From Proof Nets to Combinatorial Proofs
—
A New Approach to Hilbert's 24th Problem

8. Lecture

First-Order Combinatorial Proofs

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First-Order Combinatorial Proofs

Definition:

A *first-order combinatorial proof* of a formula A is a skew bifibration $\phi\colon \mathcal{C} \to \mathcal{G}$ where $\mathcal{C}$ is a fonet and $\mathcal{G}$ is the graph of the formula A.

First-Order Formulas

Terms: $t \;::=\; x \mid f(t_1, \ldots, t_n)$

Atoms: $a \;::=\; 1 \mid 0 \mid p(t_1, \ldots, t_n) \mid \bar{p}(t_1, \ldots, t_n)$

Formulas: $A \;::=\; a \mid A \wedge A \mid A \vee A \mid \exists x.A \mid \forall x.A$

Sequents: $\Gamma \;::=\; A_1, A_2, \ldots, A_m$

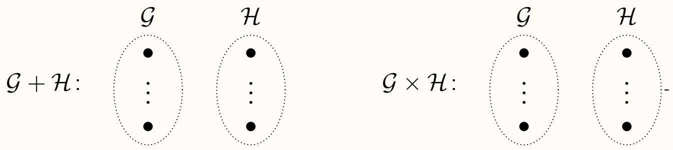
Negation:

$$\bar{1} = 0 \qquad \overline{p(t_1, \ldots, t_n)} = \bar{p}(t_1, \ldots, t_n)$$
$$\bar{0} = 1 \qquad \overline{\bar{p}(t_1, \ldots, t_n)} = p(t_1, \ldots, t_n)$$
$$\overline{\exists x.A} = \forall x.\bar{A} \qquad \overline{A \wedge B} = \bar{A} \;\vee\; \bar{B}$$
$$\overline{\forall x.A} = \exists x.\bar{A} \qquad \overline{A \vee B} = \bar{A} \;\wedge\; \bar{B}$$

Rectified Formulas: all bound variables are distinct from one another and from all free variables

Graphs of Formulas

Operations on (undirected) graphs:



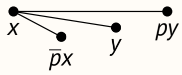
Mapping $[\![\cdot]\!]$ *from formulas to (labelled) graphs:*

$$[\![a]\!] = \bullet a \quad \text{(for any atom } a\text{)}$$

$$[\![A \vee B]\!] = [\![A]\!] + [\![B]\!] \qquad [\![\exists x.A]\!] = \bullet x \times [\![A]\!]$$

$$[\![A \wedge B]\!] = [\![A]\!] \times [\![B]\!] \qquad [\![\forall x.A]\!] = \bullet x + [\![A]\!]$$

Example: $[\![\, \exists x.(\bar{p}x \vee (\forall y.py)) \,]\!] =$



Definition: A *fograph (first-order graph)* is the graph $[\![A]\!]$ of a formula A.

Graphs of Formulas

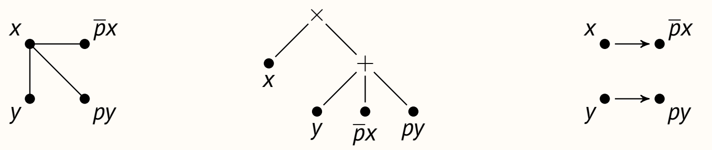
Congruence Relation $\equiv$ *on Formulas:*

$$A \wedge B \equiv B \wedge A \qquad (A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$$
$$A \vee B \equiv B \vee A \qquad (A \vee B) \vee C \equiv A \vee (B \vee C)$$
$$\forall x.\forall y.A \equiv \forall y.\forall x.A \qquad \forall x.(A \vee B) \equiv (\forall x.A) \vee B$$
$$\exists x.\exists y.A \equiv \exists y.\exists x.A \qquad \exists x.(A \wedge B) \equiv (\exists x.A) \wedge B \qquad (x \text{ not free in } B)$$

Theorem: For rectified formulas A and B we have

$$A \equiv B \iff [\![A]\!] = [\![B]\!]$$

Example: $\exists x.(\bar{p}x \vee (\forall y.py)) \equiv \exists x \forall y (py \vee \bar{p}x)$



cograph *cotree* *binding graph*

First-Order Combinatorial Proofs

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Fonets

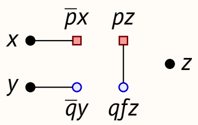
Linked Fograph: partially colored fograph, where
- each color consists of two literals with dual predicate symbols
- every literal is either 1-labelled or in a link

