Graphs (directed and undirected)

- An (undirected) graph is a pair \( G = \langle V_G, E_G \rangle \) of a (finite) set \( V_G \) of vertices and a set \( E_G \) of edges which are two-element subsets of \( V_G \).
  
  Example: \( \bullet \rightarrow \bullet \) 

  For \( x, y \in V_G \), we write \( xy \in E_G \) for \( \{x, y\} \in E_G \).

- A directed graph is a pair \( G = \langle V_G, E_G \rangle \) of a (finite) set \( V_G \) of vertices and a set \( E_G \subseteq V_G \times V_G \) of edges which are two-element subsets of \( V_G \).
  
  Example: \( \bullet \rightarrow \bullet \) \( \bullet \rightarrow \bullet \) \( \bullet \rightarrow \bullet \)

  For \( x, y \in V_G \), we write \( x \rightarrow_G y \) for \( (x, y) \in E_G \).

Examples

undirected graph

\[
\begin{array}{c}
\text{\( u \)} \\
\downarrow \\
\text{\( v \)} \\
\text{\( w \)} \\
\text{\( z \)} \\
\end{array}
\]
\[
\begin{array}{c}
\text{\( \{u, v, w, z\} \)} \\
\text{\( \{uw, uv, vw, zw\} \)} \\
\end{array}
\]

directed graph

\[
\begin{array}{c}
\text{\( u \)} \\
\text{\( v \)} \\
\text{\( w \)} \\
\text{\( z \)} \\
\end{array}
\]
\[
\begin{array}{c}
\text{\( \{u, v, w, z\} \)} \\
\text{\( \{w \rightarrow u, u \rightarrow v, v \rightarrow u, v \rightarrow w, w \rightarrow z, z \rightarrow z\} \)} \\
\end{array}
\]
Graph Homomorphisms and Isomorphisms

- A graph homomorphism $h : G \to H$ is a map $h : V_G \to V_H$ such that for all $x, y \in V_G$ we have that $xy \in E_G$ implies $h(x)h(y) \in E_H$.
- A graph isomorphism $h : G \to H$ is a bijection $h : V_G \to V_H$ such that $h$ and $h^{-1}$ are both homomorphisms.
- A directed graph homomorphism $h : G \to H$ is a map $h : V_G \to V_H$ such that for all $x, y \in V_G$ we have that $x \to_G y$ implies $h(x) \to_H h(y)$.
- A directed graph isomorphism $h : G \to H$ is a bijection $h : V_G \to V_H$ such that $h$ and $h^{-1}$ are both homomorphisms.

Homomorphisms are structure preserving maps. In the case of graphs, they are maps that preserve the graph structure.

An isomorphism makes the two structures “indistinguishable”.

Paths and Cycles

- A path $p$ in a graph $G$ is a sequence of vertices $x_0, x_1, \ldots, x_n \in V_G$ such that $x_0 x_1, x_1 x_2, \ldots, x_{n-1} x_n \in E_G$. In this the length of $p$ is $n$. The path is elementary if all $x_0, \ldots, x_n$ are pairwise distinct. A path is a cycle if $x_0 = x_n$. The cycle is elementary if all $x_1, \ldots, x_n$ are pairwise distinct (i.e., all vertices, except for $x_0 = x_n$).

Similarly for directed graphs.

Subgraphs and Induced Subgraphs

- Let $H = (V_H, E_H)$ and $G = (V_G, E_G)$ be graphs. We say that $H$ is a subgraph of $G$ if $V_H \subseteq V_G$ and $E_H \subseteq E_G$.
- We say that $H$ is an induced subgraph of $G$ if additionally $\forall u, v \in V_H. uv \in E_G \Rightarrow uv \in E_H$.
- We say that $G$ contains $H$ as induced subgraph if there is an injective homomorphism $f : H \to G$ such that $f(H)$ is an induced subgraph of $G$. If this is not the case, we say that $G$ is $H$-free.
- Example: \[ \begin{array}{c} \cdot \\
\end{array} \] is a subgraph of \[ \begin{array}{cc}
\cdot & \cdot \\
\cdot & \cdot
\end{array} \] (but not an induced subgraph).

