From Proof Nets to Combinatorial Proofs

–

A New Approach to Hilbert's 24th Problem

4. Lecture

Deep Inference

Willem Heijltjes and Lutz Straßburger

Deep inference

A method for structuring proofs

$$\begin{array}{c} A \\ \| \\ C \end{array} \quad ::= \quad a \quad | \quad \begin{array}{c} A_1 \\ \| \\ C_1 \end{array} \star \begin{array}{c} A_2 \\ \| \\ C_2 \end{array} \quad | \quad \begin{array}{c} A \\ \| \\ B_1 \\ \hline B_2 \\ \| \\ C \end{array} r$$

A *derivation* from A to C:

- *Atom a*
- *Horizontal construction* with connective $\star$
- *Vertical construction* with rule r from B_1 to B_2

A deep inference *proof system* is given by:

- A set of *connectives*
- A set of *rules*

Classical logic

$$\begin{array}{c} A \\ \| \\ C \end{array} \quad ::= \quad a \quad | \quad \begin{array}{c} A_1 \\ \| \\ C_1 \end{array} \star \begin{array}{c} A_2 \\ \| \\ C_2 \end{array} \quad | \quad \begin{array}{c} A \\ \| \\ B_1 \\ \hline B_2 \\ \| \\ C \end{array} r$$

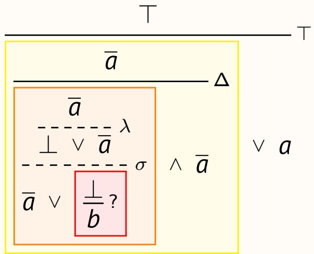
- Connectives: $\wedge$, $\vee$, $\top$, $\bot$
- Invertible rules:

$$\frac{A \vee (B \vee C)}{(A \vee B) \vee C}\,\alpha \quad \frac{A \vee B}{B \vee A}\,\sigma \quad \frac{A}{\bot \vee A}\,\lambda \qquad \frac{A \wedge (B \wedge C)}{(A \wedge B) \wedge C}\,\alpha \quad \frac{A \wedge B}{B \wedge A}\,\sigma \quad \frac{A}{\top \wedge A}\,\lambda$$

- Non-invertible rules:

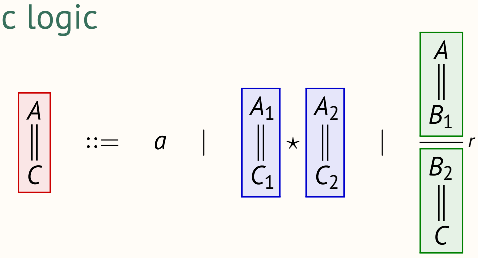
$$\frac{\top}{A \vee \bar{A}}\,\top \quad \frac{(A \vee B) \wedge C}{A \vee (B \wedge C)}\,s \quad \frac{A \wedge \bar{A}}{\bot}\,\bot \qquad \frac{\bot}{A}\,? \quad \frac{A \vee A}{A}\,\nabla \quad \frac{A}{A \wedge A}\,\Delta \quad \frac{A}{\top}\,!$$

Example: Peirce's law



$$((\bar{a} \vee b) \wedge \bar{a}) \vee a$$

Intuitionistic logic



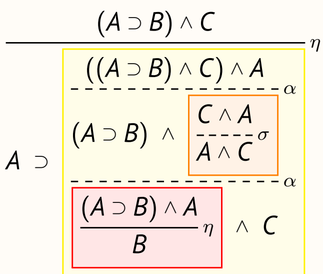
- Connectives: $\supset$, $\wedge$, $\top$
- Invertible rules:

$$\frac{A \wedge (B \wedge C)}{(A \wedge B) \wedge C}\,\alpha \qquad \frac{A \wedge B}{B \wedge A}\,\sigma \qquad \frac{A}{\top \wedge A}\,\lambda$$

- Non-invertible rules:

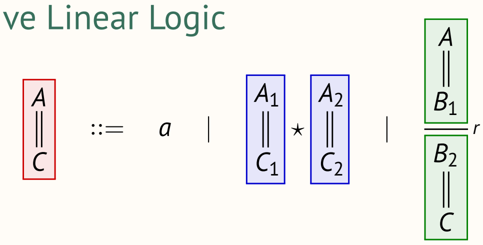
$$\frac{B}{A \supset (B \wedge A)}\,\eta \qquad \frac{(A \supset B) \wedge A}{B}\,\epsilon \qquad \frac{A}{A \wedge A}\,\Delta \qquad \frac{A}{\top}\,!$$

