From Axioms to Rules:

The Factory of Modal Proof Systems

2. Lecture

Sequent calculi for modal logic

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Is proof theory the theory of proof systems?

<u>Hilbert system</u>: many axioms, few rules

$A \to B \to A$

$\Box(A \to B) \to (\Box A \to \Box B)$

$\vdots$

$$\frac{A \quad A \to B}{B}\,\mathrm{mp} \qquad \frac{A}{\Box A}\,\mathrm{nec}$$

$\longrightarrow$ even proving $A \to A$ is tedious!

<u>Gentzen system</u>: few axioms, many rules

$\longrightarrow$ easier proof search through <u>analyticity</u>

$\longrightarrow$ fine granularity to study proofs <u>structure</u>

First some notation

1) the sequent arrow $\vdash$ (turnstyle)

2) the comma ,

$$\underbrace{A_1 , \dots , A_n}_{\Gamma_1} \vdash \underbrace{B_1 , \dots , B_p}_{\Gamma_2}$$

multisets of formulas

NOTATIONS:

1) omit multiset brackets $\{ \cdot \}$

2) write union $\Gamma_1 \cup \Gamma_2$ as Γ_1 , Γ_2

means $A_1 \wedge \dots \wedge A_n \to B_1 \vee \dots \vee B_p$

$= \mathrm{fm}(\Gamma_1 \vdash \Gamma_2)$ (formula interpretation)

Propositional sequent rules

$(\wedge \Gamma_1) \wedge a \to (\vee \Gamma_2) \vee a$

Axioms

identity $\quad \dfrac{}{\Gamma_1, a \vdash \Gamma_2, a}$

cut $\quad \dfrac{\Gamma_1 \vdash A, \Gamma_2 \qquad \Gamma_1, A \vdash \Gamma_2}{\Gamma_1 \vdash \Gamma_2}$

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conjunction $\quad \dfrac{\Gamma_1, A, B \vdash \Gamma_2}{\Gamma_1, A \wedge B \vdash \Gamma_2}\ \wedge_L \qquad \dfrac{\Gamma_1 \vdash A, \Gamma_2 \qquad \Gamma_1 \vdash B, \Gamma_2}{\Gamma_1 \vdash A \wedge B, \Gamma_2}\ \wedge_R$

PREMISS

CONCLUSION

disjunction $\quad \dfrac{\Gamma_1, A \vdash \Gamma_2 \qquad \Gamma_1, B \vdash \Gamma_2}{\Gamma_1, A \vee B \vdash \Gamma_2}\ \vee_L \qquad \dfrac{\Gamma_1 \vdash A, B, \Gamma_2}{\Gamma_1 \vdash A \vee B, \Gamma_2}\ \vee_R$

implication $\quad \dfrac{\Gamma_1 \vdash \Gamma_2, A \qquad \Gamma_1, B \vdash \Gamma_2}{\Gamma_1, A \to B \vdash \Gamma_2}\ \to_L \qquad \dfrac{\Gamma_1, A \vdash B, \Gamma_2}{\Gamma_1 \vdash A \to B, \Gamma_2}\ \to_R$

units $\quad \dfrac{}{\Gamma_1, \bot \vdash \Gamma_2}\ \bot_L \qquad \dfrac{}{\Gamma_1 \vdash \top, \Gamma_2}\ \top_R$

negation $\quad \dfrac{\Gamma_1 \vdash A, \Gamma_2}{\Gamma_1, \neg A \vdash \Gamma_2}\ \neg_L \qquad \dfrac{\Gamma_1, A \vdash \Gamma_2}{\Gamma_1 \vdash \neg A, \Gamma_2}\ \neg_R$

Modal Rules

notation : $\heartsuit\Gamma = \heartsuit A_1, \dots, \heartsuit A_n$ if $\Gamma = A_1, \dots, A_n$

$$\frac{\Gamma_1 \vdash A, \Gamma_2}{\Box\Gamma_1 \vdash \Box A, \Diamond\Gamma_2}\ \Box_R$$

$$\frac{\Gamma_1, A \vdash \Gamma_2}{\Box\Gamma_1, \Diamond A \vdash \Diamond\Gamma_2}\ \Diamond_R$$

