2. Lecture
Sequent calculi for modal logic

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Is proof theory the theory of proof systems?

**Hilbert system**: many axioms, few rules

\[
\begin{align*}
A & \rightarrow B \\
\Box (A \rightarrow B) & \rightarrow (\Box A \rightarrow \Box B)
\end{align*}
\]

\[\frac{A \quad A \rightarrow B}{B} \text{ mp}\]

\[\frac{A}{\Box A} \text{ ncc}\]

\[
\vdash \text{even proving } A \rightarrow A \text{ is tedious!}
\]

**Gentzen system**: few axioms, many rules

\[\rightarrow \text{ easier proof search through } \textit{analyticity}\]

\[\rightarrow \text{ fine granularity to study proof } \textit{structure}\]
First some notation

1) the \textit{sequent arrow} $\Gamma_1 \vdash \Gamma_2$ (turnstyle)

2) the \textit{comma} $\cdot$

\[
A_1, \ldots, A_n \vdash B_1, \ldots, B_p
\]

$\Gamma_1 \vdash \Gamma_2$

\[\text{multisets of formulas}\]

\[
\text{means} \quad A_1 \land \ldots \land A_n \rightarrow B_1 \lor \ldots \lor B_p
\]

\[= \text{fm} (\Gamma_1 \cup \Gamma_2)\]

\[\text{(formula interpretation)}\]

\textbf{Notations:}

1) omit multiset brackets $\{\cdot\}$
2) write union $\Gamma_1 \cup \Gamma_2$ as $\Gamma_1, \Gamma_2$
**Propositional sequent rules**

**Axioms**

- **Identity**
  \[ \Gamma_1, a \vdash \Gamma_2, a \]

- **Conjunction**
  \[
  \frac{\Gamma_1, A, B \vdash \Gamma_2}{\Gamma_1, A \land B \vdash \Gamma_2} \quad \text{\(\land_L\)}
  \]
  \[
  \frac{\Gamma_1 \vdash A, \Gamma_2}{\Gamma_1 \vdash A \land B, \Gamma_2} \quad \text{\(\land_R\)}
  \]

- **Disjunction**
  \[
  \frac{\Gamma_1, A \vdash \Gamma_2, \Gamma_2}{\Gamma_1, A \lor B \vdash \Gamma_2} \quad \text{\(\lor_L\)}
  \]
  \[
  \frac{\Gamma_1, B \vdash \Gamma_2}{\Gamma_1, A \lor B \vdash \Gamma_2} \quad \text{\(\lor_R\)}
  \]

- **Implication**
  \[
  \frac{\Gamma_1 \vdash \Gamma_2, A, \Gamma_2}{\Gamma_1, A \rightarrow B \vdash \Gamma_2} \quad \text{\(\rightarrow_L\)}
  \]
  \[
  \frac{\Gamma_1, B \vdash \Gamma_2}{\Gamma_1, A \rightarrow B \vdash \Gamma_2} \quad \text{\(\rightarrow_R\)}
  \]

- **Units**
  \[
  \frac{\Gamma_1 \vdash, \Gamma_2}{\bot \vdash \Gamma_2} \quad \text{\(\bot_L\)}
  \]
  \[
  \frac{\Gamma_1, \Gamma_2 \vdash}{\Gamma_1, \Gamma_2 \vdash} \quad \text{\(\bot_R\)}
  \]

- **Negation**
  \[
  \frac{\Gamma_1 \vdash A, \Gamma_2}{\Gamma_1, \neg A \vdash \Gamma_2} \quad \text{\(\neg_L\)}
  \]
  \[
  \frac{\Gamma_1, A \vdash \Gamma_2}{\Gamma_1, \neg A, \Gamma_2 \vdash} \quad \text{\(\neg_R\)}
  \]
Modal Rules

\frac{\Gamma_1 \vdash A, \Gamma_2}{\Box \Gamma_1 \vdash \Box A, \Box \Gamma_2} \quad \Box K

\frac{\Gamma_1, A \vdash \Gamma_2}{\Box \Gamma_1, \Box A \vdash \Box \Gamma_2} \quad \Box K

Structural Rules

weakening \quad \frac{\Gamma_1 \vdash \Gamma_2}{\Gamma_1 \vdash \Gamma_2, A} \quad W_L

\frac{\Gamma_1 \vdash \Gamma_2}{\Gamma_1, A \vdash \Gamma_2} \quad W_K

contraction \quad \frac{\Gamma_1, A, A \vdash \Gamma_2}{\Gamma_1, A \vdash \Gamma_2} \quad C_L

\frac{\Gamma_1 \vdash \Gamma_2, A, A}{\Gamma_1 \vdash \Gamma_2, A} \quad C_R

\text{Notation: } \Diamond \Gamma = \Diamond A_1, \ldots, \Diamond A_n \text{ if } \Gamma = A_1, \ldots, A_n
A derivation is a tree where each node is labelled by a rule and each edge is labelled by a sequent, and the root is the sequent \( \Gamma_1 \vdash \Gamma_2 \).

