From Proof Nets to Combinatorial Proofs

—

A New Approach to Hilbert's 24th Problem

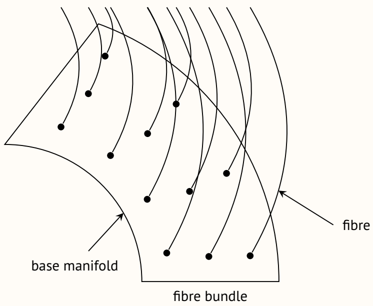
6. Lecture

Fibrations and Skew Fibrations

Willem Heijltjes and Lutz Straßburger

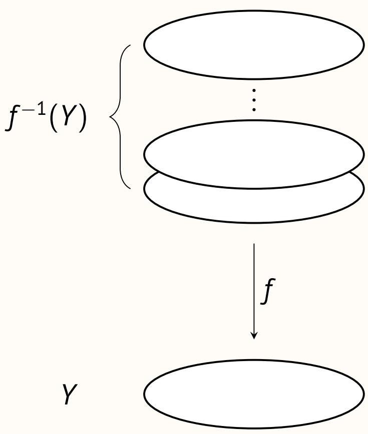
What are Fibrations?

- originate in topology as a generalization of fiber bundles:



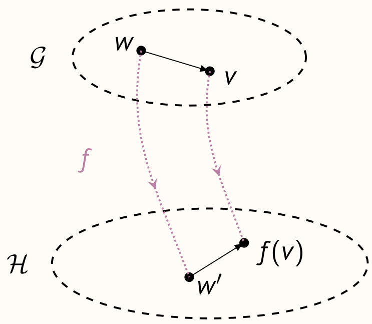
What are Fibrations?

- fibrations in topology:



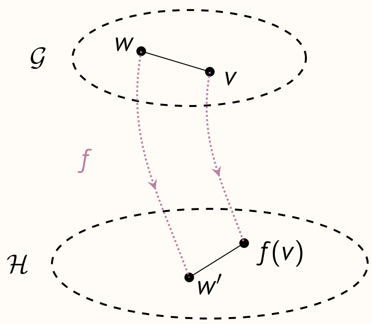
Fibrations in Directed Graphs

Definition:
A *fibration* is a graph homomorphism $f\colon \mathcal{G} \to \mathcal{H}$
such that for all $v \in V_{\mathcal{G}}$ and $w' \in V_{\mathcal{H}}$, if $w' \rightarrow_{\mathcal{H}} f(v)$
then there is a unique $w \in V_{\mathcal{G}}$ with $w \rightarrow_{\mathcal{G}} v$ and $f(w) = w'$.



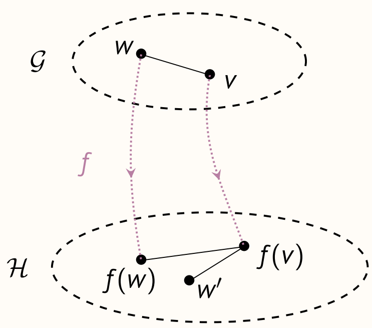
Fibrations in Undirected Graphs

Definition:
A *fibration* is a graph homomorphism $f\colon \mathcal{G} \to \mathcal{H}$
such that for all $v \in V_{\mathcal{G}}$ and $w' \in V_{\mathcal{H}}$, if $\{w', f(v)\} \in E_{\mathcal{H}}$
then there is a unique $w \in V_{\mathcal{G}}$ with $\{w, v\} \in E_{\mathcal{G}}$ and $f(w) = w'$.



Skew Fibrations in Undirected Graphs

Definition:
A *skew fibration* is a graph homomorphism $f\colon \mathcal{G} \to \mathcal{H}$
such that for all $v \in V_{\mathcal{G}}$ and $w' \in V_{\mathcal{H}}$, if $\{w', f(v)\} \in E_{\mathcal{H}}$
then there is a w with $\{w, v\} \in E_{\mathcal{G}}$ and $\{w', f(w)\} \notin E_{\mathcal{H}}$.

