From Proof Nets to Combinatorial Proofs
–
A New Approach to Hilbert's 24th Problem

10. Lecture

Normalization

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Intuitionistic composition

$$\Gamma \vdash A \quad + \quad A, \Delta \vdash B \quad \mapsto \quad \Gamma, \Delta \vdash B$$

For intuitionistic combinatorial proofs

Option 1: use a special left-implication connective $\blacktriangleright$

$$\Gamma \vdash A \quad + \quad A, \Delta \vdash B \quad \mapsto \quad \Gamma, A \blacktriangleright A, \Delta \vdash B$$

- The conclusion is still $\Gamma, \Delta \vdash B$ and $A \blacktriangleright A$ is *hidden*
- $\blacktriangleright$ is a *meta-connective* and may only occur at top level
- Cut-elimination is standard (local rewriting with *kingdoms*)

Option 2: compute the result directly

$$\Gamma \vdash A \quad + \quad A, \Delta \vdash B \quad \mapsto \quad \Gamma, \Delta \vdash B$$

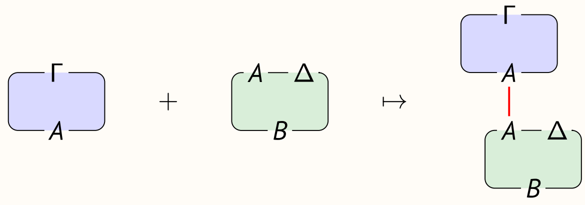
- Fixes the reduction strategy
- Game semantics and Geometry of Interaction do this

Option 1 is a standard way to add explicit cuts to proof nets in the sequent + axioms paradigm.

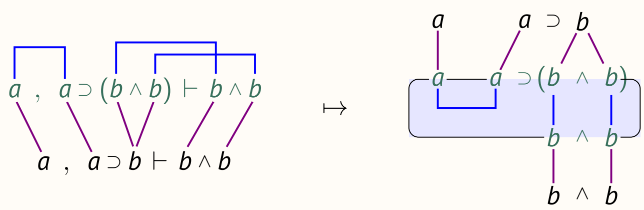
- **Exercise 10.1:** Add an explicit cut-connective to MLL proof nets in this way, and give the cut-elimination steps.

Option 3 (new!): build a tree

$$\Gamma \vdash A \quad + \quad A, \Delta \vdash B \quad \mapsto \quad \Gamma, \Delta \vdash B$$

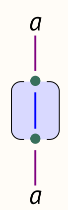


ICPs as nodes

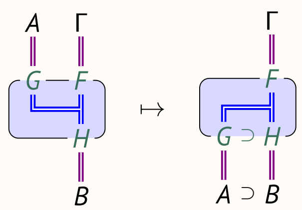


An ICP over a sequent $A_1, \ldots, A_n \vdash B$ becomes a node with premises/inputs A_1 through A_n and conclusion/output B.

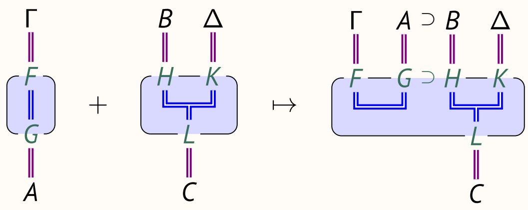
$$\frac{}{x : a \vdash x : a}\ Ax$$



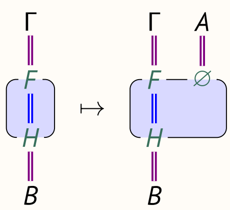
$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.\, M : A \supset B}\ {\supset}R$$



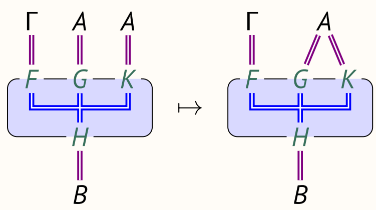
$$\frac{\Gamma \vdash M : A \qquad x : B, \Delta \vdash N : C}{\Gamma, f : A \supset B, \Delta \vdash N[f\,M/x] : C} \supset L$$



$$\frac{\Gamma \vdash M : B}{\Gamma, x : A \vdash M : B} W$$

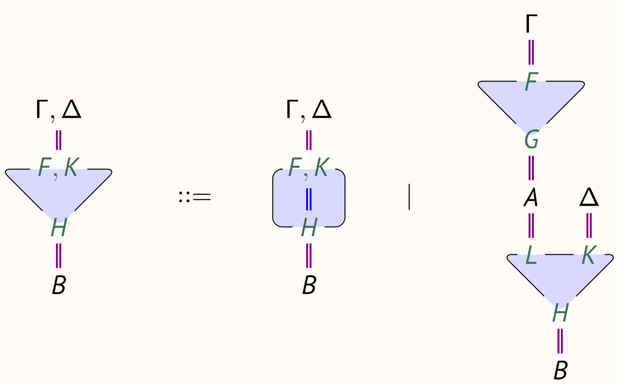


$$\frac{\Gamma, x : A, y : A \vdash M : B}{\Gamma, x : A \vdash M[x/y] : B} C$$

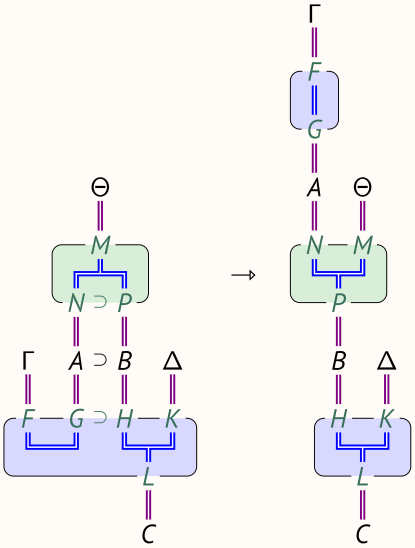


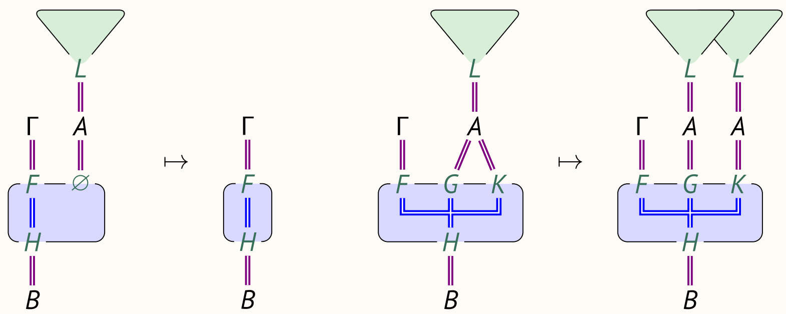
Building trees

$$\frac{\Gamma \vdash M : A \qquad x : A, \Delta \vdash N : B}{\Gamma, \Delta \vdash N[M/x] : B} Cut$$



Reduction





Theorem

ICP reduction is confluent and strongly normalizing.

Observations

- ICP reduction is sequent calculus cut-elimination, but
  - without permutations
  - with all cuts at top level
- Contraction on ICP trees requires graphs
- Abstraction on ICP trees is lambda-lifting
- ICP reduction is closed reduction (in λ-calculus: a redex $(\lambda x.M)N$ may only be reduced if it has no free variables)
- Reduction uses only sub-ICPs of those in the original tree