From Proof Nets to Combinatorial Proofs

—

A New Approach to Hilbert's 24th Problem

# 3. Lecture

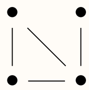
# Cographs and Handsome Proof Nets

Willem Heijltjes and Lutz Straßburger



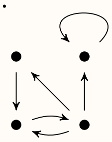
## Graphs (directed and undirected)

- An *(undirected) graph* is a pair $\mathcal{G} = \langle V_\mathcal{G}, E_\mathcal{G} \rangle$ of a (finite) set $V_\mathcal{G}$ of *vertices* and a set $E_\mathcal{G}$ of *edges* which are two-element subsets of $V_\mathcal{G}$.

  Example: 

  For $x, y \in V_\mathcal{G}$, we write $xy \in E_\mathcal{G}$ for $\{x, y\} \in E_\mathcal{G}$.

- A *directed graph* is a pair $\mathcal{G} = \langle V_\mathcal{G}, E_\mathcal{G} \rangle$ of a (finite) set $V_\mathcal{G}$ of *vertices* and a set $E_\mathcal{G} \subseteq V_\mathcal{G} \times V_\mathcal{G}$ of *edges* which are two-element subsets of $V_\mathcal{G}$.

  Example: 

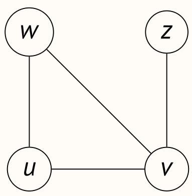
  For $x, y \in V_\mathcal{G}$, we write $x \to_\mathcal{G} y$ for $(x, y) \in E_\mathcal{G}$.



- In this course, all directed and undirected graphs are finite.
- When we say *graph*, we always mean *undirected graph*.
- When we want to speak about directed graphs, we say *directed graph*.
- Observe the notation: $\{x, y\}$ is the same as $\{y, x\}$. But $(x, y)$ is not the same as $(y, x)$.
- We often omit the index $\mathcal{G}$, when it is clear from context.

## Examples

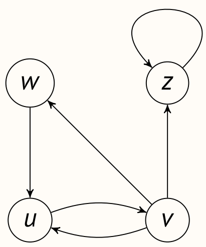
undirected graph




$V = \{u, v, w, z\}$

$E = \{uw, uv, wv, zv\}$

directed graph




$V = \{u, v, w, z\}$

$w \to u, u \to v, v \to u, v \to w, v \to z, z \to z$

## Graph Homomorphisms and Isomorphisms

- A *graph homomorphism* $h\colon \mathcal{G} \to \mathcal{H}$ is a map $h\colon V_\mathcal{G} \to V_\mathcal{H}$ such that for all $x, y \in V_\mathcal{G}$ we have that $xy \in E_\mathcal{G}$ implies $h(x)h(y) \in E_\mathcal{H}$.
- A *graph isomorphism* $h\colon \mathcal{G} \to \mathcal{H}$ is a bijection $h\colon V_\mathcal{G} \to V_\mathcal{H}$ such that $h$ and $h^{-1}$ are both homomorphisms.
- A *directed graph homomorphism* $h\colon \mathcal{G} \to \mathcal{H}$ is a map $h\colon V_\mathcal{G} \to V_\mathcal{H}$ such that for all $x, y \in V_\mathcal{G}$ we have that $x \twoheadrightarrow_\mathcal{G} y$ implies $h(x) \twoheadrightarrow_\mathcal{H} h(y)$.
- A *directed graph isomorphism* $h\colon \mathcal{G} \to \mathcal{H}$ is a bijection $h\colon V_\mathcal{G} \to V_\mathcal{H}$ such that $h$ and $h^{-1}$ are both homomorphisms.



- *Homomorphisms* are structure preserving maps. In the case of graphs, they are maps that preserve the graph structure.
- An *isomorphism* makes the two structures "indistinguishable".