Structural rules

weakening

$$\frac{\Gamma_1 \vdash \Gamma_2}{\Gamma_1 \vdash \Gamma_2, A}\ W_L \qquad \frac{\Gamma_1 \vdash \Gamma_2}{\Gamma_1, A \vdash \Gamma_2}\ W_R$$

contraction

$$\frac{\Gamma_1, A, A \vdash \Gamma_2}{\Gamma_1, A \vdash \Gamma_2}\ C_L \qquad \frac{\Gamma_1 \vdash \Gamma_2, A, A}{\Gamma_1 \vdash \Gamma_2, A}\ C_R$$

A derivation of sequent $\Gamma_1 \vdash \Gamma_2$ is a tree where each node is labelled by a rule and each edge is labelled by a sequent. and the root is the sequent $\Gamma_1 \vdash \Gamma_2$

A proof is a derivation in which each leaf is an occurence of an axiom (here: id, $\perp_L$ or $\top_R$).

Examples

$$\dfrac{\dfrac{\dfrac{?}{A , B \vdash A}}{A \vdash B \to A}\ {\to_R}}{\vdash A \to (B \to A)}\ {\to_R}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{?}{A \vdash B, A} \qquad \dfrac{?}{B, A \vdash B}}{A \to B , A \vdash B}\ {\to_L}}{\Box (A \to B), \Box A \vdash \Box B}\ {\Box_R}}{\dfrac{\Box (A \to B) \vdash \Box A \to \Box B}{\vdash \Box (A \to B) \to (\Box A \to \Box B)}\ {\to_R}}\ {\to_R}$$

Lemma (General identity)

The rule $\dfrac{}{A \vdash A}\,\mathrm{gid}$ is derivable (in previous systems)

$\downarrow$

for all A, there exists a proof of $A \vdash A$

Proof: by induction on the structure of A

$A ::= a \mid \top \mid \bot \mid (A \wedge A) \mid (A \vee A) \mid (A \to A) \mid \neg A \mid \Box A \mid \Diamond A \mid$

* $A = a$: $a \vdash a$ is derivable using id
* $A = \top$: $\top \vdash \top$ is derivable using $\top_R$
* $A = \bot$: $\bot \vdash \bot$ is derivable using $\bot_L$

$\boxed{\text{BASE CASES}}$

INDUCTIVE CASES

$\ast$ $A = B \,☆\, C$ for $☆ = \wedge, \vee$ or $\rightarrow$:

IH: $B \vdash B$ and $C \vdash C$ are derivable

e.g. for $☆ = \longrightarrow$:

$$\dfrac{\dfrac{\dfrac{B \vdash B}{B \vdash C, B}\,W_R \qquad \dfrac{C \vdash C}{C, B \vdash C}\,W_L}{B \rightarrow C, B \vdash C}\,\rightarrow_L}{B \rightarrow C \vdash B \rightarrow C}\,\rightarrow_R$$

$\ast$ $A = \heartsuit B$ for $\heartsuit = \neg, \Box$ or $\Diamond$:

IH: $B \vdash B$ is derivable

$$\dfrac{B \vdash B}{\Diamond B \vdash \Diamond B}\,\Diamond_k \qquad \dfrac{B \vdash B}{\Box B \vdash \Box B}\,\Box_k \qquad \dfrac{\dfrac{B \vdash B}{\neg B, B \vdash}\,\neg_l}{\neg B \vdash \neg B}\,\neg_R$$

Examples

general identity

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{gid}
\UnaryInfC{$A \vdash A$}
\RightLabel{W_L}
\UnaryInfC{$A, B \vdash A$}
\RightLabel{$\to_R$}
\UnaryInfC{$A \vdash B \to A$}
\RightLabel{$\to_R$}
\UnaryInfC{$\vdash A \to (B \to A)$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{gid}
\UnaryInfC{$A \vdash A$}
\RightLabel{W_R}
\UnaryInfC{$A \vdash B, A$}
\AxiomC{}
\RightLabel{gid}
\UnaryInfC{$B \vdash B$}
\RightLabel{W_L}
\UnaryInfC{$B, A \vdash B$}
\RightLabel{$\to_L$}
\BinaryInfC{$A \to B, A \vdash B$}
\RightLabel{$\Box_R$}
\UnaryInfC{$\Box(A \to B), \Box A \vdash \Box B$}
\RightLabel{$\to_R$}
\UnaryInfC{$\Box(A \to B) \vdash \Box A \to \Box B$}
\RightLabel{$\to_R$}
\UnaryInfC{$\vdash \Box(A \to B) \to (\Box A \to \Box B)$}
\end{prooftree}
$$

What about our notion of truth?

