Unit 5: Decidability

Note: this material is mostly taken from [Brü09].

What does “decidable” mean?

- A set (of natural numbers) is called **recursive**, **computable**, or **decidable** if there is an algorithm which terminates after a finite amount of time (which may depend on the input) and correctly decides whether a given number belongs to the set.

- A set (of natural numbers) is called **recursively enumerable**, r.e., or **semidecidable** if there is an algorithm that correctly decides when a number is in the set; the algorithm may give no answer (but not the wrong answer) for numbers not in the set.

  - Observation 1: If $S \subseteq \mathbb{N}$ is decidable, then so is $\mathbb{N} \setminus S$. [Exercise: Why?]

  - Observation 2: If $S \subseteq \mathbb{N}$ and $\mathbb{N} \setminus S$ are both semidecidable, then they are also decidable. [Exercise: Why?]

- Here, we talk about languages, i.e., instead of $\mathbb{N}$, we start from the set of finite words over a given alphabet.

- Example 1: The set of well-formed formulas is decidable. [Exercise: Why?]

- Example 2: The set of theorems (=provable formulas) is semidecidable. [Exercise: Why?]

- The question “Is this logic decidable?” means “Is the set of its theorems a decidable subset of the set of all finite words over some finite alphabet?”

  - Two ways are commonly used to prove decidability of a logic:
    - Either have a complete proof system and show that proof search terminates,
    - or show that the logic has the **finite model property** (FMP), i.e., if a formula is not a theorem of the logic, then there is a finite counter model (in that case the set of “non-theorems” is r.e.).

Decidability of the logics in the modal cube

**Theorem 1.** Let $X \subseteq \{d, t, b, 4, 5\}$. It is decidable whether a formula $A$ is a theorem of the modal logic $K + X$.

We use a proof search procedure to construct a finite counter model. Here is our first version:

- We work on trees of sets instead of multisets, and when applying an inference rule, we never remove a formula, i.e., the $\lor$, $\land$, and $\square$-rules become:

\[
\begin{align*}
\lor & \quad \frac{\Gamma \{ A \lor B, A, B \}}{\Gamma \{ A \lor B \}} \\
\land & \quad \frac{\Gamma \{ A \land B, A \}}{\Gamma \{ A \land B \}} \quad \frac{\Gamma \{ A \land B, B \}}{\Gamma \{ A \land B \}} \\
\square & \quad \frac{\Gamma \{ \square A, [A] \}}{\Gamma \{ \square A \}}
\end{align*}
\]

- All rules, except $\square$ and $d^\circ$, add formulas to existing sequent nodes (we reason bottom up). We only apply such an inference rule when all premises are different from the conclusion.

- The rules $\square$ and $d^\circ$ add a new child sequent node with a formula $A$ inside. We only apply these rules when no other child sequent node has $A$ inside.
• We terminate if no rule is applicable.

**Lemma 2.** For $X \subseteq \{d, t, b\}$ the algorithm described above terminates.

**Proof.** All formulas occurring in the derivation are subformulas of the endsequent, and no rule changes the "modal depth". But there are only finitely many sequents with a fixed modal depth that can be build from finitely many formulas. □

Next we show how to construct a countermodel from a failed proof search tree.

• If the proof search is failed, there is at least one leaf in the derivation tree containing a sequent to which no rule is applicable (including the axiom $id$). Let’s call it $Φ$.

• From $Φ$ construct the model $ℳ = \langle W, R, V \rangle$ as follows:
  
  - $W$ is the set of sequent nodes in $Φ$; we use letters $w, v, u$ to denote worlds in $W$ and sequent nodes in $Φ$, and we write $A \in w$ for saying that the formula $A$ occurs in the sequent node $w$.
  - We define $V$ by letting $w \models p$ if $\bar{p} \in w$;
  - For defining $R \subseteq W \times W$, we first define the relation $R_0$ to be determined by the tree structure of $Φ$;
    * if $b \in X$ then $R_1$ is the symmetric closure of $R_0$, otherwise $R_1 = R_0$;
    * if $t \in X$ then $R_2$ is the reflexive closure of $R_1$, otherwise $R_2 = R_1$;
    * if $d \in X$ then $R = R_2 \cup \{(w, w) \mid \text{there is no } w' \text{ with } (w, w') \in R_2\}$, otherwise $R = R_2$.

• We can now show that for every formula $A$ and world $w$ we have that $A \in w$ implies $w \not\models A$.

[[Exercise: Prove this (Hint: by induction on $A$).]]

• It follows that $ℳ$ contains a world (the root of $Φ$) that does not force $fm(Φ)$. Hence $ℳ$ (which is finite) is a countermodel for $Φ$.

• Since all rules (seen top-down) preserve countermodels, we have a finite countermodel for the endsequent.

This completes the proof of Theorem 1 for $X \subseteq \{d, t, b\}$.

**What about 4 and 5?**

• Or naive algorithm above does not terminate if 4 or 5 are in $X$.

• Example: $\square (\square p \lor \square \neg p)$  
  
  [[Exercise: What happens with this example?]]

• We modify of the algorithm above:
  
  We allow to apply the rules $\square$ and $\Diamond$ (the rules that create new sequent nodes) only if there is no ancestor node in the sequent tree that contains the same formulas as the current node (this is known as loop check). Otherwise we call the current sequent node looping.

**Lemma 3.** The modified algorithm terminates for all $X \subseteq \{d, t, b, 4, 5\}$.

**Proof.** There are only finitely many different sequent nodes that can be formed with the subformulas of the endsequent. This restricts the depth of the sequents to be visited during proof search. □

However, in order to make the countermodel construction work, we need that $X$ is 45-closed.

• The model $ℳ$ is defined similar as above, with the difference that:
  
  - $W$ is the set of sequent nodes in $Φ$ that are not looping.
  - We define the relation $R_0'$ to be obtained from $R_0$ by adding edges from the parent of a looping node to the ancestor that contains the same formulas (i.e., the one that causes the loop). Then we let $R$ be the smallest relation that contains $R_0'$ and has all the properties demanded by $X$.

  $d$: $\Box A \supset \Diamond A \quad \forall w. \exists v. wRv$ (serial)
  
  $t$: $(A \supset \Diamond A) \quad \forall w. wRw$ (reflexive)
  
  $b$: $(A \supset \Box \Diamond A) \quad \forall w. \forall v. wRv \supset vRw$ (symmetric)
  
  $4$: $(\Diamond \Box A \supset \Diamond A) \quad \forall w. \forall v. wRv \land vRw \supset wRv$ (transitive)
  
  $5$: $(\Box A \supset \Box \Diamond A) \quad \forall w. \forall v. wRv \land wRv \supset vRw$ (euclidean)

(one can show that this is well-defined)
• As before, we can now show that for every formula \( A \) and world \( w \) we have that \( A \in w \) implies \( w \vdash A \).
  For this step, 45-closedness is needed. [Exercise: Why?]

[Exercise: Complete the proof of Theorem 1.]
[Exercise: Show how this can be used as alternative proof of cut admissibility.]

References