## Paths and Cycles

- A *path* $p$ in a graph $\mathcal{G}$ is a sequence of vertices $x_0, x_1, \ldots, x_n \in V_\mathcal{G}$ such that $x_0x_1, x_1x_2, \ldots, x_{n-1}x_n \in E_\mathcal{G}$. In this the *length* of $p$ is $n$. The path is *elementary* if all $x_0, \ldots, x_n$ are pairwise distinct. A path is a *cycle* if $x_0 = x_n$. The cycle is *elementary* if all $x_1, \ldots, x_n$ are pairwise distinct (i.e., all vertices, except for $x_0 = x_n$).
- Similarly for directed graphs.

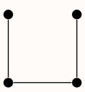
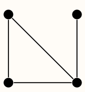


- The length of the path is its number of edges.
- The shortest possible cycle is an undirected graph has length 3. In a directed graph, the shortest possible cycle has length 1.
- In the rest of the course, all paths and cycles are elementary, but we won't say it explicitely all the time.
- **Exercise 3.1:** Give the definition of (elementary) path and cycle in directed graphs.

## Subgraphs and Induced Subgraphs

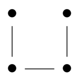
- Let $\mathcal{H} = \langle V_\mathcal{H}, E_\mathcal{H}\rangle$ and $\mathcal{G} = \langle V_\mathcal{G}, E_\mathcal{G}\rangle$ be graphs. We say that $\mathcal{H}$ is a *subgraph* of $\mathcal{G}$ if $V_\mathcal{H} \subseteq V_\mathcal{G}$ and $E_\mathcal{H} \subseteq E_\mathcal{G}$.
- We say that $\mathcal{H}$ is an *induced subgraph* of $\mathcal{G}$ if additionally

$$\forall u, v \in V_\mathcal{H}.\ uv \in E_\mathcal{G} \supset uv \in E_\mathcal{H}$$

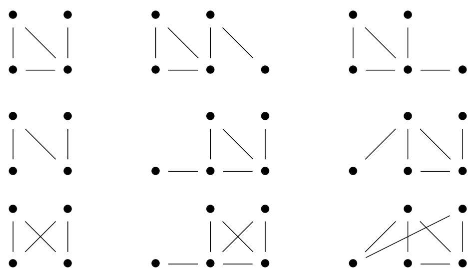
- We say that $\mathcal{G}$ *contains $\mathcal{H}$ as induced subgraph* if there is an injective homomorphism $f\colon \mathcal{H} \to \mathcal{G}$ such that $f(\mathcal{H})$ is an induced subgraph of $\mathcal{G}$. If this is not the case, we say that $\mathcal{G}$ is *$\mathcal{H}$-free*.
- Example:  is a subgraph of 

  (but not an induced subgraph).



- **Exercise 3.2:** Give the definition of subgraph and induced subgraph for directed graphs.
- For the moment, we only consider undireced graphs. Directed graphs return in Lectures 8,9, and 10.
- **Exercise 3.3:** We call $\mathbf{P_4}$ the graph  .

  Which of the following 9 graphs is $\mathbf{P_4}$-free?



## Operations on Graphs

- *Complement:*

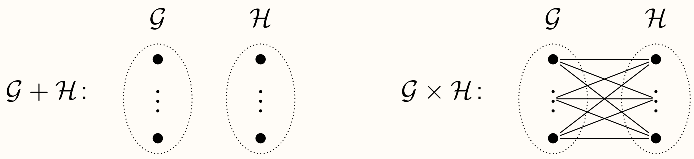
$$\bar{\mathcal{G}} = \langle V_{\mathcal{G}}, \{xy \mid x \neq y \text{ and } xy \notin E_{\mathcal{G}}\}\rangle$$

- *Disjoint Union:*

$$\mathcal{G} + \mathcal{H} = \langle V_{\mathcal{G}} \uplus V_{\mathcal{H}}, E_{\mathcal{G}} \uplus E_{\mathcal{H}}\rangle$$

- *Join:*

$$\mathcal{G} \times \mathcal{H} = \langle V_{\mathcal{G}} \uplus V_{\mathcal{H}}, E_{\mathcal{G}} \uplus E_{\mathcal{H}} \uplus \{xy \mid x \in V_{\mathcal{G}} \text{ and } y \in V_{\mathcal{H}}\}\rangle$$





