK

$$\frac{\Gamma_1 \vdash \Delta_1 \quad \ldots \quad \Gamma_n \vdash \Delta_n}{\Gamma \vdash \Delta}$$

Bipolar $\longleftrightarrow$ Bipole

Let B be a polarised negative formula.

Theorem:

- If B is bipolar, then any synthetic rule for B is a *bipole*.
- If every synthetic rule for B is a bipole, then B is *bipolar*.

This delineates precisely the scope of the relationship between axioms and rules!

- ▷ And provides the answer to Which ones?

Rules from axioms

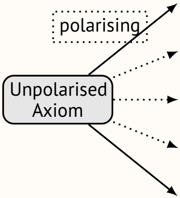
How?

Rules from axioms

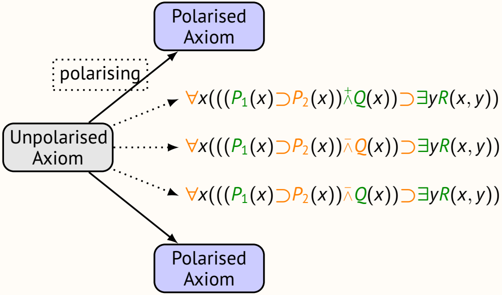
Unpolarised Axiom

$$\forall x(((P_1(x) \supset P_2(x)) \wedge Q(x)) \supset \exists y R(x,y))$$

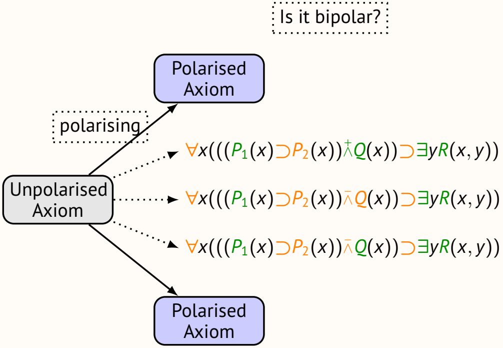
Rules from axioms



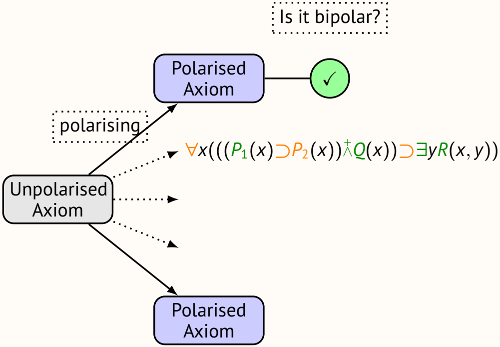
Rules from axioms



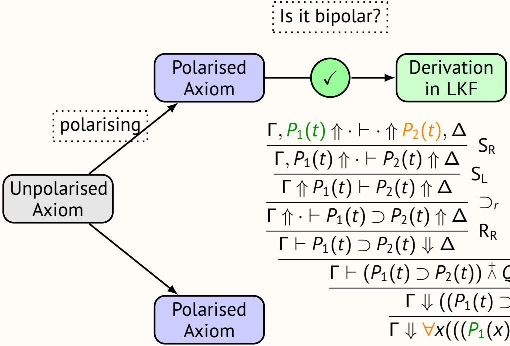
Rules from axioms



Rules from axioms



Rules from axioms

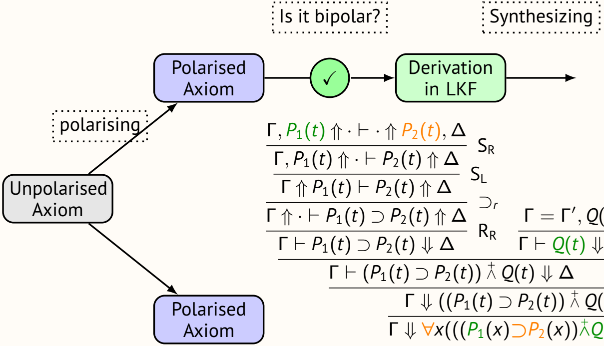


$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, P_1(t) \Uparrow \cdot \vdash \cdot \Uparrow P_2(t), \Delta}{\Gamma, P_1(t) \Uparrow \cdot \vdash P_2(t) \Uparrow \Delta}\,\mathrm{S_R}}{\Gamma \Uparrow