A proof is a derivation in which each leaf is an occurrence of an axiom (here: \( \text{id} \), \( \Gamma_L \) or \( \Gamma_R \)).
Examples

\[ A, B \vdash A \]
\[ A \vdash B \rightarrow A \]
\[ \vdash A \rightarrow (B \rightarrow A) \] → \( R \)

\[ A, B, A \vdash A \rightarrow B, A \vdash B \]
\[ \vdash A \rightarrow B, A \vdash B \rightarrow A \] \rightarrow \( L \)

\[ \Box (A \rightarrow B), \Box A \vdash \Box B \]
\[ \vdash \Box (A \rightarrow B) \rightarrow \Box A \rightarrow \Box B \] → \( R \)

\[ \vdash \Box (A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) \] → \( R \)
Lemma (General identity)  

The rule $\frac{A \vdash A}{A \vdash A}$ is derivable (in previous systems)  

for all $A$, there exists a proof of $A \vdash A$

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Proof: by induction on the structure of $A$

$A ::= a \mid T \mid \bot \mid (A \land A) \mid (A \lor A) \mid (A \rightarrow A) \mid \neg A \mid \square A \mid \diamond A$

* $A = a$: $a \vdash a$ is derivable using id
* $A = T$: $T \vdash T$ is derivable using $TR$
* $A = \bot$: $\bot \vdash \bot$ is derivable using $\bot$

$\{ \text{BASE CASES} \}$
**Inductive Cases**

\[ A = \star \quad \text{for } \star = \land, \lor, \text{ or } \rightarrow : \]

IH: \[ B \vdash B \] and \[ C \vdash C \] are derivable

e.g., for \[ \star = \rightarrow : \]

\[
\begin{align*}
\frac{B \vdash B}{B \vdash C, B} & \quad \text{WR} \\
\frac{C \vdash C}{C, B \vdash C} & \quad \text{WL} \\
\frac{B \rightarrow C, B \vdash C}{B \vdash C} & \quad \text{R}
\end{align*}
\]

\[ A = \blacklozenge B \quad \text{for } \blacklozenge = ?, \square \text{ or } \lozenge : \]

IH: \[ B \vdash B \] is derivable

\[
\begin{align*}
\frac{B \vdash B}{\blacklozenge B \vdash \blacklozenge B} & \quad \text{DR} \\
\frac{B \vdash B}{\square B \vdash \square B} & \quad \text{DR} \\
\frac{B \vdash B}{\neg B \vdash \neg B} & \quad \text{R}
\end{align*}
\]
Examples

\[
\frac{A \vdash A}{A, B \vdash A} \quad \text{\textit{WL}}
\]

\[
\frac{A \vdash B \rightarrow A}{\vdash A \rightarrow (B \rightarrow A)}
\]

\[
\frac{A \vdash A}{A \vdash B, A} \quad \text{\textit{WR}}
\]

\[
\frac{B \vdash B}{B, A \vdash B} \quad \text{\textit{WL}}
\]

\[
\frac{A \rightarrow B \text{, } A \vdash B}{\square (A \rightarrow B), \square A \vdash \square B} \quad \text{\textit{R}}
\]

\[
\frac{\square (A \rightarrow B) \vdash \square A \rightarrow \square B}{\vdash \square (A \rightarrow B) \rightarrow (\square A \rightarrow \square B)}
\]
What about our notion of truth?

**Theorem (Soundness and Completeness)**

TFAE = The Following Are Equivalent

1) Formula A is provable in modal logic K

2) Formula A is valid (in all frames)

3) There is a proof of sequent ⊢ A in the system
   \{id, cut, \perp, \top, \lozenge, \boxdot, \boxstar, \star, \lozenge, \boxstar\}
   for \(\lozenge \in \{\neg, \exists, \forall\}\) and \(\star \in \{\wedge, \vee, \rightarrow\}\)

**Proof** We take 1 \(\leftrightarrow\) 2 as given and prove 3 \(\leftrightarrow\) 1
1) Formula $A$ is provable in modal logic $K$

