From Axioms to Rules:
The Factory of Modal Proof Systems

4. Lecture

Focusing for first-order logic

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Proving is choosing

Choice of formula:

$$\frac{\vdash a(t)}{\vdash \exists x\, a(x)}\ \exists_R \qquad \frac{\vdash b(s) \vee c(s)}{\vdash \exists y\, (b(y) \vee c(y))}\ \exists_R \qquad \frac{\vdash \bar{a}(z) \wedge \bar{b}(z)}{\vdash \forall z\, (\bar{a}(z) \wedge \bar{b}(z))}\ \forall_R$$

$$\vdash \exists x\, a(x),\ \exists y\, (b(y) \vee c(y)),\ \forall z\, (\bar{a}(z) \wedge \bar{b}(z))$$

Choice of rule:

$$\frac{\vdash b(s)}{\vdash b(s) \vee c(s)}\ \vee_{R1} \qquad \frac{\vdash c(s)}{\vdash b(s) \vee c(s)}\ \vee_{R2}$$

Choice of term:

$$\frac{\vdash a(t)}{\vdash \exists x\, a(x)}$$

← t is a first order term

Invertibility

Example of rules:

$$\frac{\Gamma, B_1 \vdash \Delta}{\Gamma, B_1 \wedge B_2 \vdash \Delta} \wedge_{L1} \qquad \frac{\Gamma, B_1 \vdash B_2, \Delta}{\Gamma \vdash B_1 \supset B_2, \Delta} \supset_R$$

non-invertible — invertible

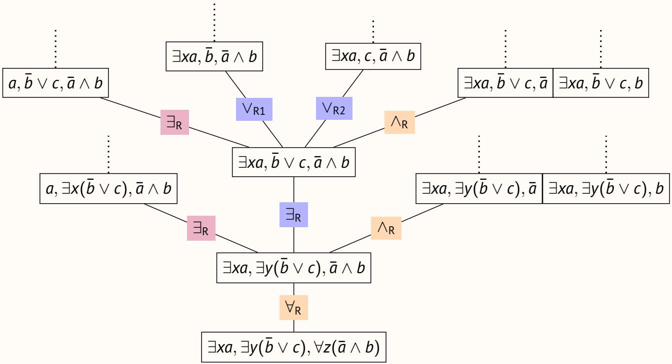
A rule $\dfrac{\text{Premiss}_1 \quad \cdots \quad \text{Premiss}_n}{\text{Conclusion}}$ is invertible if

for any $1 \leqslant k \leqslant n$, if Conclusion is provable

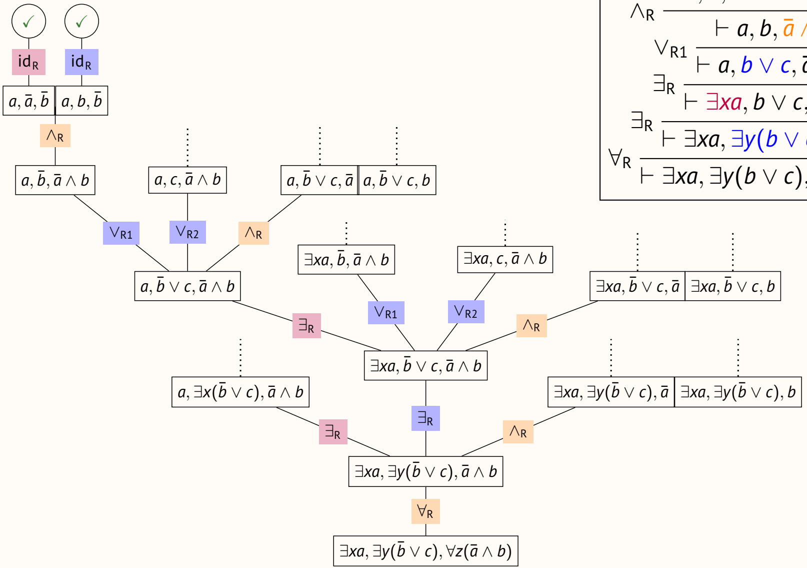
then Premiss_k is provable.

