What are we doing here?

Mathematical Logic

- Model Theory
- Proof Theory
- Set Theory
- Recursion Theory

We are studying proof theory.
And its applications to modal logic.

Overview

1. Introduction to Modal Logics (Syntax and Semantics)
2. Sequent calculus for modal logic
3. First-Order Logic
4. Introduction to Focusing
5. Synthetic Rules and Bipoles
6. Exercises
7. Labelled Sequents
8. Synthetic Rules in Labelled Sequents
9. Path Logics
10. Nested Sequents
1. Lecture
The Syntax and Semantics of Modal Logics

Sonia Marin and Lutz Straßburger

Modal Formulas

Formulas:

\[ A, B ::= \top | \bot | a | A \land B | A \lor B | A \Rightarrow B | \neg A | \Box A | \Diamond A \]

where

- \( \top \) and \( \bot \) are constants representing truth and falsum
- \( a \) is a propositional variable, aka atomic variable or atom
- \( \land \) and \( \lor \) are the symbols for conjunction and disjunction
- \( \Rightarrow \) stands for implication
- \( \neg A \) is the negation of \( A \), sometimes also written as \( \bar{A} \)
- \( \Box A \) is read as “box \( A \)" or “\( A \) is necessary”
- \( \Diamond A \) is read as “diamond \( A \)” or “\( A \) is possible”
- \( \Box \) and \( \Diamond \) are called modalities

Axioms of Modal Logic K

- Axioms for classical propositional logic (for \( \supset, \bot \)):
  - \( A \Rightarrow (B \Rightarrow A) \)
  - \( (A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)) \)
  - \( ((A \Rightarrow \bot) \Rightarrow \bot) \Rightarrow A \)
- Modal axiom k:
  - \( \Box (A \Rightarrow B) \Rightarrow (\Box A \Rightarrow \Box B) \)
- The other connectives are defined via \( \bot, \supset, \Box \):
  - \( \neg A \equiv A \Rightarrow \bot \)
  - \( \Diamond A \equiv \neg \Box \neg A \)
  - \( A \lor B \equiv \neg (\neg A \Rightarrow \neg B) \)
  - \( A \land B \equiv \neg (\neg A \lor \neg B) \)
- Inference rules:
  - \[ \frac{A \quad A \Rightarrow B}{B} \quad \text{mp} \]
  - \[ \frac{A \Rightarrow B}{\Box A} \quad \text{nec} \]

Definition: A formula is provable (or a theorem) if it is either (a substitution instance of) an axiom, or can be derived via an instance of a rule mp or nec from provable formulas.
More Axioms — More Logics

\[ k: \Box(A \supset B) \supset (\Box A \supset \Box B) \]
\[ d: \Box A \supset \Diamond A \]
\[ t: \Box A \supset A \]
\[ b: A \supset \Box \Diamond A \]
\[ 4: \Box A \supset \Box \Box A \]
\[ 5: \Diamond A \supset \Box \Diamond A \]

For example:
\[ D5 = K + d + 5 \]
\[ S4 = K + t + 4 \]
\[ S5 = K + t + 5 = S4 + b = K + b + 4 \]

More general:
\[ g_{k,i,m,n}: \Box^k A \supset \Box^m \Diamond^n A \]

These are called Geach axioms or Scott-Lemmon axioms.

The Semantics of Modal Logic K

- A **frame** is a pair \( \langle W, R \rangle \), where \( W \neq \emptyset \) and \( R \subseteq W \times W \).
- A **model** is a triple \( \langle W, R, V \rangle \), where \( \langle W, R \rangle \) is a frame and \( V: A \rightarrow 2^W \) is a function from the set \( A \) of atomic variable to the powerset of \( W \).
- **Forcing**: \( w \models a \) if \( w \in V(a) \) and \( w \not\models a \) otherwise.

\[
\begin{align*}
  w \models T & \iff \text{true (i.e., always)} \\
  w \models \bot & \iff \text{false (i.e., never)} \\
  w \models \neg A & \iff w \not\models A \\
  w \models A \land B & \iff w \models A \text{ and } w \models B \\
  w \models A \lor B & \iff w \models A \text{ or } w \models B \\
  w \models A \supset B & \iff w \models A \text{ implies } w \models B \\
  w \models \Box A & \iff \text{for all } u \in W \text{ with } wRu \text{ we have } u \models A \\
  w \models \Diamond A & \iff \text{there is a } u \in W \text{ such that } wRu \text{ and } u \models A
\end{align*}
\]

The Semantics of Modal Logic K

- A formula \( A \) is **valid in a model** \( \langle W, R, V \rangle \) iff it is forced in all worlds, i.e., for all \( w \in W \) we have \( w \models A \).
- A formula \( A \) is **valid in a frame** \( \langle W, R \rangle \) iff it is valid in all models \( \langle W, R, V \rangle \), i.e., for all valuation functions \( V \) we have \( A \) is valid in \( \langle W, R, V \rangle \).

**Definition**: A formula \( A \) is **valid** iff it is valid in all frames.

- k plus 5 axioms means *a priori* 32 different logics
- some of them coincide → only 15 logics
- they form the so-called *modal cube*
- **Exercise 1.5**: Show that \( K + t + 5 \) and \( K + b + 4 \) and \( S4 + b \) are the same logic, i.e., all three prove the same theorems.