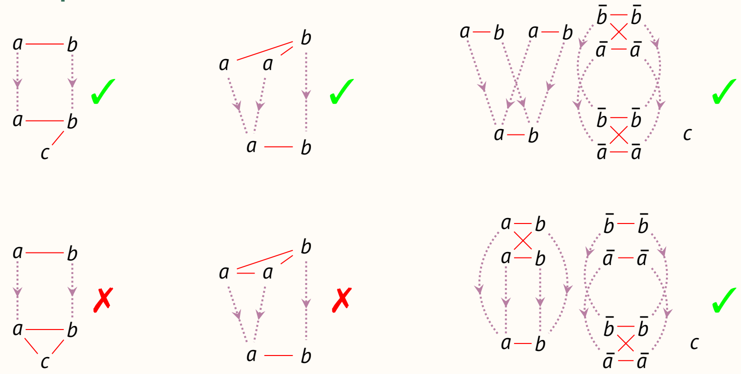


- In skew fibrations, two conditions that hold in fibrations are dropped:
  1. uniqueness of w is not demanded anymore, only existence
  2. we no longer demand that $f(w) = w'$ but only that there is no edge between the two.
- Note that in an undirected graph there are no reflexive edges. That means that $f(w) = w'$ is allowed, and a fibrations is indeed a special case of a skew fibration.
- The term *skew fibration* is due to Hughes:
  - *Dominic Hughes:* **"Proofs Without Syntax".** *Annals of Mathematics, vol. 164, no. 3, pp. 1065–1076, 2006*

Skew Fibrations in Undirected Graphs

A *skew fibration* is a graph homomorphism $f : \mathcal{G} \to \mathcal{H}$ such that for all $v \in V_{\mathcal{G}}$ and $w' \in V_{\mathcal{H}}$, if $\{w', f(v)\} \in E_{\mathcal{H}}$ then there is a w with $\{w, v\} \in E_{\mathcal{G}}$ and $\{w', f(w)\} \notin E_{\mathcal{H}}$.

Examples:



SKSg: A Deep Inference System for Classical Logic

$$\mathsf{i}{\downarrow}\,\frac{\top}{\bar{A} \vee A} \qquad \mathsf{i}{\uparrow}\,\frac{A \wedge \bar{A}}{\bot}$$

$$\mathsf{s}\,\frac{(C \vee A) \wedge B}{C \vee (A \wedge B)}$$

$$\mathsf{c}{\downarrow}\,\frac{A \vee A}{A} \qquad \mathsf{c}{\uparrow}\,\frac{A}{A \wedge A}$$

$$\mathsf{w}{\downarrow}\,\frac{\bot}{A} \qquad \mathsf{w}{\uparrow}\,\frac{A}{\top}$$

down-fragment SKSg↓ | up-fragment SKSg↑

rewriting modulo

$$A \wedge B = B \wedge A \qquad A \wedge (B \wedge C) = (A \wedge B) \wedge C \qquad A \wedge \top = A$$
$$A \vee B = B \vee A \qquad A \vee (B \wedge C) = (A \vee B) \vee C \qquad A \vee \bot = A$$

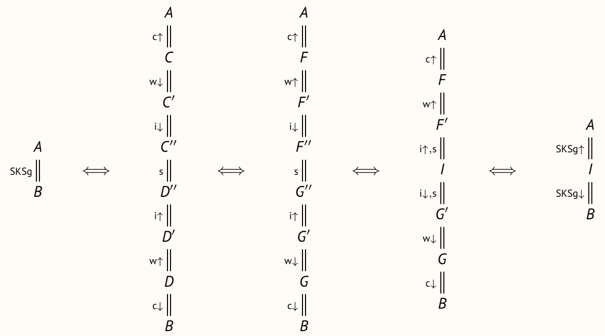
- For details on SKS consult:
  - *Kai Brünnler and Alwen Tiu:* **"A Local System for Classical Logic".** *LPAR 2001, pp. 347–361*
  - *Kai Brünnler:* **"Deep Inference and Symmetry for Classical Proofs".** *Ph.D. thesis, Technische Universität Dresden, 2003*