P_1(t) \vdash P_2(t) \Uparrow \Delta}\,\mathrm{S_L}}{\Gamma \Uparrow \cdot \vdash P_1(t) \supset P_2(t) \Uparrow \Delta}\,{\supset_r}}{\Gamma \vdash P_1(t) \supset P_2(t) \Downarrow \Delta}\,\mathrm{R_R} \qquad \dfrac{\Gamma = \Gamma', Q(t)}{\Gamma \vdash Q(t) \Downarrow \Delta}\,\mathrm{id_R}}{\Gamma \vdash (P_1(t) \supset P_2(t)) \overset{+}{\wedge} Q(t) \Downarrow \Delta}\,\overset{+}{\wedge}_r$$

$$\dfrac{\dfrac{\dfrac{\Gamma, R(t,z) \Uparrow \cdot \vdash \cdot \Uparrow \Delta}{\Gamma \Uparrow R(t,z) \vdash \cdot \Uparrow \Delta}\,\mathrm{S_L}}{\Gamma \Uparrow \exists y R(t,y) \vdash \cdot \Uparrow \Delta}\,\exists_l}{\Gamma \Downarrow \exists y R(t,y) \vdash \Delta}\,\mathrm{R_L}$$

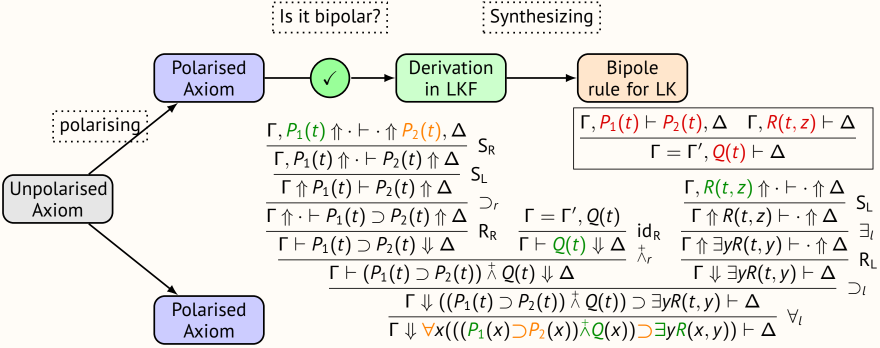
$$\dfrac{\dfrac{\Gamma \vdash (P_1(t) \supset P_2(t)) \overset{+}{\wedge} Q(t) \Downarrow \Delta \qquad \Gamma \Downarrow \exists y R(t,y) \vdash \Delta}{\Gamma \Downarrow ((P_1(t) \supset P_2(t)) \overset{+}{\wedge} Q(t)) \supset \exists y R(t,y) \vdash \Delta}\,{\supset_l}}{\Gamma \Downarrow \forall x(((P_1(x) \supset P_2(x)) \overset{+}{\wedge} Q(x)) \supset \exists y R(x,y)) \vdash \Delta}\,\forall_l$$