Theorem (Soundness and Completeness)

TFAE = The Following Are Equivalent

1) Formula A is provable in modal logic K

2) Formula A is valid (in all frames)

3) There is a proof of sequent $\vdash A$ in the system

$\{id, cut, \bot_L, \top_R, \heartsuit_R, \heartsuit_L, \star_R, \star_L\}$

for $\heartsuit \in \{\neg, \Box, \Diamond\}$ and $\star \in \{\wedge, \vee, \rightarrow\}$

Proof We take $1 \longleftrightarrow 2$ as given and prove $3 \longleftrightarrow 1$

1) Formula A is provable in modal logic K

$\underset{\sim\sim\sim}{3 \longrightarrow 1}$

It is enough to prove that

$\left(\bigwedge \text{fm} \text{ (premisses)} \right) \longrightarrow \text{fm (conclusion)}$ is provable in K

formula interpretation

$\underset{\sim\sim\sim}{1 \longrightarrow 3}$

It is enough to prove that the axioms and rules of K are derivable in the sequent system

3) $\boxed{\text{There is a proof}}$ of sequent $\vdash A$ in the system

$\{ id, cut, \bot_L, \top_R, \heartsuit_R, \heartsuit_L, \star_R, \star_L \}$

for $\heartsuit \in \{\neg, \Box, \Diamond\}$ and $\star \in \{\wedge, \vee, \rightarrow\}$

$\underset{\sim\sim}{1 \to 3}$:

$$\dfrac{\overbrace{\Gamma_1 \vdash A, B, \Gamma_2}^{(\bigwedge\Gamma_1) \to ((A \vee B) \vee (\bigvee \Gamma_2))}}{\Gamma_1 \vdash A \vee B, \Gamma_2} \vee_R$$

$$\dfrac{\overbrace{\Gamma_1 \vdash A, \Gamma_2}^{(\bigwedge\Gamma_1) \to (A \vee (\bigvee \Gamma_2))} \qquad \overbrace{\Gamma_1 \vdash B, \Gamma_2}^{(\bigwedge\Gamma_1) \to (B \vee (\bigvee \Gamma_2))}}{\underbrace{\Gamma_1 \vdash A \wedge B, \Gamma_2}_{(\bigwedge\Gamma_1) \to ((A \wedge B) \vee (\bigvee \Gamma_2))}} \wedge_R$$

via mp on $((C \to A) \wedge (C \to B)) \to (C \to A \wedge B)$

$$\dfrac{\overbrace{\Gamma_1 \vdash A, \Gamma_2}^{(\bigwedge\Gamma_1) \to (A \vee (\bigvee \Gamma_2))}}{\underbrace{\Box\Gamma_1 \vdash \Box A, \Diamond\Gamma_2}_{(\bigwedge \Box\Gamma_1) \to (\Box A \vee (\bigvee \Diamond\Gamma_2))}} \Box k$$

$(\bigwedge\Gamma_1) \to (A \to \overline{(\bigvee \Gamma_2)})$

$\Box((\bigwedge\Gamma_1) \to (A \to (\bigwedge \overline{\Gamma_2})))$ via nec

$\Box(\bigwedge\Gamma_1) \to (\Box A \to \Box(\bigwedge \overline{\Gamma_2}))$ via mp on k twice

$\bigwedge(\Box\Gamma_1) \to (\Box A \vee \bigvee \Diamond\Gamma_2)$ via $\bigwedge \Box \Gamma_1 \equiv \Box(\bigwedge\Gamma_1)$

and $\bigvee \Diamond \Gamma_2 \equiv \Diamond(\bigvee\Gamma_2)$

$\underset{\sim\sim}{3 \to 1}$: we already showed how to derive $A \to (B \to A)$ and $\Box(A \to B) \to (\Box A \to \Box B)$

for nec $\dfrac{A}{\Box A}$: suppose $\vdash A$ is derivable then $\dfrac{\vdash A}{\vdash \Box A} \Box k$

for mp $\dfrac{A \quad A \to B}{B}$: suppose $\begin{cases} \vdash A \\ A \vdash B \end{cases}$ derivable then $\dfrac{\vdash A \qquad A \vdash B}{\vdash B}$ cut

