From Proof Nets to Combinatorial Proofs
–
A New Approach to Hilbert's 24th Problem

2. Lecture

Multiplicative Linear Logic: Sequent proofs and Proof Nets

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The Curry–Howard Isomorphism

- Typed λ-terms are (isomporphic to) intuitionistic natural-deduction proofs

$$\dfrac{\begin{matrix}[x:A]^x\\ \vdots\\ M:B\end{matrix}}{\lambda x.\,M:A\supset B}\,{\supset}I,x \qquad \dfrac{M:A\supset B \qquad N:A}{M\,N:B}\,{\supset}E$$

- This includes pairs/products/conjunction

$$\dfrac{M:A \qquad M:B}{\langle M,N\rangle:A\wedge B}\,{\wedge}I \qquad \dfrac{M:A\wedge B}{\pi_1(M):A}\,{\wedge}E,1 \qquad \dfrac{M:A\wedge B}{\pi_2(M):B}\,{\wedge}E,2$$

The *Curry–Howard isomorphism* (or *correspondence* where it is less strong) connects proof theory and computation. It makes proof-theoretical constructions relevant to typed functional programming, where they may convey guarantees of termination, safety, or other good properties.

- **Exercise 2.1:** One case of the definition of (typed) λ-terms is missing: the *variable* case. What should it look like?
- **Exercise 2.2:** The isomorphism also includes *sums* (or *coproduct*, or *disjuction*). What are the term constructors and typing rules for it?

Beta-reduction

- For implication:

$$\dfrac{\dfrac{\begin{matrix}[x:A]^x\\ \vdots\\ M:B\end{matrix}}{\lambda x.\,M:A\supset B}\,{\supset}I,x \qquad \begin{matrix}\vdots\\ N:A\end{matrix}}{(\lambda x.\,M)\,N:B}\,{\supset}E \quad\longrightarrow\quad \begin{matrix}\left[\begin{matrix}\vdots\\ N:A\end{matrix}\right]\\ \vdots\\ M[N/x]:B\end{matrix}$$

- For conjunction:

$$\dfrac{\dfrac{M:A \qquad N:B}{\langle M,N\rangle:A\wedge B}\,{\wedge}I}{\pi_1\langle M,N\rangle:A}\,{\wedge}E,1 \;\longrightarrow\; M:A \qquad\qquad \dfrac{\dfrac{M:A \qquad N:B}{\langle M,N\rangle:A\wedge B}\,{\wedge}I}{\pi_2\langle M,N\rangle:B}\,{\wedge}E,2 \;\longrightarrow\; N:B$$

- $(\supset/\wedge)$–*Intuitionistic natural deduction* is ideal:
  - Reduction is *confluent* and *strongly normalizing*
  - *Normal forms* represent the *meaning* of a term

The correspondence relates *formulae* to *types*, *proofs* to *programs*, and here, *normalization* to *computation* (β-reduction).

- **Exercise 2.3:** Give also the reductions for disjunction.
- **Exercise 2.4:** The rules for disjunction give permutations, and do not give unique normal forms. Construct an example of this, or find one in the literature.

Sequent calculus

$$\frac{}{x\colon A \vdash x\colon A}\,Ax \qquad \frac{\Gamma \vdash M\colon B}{\Gamma, x\colon A \vdash M\colon B}\,W \qquad \frac{\Gamma, x\colon A, y\colon A \vdash M\colon B}{\Gamma, x\colon A \vdash M[x/y]\colon B}\,C$$

$$\frac{\Gamma, x\colon A \vdash M\colon B}{\Gamma \vdash \lambda x.\,M\colon A \supset B}\,{\supset}R \qquad \frac{\Gamma \vdash M\colon A \qquad x\colon B, \Delta \vdash N\colon C}{\Gamma, f\colon A \supset B, \Delta \vdash N[f\,M/x]\colon C}\,{\supset}L$$

$$\frac{\Gamma, x\colon A \vdash M\colon C}{\Gamma, y\colon A \wedge B \vdash M[\pi_1(y)/x]\colon C}\,{\wedge}L,1 \qquad \frac{\Gamma, x\colon B \vdash M\colon C}{\Gamma, y\colon A \wedge B \vdash M[\pi_2(y)/x]\colon C}\,{\wedge}L,2$$

$$\frac{\Gamma \vdash M\colon A \qquad \Delta \vdash N\colon B}{\Gamma, \Delta \vdash \langle M, N\rangle\colon A \wedge B}\,{\wedge}R \qquad \frac{\Gamma \vdash M\colon A \qquad x\colon A, \Delta \vdash N\colon B}{\Gamma, \Delta \vdash N[M/x]\colon B}\,Cut$$

- *Not* an isomorphism between terms and proofs
- Builds terms in *normal form* (except the cut-rule)

This is a standard annotation of sequent proofs with λ-terms. The implication and conjunctions are formulated with *multiplicative* contexts, and weakening and contraction are explicit.