Rules from axioms

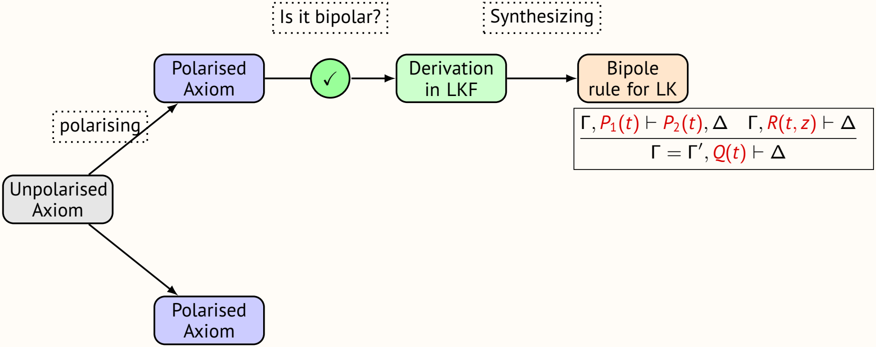


$$
\dfrac{
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, P_1(t) \Uparrow \cdot \vdash \cdot \Uparrow P_2(t), \Delta}{\Gamma, P_1(t) \Uparrow \cdot \vdash P_2(t) \Uparrow \Delta}\,S_R}{\Gamma \Uparrow P_1(t) \vdash P_2(t) \Uparrow \Delta}\,S_L}{\Gamma \Uparrow \cdot \vdash P_1(t) \supset P_2(t) \Uparrow \Delta}\,\supset_r}{\Gamma \vdash P_1(t) \supset P_2(t) \Downarrow \Delta}\,R_R
\qquad
\dfrac{\Gamma = \Gamma', Q(t)}{\Gamma \vdash Q(t) \Downarrow \Delta}\,id_R
}{\Gamma \vdash (P_1(t) \supset P_2(t)) \overset{+}{\wedge} Q(t) \Downarrow \Delta}\,\overset{+}{\wedge}_r
\qquad
\dfrac{\dfrac{\dfrac{\Gamma, R(t,z) \Uparrow \cdot \vdash \cdot \Uparrow \Delta}{\Gamma \Uparrow R(t,z) \vdash \cdot \Uparrow \Delta}\,S_L}{\Gamma \Uparrow \exists y R(t,y) \vdash \cdot \Uparrow \Delta}\,\exists_l}{\Gamma \Downarrow \exists y R(t,y) \vdash \Delta}\,R_L
}{
\dfrac{\Gamma \Downarrow ((P_1(t) \supset P_2(t)) \overset{+}{\wedge} Q(t)) \supset \exists y R(t,y) \vdash \Delta}{\Gamma \Downarrow \forall x(((P_1(x) \supset P_2(x)) \overset{+}{\wedge} Q(x)) \supset \exists y R(x,y)) \vdash \Delta}\,\forall_l
}\,\supset_l
$$

