From Proof Nets to Combinatorial Proofs
—
A New Approach to Hilbert's 24th Problem

5. Lecture

Proof Nets for Additive Linear Logic

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Additive linear logic

Formulae:

$$A, B, C ::= a \mid A \oplus B \mid A \mathbin{\&} B$$

Sequents:

$$A \vdash C \quad \text{or} \quad \vdash \bar{A}, C$$

Sequent calculus:

$$\frac{}{A \vdash A}\,Ax \qquad \frac{A \vdash C \quad B \vdash C}{A \oplus B \vdash C}\,\oplus L \qquad \frac{A \vdash C_i}{A \vdash C_1 \oplus C_2}\,\oplus R$$

$$\frac{A \vdash B \quad B \vdash C}{A \vdash C}\,Cut \qquad \frac{A_i \vdash C}{A_1 \mathbin{\&} A_2 \vdash C}\,\& L \qquad \frac{A \vdash B \quad A \vdash C}{A \vdash B \mathbin{\&} C}\,\& R$$

Sum and product

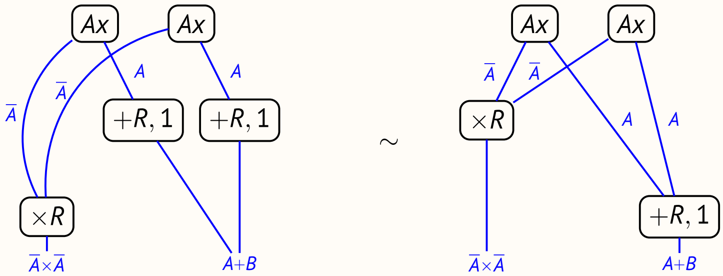
$$\begin{tikzcd} & A \arrow[dl] \arrow[d, dashed, "!"] \arrow[dr] & \\ B & B \mathbin{\&} C \arrow[l] \arrow[r] & C \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r] \arrow[dr] & B \oplus C \arrow[d, dashed, "!"] & C \arrow[l] \arrow[dl] \\ & D & \end{tikzcd}$$

$$\frac{B \vdash D}{B \mathbin{\&} C \vdash D}\,\& L \qquad \frac{C \vdash D}{B \mathbin{\&} C \vdash D}\,\& L \qquad \frac{B \vdash D \quad C \vdash D}{B \oplus C \vdash D}\,\oplus L$$

$$\frac{A \vdash B \quad A \vdash C}{A \vdash B \mathbin{\&} C}\,\& R \qquad \frac{A \vdash B}{A \vdash B \oplus C}\,\oplus R \qquad \frac{A \vdash C}{A \vdash B \oplus C}\,\oplus R$$

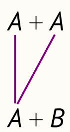
Proof nets: graph–of–rules

$$\dfrac{\dfrac{\dfrac{}{\vdash \overline{A}, A}\,{\scriptstyle Ax}}{\vdash \overline{A}, A+B}\,{\scriptstyle +R,1} \qquad \dfrac{\dfrac{}{\vdash \overline{A}, A}\,{\scriptstyle Ax}}{\vdash \overline{A}, A+B}\,{\scriptstyle +R,1}}{\vdash \overline{A}\times\overline{A}, A+B}\,{\scriptstyle \times R} \quad\sim\quad \dfrac{\dfrac{\dfrac{}{\vdash \overline{A}, A}\,{\scriptstyle Ax} \qquad \dfrac{}{\vdash \overline{A}, A}\,{\scriptstyle Ax}}{\vdash \overline{A}\times\overline{A}, A}\,{\scriptstyle \times R}}{\vdash \overline{A}\times\overline{A}, A+B}\,{\scriptstyle +R,1}$$

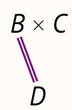


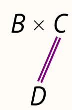
Proof nets: (two-sided) sequent + axioms

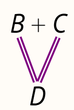
$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,{\scriptstyle Ax}}{A \vdash A+B}\,{\scriptstyle +R,1} \qquad \dfrac{\dfrac{}{A \vdash A}\,{\scriptstyle Ax}}{A \vdash A+B}\,{\scriptstyle +R,1}}{A+A \vdash A+B}\,{\scriptstyle \times R} \quad\sim\quad \dfrac{\dfrac{\dfrac{}{A \vdash A}\,{\scriptstyle Ax} \qquad \dfrac{}{A \vdash A}\,{\scriptstyle Ax}}{A+A \vdash A}\,{\scriptstyle \times R}}{A+A \vdash A+B}\,{\scriptstyle +R,1}$$



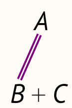
$$\dfrac{}{A \vdash A}\,{\scriptstyle Ax} \qquad \dfrac{B \vdash D}{B \times C \vdash D}\,{\scriptstyle \times L} \qquad \dfrac{C \vdash D}{B \times C \vdash D}\,{\scriptstyle \times L} \qquad \dfrac{B \vdash D \quad C \vdash D}{B + C \vdash D}\,{\scriptstyle +L}$$

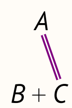


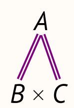




$$\dfrac{A \vdash B \quad B \vdash C}{A \vdash C}\,{\scriptstyle Cut} \qquad \dfrac{A \vdash B}{A \vdash B + C}\,{\scriptstyle +R} \qquad \dfrac{A \vdash C}{A \vdash B + C}\,{\scriptstyle +R} \qquad \dfrac{A \vdash B \quad A \vdash C}{A \vdash B \times C}\,{\scriptstyle \times R}$$

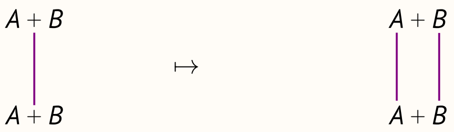






Eta-expansion

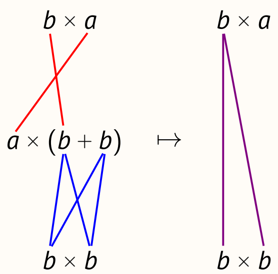
$$\dfrac{}{A + B \vdash A + B}\,Ax \quad \mapsto \quad \dfrac{\dfrac{\dfrac{}{A \vdash A}\,Ax}{A \vdash A + B}\,+R,1 \qquad \dfrac{\dfrac{}{B \vdash B}\,Ax}{B \vdash A + B}\,+R,2}{A + B \vdash A + B}\,+L$$



- Links may be restricted to atoms

Composition

- Composition is *relational composition*
- Requires *eta-expansion*

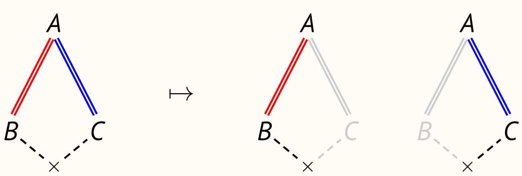


- Give a simple example to show that relational composition requires eta-expanded proof nets.

Correctness

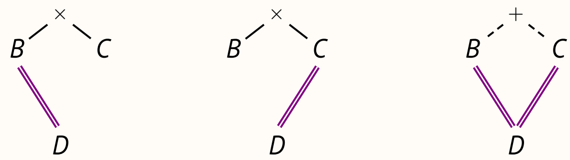
- A product $\times$ (and a sum $+$ on the left) is *switched*
- A switching deletes one entire subtree (not only the edge)

$$\dfrac{A \vdash B \quad A \vdash C}{A \vdash B \times C}\,\times R$$

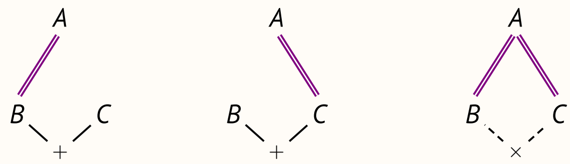


De-sequentialization and switching

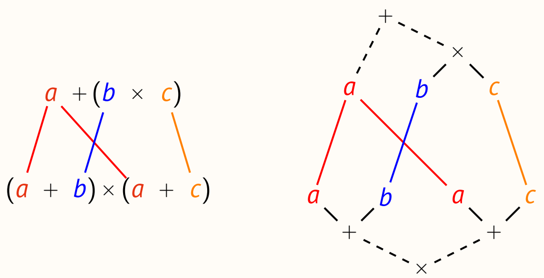
$$\frac{B \vdash D}{B \times C \vdash D}\times L \qquad \frac{C \vdash D}{B \times C \vdash D}\times L \qquad \frac{B \vdash D \quad C \vdash D}{B + C \vdash D}+L$$



$$\frac{A \vdash C}{A \vdash B + C}+R \qquad \frac{A \vdash C}{A \vdash B + C}+R \qquad \frac{A \vdash B \quad A \vdash C}{A \vdash B \times C}\times R$$