$$\text{cut}\ \dfrac{\Gamma \vdash B_1 \supset B_2, \Delta \qquad \dfrac{\overline{\overline{\Gamma, B_1 \vdash B_1, B_2, \Delta}} \qquad \overline{\overline{\Gamma, B_2, B_1 \vdash B_2, \Delta}}}{\Gamma, B_1 \supset B_2, B_1 \vdash B_2, \Delta} \supset_L}{\Gamma \quad B_1 \quad \vdash B_2 \quad \Delta}$$

Proof search space

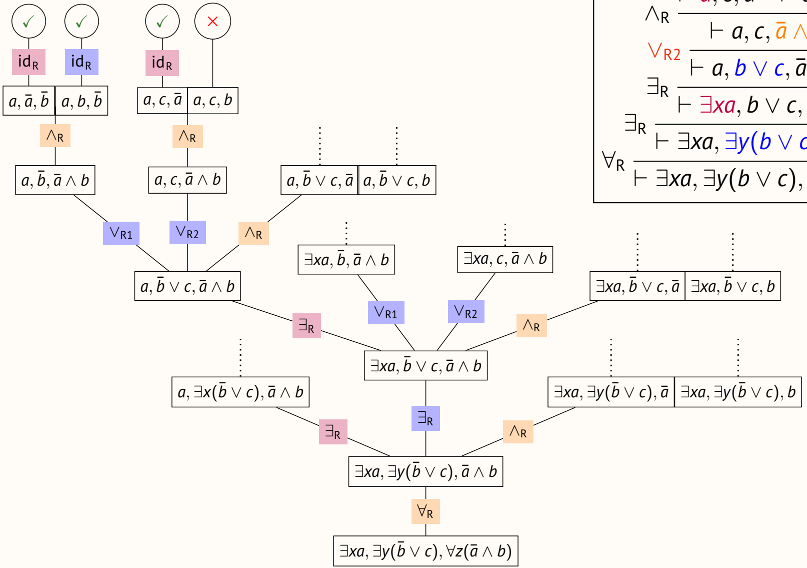


Proof search space



$$
\forall_R\,\dfrac{\exists_R\,\dfrac{\exists_R\,\dfrac{\vee_{R1}\,\dfrac{\wedge_R\,\dfrac{\mathrm{id}_R\,\dfrac{}{\vdash a, b, \bar{a}} \qquad \mathrm{id}_R\,\dfrac{}{\vdash a, b, \bar{b}}}{\vdash a, b, \bar{a} \wedge \bar{b}}}{\vdash a, b \vee c, \bar{a} \wedge \bar{b}}}{\vdash \exists xa, b \vee c, \bar{a} \wedge \bar{b}}}{\vdash \exists xa, \exists y(b \vee c), \bar{a} \wedge \bar{b}}}{\vdash \exists xa, \exists y(b \vee c), \forall z(\bar{a} \wedge \bar{b})}
$$

Proof search space



$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash a, c, \bar{a}}\,\mathrm{id}_R \qquad \dfrac{\times}{\vdash a, c, \bar{b}}}{\vdash a, c, \bar{a} \wedge \bar{b}}\,\wedge_R}{\vdash a, b \vee c, \bar{a} \wedge \bar{b}}\,\vee_{R2}}{\vdash \exists x a, b \vee c, \bar{a} \wedge \bar{b}}\,\exists_R}{\vdash \exists x a, \exists y(b \vee c), \bar{a} \wedge \bar{b}}\,\exists_R}{\vdash \exists x a, \exists y(b \vee c), \forall z(\bar{a} \wedge \bar{b})}\,\forall_R
$$

An organisational tool

Focusing provides a way to restrict the proof search space

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- Identify and always apply invertible introduction rules;

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- Chain together the other rules
  (non-invertible/consuming external information).