Dualizer: substitution unifying all the links

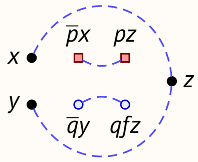
Dependency: pair $\{\bullet x, \bullet y\}$ such that most general dualizer assigns to the existential x a term containing the universal y

Leap Graph: vertices: as in the fograph
edges: links and dependencies

Example:



$[x/z, y/fz]$

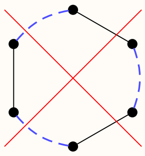


linked fograph *dualizer* *leap graph*

Fonets

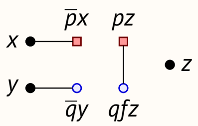
Definition: A set of vertices *induces a bimatching* if it induces a matching in the fograph and a matching in the leap graph.

Definition: A *fonet* (or *first-order net*) is a linked fograph which has a dualizer but no induced bimatching.

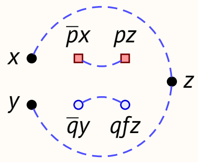


forbidden

Example:



$[x/z, y/fz]$



linked fograph *dualizer* *leap graph*

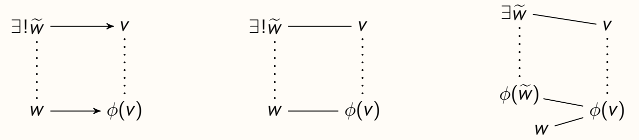
- Observe that "no bimatching" is the same condition as "critically chorded" for RB-cograph. The difference is that here the "blue" edges (the edges of the leap graph) do not form a perfect matching for the vertex set.
- **Exercise 8.1:** Show that Retoré's "handsome proof nets" are a special case of fonets.

First-Order Combinatorial Proofs

Definition:

A *first-order combinatorial proof* of a formula A is a skew bifibration $\phi\colon \mathcal{C} \to \mathcal{G}$ where $\mathcal{C}$ is a fonet and $\mathcal{G}$ is the graph of the formula A.

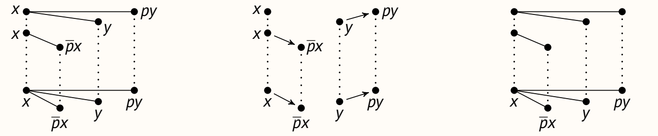
Skew Bifibrations



fibration (directed graphs) — *fibration (undirected graphs)* — *skew fibration (undirected graphs)*

Definition: Let $\mathcal{G}$ and $\mathcal{H}$ be fographs. A *skew bifibration* $\phi\colon \mathcal{G} \to \mathcal{H}$ is a label- and existential-preserving graph homomorphism that is a skew fibration on the underlying cographs and a fibration on the binding graphs.

Example:



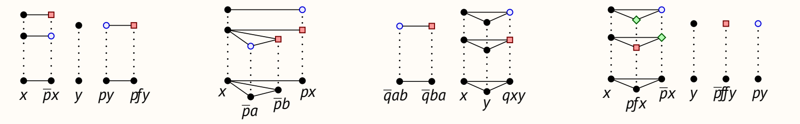
cograph skew fibration — *binding graph fibration* — *omitting labels upstairs*

First-Order Combinatorial Proofs

Definition:

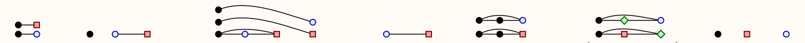
A *first-order combinatorial proof (focp)* of A is a skew bifibration $\phi\colon \mathcal{C} \to \mathcal{G}$ where $\mathcal{C}$ is a fonet and $\mathcal{G}$ is the graph of the formula A.

Examples:



$(\forall x\, px) \supset \forall y\,(py \wedge \bar{p}fy)$ — $\exists x\,(pa \vee pb \supset px)$ — $qab \vee qba \supset \exists x\, \exists y\, qxy$ — $(\forall x(pfx \supset px)) \supset \forall y\,(pffy \supset py)$

More Compact Notation:



$(\forall x\, px) \supset \forall y\,(py \wedge \bar{p}fy)$ — $\exists x\,(pa \vee pb \supset px)$ — $qab \vee qba \supset \exists x\, \exists y\, qxy$ — $(\forall x(pfx \supset px)) \supset \forall y\,(pffy \supset py)$

Theorem:

First-order combinatorial proofs are sound and complete for first-order logic.