The length of the path is its number of edges.

The shortest possible cycle is an undirected graph has length 3. In a directed graph, the shortest possible cycle has length 1.

In the rest of the course, all paths and cycles are elementary, but we won’t say it explicitly all the time.

Exercise 3.1: Give the definition of (elementary) path and cycle in directed graphs.

Exercise 3.2: Give the definition of subgraph and induced subgraph for directed graphs.

For the moment, we only consider undirected graphs. Directed graphs return in Lectures 8, 9, and 10.

Exercise 3.3: We call $P_4$ the graph \[ \begin{array}{cc}
\cdot & \cdot \\
\cdot & \cdot
\end{array} \] .

Which of the following 9 graphs is $P_4$-free?
Operations on Graphs

- **Complement:**
  \[ \overline{G} = (V_G, \{xy \mid x \neq y \text{ and } xy \notin E_G\}) \]

- **Disjoint Union:**
  \[ G + H = (V_G \uplus V_H, E_G \uplus E_H) \]

- **Join:**
  \[ G \times H = (V_G \uplus V_H, E_G \uplus E_H \uplus \{xy \mid x \in V_G \text{ and } y \in V_H\}) \]

Cographs

- A **cograph** is a graph that can be constructed from single vertex graphs using the operations + and ×.
- A **cotree** is the term tree constructing the cograph.

Example: cograph cotree

\[
\begin{align*}
\text{cograph:} & \quad & \text{cotree:} \\
& \downarrow & \downarrow \\
& a & b \\
& \times & + \\
& & & c \quad d
\end{align*}
\]

**Theorem:** A graph is a cograph if and only if it is P₄-free.

( where P₄ is the graph \( \bullet \longrightarrow \bullet \longrightarrow \bullet \rightarrow \bullet \) )

Graphs of Formulas

**Mapping \([\cdot]\) from MLL formulas to (labelled) graphs:**

- \([a] = \bullet a\) (single vertex labelled \(a\))
- \([a^+] = \bullet a^+\) (single vertex labelled \(a^-\))

\[
\begin{align*}
[A \otimes B] &= [A] + [B] \\
[A \otimes B] &= [A] \times [B]
\end{align*}
\]

**Example:** \(\llbracket c \otimes ((a \otimes b) \otimes d) \rrbracket = \)

![Diagram](image)

**Equivalence of formulas:**

\[
\begin{align*}
A \otimes B &\equiv B \otimes A \\
(A \otimes B) \otimes C &\equiv A \otimes (B \otimes C) \\
A \otimes B &\equiv B \otimes A \\
(A \otimes B) \otimes C &\equiv A \otimes (B \otimes C)
\end{align*}
\]

**Theorem:** \([A] = [B] \iff A \equiv B\)
### Perfect Matchings

- A **matching** $M$ in a graph $G = (V_G, E_G)$ is a subset $M \subseteq E_G$ of pairwise disjoint edges, i.e., no two edges in $M$ are adjacent (share a common vertex).
- A matching $M$ is **perfect** if for all $v \in V_G$ there is a $w \in V_G$ such that $vw \in E_G$ (i.e., every vertex is incident to a matching edge).

Example:

```
                      .
          .               .
          .               .
          .               .
```

**Attention:** In the textbook definition of matching (the one we give here), the matching is part of the graph, i.e., every matching edge is also an edge in the graph. But in this course, we often consider cases where the matching not part of a certain graph, i.e., formally, we have two graphs with the same vertex set: one is a graph with a certain property, very often a cograph, and the other is a perfect matching, i.e., a graph in which every edge participates in the matching.