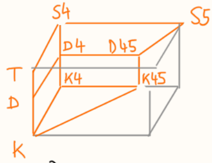
Modal Rules (Extended)

$$\frac{\Gamma_1 \vdash A, \Gamma_2}{\Box\Gamma_1 \vdash \Box A, \Diamond\Gamma_2}\ \Box k \qquad\qquad \frac{\Gamma_1, A \vdash \Gamma_2}{\Box\Gamma_1, \Diamond A \vdash \Diamond\Gamma_2}\ \Diamond k$$

$$\frac{\Gamma_1, \Box\Gamma_1 \vdash A, \Gamma_2, \Diamond\Gamma_2}{\Box\Gamma_1 \vdash \Box A, \Diamond\Gamma_2}\ \Box k4 \qquad\qquad \frac{\Gamma_1, \Box\Gamma_1, A \vdash \Gamma_2, \Diamond\Gamma_2}{\Box\Gamma_1, \Diamond A \vdash \Diamond\Gamma_2}\ \Diamond k4$$

$$\frac{\Gamma_1, \Box\Gamma_1, \Diamond\Gamma_3 \vdash A, \Gamma_2, \Diamond\Gamma_2, \Box\Gamma_4}{\Box\Gamma_1, \Diamond\Gamma_3 \vdash \Box A, \Diamond\Gamma_2, \Box\Gamma_4}\ \Box k45 \qquad\qquad \frac{\Gamma_1, \Box\Gamma_1, \Diamond\Gamma_3, A \vdash \Gamma_2, \Diamond\Gamma_2, \Box\Gamma_4}{\Box\Gamma_1, \Diamond\Gamma_3, \Diamond A \vdash \Diamond\Gamma_2, \Box\Gamma_4}\ \Diamond k45$$

$$\frac{\Gamma_1 \vdash \Gamma_2}{\Box\Gamma_1 \vdash \Diamond\Gamma_2}\ \Diamond d \qquad \frac{\Gamma_1, A \vdash \Gamma_2}{\Gamma_1, \Box A \vdash \Gamma_2}\ \Box t \qquad \frac{\Gamma_1 \vdash A, \Gamma_2}{\Gamma_1 \vdash \Diamond A, \Gamma_2}\ \Diamond t$$



Theorem (Soundness and completeness) TFAE

- Formula A is provable in logic $\begin{cases} K \text{ or } D \text{ or } T \\ K4 \text{ or } D4 \text{ or } S4 \\ K45 \text{ or } D45 \text{ or } S5 \end{cases}$
- Sequent $\vdash A$ is provable in system ... $+ \begin{Bmatrix} \Box k + \Diamond k \\ \Box k4 + \Diamond k4 \\ \Box k45 + \Diamond k45 \end{Bmatrix} + \Diamond d + \{\Box t + \Diamond t\}$

with cut!

Issues

(i) locality and separation :
$$\frac{\Gamma_1 \vdash A, \Gamma_2}{\Box\Gamma_1 \vdash \Box A, \Diamond\Gamma_2}\ \Box k \qquad \frac{\Gamma_1, A \vdash \Gamma_2}{\Box\Gamma_1, \Diamond A \vdash \Diamond\Gamma_2}\ \Diamond k$$

(ii) modularity : need to swap $\Box k$ for $\Box k4$ or $\Box k5$

would like : keep $\Box k$ and add $\Box 4$ and/or $\Box 5$

(iii) generality : no complete system for $\begin{cases} K5 \text{ or } D5 \\ KB \text{ or } DB \text{ or } TB \end{cases}$

and more Scott-Lemmon axioms

(iv) analyticity :

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Box A \vdash \Box A}\ \text{gid}}{\Diamond\Box A \vdash \Box A}\ \Diamond k45}{\Diamond\Box A \vdash \Box A, A}\ W_R \qquad \dfrac{\dfrac{\dfrac{}{A \vdash A}\ \text{gid}}{\Diamond\Box A, A \vdash A}\ W_L}{\Diamond\Box A, \Box A \vdash A}\ \Box t}{\dfrac{\Diamond\Box A \vdash A}{\vdash \Diamond\Box A \to A}\ \to_R}\ \text{cut}$$

$\Rightarrow$ HOW TO design sequent systems for modal logics that preserve these good properties?


Questions?