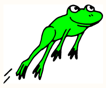
From Axioms to Rules:
The Factory of Modal Proof Systems

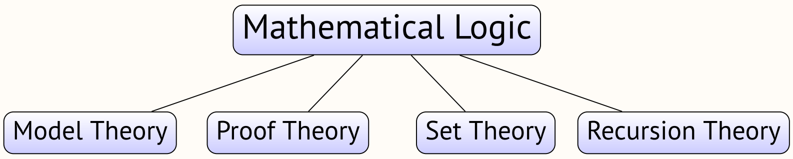
Sonia Marin and Lutz Straßburger

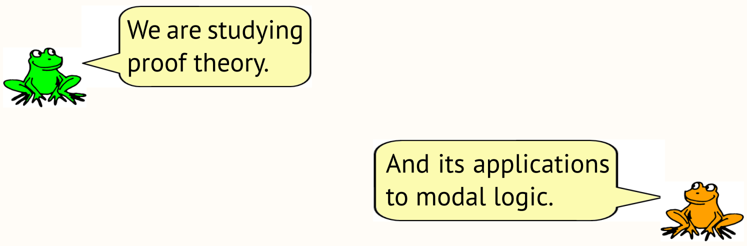
ESSLLI 2022 — Galway — August 15-19, 2022



These are the slides and notes for the course *"From Axioms to Rules: The Factory of Modal Proof System"* given at ESSLLI 2022, held from August 8 to 19, 2022, in Galway, Ireland.

What are we doing here?





Overview

1. Introduction to Modal Logics (Syntax and Semantics)
2. Sequent calculus for modal logic
3. First-Order Logic
4. Introduction to Focusing
5. Synthetic Rules and Bipoles
6. Exercises
7. Labelled Sequents
8. Synthetic Rules in Labelled Sequents
9. Path Logics
10. Nested Sequents

5 x 90 min $\Longrightarrow$ 10 x 45 min

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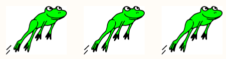
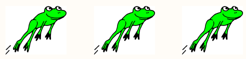


1. Lecture

The Syntax and Semantics of Modal Logics



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Modal Formulas

Formulas:

$$A, B ::= \top \mid \bot \mid a \mid A \wedge B \mid A \vee B \mid A \supset B \mid \neg A \mid \Box A \mid \Diamond A$$

where

- $\top$ and $\bot$ are constants representing *truth* and *falsum*
- a is a *propositional variable*, aka *atomic variable* or *atom*
- $\wedge$ and $\vee$ are the symbols for *conjunction* and *disjunction*
- $\supset$ stands for *implication*
- $\neg A$ is the *negation* of A, sometimes also written as $\bar{A}$
- $\Box A$ is read as *"box A"* or *"A is necessary"*
- $\Diamond A$ is read as *"diamond A"* or *"A is possible"*
- $\Box$ and $\Diamond$ are called *modalities*

- We are discussing here only monomodal logics, i.e., there is only one $\Box$ and one $\Diamond$. In multimodal logics, one can have many $\Box$-$\Diamond$ pairs.
- Besides the "necessary/possible" reading, there are many other possible interpretations, e.g. temporal, where $\Box A$ means "A holds in all futures" and $\Diamond A$ means "A holds in some futures". But we will not discuss the different readings of the modalities in this course.

Axioms of Modal Logic **K**

- Axioms for classical propositional logic (for $\supset, \bot$):
  - $A \supset (B \supset A)$
  - $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
  - $((A \supset \bot) \supset \bot) \supset A$
- Modal axiom k:
  - $\Box(A \supset B) \supset (\Box A \supset \Box B)$
- The other connectives are defined via $\bot, \supset, \Box$:
  - $\neg A \equiv A \supset \bot$ $\quad$ $\Diamond A \equiv \neg \Box \neg A$
  - $A \vee B \equiv \neg A \supset B$ $\quad$ $A \wedge B \equiv \neg(\neg A \vee \neg B)$
- Inference rules:

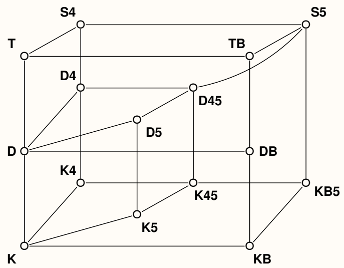
$$\mathsf{mp}\ \frac{A \qquad A \supset B}{B} \qquad\qquad \mathsf{nec}\ \frac{A}{\Box A}$$

Definition: A formula is *provable* (or *a theorem*) if it is either (a substitution instance of) an axiom, or can be derived via an instance of a rule mp or nec from provable formulas.