- This definition of *first-order combinatorial proofs* and this theorem are due to Hughes:
  - *Dominic Hughes:* **"First-order Proofs Without Syntax".** *arXiv preprint arXiv:1906.11236, 2019*
- **Exercise 8.2:** Show that the combinatorial proofs for classical propostional logic are indeed a special case of the first-order combinatorial proofs.

First-Order Sequent Calculus

$$\mathsf{ax}\ \frac{}{\vdash a, \bar{a}} \qquad \vee\ \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B} \qquad \wedge\ \frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \wedge B, \Delta}$$

$$1\ \frac{}{\vdash 1} \qquad \mathsf{mix}\ \frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}$$

$$\exists\ \frac{\vdash \Gamma, A[x/t]}{\vdash \Gamma, \exists x.A} \qquad \forall\ \frac{\vdash \Gamma, A}{\vdash \Gamma, \forall x.A}\ (x \text{ not free in } \Gamma)$$

$$\mathsf{cont}\ \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \qquad \mathsf{weak}\ \frac{\vdash \Gamma}{\vdash \Gamma, A}$$

LK1: sequent calculus for first-order logic (all rules)

MLL1: linear fragment (dashed box)

Relation between LK1 and focp:

MLL1 proof	→	fonet	✓
LK1 proof	→	focp	✓
fonet	→	MLL1 proof	✓
focp	↛	LK1 proof	✗

- **Exercise 8.3:** Are there sequent calculus derivations corresponding to the example of combinatorial proofs on the previous slide? If yes, give them.

First-Order Deep Inference System

$$\forall\,\frac{1}{\forall x.1} \qquad \mathsf{ai}\,\frac{1}{a\vee\bar a} \qquad 1\,\frac{A}{A\wedge 1} \qquad \mathsf{w}\,\frac{A}{A\vee B} \qquad \mathsf{m}\,\frac{(A\wedge C)\vee(B\wedge D)}{(A\vee B)\wedge(C\vee D)} \qquad \mathsf{ac}\,\frac{a\vee a}{a}$$

$$\mathsf{s}\,\frac{A\wedge(B\vee C)}{(A\wedge B)\vee C} \qquad \mathsf{mix}\,\frac{A\wedge B}{A\vee B} \qquad \mathsf{m}_\forall\,\frac{(\forall x.A)\vee(\forall x.B)}{\forall x.(A\vee B)} \qquad \mathsf{m}_\exists\,\frac{(\exists x.A)\vee(\exists x.B)}{\exists x.(A\vee B)}$$

$$\exists\,\frac{A[x/t]}{\exists x.A} \qquad \equiv\,\frac{B}{A}\ (\text{where } A\equiv B) \qquad \mathsf{w}_\forall\,\frac{A}{\forall x.A}\ (x \text{ not free in } A) \qquad \mathsf{c}_\forall\,\frac{\forall x.\forall x.A}{\forall x.A}$$

KS1: deep inference system for first-order logic (all rules)
MLS1: linear fragment (dashed box)

Relation between KS1 and focp:

MLS1 proof	→	fonet	✓
KS1 proof	→	focp	✓
fonet	→	MLS1 proof	✓
focp	→	KS1 proof	✓

First-Order Deep Inference System

$$\forall\,\frac{1}{\forall x.1} \qquad \mathsf{ai}\,\frac{1}{a\vee\bar a} \qquad 1\,\frac{A}{A\wedge 1}$$

$$\mathsf{s}\,\frac{A\wedge(B\vee C)}{(A\wedge B)\vee C} \qquad \mathsf{mix}\,\frac{A\wedge B}{A\vee B}$$

$$\exists\,\frac{A[x/t]}{\exists x.A} \qquad \equiv\,\frac{B}{A}\ (\text{where } A\equiv B)$$

$$\mathsf{w}\,\frac{A}{A\vee B} \qquad \mathsf{m}\,\frac{(A\wedge C)\vee(B\wedge D)}{(A\vee B)\wedge(C\vee D)} \qquad \mathsf{ac}\,\frac{a\vee a}{a}$$