Example

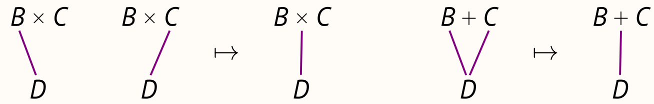


Definition

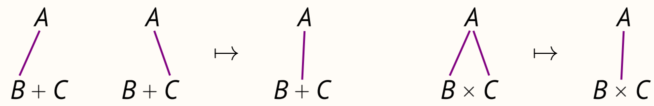
- A *linking* for an additive sequent $A \vdash B$ is a relation $L \subseteq sub(A) \times sub(B)$ where $sub(A)$ are the subformula occurrences of A. An *axiom* linking is one where every link relates two occurrences of the same formula.
- A *switching* s of A is a choice of B or C for every product $B \times C$ in $sub(A)$. A *co-switching* chooses on sums $B + C$.
- The *resolution* $s(A) \subseteq sub(A)$ given by a switching s consists of those subformula occurrences C where for every $B_1 \times B_2$ in A, if C is in B_i then s chooses B_i. A *co-resolution* is defined analogously for a *co-switching*.
- A *resolution* of a linking L for $A \vdash B$ is the restriction of L to $r(A) \times s(B)$ for some co-switching r of A and switching s of B.
- An *additive proof net* is a sequent $A \vdash B$ with an axiom linking L such that every resolution is a singleton.

Coalescence: correctness by rewriting

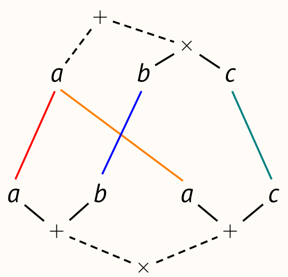
$$\frac{B \vdash D}{B \times C \vdash D}\times L \qquad \frac{C \vdash D}{B \times C \vdash D}\times L \qquad \frac{B \vdash D \quad C \vdash D}{B + C \vdash D}+L$$



$$\frac{A \vdash B}{A \vdash B + C}+R \qquad \frac{A \vdash C}{A \vdash B + C}+R \qquad \frac{A \vdash B \quad A \vdash C}{A \vdash B \times C}\times R$$



Example again



- Apply coalescence steps to show correctness of this example.

Theorem

A linking is a proof net if and only if it coalesces, if and only if it sequentializes.

Theorem

Coalescence for a linking over $A \vdash B$ is decidable in $\mathcal{O}(|A| \times |B|)$.

Deep inference: classical logic

$$\boxed{\begin{array}{c} A \\ \| \\ C \end{array}} \quad ::= \quad a \quad \mid \quad \boxed{\begin{array}{c} A_1 \\ \| \\ C_1 \end{array}} \star \boxed{\begin{array}{c} A_2 \\ \| \\ C_2 \end{array}} \quad \mid \quad \dfrac{\boxed{\begin{array}{c} A \\ \| \\ B_1 \end{array}}}{\boxed{\begin{array}{c} B_2 \\ \| \\ C \end{array}}}\, r$$

- Connectives: $\wedge$, $\vee$, $\top$, $\bot$
- Invertible rules:

$$\frac{A \vee (B \vee C)}{(A \vee B) \vee C}\,\alpha \quad \frac{A \vee B}{B \vee A}\,\sigma \quad \frac{A}{\bot \vee A}\,\lambda \qquad \frac{A \wedge (B \wedge C)}{(A \wedge B) \wedge C}\,\alpha \quad \frac{A \wedge B}{B \wedge A}\,\sigma \quad \frac{A}{\top \wedge A}\,\lambda$$

- Non-invertible rules:

$$\frac{\top}{A \vee \bar{A}}\,\top \quad \frac{(A \vee B) \wedge C}{A \vee (B \wedge C)}\,s \quad \frac{A \wedge \bar{A}}{\bot}\,\bot \qquad \frac{\bot}{A}\,? \quad \frac{A \vee A}{A}\,\nabla \quad \frac{A}{A \wedge A}\,\Delta \quad \frac{A}{\top}\,!$$

Deep inference: additive linear logic

$$\boxed{\begin{array}{c} A \\ \| \\ C \end{array}} \quad ::= \quad a \quad \mid \quad \boxed{\begin{array}{c} A_1 \\ \| \\ C_1 \end{array}} \star \boxed{\begin{array}{c} A_2 \\ \| \\ C_2 \end{array}} \quad \mid \quad \dfrac{\boxed{\begin{array}{c} A \\ \| \\ B_1 \end{array}}}{\boxed{\begin{array}{c} B_2 \\ \| \\ C \end{array}}}\, r$$