- Proof systems that use axioms and inference rules in this way are called *Hilbert systems* or *Frege systems*
- The axioms for classical logic presented here are due to Church. There are many other complete sets. See e.g., https://en.wikipedia.org/wiki/List_of_Hilbert_systems#Classical_propositional_calculus_systems for other possibilities.
- The axioms we present on this page define the *modal logic* **K**.
- **Exercise 1.1:** Define the connectives $\Diamond, \wedge, \vee$ only with $\bot, \supset, \Box$.
- **Exercise 1.2:** Show that $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$ is provable.
- **Exercise 1.3:** Is $\Diamond(A \supset B) \supset (\Diamond A \supset \Diamond B)$ also provable?
- **Exercise 1.4:** What about $\Diamond(A \supset B) \supset (\Box A \supset \Box B)$?

More Axioms — More Logics

k: $\Box(A \supset B) \supset (\Box A \supset \Box B)$
d: $\Box A \supset \Diamond A$
t: $\Box A \supset A$
b: $A \supset \Box\Diamond A$
4: $\Box A \supset \Box\Box A$
5: $\Diamond A \supset \Box\Diamond A$



For example: $\textbf{D5} = \textbf{K} + \text{d} + 5$
$\textbf{S4} = \textbf{K} + \text{t} + 4$
$\textbf{S5} = \textbf{K} + \text{t} + 5 = \textbf{S4} + \text{b} = \textbf{K} + \text{b} + 4$

More general:

$$g_{k,l,m,n}: \; \Diamond^k \Box^l A \supset \Box^m \Diamond^n A$$

These are called *Geach axioms* or *Scott-Lemmon axioms*.

- k plus 5 axioms means *a priori* 32 different logics
- some of them coincide
  $\rightarrow$ only 15 logics
- they form the so-called *modal cube*
- **Exercise 1.5:** Show that $\textbf{K} + \text{t} + 5$ and $\textbf{K} + \text{b} + 4$ and $\textbf{S4} + \text{b}$ are the same logic, i.e., all three prove the same theorems.

The Semantics of Modal Logic K

- A *frame* is a pair $\langle W, R\rangle$, where $W \neq \emptyset$ and $R \subseteq W \times W$.
- A *model* is a triple $\langle W, R, V\rangle$, where $\langle W, R\rangle$ is a frame and $V\colon \mathcal{A} \to 2^W$ is a function from the set $\mathcal{A}$ of atomic variable to the powerset of W.
- *Forcing*: $w \Vdash a$ if $w \in V(a)$ and $w \nVdash a$ otherwise.

$w \Vdash \top$	iff	true (i.e., always)
$w \Vdash \bot$	iff	false (i.e., never)
$w \Vdash \neg A$	iff	$w \nVdash A$
$w \Vdash A \wedge B$	iff	$w \Vdash A$ and $w \Vdash B$
$w \Vdash A \vee B$	iff	$w \Vdash A$ or $w \Vdash B$
$w \Vdash A \supset B$	iff	$w \Vdash A$ implies $w \Vdash B$
$w \Vdash \Box A$	iff	for all $u \in W$ with wRu we have $u \Vdash A$
$w \Vdash \Diamond A$	iff	there is a $u \in W$ such that wRu and $u \Vdash A$

- This models are due to Saul Kripke. That's why they are called *Kripke models*.
- The elements of W are called *worlds*.
- V is called the *valuation function* assigning each atomic variable a set of worlds. Sometimes it is given as a function $\mathcal{A} \times W \to \{0,1\}$ or as a function $W \to 2^{\mathcal{A}}$.
- Basic idea: each world behaves individually like a model for Boolean logic. The binary relation R is called *accessibility relation* and is needed for interpreting the modalites.

The Semantics of Modal Logic K

- A formula A is *valid in a model* $\langle W, R, V\rangle$ iff
  it is forced in all worlds, i.e.,
  for all $w \in W$ we have $w \Vdash A$.

- A formula A is *valid in a frame* $\langle W, R\rangle$ iff
  it is valid in all models $\langle W, R, V\rangle$, i.e.,
  for all valuation functions V we have A is valid in $\langle W, R, V\rangle$

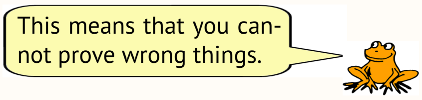
Definition: A formula A is *valid* iff it is valid in all frames.

- **Exercise 1.6:** Show that $\Diamond A \implies \neg\Box\neg A$ is valid.
- **Exercise 1.7:** Show that $\Box A \vee \Box\neg A$ is not valid.
- **Exercise 1.8:** Show that the k-axiom is valid.
- **Exercise 1.9:** Show that the axioms d, t, b, 4, and 5 are not valid.

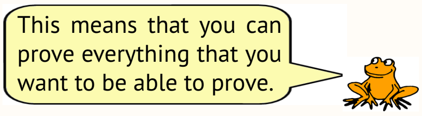
Soundness and Completeness

Theorem: A formula A is provable in modal logic **K** iff it is valid.