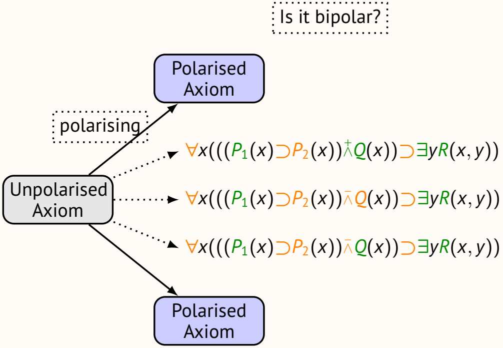
Rules from axioms



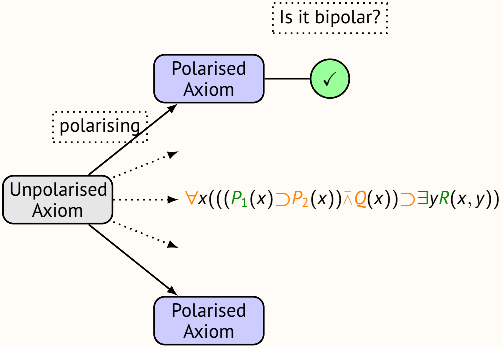
Rules from axioms



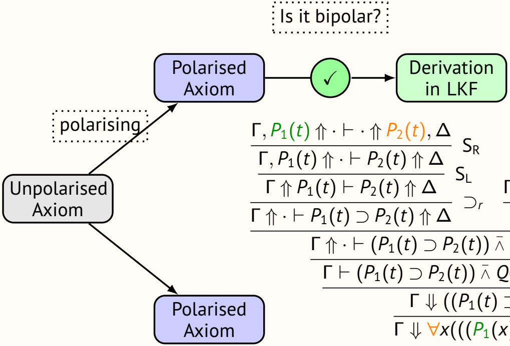
Rules from axioms



Rules from axioms

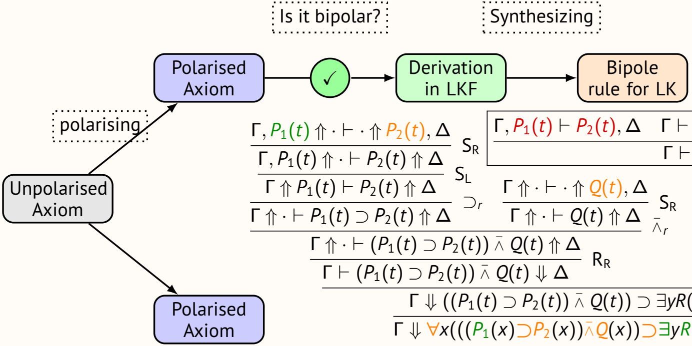


Rules from axioms



$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, P_1(t) \Uparrow \cdot \vdash \cdot \Uparrow P_2(t), \Delta}{\Gamma, P_1(t) \Uparrow \cdot \vdash P_2(t) \Uparrow \Delta}\,S_R}{\Gamma \Uparrow P_1(t) \vdash P_2(t) \Uparrow \Delta}\,S_L}{\Gamma \Uparrow \cdot \vdash P_1(t) \supset P_2(t) \Uparrow \Delta}\supset_r \quad \dfrac{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow Q(t), \Delta}{\Gamma \Uparrow \cdot \vdash Q(t) \Uparrow \Delta}\,S_R}{\Gamma \Uparrow \cdot \vdash (P_1(t) \supset P_2(t)) \bar{\wedge} Q(t) \Uparrow \Delta}\,\bar{\wedge}_r}{\Gamma \vdash (P_1(t) \supset P_2(t)) \bar{\wedge} Q(t) \Downarrow \Delta}\,R_R \quad \dfrac{\dfrac{\dfrac{\Gamma, R(t,z) \Uparrow \cdot \vdash \cdot \Uparrow \Delta}{\Gamma \Uparrow R(t,z) \vdash \cdot \Uparrow \Delta}\,S_L}{\Gamma \Uparrow \exists y R(t,y) \vdash \cdot \Uparrow \Delta}\,\exists_l}{\Gamma \Downarrow \exists y R(t,y) \vdash \Delta}\,R_L}{\dfrac{\Gamma \Downarrow ((P_1(t) \supset P_2(t)) \bar{\wedge} Q(t)) \supset \exists y R(t,y) \vdash \Delta}{\Gamma \Downarrow \forall x(((P_1(x) \supset P_2(x)) \bar{\wedge} Q(x)) \supset \exists y R(x,y)) \vdash \Delta}\,\forall_l}\supset_l$$

Rules from axioms

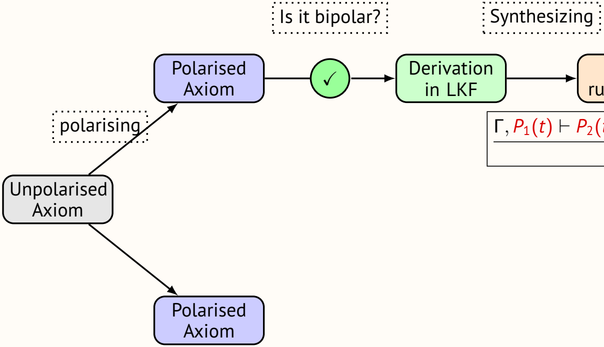


$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, P_1(t) \Uparrow \cdot \vdash \cdot \Uparrow P_2(t), \Delta}{\Gamma, P_1(t) \Uparrow \cdot \vdash P_2(t) \Uparrow \Delta}\, S_R}{\Gamma \Uparrow P_1(t) \vdash P_2(t) \Uparrow \Delta}\, S_L}{\Gamma \Uparrow \cdot \vdash P_1(t) \supset P_2(t) \Uparrow \Delta}\, \supset_r \quad \dfrac{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow Q(t), \Delta}{\Gamma \Uparrow \cdot \vdash Q(t) \Uparrow \Delta}\, S_R}{\dfrac{\Gamma \Uparrow \cdot \vdash (P_1(t) \supset P_2(t)) \bar{\wedge} Q(t) \Uparrow \Delta}{\Gamma \vdash (P_1(t) \supset P_2(t)) \bar{\wedge} Q(t) \Downarrow \Delta}\, R_R}\, \bar{\wedge}_r \quad \dfrac{\dfrac{\dfrac{\Gamma, R(t,z) \Uparrow \cdot \vdash \cdot \Uparrow \Delta}{\Gamma \Uparrow R(t,z) \vdash \cdot \Uparrow \Delta}\, S_L}{\Gamma \Uparrow \exists y R(t,y) \vdash \cdot \Uparrow \Delta}\, \exists_l}{\Gamma \Downarrow \exists y R(t,y) \vdash \Delta}\, R_L$$

$$\dfrac{\dfrac{\cdots}{\Gamma \Downarrow ((P_1(t) \supset P_2(t)) \bar{\wedge} Q(t)) \supset \exists y R(t,y) \vdash \Delta}\, \supset_l}{\Gamma \Downarrow \forall x(((P_1(x) \supset P_2(x)) \bar{\wedge} Q(x)) \supset \exists y R(x,y)) \vdash \Delta}\, \forall_l$$

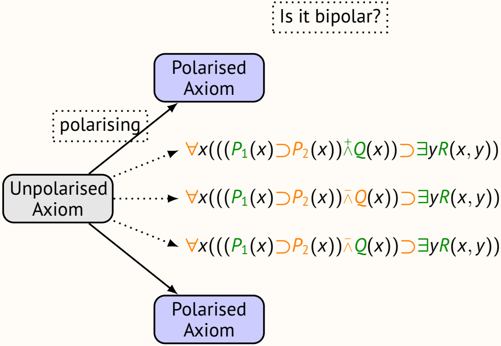
$$\dfrac{\Gamma, P_1(t) \vdash P_2(t), \Delta \quad \Gamma \vdash Q(t), \Delta \quad \Gamma, R(t,z) \vdash \Delta}{\Gamma \vdash \Delta}$$

Rules from axioms

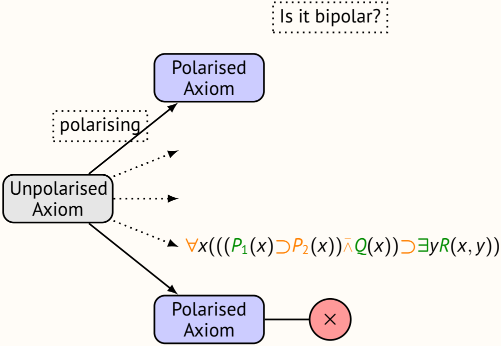


$$\frac{\Gamma, P_1(t) \vdash P_2(t), \Delta \quad \Gamma \vdash Q(t), \Delta \quad \Gamma, R(t,z) \vdash \Delta}{\Gamma \vdash \Delta}$$