It is enough to prove that

$(\land \text{fm (premises)})$ is provable in $K$

formula interpretation

3) There is a proof of sequent $\vdash A$ in the system

$\{ \text{id, cut, } L, R, \land, \lor, \neg \}$

for $\land \in \{\land, \lor, \lor\}$ and $\neg \in \{\land, \lor, \rightarrow\}$
\[
1 \rightarrow 3:
\]
\[
\begin{align*}
\Gamma_1 \vdash A, B, \Gamma_2 & \quad \vdash A \lor B, \Gamma_2 \\
\Gamma_1 \vdash A \lor B, \Gamma_2 & \quad \Gamma_1 \vdash A, \Gamma_2 & \quad \Gamma_1 \vdash B, \Gamma_2
\end{align*}
\]

\[
\text{via mp on } ((C \rightarrow A) \land (C \rightarrow B)) \rightarrow (C \rightarrow A \land B)
\]

\[
\begin{align*}
(\vdash \Gamma_1) \rightarrow (A \lor (\Gamma_2)) & \quad \vdash \Gamma_1 \rightarrow (A \rightarrow (\Gamma_2)) \\
\Gamma_1 \vdash A, \Gamma_2 & \quad \square (\vdash \Gamma_1) \rightarrow (A \rightarrow (\Gamma_2)) \quad \text{via nec} \\
\square \Gamma_1 \vdash \square A, \square \Gamma_2 & \quad \square (\vdash \Gamma_1) \rightarrow (\square A \rightarrow (\square \Gamma_2)) \quad \text{via mp on k twice} \\
(\square \Gamma_1) \rightarrow (\square A \lor (\square \Gamma_2)) & \quad \land (\square \Gamma_1) \rightarrow (\square A \lor (\square \Gamma_2)) \quad \text{via } \land \square \Gamma_1 \equiv \square (\land \Gamma_1)
\end{align*}
\]

\[
\vdash (\land \Gamma_1) \rightarrow (A \rightarrow (\land \Gamma_2))
\]

\[
3 \rightarrow 1: \text{ we already showed how to derive } A \rightarrow (B \rightarrow A) \text{ and } \square (A \rightarrow B) \rightarrow (\square A \rightarrow \square B)
\]

\[
\text{for nec } \square A \quad : \text{ suppose } \vdash A \text{ is derivable then } \vdash \square A \quad \text{Ok}
\]

\[
\text{for mp } A \rightarrow B \quad : \text{ suppose } \{\vdash A, A \rightarrow B\} \text{ derivable then } \vdash A \rightarrow B \quad \text{cut}
\]

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**Modal Rules (Extended)**

\[
\frac{\Gamma_1 \vdash A, \Gamma_2}{\Box \Gamma_1 \vdash \Box A, \Box \Gamma_2} \quad \Box R
\]

\[
\frac{\Gamma_1, \Box \Gamma_1 \vdash \Gamma_2, \Box \Gamma_2}{\Box \Gamma_1, \Box A \vdash \Box \Gamma_2} \quad \Box K_4
\]

\[
\frac{\Gamma_1, \Box \Gamma_1, A \vdash \Gamma_2, \Box \Gamma_2}{\Box \Gamma_1, \Box A \vdash \Box \Gamma_2} \quad \Box K_4
\]

\[
\frac{\Gamma_1, \Box \Gamma_1, \Box \Gamma_3, A \vdash \Gamma_2, \Box \Gamma_3, \Box \Gamma_4}{\Box \Gamma_1, \Box \Gamma_3, \Box \Gamma_4 \vdash \Box \Gamma_2, \Box \Gamma_4} \quad \Box K_{45}
\]

**Theorem (Soundness and completeness)**

- Formula \( A \) is provable in logic \( K \) or \( D \) or \( T \)
- \( K_4 \) or \( D_4 \) or \( S_4 \)
- \( K_{45} \) or \( D_{45} \) or \( S_5 \)

- Sequent \( \vdash A \) is provable in system ... with cut!
**Issues**

(i) **locality and separation**: \[ \Gamma_1 \vdash A \vdash \Gamma_2 \]
\[ \square \Gamma_1 \vdash \square A, \square \Gamma_2 \]
\[ \square \Gamma_1, \square A \vdash \square \Gamma_2 \]
\[ \square \Gamma_1, \square A \vdash \square \Gamma_2 \]

(ii) **modularity**: need to swap \( \square \Gamma_4 \) for \( \square \Gamma_5 \)
would like: keep \( \square \Gamma_k \) and add \( \square \Gamma_4 \) and/or \( \square \Gamma_5 \)

(iii) **generality**: no complete system for \{ K5, D5 \}
\{ KB, DB, TB \}
and more Scott-Lemmon axioms

(iv) **analyticity**: \[ \square \square A \rightarrow A \]
\[ \square \square A \rightarrow A, \square \square A \]
\[ \square \square A \rightarrow A, \square \square A \]
\[ \square \square A \rightarrow A, \square \square A \]
\[ \square \square A \rightarrow A, \square \square A \]
\[ \square \square A \rightarrow A, \square \square A \]
\[ \square \square A \rightarrow A, \square \square A \]
\[ \square \square A \rightarrow A, \square \square A \]

→ **How to design sequent systems for modal logics that preserve these good properties?**
Questions?