- **Exercise 2.5:** Try to give an additive (context-sharing) formulation of intuitionistic sequent calculus. The annoying rule is the implication-left one (${\supset}L$). Test it by typing the Church numeral three, $\lambda f.\lambda x.f(f(f\,x))$. Conclude that the multiplicative formulation is nicer.

For the axiom and right-rules (Ax, ${\supset}R$, ${\wedge}R$) terms and proofs are constructed correspondingly, as they are in natural deduction. But for the left-rules (${\supset}L$, ${\wedge}L$), the structural rules (w, c), and the cut-rule, term construction and proof construction are dissimilar.

- **Exercise 2.6:** Prove that cut-free proofs construct λ-terms in normal form.

- Sequent calculus is a *meta-calculus*: it describes the *construction* of natural-deduction proofs

$$\frac{\Gamma, x\colon A \vdash M\colon B}{\Gamma \vdash \lambda x.\,M\colon A \supset B}\,{\supset}R$$

$$\begin{array}{c}[x\colon A]\\ \vdots \\ M\colon B\end{array} \quad\mapsto\quad \dfrac{\begin{array}{c}[x\colon A]^x\\ \vdots \\ M\colon B\end{array}}{\lambda x.\,M\colon A \supset B}\,{\supset}I,x$$

The sequent rule above expresses that a natural deduction proof of B with open assumptions A, as below left, can be transformed into one of $A \supset B$ where the assumptions A are closed.
All sequent rules can be interpreted as constructing natural deduction proofs, in this way, and in fact Gentzen explicitly introduced sequent calculus from this perspective in his 1934/1935 papers.

$$\frac{\Gamma \vdash M\colon A \qquad x\colon B, \Delta \vdash N\colon C}{\Gamma, f\colon A \supset B, \Delta \vdash N[f\,M/x]\colon C}\,{\supset}L$$

$$\begin{array}{c}\vdots \\ M\colon A\end{array} \quad + \quad \begin{array}{c}[x\colon B]\\ \vdots \\ N\colon C\end{array} \quad\mapsto\quad \begin{array}{c}\left[\dfrac{f\colon A \supset B \qquad \begin{array}{c}\vdots \\ M\colon A\end{array}}{f\,M\colon B}\,{\supset}E\right]\\ \vdots \\ N[f\,M/x]\colon C\end{array}$$

- Note: this may duplicate or delete M.
- Possible *exponential* growth from sequent proof to λ-term

Sequent-calculus right-rules build up a natural deduction proof at its conclusion, as ${\supset}R$ on the previous slide. Left-rules build up a proof at its assumptions, as ${\supset}L$ here.

- **Exercise 2.7:** Complete the demonstration of how sequent rules construct natural deduction proofs for the other rules. Include also disjunction if you have given these rules previously. This makes the translation from sequent calculus to natural deduction explicit (it was implicit already in the annotation with λ-terms).
- **Exercise 2.8:** Give also a reverse translation, from natural deduction to sequent calculus.

Permutations

- The same λ-term can be constructed in multiple ways.
- This gives rise to *permutations* in sequent proofs.

$$\frac{\Gamma \vdash M : A \qquad \frac{x : B, \Delta, y : C \vdash N : D}{x : B, \Delta \vdash \lambda y.\, N : C \supset D}\,{\supset}R}{\Gamma, f : A \supset B, \Delta \vdash \lambda y.\, N[f\,M/x] : C \supset D}\,{\supset}L \quad \sim \quad \frac{\frac{\Gamma \vdash M : A \qquad x : B, \Delta \vdash N : D}{\Gamma, f : A \supset B, \Delta, y : C \vdash N[f\,M/x] : D}\,{\supset}L}{\Gamma, f : A \supset B, \Delta \vdash \lambda y.\, N[f\,M/x] : C \supset D}\,{\supset}R$$

$$\begin{array}{c}\vdots\\ M : A\end{array} \;+\; \begin{array}{c}[x : B]\;[y : C]\\ \vdots\\ N : D\end{array} \;\mapsto\; \frac{\begin{array}{c}\left[\frac{f : A \supset B \qquad \begin{array}{c}\vdots\\ M : A\end{array}}{f\,M : B}\,{\supset}E\right]\;[y : C]^{y}\\ \vdots\\ N[f\,M/x] : D\end{array}}{\lambda y.\, N[f\,M/x] : C \supset D}\,{\supset}I, y$$

- **Exercise 2.9:** Show three more permutations.
- **Exercise 2.10:** Find a way to count, list, or organize all permutations without having to actually write them out fully.

Intuitionistic logic summary

Natural deduction:

- Object-level calculus
- Isomorphic with simply-typed λ-calculus
- Does not need permutations (for $\wedge$ and $\supset$)

Sequent calculus:

- Meta-level calculus
- Describes construction of natural-deduction proofs
- Needs permutations
- Direct characterization of normal forms as cut-free

Natural deduction factors out the permutations of sequent calculus

Linear logic: the idea

In classical logic:

$$A \supset B \;=\; \overline{A} \vee B$$

But this breaks computational interpretations.

Enter linear logic:

$$A \supset B \;=\; ?\overline{A} \parr B$$

- $A \parr B$ is a *linear* disjunction
- $?A$ *non-linearizes* A
- $\overline{A}$ is an *involutive* (i.e. classical) negation

This is computational! (Or at least, not un-computational.)

The origins of linear logic are in its semantics of *coherence spaces*, which are an alternative by Girard to Scott's *domains* as a semantics of (untyped) λ-calculus. However, what makes linear logic convincing is its simple and natural sequent calculus, the intuitive idea of linearly-used resources, and its intriguing decomposition of intuitionistic implication.
With the *linear* implication $\multimap$, the encoding is

$$A \supset B \;=\; !A \multimap B$$

where ! is the dual to ?.

Linear logic

$$\dfrac{}{\vdash A, \overline{A}}\,Ax \qquad \dfrac{\vdash \Gamma, A \qquad \vdash \overline{A}, \Delta}{\vdash \Gamma, \Delta}\,Cut$$

Multiplicatives:

$$\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,\parr \qquad \dfrac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\,\otimes$$

Additives:

$$\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\,\oplus,1 \qquad \dfrac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\,\oplus,2 \qquad \dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\,\&$$

Exponentials:

$$\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,! \qquad \dfrac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\,? \qquad \dfrac{\vdash \Gamma}{\vdash \Gamma, ?A}\,w \qquad \dfrac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\,c$$

The connectives are called: *par* $\parr$, *tensor* $\otimes$, *plus* $\oplus$, *with* &, *bang* !, and *why not* ?.
The *modality* ? governs whether formulas can be weakened and contracted or not. The rule ! for the *bang* requires that the context $?\Gamma = ?A_1, \ldots, ?A_n$ contains only formulae that can be contracted. This is necessary for cut-elimination to work.

- **Exercise 2.11:** Investigate this by giving a cut-reduction step for:

$$\dfrac{\vdash ?\Gamma, !A \qquad \dfrac{\vdash ?\overline{A}, ?\overline{A}, \Delta}{\vdash ?\overline{A}, \Delta}\,c}{\vdash ?\Gamma, \Delta}\,Cut$$

Is the ? on the context $?\Gamma$ necessary here?

The dream of the 90s

New grounds in typed functional programming:

- Linearity gives *destructive updates* (i.e. mutable state)
- Multiplicatives $\otimes$ and $\parr$ give *deadlock-free concurrency*

But:

- The proof system is a *sequent calculus*!
- What about *natural deduction*?

Linear logic led to a great diversification of research in the areas around proof theory and computation in the 1990s, going into the 2000s and 2010s. Early new entrants were *game semantics*, *interaction nets*, and *geometry of interaction*.
From the start, the hope was that linear logic could give a proof-theoretic (and thus *typed*, and thus *safe*) account of various computational phenomena, such as destructive updates and concurrency. The programming language *Clean*, developed independently and simultaneously with linear logic, implements the former idea. Work towards the latter continues in the area of *session types*.