Example: switch



$$\frac{(A \supset B) \wedge C}{A \supset (B \wedge C)}$$

Multiplicative Linear Logic



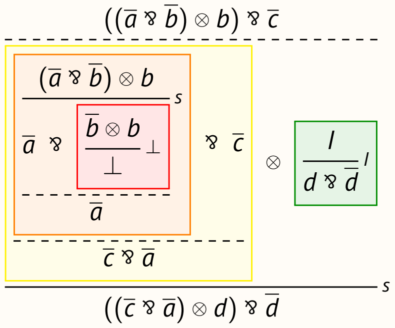
- Connectives: $\otimes$, $\parr$, I, $\bot$
- Invertible rules:

$$\frac{A \parr (B \parr C)}{(A \parr B) \parr C}\,\alpha \quad \frac{A \parr B}{B \parr A}\,\sigma \quad \frac{A}{\bot \parr A}\,\lambda \qquad \frac{A \otimes (B \otimes C)}{(A \otimes B) \otimes C}\,\alpha \quad \frac{A \otimes B}{B \otimes A}\,\sigma \quad \frac{A}{I \otimes A}\,\lambda$$

- Non-invertible rules:

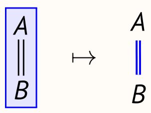
$$\frac{I}{A \parr \overline{A}}\,I \quad \frac{(A \parr B) \otimes C}{A \parr (B \otimes C)}\,s \quad \frac{A \otimes \overline{A}}{\bot}\,\bot$$

Example



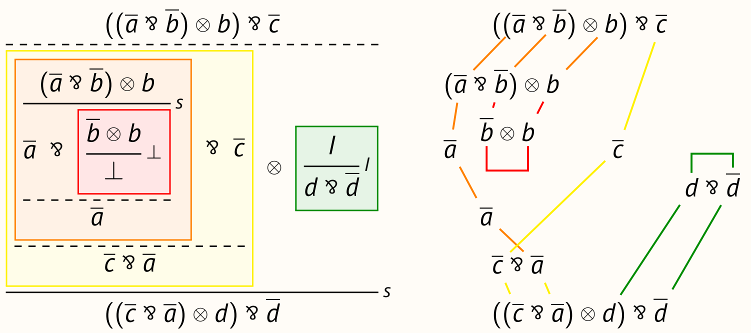
From deep inference to proof nets

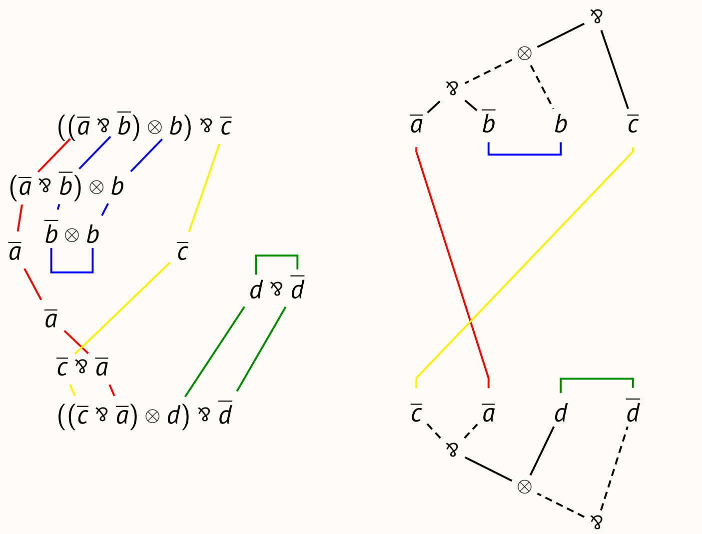
A derivation from A to B becomes a two-sided proof net:



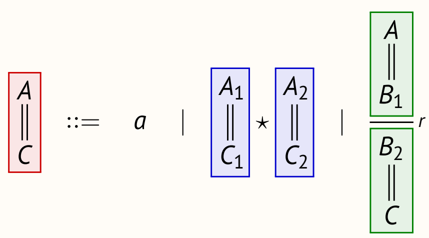
- Informally: by tracing atoms through a derivation
- Formally: by defining horizontal and vertical construction on proof nets

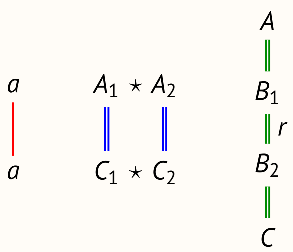
Tracing atoms





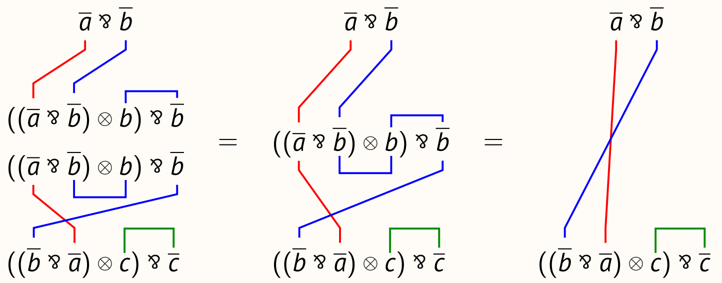
Inductive translation





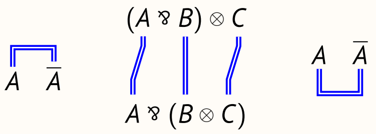
Composition

- Composition is path composition across the common formula

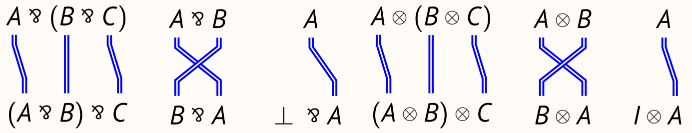


Rules

$$\frac{I}{A \parr \bar{A}}\, I \qquad \frac{(A \parr B) \otimes C}{A \parr (B \otimes C)}\, s \qquad \frac{A \otimes \bar{A}}{\bot}\, \bot$$

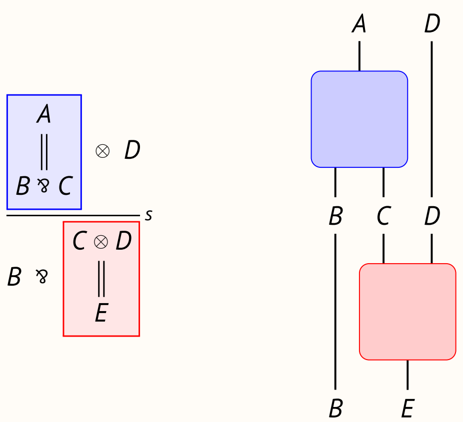


$$\frac{A \parr (B \parr C)}{(A \parr B) \parr C}\, \alpha \quad \frac{A \parr B}{B \parr A}\, \sigma \quad \frac{A}{\bot \parr A}\, \lambda \quad \frac{A \otimes (B \otimes C)}{(A \otimes B) \otimes C}\, \alpha \quad \frac{A \otimes B}{B \otimes A}\, \sigma \quad \frac{A}{I \otimes A}\, \lambda$$



Switch

$$\frac{(B \parr C) \otimes D}{B \parr (C \otimes D)}\, s$$



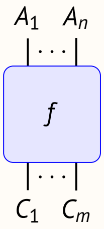
MLL is the logic of undirected, connected, acyclic networks.

- We can add the mix rule to allow disconnected networks

$$\frac{A \otimes B}{A \parr B}\,mix$$

- We can drop the axiom and cut to get directed networks (this is called weakly or linearly distributive logic)
- We can add primitives (with inputs and outputs)

$$\frac{A_1 \otimes \ldots \otimes A_m}{C_1 \parr \ldots \parr C_m}\,f$$



From sequent calculus

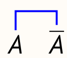
$$\frac{}{\vdash A, \overline{A}}\,Ax \quad \mapsto \quad \frac{I}{A \parr \overline{A}}\,I$$

$$\frac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\,\otimes \quad \mapsto \quad \frac{(\Gamma \parr A) \otimes (B \parr \Delta)}{\Gamma \parr \dfrac{A \otimes (B \parr \Delta)}{(A \otimes B) \parr \Delta}\,s}\,s$$