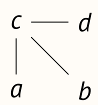

- **Exercise 3.4:** Show that

$$\mathcal{G} \times \mathcal{H} = \overline{\bar{\mathcal{G}} + \bar{\mathcal{H}}} \quad \text{and} \quad \bar{\bar{\mathcal{G}}} = \mathcal{G}$$

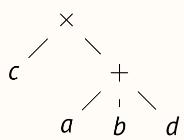
## Cographs

- A *cograph* is a graph that can be constructed from single vertex graphs using the operations $+$ and $\times$.
- A *cotree* is the term tree constructing the cograph.

Example: cograph




cotree

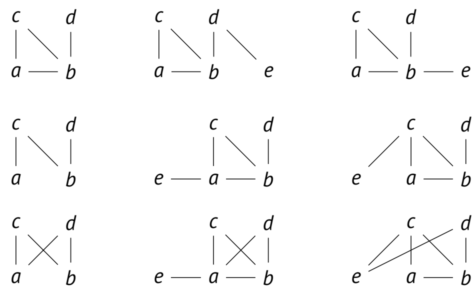



**Theorem:** A graph is a cograph if and only if it is $P_4$-free.

( where $P_4$ is the graph $\bullet - \bullet - \bullet - \bullet$ )



- The cographs is the smallest class of graph containing the the singletons, and being closed under complement and disjoint union.
- **Exercise 3.5:** Which of these graphs are cographs? Give the corresponding cotrees.




## Graphs of Formulas

*Mapping $[\![\cdot]\!]$ from MLL formulas to (labelled) graphs:*

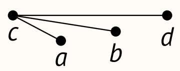
$$[\![a]\!] = \bullet a \quad \text{(single vertex labelled } a\text{)}$$

$$[\![a^{\perp}]\!] = \bullet a^{\perp} \quad \text{(single vertex labelled } a^{\perp}\text{)}$$

$$[\![A \parr B]\!] = [\![A]\!] + [\![B]\!]$$

$$[\![A \otimes B]\!] = [\![A]\!] \times [\![B]\!]$$

*Example:* $[\![\, c \otimes ((a \parr b) \parr d) \,]\!] =$




*Equivalence of formulas:*

$$A \parr B \equiv B \parr A \qquad (A \parr B) \parr C \equiv A \parr (B \parr C)$$

$$A \otimes B \equiv B \otimes A \qquad (A \otimes B) \otimes C \equiv A \otimes (B \otimes C)$$

**Theorem:** $[\![A]\!] = [\![B]\!]$ iff $A \equiv B$

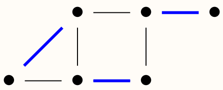


- **Exercise 3.6:** Draw the graphs of the following formulas:
  - $(a \parr a^{\perp}) \otimes (b \parr b^{\perp})$
  - $(a \otimes b) \parr (a^{\perp} \otimes b^{\perp})$
  - $(a \parr (a^{\perp} \otimes (b \parr b^{\perp}))$
  - $(a \otimes a^{\perp}) \parr ((a \parr a^{\perp}) \otimes (a \parr a^{\perp})) \parr (a \otimes a^{\perp})$
- **Exercise 3.7:** Prove the theorem.

## Perfect Matchings

- A *matching* $M$ in a graph $\mathcal{G} = \langle V_{\mathcal{G}}, E_{\mathcal{G}} \rangle$ is subset $M \subset E_{\mathcal{G}}$ of pairwise disjoint edges, i.e., no two edges in $M$ are adjacent (share a common vertex).
- A matching $M$ is *perfect* if for all $v \in V_{\mathcal{G}}$ there is a $w \in V_{\mathcal{G}}$ such that $vw \in E_{\mathcal{G}}$ (i.e., every vertex is incident to a matching edge).

  Example:





- **Attention:** In the textbook definition of matching (the one we give here), the matching is part of the graph, i.e., every matching edge is also an edge in the graph. But in this course, we often consider cases where the matching not part of a certain graph, i.e., formally, we have two graphs with the same vertex set: one is a graph with a certain property, very often a cograph, and the other is a perfect matching, i.e., a graph in which every edge participates in the matching.