Proof nets

Idea: remove the contexts Γ, Δ from the rules.

$$\dfrac{}{\vdash A, \overline{A}}\,Ax \quad\mapsto\quad \overline{A \quad \overline{A}}$$

$$\dfrac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\,\otimes \quad\mapsto\quad \dfrac{A \quad B}{A \otimes B}$$

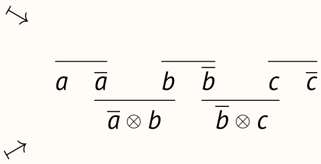
$$\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,\parr \quad\mapsto\quad \begin{array}{c} A \quad B \\ \hdashline A \parr B \end{array}$$

$$\dfrac{\vdash \Gamma, A \qquad \vdash \overline{A}, \Delta}{\vdash \Gamma, \Delta}\,Cut \quad\mapsto\quad \underline{A \quad \overline{A}}$$

- inferences become *nodes*, called *links*,
- in a *graph* called a *proof structure*

Example

$$\dfrac{\dfrac{}{\vdash a,\overline{a}}\,Ax \qquad \dfrac{\dfrac{}{\vdash b,\overline{b}}\,Ax \qquad \dfrac{}{\vdash c,\overline{c}}\,Ax}{\vdash b,\overline{b}\otimes c,\overline{c}}\,\otimes}{\vdash a,\overline{a}\otimes b,\overline{b}\otimes c,\overline{c}}\,\otimes$$



$$\wr$$

$$\dfrac{\dfrac{\dfrac{}{\vdash a,\overline{a}}\,Ax \qquad \dfrac{}{\vdash b,\overline{b}}\,Ax}{\vdash a,\overline{a}\otimes b,\overline{b}}\,\otimes \qquad \dfrac{}{\vdash c,\overline{c}}\,Ax}{\vdash a,\overline{a}\otimes b,\overline{b}\otimes c,\overline{c}}\,\otimes$$

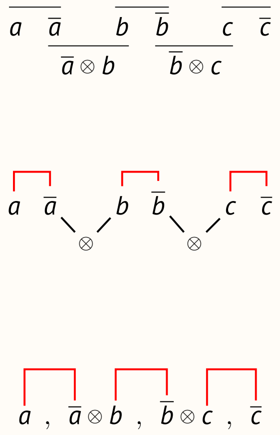
Two perspectives

Graph–of–rules:

- Nodes/links represent inferences
- Natural deduction–like
- Focus: computation

Sequent + axioms:

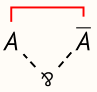
- Nodes represent connectives
- String diagram–like
- Focus: canonicity



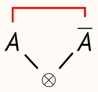
Correctness

Which proof structures come from proofs?

correct:



incorrect:

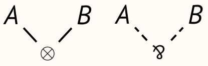


All rules except ⅋ create *connected acyclic graphs*

$$\dfrac{}{\vdash A,\overline{A}}\,Ax$$

$$\dfrac{\vdash \Gamma, A \qquad \vdash \overline{A}, \Delta}{\vdash \Gamma, \Delta}\,Cut$$

$$\dfrac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, A\otimes B, \Delta}\,\otimes$$



$$\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \mathbin{⅋} B}\,⅋$$

The premises of ⅋ must already be connected

MLL proof nets

Definition

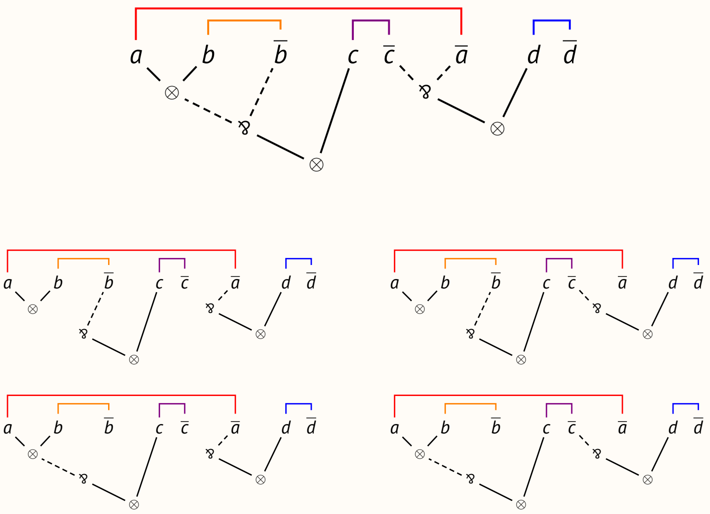
A *proof net* is a sequent Γ with a correct linking, where:

- A *linking* on a sequent Γ is a partitioning of its atoms into dual, unordered pairs
- A *switching* of Γ is a choice of left/right for each ⅋
- A *switching graph* for a linking and a switching on Γ is the undirected graph where nodes are connectives and atoms of Γ and edges connect:
  - each $\otimes$ to both children
  - each ⅋ to the child indicated by the switching
  - each pair of atoms in the linking
- A linking for Γ is *correct* if each switching graph is acyclic and connected

This is the *sequent + axioms* definition.