### RB-graphs and RB-cographs

- An **RB-graph** is a triple $G = (V_G, R_G, B_G)$, where $(V_G, R_G)$ is a graph and $B_G$ is a perfect matching in the graph $(V_G, B_G)$.
- An **RB-cograph** is an RB-graph $G = (V_G, R_G, B_G)$, where $(V_G, R_G)$ is a cograph.

Examples:

```

```

### Æ-Paths and Æ-Cycles

- An **alternating elementary path** (or Æ-path) in an RB-graph $G = (V_G, R_G, B_G)$ is an elementary path in $(V_G, R_G \uplus B_G)$, such that the edges of the path are alternating in $R_G$ and $B_G$.
- An **alternating elementary cycle** (or Æ-cycle) in an RB-graph $G = (V_G, R_G, B_G)$ is an elementary cycle in $(V_G, R_G \uplus B_G)$, such that the edges of the cycle are alternating in $R_G$ and $B_G$.
- A **chord** in an Æ-path $v_0v_1, \ldots, v_n$ (or in an Æ-cycle $v_0v_1, \ldots, v_n$ with $v_0 = v_n$) is an edge $v_iv_j \in R_G$ (with $0 \leq i \neq j \leq n$) that is not part of the Æ-path (or Æ-cycle). An Æ-path (Æ-cycle) in an RB-graph is **chordless** if it has no chord.
- An RB-graph is Æ-**connected** if any two vertices are connected by a chordless Æ-path.
- An RB-graph is Æ-**acyclic** if it has no chordless Æ-cycle.

**Attention:** In the textbook definition of matching (the one we give here), the matching is part of the graph, i.e., every matching edge is also an edge in the graph. But in this course, we often consider cases where the matching not part of a certain graph, i.e., formally, we have two graphs with the same vertex set: one is a graph with a certain property, very often a cograph, and the other is a perfect matching, i.e., a graph in which every edge participates in the matching.

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Christian Retoré: "Handsome proof-nets: R&B-graphs, perfect matchings and series-parallel graphs", *Rapport de Recherche RR-3652, INRIA*

These definitions come from Christian Retoré’s work.

Observe that an Æ-cycle always has even length.
What does all this have to do with proof nets?

Terminology:
- prenet = graph constructed from the sequent forest $\Gamma$ and axiom edges $\ell$; denoted as $\pi(\Gamma, \ell)$
- proof net = a prenet that is correct, i.e., every switching is acyclic and connected

Recall: MLL-proof net is a formula tree with a linking on the atoms, such that a correctness criterion is obeyed.
Today we see two ways to translate an MLL-proof net into an RB-graph.
In both cases we obtain an RB-cograph.
In both cases we get the same correctness criterion: An RB-cograph $G$ is the translation of a proof net if and only if it is $\alpha$-connected and $\alpha$-acyclic.

Translating MLL-Prenets into RB-graphs (Method 1)

Let a sequent $\Gamma$ and an axiom linking $\ell$ be given.

1. Translate every formula $A$ in $\Gamma$ into an RB-tree $T_{RB}(A)$:

   - $T_{RB}(A \otimes B)$
   - $T_{RB}(A \otimes B)$
   - $T_{RB}(A \otimes B)$

2. For each linking edge in $\ell$ add a matching edge between the corresponding atoms.

The result is an RB-cograph and called the (tree-like) RB-prenet $\tau(\Gamma, \ell)$

Theorem: $\pi(\Gamma, \ell)$ is correct iff $\tau(\Gamma, \ell)$ is $\alpha$-acyclic and $\alpha$-connected.
In that case $\tau(\Gamma, \ell)$ is called a (tree-like) RB-proof net.

Exercise 3.8: Show that if $\pi(\Gamma, \ell)$ is a MLL-prenet, then $\tau(\Gamma, \ell)$ is indeed an RB-cograph.
Exercise 3.9: Show that in $\tau(\Gamma, \ell)$, no $\alpha$-path or $\alpha$-cycle has a chord.