## RB-graphs and RB-cographs

- An *RB-graph* is a triple $\mathcal{G} = \langle V_{\mathcal{G}}, R_{\mathcal{G}}, B_{\mathcal{G}} \rangle$, where $\langle V_{\mathcal{G}}, R_{\mathcal{G}} \rangle$ is a graph and $B_{\mathcal{G}}$ is a perfect matching in the graph $\langle V_{\mathcal{G}}, B_{\mathcal{G}} \rangle$.
- An *RB-cograph* is an RB-graph $\mathcal{G} = \langle V_{\mathcal{G}}, R_{\mathcal{G}}, B_{\mathcal{G}} \rangle$, where $\langle V_{\mathcal{G}}, R_{\mathcal{G}} \rangle$ is a cograph.

  Examples:



- An RB-graph can also be seen as a *multigraph* $\mathcal{G} = \langle V_{\mathcal{G}}, R_{\mathcal{G}} \uplus B_{\mathcal{G}} \rangle$ (i.e., a graph which can have more than one edge between two vertices), in which $B_{\mathcal{G}}$ is a perfect matching.
- Following the work of Christian Retoré, we draw the $R$-edges of RB-graphs in red/regular and the $B$-edges in blue/bold.
  - *Christian Retoré:* **"Handsome proof-nets: perfect matchings and cographs".** *Theoretical Computer Science 294 (2003) 473–488*
  - *Christian Retoré:* **"Handsome proof-nets: R&B-graphs, perfect matchings and series-parallel graphs".** *Rapport de Recherche RR-3652, INRIA*

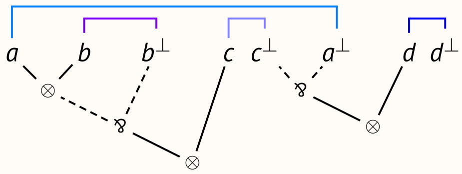
## Æ-Paths and Æ-Cycles

- An *alternating elementary path* (or *æ-path*) in an RB-graph $\mathcal{G} = \langle V_{\mathcal{G}}, R_{\mathcal{G}}, B_{\mathcal{G}} \rangle$ is an elementary path in $\langle V_{\mathcal{G}}, R_{\mathcal{G}} \uplus B_{\mathcal{G}} \rangle$, such that the edges of the path are alternating in $R_{\mathcal{G}}$ and $B_{\mathcal{G}}$.
- An *alternating elementary cycle* (or *æ-cycle*) in an RB-graph $\mathcal{G} = \langle V_{\mathcal{G}}, R_{\mathcal{G}}, B_{\mathcal{G}} \rangle$ is an elementary cycle in $\langle V_{\mathcal{G}}, R_{\mathcal{G}} \uplus B_{\mathcal{G}} \rangle$, such that the edges of the cycle are alternating in $R_{\mathcal{G}}$ and $B_{\mathcal{G}}$.
- A *chord* in an æ-path $v_0 v_1, \ldots, v_n$ (or in an -cycle $v_0 v_1, \ldots, v_n$ with $v_0 = v_n$) is an edge $v_i v_j \in R_{\mathcal{G}}$ (with $0 \leq i \neq j \leq n$) that is not part of the æ-path (or æ-cycle). An æ-path (æ-cycle) in an RB-graph is *chordless* if it has no chord.
- An RB-graph is *æ-connected* if any two vertices are connected by a chordless æ-path.
- An RB-graph is *æ-acyclic* if it has no chordless æ-cycle.



- These definitions come from Christian Retoré's work.
- Observe that an æ-cycle always has even length.