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Unfocused

An organisational tool

Focusing provides a way to restrict the proof search space

- Identify and always apply invertible introduction rules;
- Chain together the other rules
  (non-invertible/consuming external information).

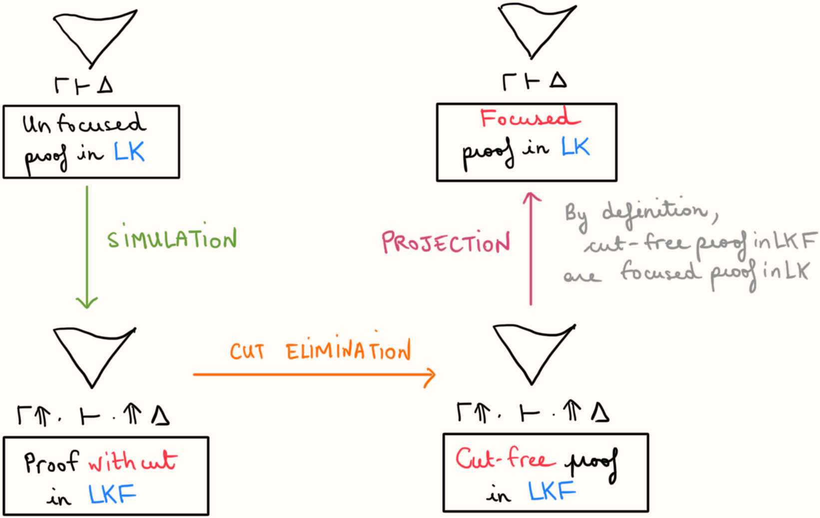
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash a, b, \bar{a}}\,\text{id} \qquad \dfrac{}{\vdash a, b, \bar{b}}\,\text{id}}{\vdash a, b, \bar{a} \wedge \bar{b}}\,\wedge_{\mathrm{R}}}{\vdash a, b \vee c, \bar{a} \wedge \bar{b}}\,\vee_{\mathrm{R1}}}{\vdash \exists x a, b \vee c, \bar{a} \wedge \bar{b}}\,\exists_{\mathrm{R}}}{\vdash \exists x a, \exists y(b \vee c), \bar{a} \wedge \bar{b}}\,\exists_{\mathrm{R}}}{\vdash \exists x a, \exists y(b \vee c), \forall z(\bar{a} \wedge \bar{B})}\,\forall_{\mathrm{R}}$$

Unfocused → Focused

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash a, \exists y(b \vee c), \bar{a}}\,\text{id}}{\vdash \exists x a, \exists y(b \vee c), \bar{a}}\,\exists_{\mathrm{R}} \qquad \dfrac{\dfrac{\dfrac{}{\vdash \exists x a, b, \bar{b}}\,\text{id}}{\vdash \exists x a, b \vee c, \bar{b}}\,\vee_{\mathrm{R1}}}{\vdash \exists x a, \exists y(b \vee c), \bar{b}}\,\exists_{\mathrm{R}}}{\vdash \exists x a, \exists y(b \vee c), \bar{a} \wedge \bar{b}}\,\vee_{\mathrm{R}}}{\vdash \exists x a, \exists y(b \vee c), \forall z(\bar{a} \wedge \bar{b})}\,\forall_{\mathrm{R}}$$

The focusing theorem

Theorem. [Andreoli, 90]
If a formula is provable, then it has a focused proof.



Polarities of connectives

First-order classical language:

$$A ::= p(t_1, \dots, t_n) \mid A \wedge A \mid \top \mid A \vee A \mid \bot \mid A \supset A \mid \exists x\, A \mid \forall x\, A$$

Polarised connectives:

- positive and negative versions of the propositional connectives and constants:

  $$\overset{-}{\wedge}, \overset{+}{\wedge}, \overset{-}{\top}, \overset{+}{\top}, \overset{-}{\vee}, \overset{+}{\vee}, \overset{-}{\bot}, \overset{+}{\bot}$$

  Except $\supset$ which is negative.
- first-order quantifiers: $\exists$ positive and $\forall$ negative.

Important: Even atomic formulas are polarised!

Example: $\exists x\, a(x) \vee \forall z\,(\bar{a}(z) \wedge \bar{b}(z))$

	$\overset{+}{\vee}$	$\overset{-}{\vee}$
$\overset{+}{\wedge}$	$\exists x\, a(x) \overset{+}{\vee} \forall z\,(\bar{a}(z) \overset{+}{\wedge} \bar{b}(z))$	$\exists x\, a(x) \overset{-}{\vee} \forall z\,(\bar{a}(z) \overset{+}{\wedge} \bar{b}(z))$
$\overset{-}{\wedge}$	$\exists x\, a(x) \overset{+}{\vee} \forall z\,(\bar{a}(z) \overset{-}{\wedge} \bar{b}(z))$	$\exists x\, a(x) \overset{-}{\vee} \forall z\,(\bar{a}(z) \overset{-}{\wedge} \bar{b}(z))$

Dual atoms are polarized dually:

$$\exists x\, a(x) \vee \forall z\,(\bar{a}(z) \wedge \bar{b}(z))$$

LKF [Liang and Miller, 07]

Two kinds of focused sequents

- $\Downarrow$ sequents to decompose the formula under focus

$$\Gamma \Downarrow B \vdash \Delta \text{ with a left focus on } B$$
$$\Gamma \vdash B \Downarrow \Delta \text{ with a right focus on } B$$

  When the conclusion of a rule, then that rule introduced B.

- $\Uparrow$ sequents for invertible introduction rules

$$\Gamma_1 \Uparrow \Gamma_2 \vdash \Delta_1 \Uparrow \Delta_2$$

Sequent derivations are organised into synchronous / asynchronous phases.

The dynamic of proof search:

- A formula is put under focus

Decide: $$\frac{\Gamma, N \Downarrow N \vdash \Delta}{\Gamma, N \Uparrow \cdot \vdash \cdot \Uparrow \Delta}\, D_l \qquad \frac{\Gamma \vdash P \Downarrow \Delta}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow P, \Delta}\, D_r$$

The dynamic of proof search:

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  Decide: $$\frac{\Gamma, N \Downarrow N \vdash \Delta}{\Gamma, N \Uparrow \cdot \vdash \cdot \Uparrow \Delta} D_l \qquad \frac{\Gamma \vdash P \Downarrow \Delta}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow P, \Delta} D_r$$

- Focus is transferred from conclusion to premises until

The dynamic of proof search:

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- Focus is transferred from conclusion to premises until
  - either the focused phase ends

  Release: $$\frac{\Gamma \Uparrow P \vdash \cdot \Uparrow \Delta}{\Gamma \Downarrow P \vdash \Delta}\, R_l \qquad \frac{\Gamma \Uparrow \cdot \vdash N \Uparrow \Delta}{\Gamma \vdash N \Downarrow \Delta}\, R_r$$

The dynamic of proof search:

- A formula is put under focus

  Decide: $$\frac{\Gamma, N \Downarrow N \vdash \Delta}{\Gamma, N \Uparrow \cdot \vdash \cdot \Uparrow \Delta} D_l \qquad \frac{\Gamma \vdash P \Downarrow \Delta}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow P, \Delta} D_r$$

- Focus is transferred from conclusion to premises until
  - either the focused phase ends

    Release: $$\frac{\Gamma \Uparrow P \vdash \cdot \Uparrow \Delta}{\Gamma \Downarrow P \vdash \Delta} R_l \qquad \frac{\Gamma \Uparrow \cdot \vdash N \Uparrow \Delta}{\Gamma \vdash N \Downarrow \Delta} R_r$$

  - or the derivation ends

    Initial: $$\frac{}{\Gamma \Downarrow n \vdash n, \Delta} I_l \qquad \frac{}{\Gamma, p \vdash p \Downarrow \Delta} I_r$$

The dynamic of proof search:

- A formula is put under focus

  Decide: $$\frac{\Gamma, N \Downarrow N \vdash \Delta}{\Gamma, N \Uparrow \cdot \vdash \cdot \Uparrow \Delta} D_l \qquad \frac{\Gamma \vdash P \Downarrow \Delta}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow P, \Delta} D_r$$

- Focus is transferred from conclusion to premises until
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    Release: $$\frac{\Gamma \Uparrow P \vdash \cdot \Uparrow \Delta}{\Gamma \Downarrow P \vdash \Delta} R_l \qquad \frac{\Gamma \Uparrow \cdot \vdash N \Uparrow \Delta}{\Gamma \vdash N \Downarrow \Delta} R_r$$

  - or the derivation ends

    Initial: $$\frac{}{\Gamma \Downarrow n \vdash n, \Delta} I_l \qquad \frac{}{\Gamma, p \vdash p \Downarrow \Delta} I_r$$

- Once the focus is released, the formula is eagerly decomposed into subformulas,

The dynamic of proof search:

- A formula is put under focus

  Decide: $$\frac{\Gamma, N \Downarrow N \vdash \Delta}{\Gamma, N \Uparrow \cdot \vdash \cdot \Uparrow \Delta} D_l \qquad \frac{\Gamma \vdash P \Downarrow \Delta}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow P, \Delta} D_r$$

- Focus is transferred from conclusion to premises until
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    Release: $$\frac{\Gamma \Uparrow P \vdash \cdot \Uparrow \Delta}{\Gamma \Downarrow P \vdash \Delta} R_l \qquad \frac{\Gamma \Uparrow \cdot \vdash N \Uparrow \Delta}{\Gamma \vdash N \Downarrow \Delta} R_r$$

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    Initial: $$\frac{}{\Gamma \Downarrow n \vdash n, \Delta} I_l \qquad \frac{}{\Gamma, p \vdash p \Downarrow \Delta} I_r$$

- Once the focus is released, the formula is eagerly decomposed into subformulas,
  - which are ultimately stored in the context.

    Store: $$\frac{\Gamma_1, P \Uparrow \Gamma_2 \vdash \Delta_1 \Uparrow \Delta_2}{\Gamma_1 \Uparrow \Gamma_2, P \vdash \Delta_1 \Uparrow \Delta_2} S_l \qquad \frac{\Gamma \Uparrow \cdot \vdash \Delta_1 \Uparrow N, \Delta_2}{\Gamma \Uparrow \cdot \vdash N, \Delta_1 \Uparrow \Delta_2} S_r$$

The logical rules are similar to the non focused versions but split into:

asynchronous
||
→ those that can be applied during the inversion phase

- negative formulas on the right

e.g. $$\frac{\Gamma_1 \Uparrow \Gamma_2, A \vdash \Delta_1, B \Uparrow \Delta_2}{\Gamma_1 \Uparrow \Gamma_2 \vdash \Delta_1, A \supset B \Uparrow \Delta_2} \supset_R$$

- positive formulas on the left

e.g. $$\frac{\Gamma_1 \Uparrow \Gamma_2, A, B \vdash \Delta_1 \Uparrow \Delta_2}{\Gamma_1 \Uparrow \Gamma_2, A \overset{+}{\vee} B \vdash \Delta_1 \Uparrow \Delta_2} \overset{+}{\vee}_L$$

synchronous

||

→ those that can be applied during the focused phase

- positive formulas on the right

e.g. $$\frac{\Gamma \vdash A \Downarrow \Delta}{\Gamma \vdash A \overset{+}{\vee} B \Downarrow \Delta}\ \overset{+}{\vee}R_1$$

- negative formulas on the left

e.g. $$\frac{\Gamma \vdash A \Downarrow \Delta \qquad \Gamma \Downarrow B \vdash \Delta}{\Gamma \Downarrow A \supset B \vdash \Delta}\ \supset_L$$

note that the focus changes sides


Questions?