- *Soundness:* If it is provable then it is valid.



- *Completeness:* If it is valid then it is provable.



- **Exercise 1.10:** Show soundness for **K**.
  - First show that all axioms are valid.
  - Then show that the inference rules mp and nec preserve validity.
- Completeness is shown later in this course.

More Logics — More Models

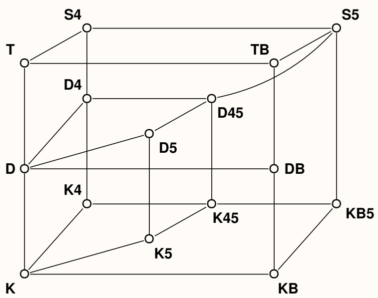
Definition: A frame $\langle W, R\rangle$ is

- *serial* iff for all $w \in W$ there is a $v \in W$ such that wRv;
- *reflexive* iff for all $w \in W$ we have wRw;
- *symmetric* iff for all $w, v \in W$, if wRv then vRw;
- *transitive* iff for all $w, v, u \in W$, if wRv and vRu then wRu;
- *euclidean* iff for all $w, v, u \in W$, if wRv and wRu then vRu.

Definition: A model $\langle W, R, V\rangle$ is *serial* (resp. *reflexive*, *symmetric*, *transitive*, *euclidean*) if the frame $\langle W, R\rangle$ is.

Modal Cube again — Axioms and Frame Conditions

	axiom	frame condition	name
d	$\Box A \supset \Diamond A$	$\forall w. \exists v. wRv$	serial
t	$\Box A \supset A$	$\forall w. wRw$	reflexive
b	$\Diamond\Box A \supset A$	$\forall w. \forall v. wRv \supset vRw$	symmetric
4	$\Box A \supset \Box\Box A$	$\forall w. \forall v. \forall u. wRv \wedge vRu \supset wRu$	transitive
5	$\Diamond A \supset \Box\Diamond A$	$\forall w. \forall v. \forall u. wRv \wedge wRu \supset vRu$	Euclidean



$$
\begin{aligned}
\textbf{KABC} &= \textbf{K} + \mathsf{a} + \mathsf{b} + \mathsf{c} \\
\textbf{ABC} &= \textbf{K} + \mathsf{a} + \mathsf{b} + \mathsf{c} \\
\textbf{S4} &= \textbf{KT4} \\
\textbf{S5} &= \textbf{S4} + 5 = \textbf{KT5} = \textbf{KB4}
\end{aligned}
$$

Modal Cube again — Soundness and Completeness

Definition: Let A be a formula and **KX** be a logic in the modal cube.

- We say A is *provable* in **KX**, written as $\vdash_{\mathbf{KX}} A$, if A is derivable from the axioms of **KX** via the rules mp and nec.
- We say A is *valid* in **KX**, written as $\models_{\mathbf{KX}} A$, if A is valid in every model obeying the frame conditions corresponding to **KX**.

Theorem: For all formulas A and logics **KX** in the modal cube:

$$\vdash_{\mathbf{KX}} A \quad \iff \quad \models_{\mathbf{KX}} A$$

- **Exercise 1.11:** Show that **KT5** = **KB4** using the frame conditions. I.e., show that every frame that is reflexive and Euclidean is also transitive and symmetric.
- **Exercise 1.12:** Prove soundness.
- **Exercise 1.13:** What has to be done to prove completeness?
- **Exercise 1.14:** Is $\Box(A \vee \Diamond B) \supset (\Box A \vee \Diamond B)$ provable in **T**? In **K4**? In **S4**? In **B**? In **S5**?
- **Exercise 1.15:** In which logics is the formula $A \supset \Box\Diamond A$ provable?
- **Exercise 1.16:** What would be the corresponding frame conditions for the axioms $\Diamond^k\Box^l A \supset \Box^m\Diamond^n A$?

Axioms versus Models

Two ways to check if a formula is "true":

1. Is is provable from *axioms*?
   - Easy to check when it is probable (show the proof)
   - Hard to check when it is not provable (check all proofs?)
2. Is is valid in all *models*?
   - Easy to check when it is not valid (show a countermodel)
   - Hard to check when it is valid (check all models?)

- The "easy" does not necessarily mean easy, as it can be very hard to find a proof or a countermodel. But once it is found, it is easy to check. But the "hard" really means hard. There is no easy way to check that something is not provable from a set of axioms, using only the axioms and the inference rules given so far, and there is no easy way to check that someting is valid in all models, using only the notion of model presented so far.
- This course is about presenting methods of modern proof theory that unify the axiomatic and the model theoretic side, such that
  1. searching for a proof and searching for a countermodel becomes a bit easier, and
  2. the same method is used for proof search and countermodel search.