## What does all this have to do with proof nets?




Terminology:

- *prenet* = graph constructed from the sequent forest $\Gamma$ and axiom edges $\ell$; denoted as $\pi(\Gamma, \ell)$
- *proof net* = a prenet that is correct, i.e., every switching is acyclic and connected

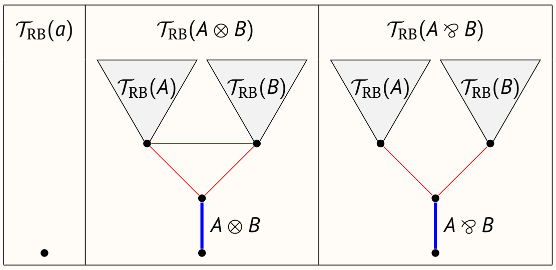


- Recall: MLL-proof net is a formula tree with a linking on the atoms, such that a correctness criterion is obeyed
- Today we see two ways to translate an MLL-proof net into an RB-graph
- In both cases we obtain an RB-cograph
- In both cases we get the same correctness criterion: *An RB-cograph $\mathcal{G}$ is the translation of a proof net if and only if it is æ-connected and æ-acyclic.*

## Translating MLL-Prenets into RB-graphs (Method 1)

Let a sequent $\Gamma$ and an axiom linking $\ell$ be given.

1. Translate every formula $A$ in $\Gamma$ into an RB-tree $\mathcal{T}_{\mathrm{RB}}(A)$:



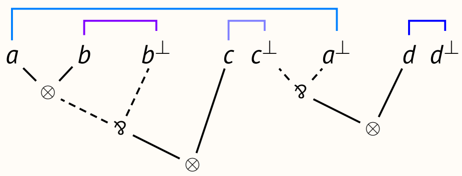

2. For each linking edge in $\ell$ add a matching edge between the corresponding atoms

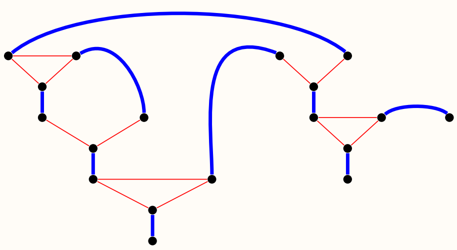
The result is an RB-cograph and called the *(tree-like) RB-prenet* $\tau(\Gamma, \ell)$



- Graph theoretically, $\mathcal{T}_{\mathrm{RB}}(A)$ is not a tree, but it carries the structure of the formula tree.
- **Exercise 3.8:** Show that if $\pi(\Gamma, \ell)$ is a MLL-prenet, then $\tau(\Gamma, \ell)$ is indeed an RB-cograph.
- **Exercise 3.9:** Show that in $\tau(\Gamma, \ell)$, no æ-path or æ-cycle has a chord.

## Example



**Theorem:** $\pi(\Gamma, \ell)$ is correct iff $\tau(\Gamma, \ell)$ is æ-acyclic and æ-connected.

In that case $\tau(\Gamma, \ell)$ is called a *(tree-like) RB-proof net*.



- Observe that in a tree-like RB-prenet, every æ-path and every æ-cycle is chordless. Therefore, checking for æ-acyclicity amounts to checking that there is no æ-cycle.
- **Exercise 3.10:** Prove the theorem. More precisely, show the following:
  1. $\tau(\Gamma, \ell)$ is æ-acyclic iff every switching of $\pi(\Gamma, \ell)$ is acyclic.
  2. $\tau(\Gamma, \ell)$ is æ-connected iff every switching of $\pi(\Gamma, \ell)$ is connected.

## Translating MLL-Prenets into RB-graphs (Method 2)

Let a sequent $\Gamma = A_1, A_2, \ldots, A_n$ and an axiom linking $\ell$ be given.