$$\mathsf{m}_\forall\,\frac{(\forall x.A)\vee(\forall x.B)}{\forall x.(A\vee B)} \qquad \mathsf{m}_\exists\,\frac{(\exists x.A)\vee(\exists x.B)}{\exists x.(A\vee B)}$$

$$\mathsf{w}_\forall\,\frac{A}{\forall x.A}\ (x \text{ not free in } A) \qquad \mathsf{c}_\forall\,\frac{\forall x.\forall x.A}{\forall x.A}$$

$$\mathsf{c}\,\frac{A\vee A}{A}$$

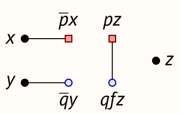
Theorem:

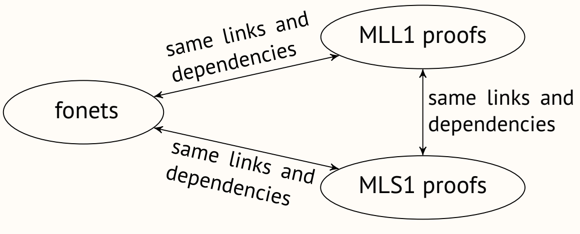
$$\begin{array}{c} 1 \\ \mathsf{KS1}\,\Vert \\ A \end{array} \quad\leftrightarrow\quad \begin{array}{c} 1 \\ \mathsf{MLS1}\,\Vert \\ A_3 \\ \{\mathsf{w},\mathsf{c},\equiv\}\,\Vert \\ A \end{array} \quad\leftrightarrow\quad \begin{array}{c} 1 \\ \{\forall,\mathsf{ai},1\}\,\Vert \\ A_5 \\ \{\mathsf{s},\mathsf{mix},\equiv\}\,\Vert \\ A_4 \\ \{\exists,\equiv\}\,\Vert \\ A_3 \\ \{\mathsf{m},\mathsf{m}_\forall,\mathsf{m}_\exists,\equiv\}\,\Vert \\ A_2 \\ \{\mathsf{ac},\mathsf{c}_\forall\}\,\Vert \\ A_1 \\ \{\mathsf{w},\mathsf{w}_\forall,\equiv\}\,\Vert \\ A \end{array}$$

First-Order Linear Proofs

Theorems:

$$\forall\,\dfrac{\exists\,\dfrac{\exists\,\dfrac{\wedge\,\dfrac{\mathsf{ax}\,\dfrac{}{\vdash pz,\bar pz} \qquad \mathsf{ax}\,\dfrac{}{\vdash qfz,\bar qfz}}{\vdash pz\wedge qfz,\bar pz,\bar qfz}}{\vdash pz\wedge qfz,\bar pz,\exists\bar qy}}{\vdash pz\wedge qfz,\exists x.\bar px,\exists\bar qy}}{\vdash \forall z.pz\wedge qfz,\exists x.\bar px,\exists\bar qy}$$





$$\begin{array}{rc} & 1 \\ 1,\forall & \overline{\forall z.(1\wedge 1)} \\ \mathsf{ai}{\downarrow},\mathsf{ai}{\downarrow} & \overline{\forall z.((pz\vee\bar pz)\wedge(qfz\vee\bar qfz))} \\ \mathsf{s} & \overline{\forall z.(((pz\vee\bar pz)\wedge qfz)\vee\bar qfz)} \\ \mathsf{s},\equiv & \overline{(\forall z.pz\wedge qfz)\vee\bar pz\vee\bar qfz} \\ \exists,\exists & \overline{(\forall z.pz\wedge qfz)\vee(\exists x.\bar px)\vee(\exists\bar qy)} \end{array}$$

- **Exercise 8.4:** Give the deep inference derivations for the four compinatorial proofs above.