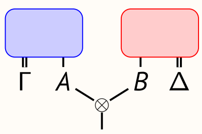
$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,\parr \quad \mapsto \quad \Gamma \parr (A \parr B)$$

$$\frac{\vdash \Gamma, A \qquad \vdash \overline{A}, \Delta}{\vdash \Gamma, \Delta}\,Cut \quad \mapsto \quad \frac{(\Gamma \parr A) \otimes (\overline{A} \parr \Delta)}{\Gamma \parr \dfrac{\dfrac{A \otimes (\overline{A} \parr \Delta)}{\dfrac{A \otimes \overline{A}}{\bot}\,\bot \;\parr\; \Delta}\,s}{\Delta}\,\lambda}\,s$$

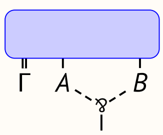
$$\frac{}{\vdash A, \overline{A}}\,Ax \quad \mapsto$$



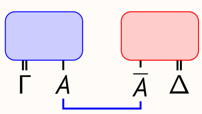
$$\frac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\,\otimes \quad \mapsto$$



$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,\parr \quad \mapsto$$

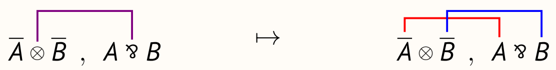


$$\frac{\vdash \Gamma, A \qquad \vdash \overline{A}, \Delta}{\vdash \Gamma, \Delta}\,Cut \quad \mapsto$$



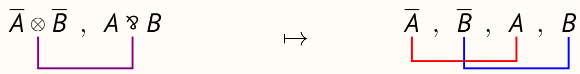
Eta-expansion

$$\dfrac{}{\vdash \overline{A} \otimes \overline{B}, A \parr B}\,Ax \quad \mapsto \quad \dfrac{\dfrac{\dfrac{}{\vdash \overline{A}, A}\,Ax \qquad \dfrac{}{\vdash \overline{B}, B}\,Ax}{\vdash \overline{A} \otimes \overline{B}, A, B}\,\otimes}{\vdash \overline{A} \otimes \overline{B}, A \parr B}\,\parr$$

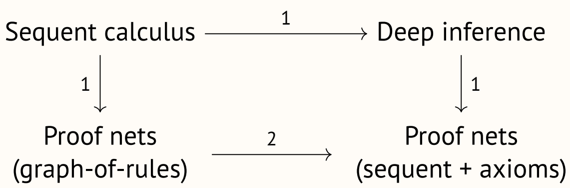


Cut-elimination

$$\dfrac{\dfrac{\vdash \Gamma, \overline{A} \qquad \vdash \Delta, \overline{B}}{\vdash \Gamma, \Delta, \overline{A} \otimes \overline{B}}\,\otimes \qquad \dfrac{\vdash \Theta, A, B}{\vdash \Theta, A \parr B}\,\parr}{\vdash \Gamma, \Delta, \Theta}\,Cut \quad \mapsto \quad \dfrac{\vdash \Gamma, \overline{A} \qquad \dfrac{\vdash \Delta, \overline{B} \qquad \vdash \Theta, A, B}{\vdash \Delta, \Theta, A}\,Cut}{\vdash \Gamma, \Delta, \Theta}\,Cut$$



Big picture



1. Translations
2. Cut-elimination and eta-expansion

Bibliography

Guglielmi, *Deep Inference webpages*,
`http://alessio.guglielmi.name/res/cos/`

Blute, Cockett, Seely & Trimble, *Natural deduction and coherence for weakly distributive categories*, Journal of Pure and Applied Algebra 113, pp. 229–296, 1996

Lamarche & Straßburger, *From Proof Nets to the Free *-Autonomous Category*, Logical Methods in Computer Science 2(4:3), pp. 1–44, 2006

Brünnler & McKinley, *An Algorithmic Interpretation of a Deep Inference System*, LPAR 2008

Guglielmi, Gundersen & Parigot, *A proof calculus which reduces syntactic bureaucracy*, RTA 2010

Hughes, *Simple free star-autonomous categories and full coherence*, Journal of Pure and Applied Algebra 216(11), pp. 2386–2410, 2012