1. Mapping $[\![\cdot]\!]$ from MLL formulas to (labelled) graphs:

$$[\![a]\!] = \bullet a \quad \text{(single vertex labelled } a\text{)}$$
$$[\![a^\perp]\!] = \bullet a^\perp \quad \text{(single vertex labelled } a^\perp\text{)}$$
$$[\![A \parr B]\!] = [\![A]\!] + [\![B]\!]$$
$$[\![A \otimes B]\!] = [\![A]\!] \times [\![B]\!]$$

   Let $[\![\Gamma]\!] = [\![A_1]\!] + [\![A_2]\!] + \cdots + [\![A_n]\!]$

2. For each linking edge in $\ell$ add a matching edge between the corresponding atoms

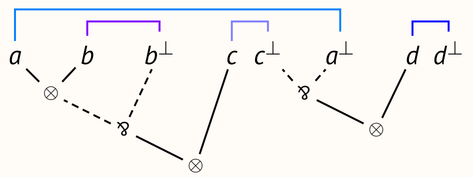
The result is an RB-cograph and called the *handsome RB-prenet* $\rho(\Gamma, \ell)$

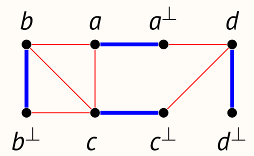
Every RB-cograph can be obtained in this way from a sequent $\Gamma$ and an axiom linking $\ell$.



- **Exercise 3.11:** Show that if $\pi(\Gamma, \ell)$ is a MLL-prenet, then $\rho(\Gamma, \ell)$ is indeed an RB-cograph.
- **Exercise 3.12:** Prove the converse, i.e., prove that every RB-cograph can be labelled such that it is a handsome prenet $\rho(\Gamma, \ell)$ for some sequent $\Gamma$ and some linking $\ell$.

## Example




**Theorem:**
$\pi(\Gamma, \ell)$ is correct iff $\rho(\Gamma, \ell)$ is æ-acyclic and æ-connected.

In that case, $\rho(\Gamma, \ell)$ is called a *handsome proof net*.

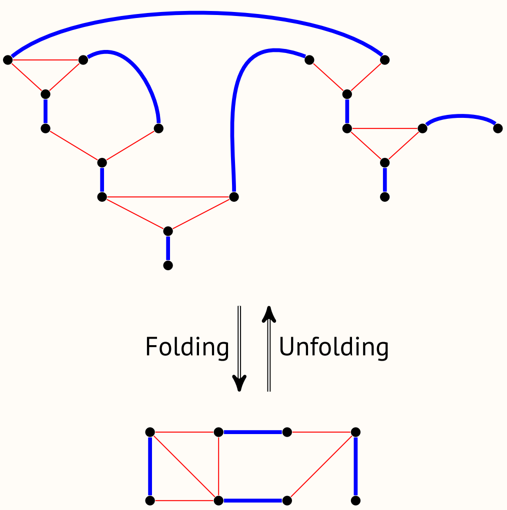


- Observe that in handsome RB-prenets, æ-paths and æ-cycles do have chords.
- **Exercise 3.13:** Convince yourself of the need of the "chordless" condition. For this, consider the following two sequents:

$$\vdash a \otimes b, (a^\perp \parr b^\perp) \otimes (c^\perp \parr d^\perp), c \otimes d$$
$$\vdash a \otimes b, (a^\perp \otimes c^\perp) \parr (b^\perp \otimes d^\perp), c \otimes d$$

  Note that for both sequents there is a unique axiom linking. Try to prove both in the sequent calculus for MLL, and draw the handsome RB-prenet for both. Both contain an æ-cycle, but in only one it is chordless.
- **Exercise 3.14:** (Difficult) Show the following:
  1. $\rho(\Gamma, \ell)$ is æ-acyclic iff $\tau(\Gamma, \ell)$ is æ-acyclic.
  2. $\rho(\Gamma, \ell)$ is æ-connected iff $\tau(\Gamma, \ell)$ is æ-connected.
- The term RB-proof net is used for both, the tree-like and the handsome proof nets.
- Retoré also uses the term *critically chorded* for a RB-cograph that is æ-acyclic and æ-connected. In other words, a handsome proof net is a critically chorded RB-cograph. And every RB-cograph is a handsome prenet.

## Folding and Unfolding






- The folding and unfolding are performed by small steps that preserve chordless æ-paths and chordless æ-cycles. Details can be found here:
  - *Christian Retoré:* **"Handsome proof-nets: R&B-graphs, perfect matchings and series-parallel graphs".** *Rapport de Recherche RR-3652, INRIA*