Resource Management in First-Order Proofs

Theorem:

$$\begin{array}{c} A \\ \{\mathsf{w},\mathsf{c},\equiv\}\,\Big\| \\ B \end{array} \quad\Longleftrightarrow\quad \begin{array}{c} A \\ \{\mathsf{m},\mathsf{m}_\forall,\mathsf{m}_\exists,\equiv\}\,\Big\| \\ A' \\ \{\mathsf{ac},\mathsf{c}_\forall\}\,\Big\| \\ B' \\ \{\mathsf{w},\mathsf{w}_\forall,\equiv\}\,\Big\| \\ B \end{array} \quad\Longleftrightarrow\quad \begin{array}{c} [\![A]\!] \\ \Big\downarrow \text{skew bifibration} \\ [\![B]\!] \end{array}$$

$$\mathsf{m}\,\frac{(A\wedge C)\vee(B\wedge D)}{(A\vee B)\wedge(C\vee D)} \qquad \mathsf{m}_\forall\,\frac{(\forall x.A)\vee(\forall x.B)}{\forall x.(A\vee B)} \qquad \mathsf{m}_\exists\,\frac{(\exists x.A)\vee(\exists x.B)}{\exists x.(A\vee B)}$$

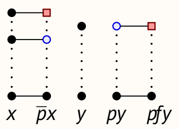
$$\mathsf{c}\,\frac{A\vee A}{A} \qquad \mathsf{ac}\,\frac{a\vee a}{a} \qquad \mathsf{c}_\forall\,\frac{\forall x.\forall x.A}{\forall x.A} \qquad \mathsf{w}\,\frac{A}{A\vee B} \qquad \mathsf{w}_\forall\,\frac{A}{\forall x.A}\ (x \text{ not free in } A)$$

Example

$$\begin{array}{rc}
\mathsf{ax},\ \mathsf{ax} & \vdash \bar{p}y, py \qquad \vdash \bar{p}fy, pfy \\ \hline
\wedge & \vdash \bar{p}y, \bar{p}fy, py\wedge pfy \\ \hline
\exists & \vdash \bar{p}y, \exists x.\bar{p}x, py\wedge pfy \\ \hline
\exists & \vdash \exists x.\bar{p}x, \exists x.\bar{p}x, py\wedge pfy \\ \hline
\forall & \vdash \exists x.\bar{p}x, \exists x.\bar{p}x, \forall y.(py\wedge pfy) \\ \hline
\mathsf{cont} & \vdash \exists x.\bar{p}x, \forall y.(py\wedge pfy) \\ \hline
\vee & \vdash (\exists x.\bar{p}x)\vee(\forall y.(py\wedge pfy))
\end{array}$$

$$\begin{array}{rc}
 & 1 \\ \hline
\forall & \forall y.1 \\ \hline
1 & \forall y.(1\wedge 1) \\ \hline
\mathsf{ai} & \forall y.((\bar{p}y\vee py)\wedge 1) \\ \hline
\mathsf{ai} & \forall y.((\bar{p}y\vee py)\wedge(pfy\vee\bar{p}fy)) \\ \hline
\equiv & \forall y.(\bar{p}y\vee(py\wedge(pfy\vee\bar{p}fy))) \\ \hline
\mathsf{s} & \forall y.(\bar{p}y\vee((py\wedge pfy)\vee\bar{p}fy)) \\ \hline
\equiv & \forall y.((\bar{p}y\vee\bar{p}fy)\vee(py\wedge pfy)) \\ \hline
\exists & \forall y.((\bar{p}y\vee(\exists x.\bar{p}x))\vee(py\wedge pfy)) \\ \hline
\exists & \forall y.(((\exists x.\bar{p}x)\vee(\exists x.\bar{p}x))\vee(py\wedge pfy)) \\ \hline
\equiv & (\exists x.\bar{p}x)\vee(\exists x.\bar{p}x)\vee(\forall y.(py\wedge pfy)) \\ \hline
\mathsf{m}_\exists & (\exists x.\bar{p}x\vee\bar{p}x)\vee(\forall y.(py\wedge pfy)) \\ \hline
\mathsf{ac} & (\exists x.\bar{p}x)\vee(\forall y.(py\wedge pfy))
\end{array}$$

$\searrow \qquad \swarrow$



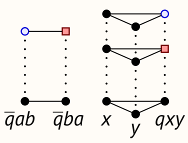
$(\forall x\, px) \supset \forall y\,(py\wedge pfy)$