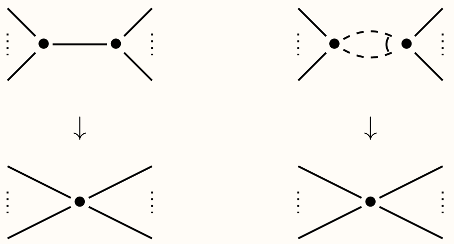
- **Exercise 2.12:** Give also the *graph–of–rules* definition.
- **Exercise 2.13:** Prove that the two definitions are isomorphic.

Example



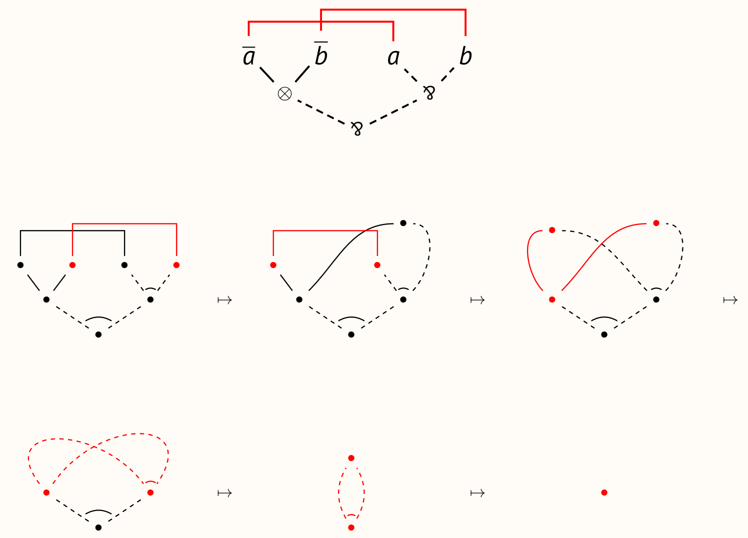
Contractibility

1. Start from an unlabelled graph with paired ⅋-edges
2. Contract by:



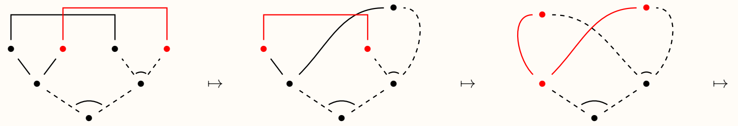
3. Correct ⇔ contracts to a single point

Implemented in linear time via union–find



$\overline{b}, b$

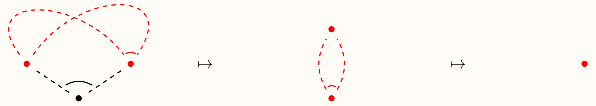
$\overline{a}, a \qquad \overline{b}, b$



$$\dfrac{\overline{a}, a \qquad \overline{b}, b}{(\overline{a} \otimes \overline{b}), a, b}$$

$$\dfrac{\dfrac{\overline{a}, a \qquad \overline{b}, b}{(\overline{a} \otimes \overline{b}), a, b}}{(\overline{a} \otimes \overline{b}), a \parr b}$$

$$\dfrac{\dfrac{\dfrac{\overline{a}, a \qquad \overline{b}, b}{(\overline{a} \otimes \overline{b}), a, b}}{(\overline{a} \otimes \overline{b}), a \parr b}}{(\overline{a} \otimes \overline{b}) \parr a \parr b}$$



Sequentialization: proof nets ↦ sequent proofs

De-sequentialization: sequent proofs ↦ proof nets

Theorem

Sequentialization and de-sequentialization are inverses (up to permutations).

Theorem

A linking is correct if and only if it contracts, if and only if it sequentializes.

Theorem

Two sequent proofs are equivalent under permutations if and only if they translate to the same proof net.

The proofs of these theorems are a little tricky, but not infeasible. You can attempt them yourself.

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