- This models are due to Saul Kripke. That’s why they are called *Kripke models*.
- The elements of \( W \) are called *worlds*.
- \( V \) is called the *valuation function* assigning each atomic variable a set of worlds. Sometimes it is given as a function \( A \times W \rightarrow \{0, 1\} \) or as a function \( W \rightarrow 2^A \).
- Basic idea: each world behaves individually like a model for Boolean logic. The binary relation \( R \) is called *accessibility relation* and is needed for interpreting the modalites.

- **Exercise 1.6**: Show that \( \Diamond A \implies \neg \Diamond \neg A \) is valid.
- **Exercise 1.7**: Show that \( \Box A \lor \Box \neg A \) is not valid.
- **Exercise 1.8**: Show that the \( k \)-axiom is valid.
- **Exercise 1.9**: Show that the axioms \( d, t, b, 4, \) and \( 5 \) are not valid.
Soundness and Completeness

**Theorem:** A formula $A$ is provable in modal logic $K$ iff it is valid.

- **Soundness:** If it is provable then it is valid.
  
  This means that you cannot prove wrong things.

- **Completeness:** If it is valid then it is provable.
  
  This means that you can prove everything that you want to be able to prove.

Exercise 1.10: Show soundness for $K$.
- First show that all axioms are valid.
- Then show that the inference rules mp and nec preserve validity.
- Completeness is shown later in this course.

More Logics — More Models

**Definition:** A frame $\langle W, R \rangle$ is
- **serial** iff for all $w \in W$ there is a $v \in W$ such that $wRv$;
- **reflexive** iff for all $w \in W$ we have $wRw$;
- **symmetric** iff for all $w, v \in W$, if $wRv$ then $vRw$;
- **transitive** iff for all $w, v, u \in W$, if $wRv$ and $vRu$ then $wRu$;
- **euclidean** iff for all $w, v, u \in W$, if $wRv$ and $wRu$ then $vRu$.

**Definition:** A model $\langle W, R, V \rangle$ is **serial** (resp. **reflexive**, **symmetric**, **transitive**, **euclidean**) if the frame $\langle W, R \rangle$ is.

Modal Cube again — Axioms and Frame Conditions

<table>
<thead>
<tr>
<th>axiom</th>
<th>frame condition</th>
<th>name</th>
</tr>
</thead>
<tbody>
<tr>
<td>d</td>
<td>$\Box A \supset \Box A$</td>
<td>$\forall w. \exists v. wRv$</td>
</tr>
<tr>
<td>t</td>
<td>$\Box A \supset \forall w. \Box A$</td>
<td>$\forall w. wRw$</td>
</tr>
<tr>
<td>b</td>
<td>$\Diamond A \subset A$</td>
<td>$\forall w. \forall v. wRv \supset vRw$</td>
</tr>
<tr>
<td>4</td>
<td>$\Box A \supset \Box A$</td>
<td>$\forall w. \forall v. wRv \land vRu \supset wRu$</td>
</tr>
<tr>
<td>5</td>
<td>$\Diamond A \supset \Diamond A$</td>
<td>$\forall w. \forall v. wRv \land wRu \supset vRu$</td>
</tr>
</tbody>
</table>

$KABC = K + a + b + c$

$ABC = K + a + b + c$

$S4 = KT4$

$S5 = S4 + S = KT5 = KB4$
Modal Cube again — Soundness and Completeness

**Definition:** Let \( A \) be a formula and \( KX \) be a logic in the modal cube.

- We say \( A \) is **provable** in \( KX \), written as \( \vdash_{KX} A \), if \( A \) is derivable from the axioms of \( KX \) via the rules mp and nec.

- We say \( A \) is **valid** in \( KX \), written as \( \models_{KX} A \), if \( A \) is valid in every model obeying the frame conditions corresponding to \( KX \).

**Theorem:** For all formulas \( A \) and logics \( KX \) in the modal cube:

\[
\vdash_{KX} A \iff \models_{KX} A
\]

### Exercises

- **Exercise 1.11:** Show that \( KT5 = KB4 \) using the frame conditions. I.e., show that every frame that is reflexive and Euclidean is also transitive and symmetric.

- **Exercise 1.12:** Prove soundness.

- **Exercise 1.13:** What has to be done to prove completeness?

- **Exercise 1.14:** Is \( (A \lor \lozenge B) \supset (\lozenge A \lor \lozenge B) \) provable in \( T \)? In \( K4 \)? In \( S4 \)? In \( B \)? In \( S5 \)?

- **Exercise 1.15:** In which logics is the formula \( A \lor \lozenge A \) provable?

- **Exercise 1.16:** What would be the corresponding frame conditions for the axioms \( \lozenge \lozenge A \lor \lozenge \lozenge B \)?

### Axioms versus Models

Two ways to check if a formula is “true”:

1. **Is is provable from axioms?**
   - **Easy** to check when it is provable (show the proof)
   - **Hard** to check when it is not provable (check all proofs?)

2. **Is is valid in all models?**
   - **Easy** to check when it is not valid (show a countermodel)
   - **Hard** to check when it is valid (check all models?)

**Proof theory:** exploring methods from both sides.

### Can we do better?

- The “easy” does not necessarily mean easy, as it can be very hard to find a proof or a countermodel. But once it is found, it is easy to check. But the “hard” really means hard. There is no easy way to check that something is not provable from a set of axioms, using only the axioms and the inference rules given so far, and there is no easy way to check that something is valid in all models, using only the notion of model presented so far.

- This course is about presenting methods of modern proof theory that unify the axiomatic and the model theoretic side, such that
  1. searching for a proof and searching for a countermodel becomes a bit easier, and
  2. the same method is used for proof search and countermodel search.