Example

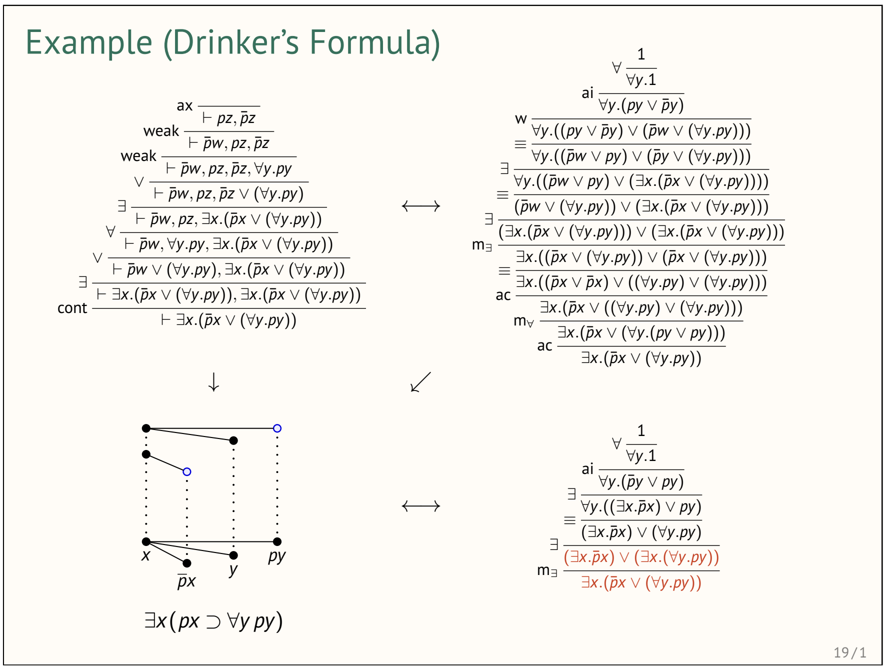
$$\begin{array}{rc}
\mathsf{ax},\ \mathsf{ax} & \vdash \bar{q}ab, qab \qquad \vdash \bar{q}ba, qba \\ \hline
\wedge & \vdash \bar{q}ab\wedge\bar{q}ba, qab, qba \\ \hline
\exists,\exists & \vdash \bar{q}ab\wedge\bar{q}ba, \exists x.\exists y.qxy, \exists x.\exists y.qxy \\ \hline
\mathsf{cont} & \vdash \bar{q}ab\wedge\bar{q}ba, \exists x.\exists y.qxy \\ \hline
\vee & \vdash (\bar{q}ab\wedge\bar{q}ba)\vee(\exists x.\exists y.qxy)
\end{array}$$

$$\begin{array}{rc}
 & 1 \\ \hline
\mathsf{ai} & qab\vee\bar{q}ab \\ \hline
1 & (qab\vee\bar{q}ab)\wedge 1 \\ \hline
\mathsf{ai} & (qab\vee\bar{q}ab)\wedge(\bar{q}ba\vee qba) \\ \hline
\mathsf{s} & ((qab\vee\bar{q}ab)\wedge\bar{q}ba)\vee qba \\ \hline
\equiv & (\bar{q}ba\wedge(qab\vee\bar{q}ab))\vee qba \\ \hline
\mathsf{s} & ((\bar{q}ba\wedge\bar{q}ab)\vee qab)\vee qba \\ \hline
\equiv & (\bar{q}ab\wedge\bar{q}ba)\vee(qab\vee qba) \\ \hline
\exists,\exists & (\bar{q}ab\wedge\bar{q}ba)\vee((\exists x.\exists y.qxy)\vee(\exists x.\exists y.qxy)) \\ \hline
\mathsf{m}_\exists & (\bar{q}ab\wedge\bar{q}ba)\vee(\exists x.((\exists y.qxy)\vee(\exists y.qxy))) \\ \hline
\mathsf{m}_\exists & (\bar{q}ab\wedge\bar{q}ba)\vee(\exists x.\exists y.(qxy\vee qxy)) \\ \hline
\mathsf{ac} & (\bar{q}ab\wedge\bar{q}ba)\vee(\exists x.\exists y.qxy)
\end{array}$$

$\searrow \qquad \swarrow$



$qab\vee qba \ \supset\ \exists x\,\exists y\, qxy$



- This formula is called the *drinker's formula* because it can be read as "In every bar there is a person x, such that whenever that person is drinking then everyone else is also dinking": $\exists x(px \supset \forall y\, py)$.
- That formula is interesting for several reasons:
  - It is not valid intuitionistically.
  - In the sequent calculus the whole formula needs to be duplicated first. This can be simulated in deep inference. However, in deep inference and in combinatorial proofs, this duplication is not necessary.