SKSg: A Deep Inference System for Classical Logic

$$\mathsf{i}{\downarrow}\,\frac{\top}{\bar{A} \vee A} \qquad \mathsf{s}\,\frac{(C \vee A) \wedge B}{C \vee (A \wedge B)} \qquad \mathsf{i}{\uparrow}\,\frac{A \wedge \bar{A}}{\bot}$$

$$\mathsf{c}{\downarrow}\,\frac{A \vee A}{A} \qquad \mathsf{w}{\downarrow}\,\frac{\bot}{A} \qquad \mathsf{w}{\uparrow}\,\frac{A}{\top} \qquad \mathsf{c}{\uparrow}\,\frac{A}{A \wedge A}$$

Theorem: (Decomposition and Interpolation)



Decomposition for the Down-Fragment

$$\mathsf{SKSg}{\downarrow}: \qquad \mathsf{i}{\downarrow}\,\frac{\top}{\bar{A} \vee A} \qquad \mathsf{s}\,\frac{(C \vee A) \wedge B}{C \vee (A \wedge B)} \qquad \mathsf{w}{\downarrow}\,\frac{\bot}{A} \qquad \mathsf{c}{\downarrow}\,\frac{A \vee A}{A}$$

Theorem: (Decomposition)

$$\begin{array}{c} \top \\ \mathsf{SKSg}{\downarrow}\,\Big\Vert \\ A \end{array} \quad \iff \quad \begin{array}{c} \top \\ \{\mathsf{i}{\downarrow},\mathsf{s}\}\,\Big\Vert \\ A' \\ \{\mathsf{c}{\downarrow},\mathsf{w}{\downarrow}\}\,\Big\Vert \\ A \end{array} \quad \leftarrow \text{ this is MLL}$$

Decomposition for the Down-Fragment (without units)

$$\mathsf{SKSg}'{\downarrow}: \qquad \mathsf{i}{\downarrow}\,\frac{}{\bar{A} \vee A} \qquad \mathsf{i}{\downarrow}\,\frac{B}{B \wedge (\bar{A} \vee A)} \qquad \mathsf{s}\,\frac{(C \vee A) \wedge B}{C \vee (A \wedge B)} \qquad \mathsf{w}{\downarrow}\,\frac{B}{B \vee A} \qquad \mathsf{c}{\downarrow}\,\frac{A \vee A}{A}$$

Theorem: (Decomposition)

$$\begin{array}{c} \mathsf{SKSg}'{\downarrow}\,\Big\Vert \\ A \end{array} \quad \iff \quad \begin{array}{c} \{\mathsf{i}{\downarrow},\mathsf{s}\}\,\Big\Vert \\ A' \\ \{\mathsf{c}{\downarrow},\mathsf{w}{\downarrow}\}\,\Big\Vert \\ A \end{array} \quad \leftarrow \text{ this is unit-free MLL}$$

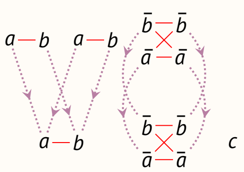
- Recall from earlier this week, that proofs in unit-free MLL can be described by proof nets and critically chorded RB-cographs.

First Skew Fibration Theorem

Theorem: Let A and B be (classical logic) formulas.

Then there is a skew fibration $f\colon [\![A]\!] \to [\![B]\!]$ iff $\begin{array}{c} A \\ \{\mathsf{c}{\downarrow},\mathsf{w}{\downarrow}\}\,\Big\Vert \\ B \end{array}$

Examples:

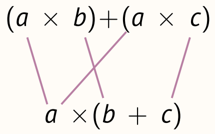


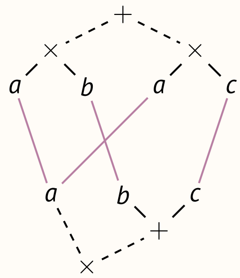
$$\mathsf{c}{\downarrow}\,\dfrac{\mathsf{w}{\downarrow}\,\dfrac{(a \wedge b) \vee (a \wedge b) \vee ((\bar{a} \vee \bar{b}) \wedge (\bar{a} \vee \bar{b}))}{(a \wedge b) \vee (a \wedge b) \vee ((\bar{a} \vee \bar{b}) \wedge (\bar{a} \vee \bar{b})) \vee c}}{(a \wedge b) \vee ((\bar{a} \vee \bar{b}) \wedge (\bar{a} \vee \bar{b})) \vee c}$$

