Intuitionistic composition

\[ \Gamma \vdash A + A, \Delta \vdash B \mapsto \rightarrow \Gamma, A \triangleright A, \Delta \vdash B \]

For intuitionistic combinatorial proofs

**Option 1:** use a special left-implication connective \(\triangleright\)

\[ \Gamma \vdash A + A, \Delta \vdash B \mapsto \rightarrow \Gamma, A \triangleright A, \Delta \vdash B \]

- The conclusion is still \(\Gamma, \Delta \vdash B\) and \(A \triangleright A\) is hidden
- \(\triangleright\) is a *meta-connective* and may only occur at top level
- Cut-elimination is standard (local rewriting with *kingdoms*)

**Option 2:** compute the result directly

\[ \Gamma \vdash A + A, \Delta \vdash B \mapsto \rightarrow \Gamma, \Delta \vdash B \]

- Fixes the reduction strategy
- Game semantics and Geometry of Interaction do this

Option 1 is a standard way to add explicit cuts to proof nets in the sequent + axioms paradigm.

**Exercise 10.1:** Add an explicit cut-connective to MLL proof nets in this way, and give the cut-elimination steps.
Option 3 (new!): build a tree

\[ \Gamma \vdash A + A, \Delta \vdash B \implies \Gamma, \Delta \vdash B \]

ICPs as nodes

\[ a, a \supset (b \land b) \vdash b \land b \]

An ICP over a sequent \( A_1, \ldots, A_n \vdash B \) becomes a node with premises/inputs \( A_1 \) through \( A_n \) and conclusion/output \( B \).
\[
\Gamma, M : A, x : B, \Delta \vdash N : C \\
\Gamma, f : A \supset B, \Delta \vdash N[f/M] : C
\]

\[
\Gamma \\
\begin{array}{c}
\begin{array}{c}
F \quad G \\
\hline
A
\end{array}
\end{array}
\quad + \\
\begin{array}{c}
\begin{array}{c}
B \quad \Delta \\
\hline
H \quad K
\end{array}
\end{array}
\quad \mapsto \\
\begin{array}{c}
\begin{array}{c}
\Gamma \quad A \supset B \quad \Delta \\
\hline
G \quad H \quad K
\end{array}
\end{array}
\]

\[
\Gamma \vdash M : B \\
\Gamma, x : A \vdash M : B
\]

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
F \quad G \\
\hline
H \quad K \\
\hline
B
\end{array}
\end{array}
\end{array}
\quad \mapsto \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
F \quad G \\
\hline
H \quad K
\end{array}
\end{array}
\end{array}
\end{array}
\]

\[
\Gamma, x : A, y : A \vdash M : B \\
\Gamma, x : A \vdash M[x/y] : B
\]

Building trees

\[
\Gamma \vdash M : A, x : A, \Delta \vdash N : B \\
\Gamma, \Delta \vdash N[M/x] : B
\]
Theorem

ICP reduction is confluent and strongly normalizing.
Observations

- ICP reduction is sequent calculus cut-elimination, but
  - without permutations
  - with all cuts at top level
- Contraction on ICP trees requires graphs
- Abstraction on ICP trees is lambda-lifting
- ICP reduction is closed reduction (in λ-calculus: a redex 
  \((\lambda x.M)N\) may only be reduced if it has no free variables)
- Reduction uses only sub-ICPs of those in the original tree