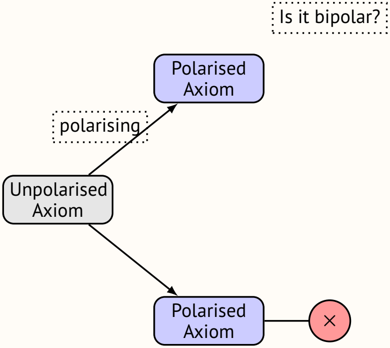
Rules from axioms



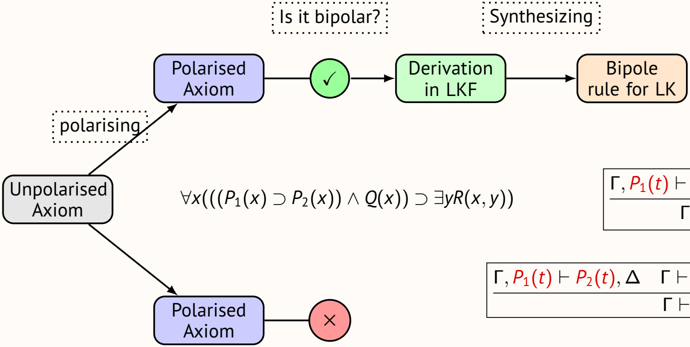
Rules from axioms



Rules from axioms



Rules from axioms

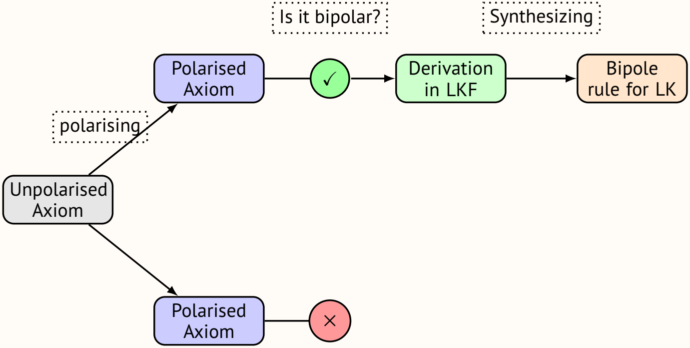


$$\forall x(((P_1(x) \supset P_2(x)) \wedge Q(x)) \supset \exists y R(x,y))$$

$$\frac{\Gamma, P_1(t) \vdash P_2(t), \Delta \quad \Gamma, R(t,z) \vdash \Delta}{\Gamma = \Gamma', Q(t) \vdash \Delta}$$

$$\frac{\Gamma, P_1(t) \vdash P_2(t), \Delta \quad \Gamma \vdash Q(t), \Delta \quad \Gamma, R(t,z) \vdash \Delta}{\Gamma \vdash \Delta}$$

Rules from axioms



Cut admissibility

Let $\mathcal{T}$ be a set of bipolar formulas.

LK$\mathcal{T}$ denotes the extension of LK with the synthetic inference rules corresponding to a bipole for each $B \in \mathcal{T}$.

Theorem: The cut rule is admissible for the proof system LK$\mathcal{T}$.

Cut admissibility

Let $\mathcal{T}$ be a set of bipolar formulas.

LK$\mathcal{T}$ denotes the extension of LK with the synthetic inference rules corresponding to a bipole for each $B \in \mathcal{T}$.

Theorem: The cut rule is admissible for the proof system LK$\mathcal{T}$.

Note: the proof is simple!

It is a direct consequence of cut admissibility in LKF.

$$\frac{\Gamma \Uparrow \cdot \vdash B \Uparrow \Delta \qquad \Gamma \Uparrow B \vdash \cdot \Uparrow \Delta}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow \Delta}\ \mathit{Cut}$$

Cut admissibility

Let $\mathcal{T}$ be a set of bipolar formulas.

LK$\mathcal{T}$ denotes the extension of LK with the synthetic inference rules corresponding to a bipole for each $B \in \mathcal{T}$.

Theorem: The cut rule is admissible for the proof system LK$\mathcal{T}$.

Note: the proof is simple!

It is a direct consequence of cut admissibility in LKF.

$$\frac{\Gamma \Uparrow \cdot \vdash B \Uparrow \Delta \qquad \Gamma \Uparrow B \vdash \cdot \Uparrow \Delta}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow \Delta}\ \mathit{Cut}$$

This is why bipoles live in harmony within the sequent framework.


Questions?