- Connectives: $\times$, $+$, 1, 0
- Invertible rules:

$$\frac{A + (B + C)}{(A + B) + C}\,\alpha \quad \frac{A + B}{B + A}\,\sigma \quad \frac{A}{0 + A}\,\lambda \qquad \frac{A \times (B \times C)}{(A \times B) \times C}\,\alpha \quad \frac{A \times B}{B \times A}\,\sigma \quad \frac{A}{1 \times A}\,\lambda$$

- Non-invertible rules:

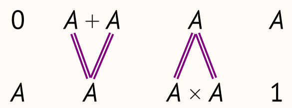
$$\frac{0}{A}\,? \quad \frac{A + A}{A}\,\nabla \quad \frac{A}{A \times A}\,\Delta \quad \frac{A}{1}\,!$$

Inductive translation

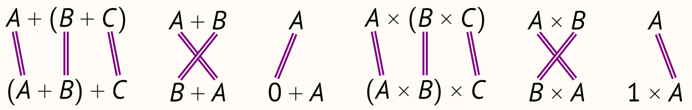
$$\boxed{\begin{array}{c} A \\ \| \\ C \end{array}} \quad ::= \quad a \quad \mid \quad \boxed{\begin{array}{c} A_1 \\ \| \\ C_1 \end{array}} \star \boxed{\begin{array}{c} A_2 \\ \| \\ C_2 \end{array}} \quad \mid \quad \dfrac{\boxed{\begin{array}{c} A \\ \| \\ B_1 \end{array}}}{\boxed{\begin{array}{c} B_2 \\ \| \\ C \end{array}}}\, r$$

$$\begin{array}{c} a \\ | \\ a \end{array} \qquad \begin{array}{ccc} A_1 & \star & A_2 \\ \| & & \| \\ C_1 & \star & C_2 \end{array} \qquad \begin{array}{c} A \\ \| \\ B_1 \\ \| \, r \\ B_2 \\ \| \\ C \end{array}$$

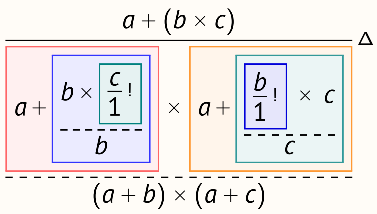
$$\frac{0}{A}\,? \qquad \frac{A+A}{A}\,\nabla \qquad \frac{A}{A\times A}\,\Delta \qquad \frac{A}{1}\,!$$

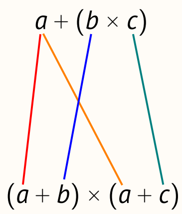


$$\frac{A+(B+C)}{(A+B)+C}\,\alpha \qquad \frac{A+B}{B+A}\,\sigma \qquad \frac{A}{0+A}\,\lambda \qquad \frac{A\times(B\times C)}{(A\times B)\times C}\,\alpha \qquad \frac{A\times B}{B\times A}\,\sigma \qquad \frac{A}{1\times A}\,\lambda$$

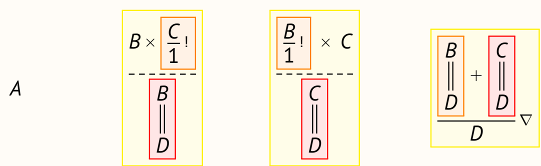


Example again again

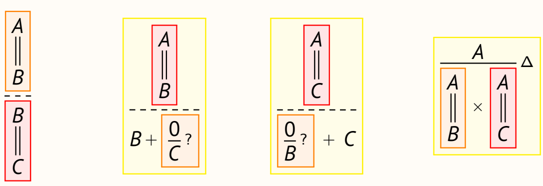




$$\frac{}{A\vdash A}\,Ax \qquad \frac{B\vdash D}{B\times C\vdash D}\,\times L \qquad \frac{C\vdash D}{B\times C\vdash D}\,\times L \qquad \frac{B\vdash D \quad C\vdash D}{B+C\vdash D}\,+L$$



$$\frac{A\vdash B \quad B\vdash C}{A\vdash C}\,Cut \qquad \frac{A\vdash B}{A\vdash B+C}\,+R \qquad \frac{A\vdash C}{A\vdash B+C}\,+R \qquad \frac{A\vdash B \quad A\vdash C}{A\vdash B\times C}\,\times R$$



Big picture

$$
\begin{tikzcd}
\text{Sequent calculus} \arrow[rr, "1"] \arrow[dr, "2"'] & & \text{Deep inference} \arrow[dl, "1"] \\
& \begin{array}{c}\text{Proof nets}\\ \text{(sequent + axioms)}\end{array} &
\end{tikzcd}
$$

1. Translation
2. Translation with eta-expansion and cut-elimination

Bibliography

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