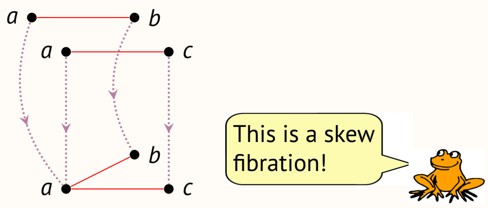
- This theorem has first been observed by Hughes:
  - *Dominic Hughes:* **“Towards HilbertâĂŹs 24th Problem: Combinatorial Proof Invariants:(preliminary version)”.** *Electronic Notes in Theoretical Computer Science, vol. 165, pp. 37–63, 2006*

  An alternative proof using deep inference can be found here:
  - *Lutz StraÃ§burger:* **“A Characterization of Medial as Rewriting Rule”.** *in International Conference on Rewriting Techniques and Applications (RTA 2007), pp. 344–358, 2007*

ALL Proof Nets and Skew Fibrations

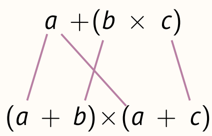


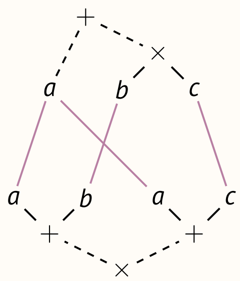


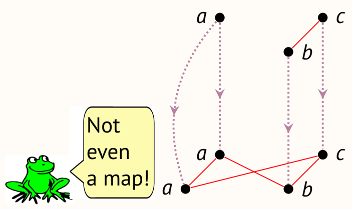


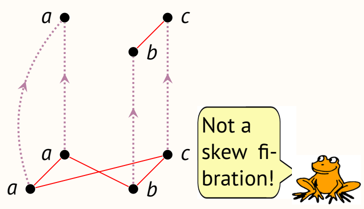
- **Exercise 6.1:** Show that every skew fibration defines an ALL proof net.

Another ALL Proof Net









- Obviously not every ALL proof net is a skew fibration.

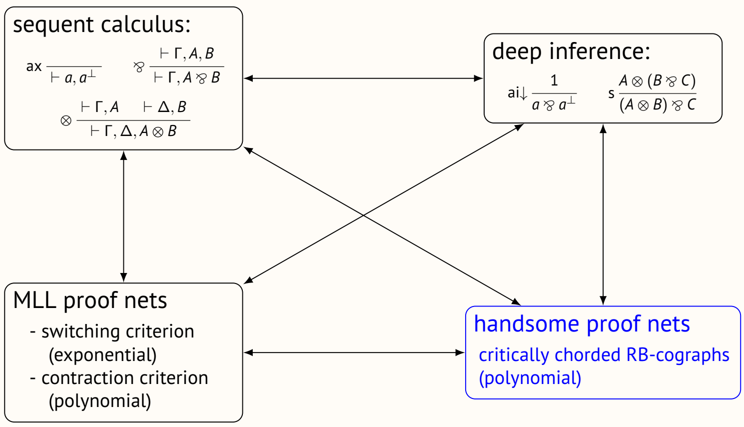
Second Skew Fibration Theorem

Theorem: Let A and B be formulas. Then the following are equivalent:

1. There is an ALL proof net for $A \vdash B$.
2. There is a formula C such that there are skew fibrations $f\colon [\![C]\!] \to [\![B]\!]$ and $g\colon [\![\bar{C}]\!] \to [\![\bar{A}]\!]$.

- **Exercise 6.2:** (Difficult) Prove the theorem.

Summary (Multiplicative Part)



Summary (Additive Part)