Example

Observe that in a tree-like RB-prenet, every $\alpha$-path and every $\alpha$-cycle is chordless. Therefore, checking for $\alpha$-acyclicity amounts to checking that there is no $\alpha$-cycle.

Exercise 3.10: Prove the theorem. More precisely, show the following:
1. $\tau(\Gamma, \ell)$ is $\alpha$-acyclic iff every switching of $\pi(\Gamma, \ell)$ is acyclic.
2. $\tau(\Gamma, \ell)$ is $\alpha$-connected iff every switching of $\pi(\Gamma, \ell)$ is connected.
Translating MLL-Prenets into RB-graphs (Method 2)

Let a sequent \( \Gamma = A_1, A_2, \ldots, A_n \) and an axiom linking \( \ell \) be given.

1. Mapping \( \cdot \) from MLL formulas to (labelled) graphs:
   
   \[
   \begin{align*}
   [a] & = \bullet a \quad \text{(single vertex labelled } a) \\
   [a^\perp] & = \bullet a^\perp \quad \text{(single vertex labelled } a^\perp) \\
   [A \otimes B] & = [A] + [B] \\
   [A \oplus B] & = [A] \times [B]
   \end{align*}
   \]

   Let \( [\Gamma] = [A_1] + [A_2] + \cdots + [A_n] \)

2. For each linking edge in \( \ell \) add a matching edge between the corresponding atoms

The result is an RB-cograph and called the \( \text{handsome RB-prenet} \rho(\Gamma, \ell) \)

Every RB-cograph can be obtained in this way from a sequent \( \Gamma \) and an axiom linking \( \ell \).

Example

![Diagram of a prenet example](image)

Theorem:

\( \pi(\Gamma, \ell) \) is correct iff \( \rho(\Gamma, \ell) \) is \( \omega \)-acyclic and \( \omega \)-connected.

In that case, \( \rho(\Gamma, \ell) \) is called a \( \text{handsome proof net} \).

Folding and Unfolding

![Diagram of folding and unfolding](image)

- **Exercise 3.11:** Show that if \( \pi(\Gamma, \ell) \) is a MLL-prenet, then \( \rho(\Gamma, \ell) \) is indeed an RB-cograph.
- **Exercise 3.12:** Prove the converse, i.e., prove that every RB-cograph can be labelled such that it is a handsome prenet \( \rho(\Gamma, \ell) \) for some sequent \( \Gamma \) and some linking \( \ell \).

- **Exercise 3.13:** Convince yourself of the need of the “chordless” condition. For this, consider the following two sequents:

\[
\begin{align*}
\Gamma & = \vdash a \otimes (b^\perp \otimes c^\perp \otimes d^\perp), c \otimes d \\
\Gamma & = \vdash a \otimes (b^\perp \otimes c^\perp \otimes (b^\perp \otimes d^\perp)), c \otimes d
\end{align*}
\]

Note that for both sequents there is a unique axiom linking. Try to prove both in the sequent calculus for MLL, and draw the handsome RB-prenet for both. Both contain an \( \omega \)-cycle, but in only one it is chordless.

- **Exercise 3.14:** (Difficult) Show the following:

1. \( \rho(\Gamma, \ell) \) is \( \omega \)-acyclic iff \( \tau(\Gamma, \ell) \) is \( \omega \)-acyclic.
2. \( \rho(\Gamma, \ell) \) is \( \omega \)-connected iff \( \tau(\Gamma, \ell) \) is \( \omega \)-connected.

- The term RB-proof net is used for both, the tree-like and the handsome proof nets.

- Retoré also uses the term \( \text{critically chorded} \) for a RB-cograph that is \( \omega \)-acyclic and \( \omega \)-connected. In other words, a handsome proof net is a critically chorded RB-cograph. And every RB-cograph is a handsome prenet.

- The folding and unfolding are performed by small steps that preserve chordless \( \omega \)-paths and chordless \( \omega \)-cycles. Details can be found here: