Unit 4: Nested Sequents

Note: this material is mostly taken from [Brü09].

Motivation

- Want: “sequent” systems for all logics in the modal cube.

  - Modal axioms:
    
    \begin{align*}
    d: \quad & \Box A \supset \Diamond A \quad \forall w. \exists v. wRv \\
    t: \quad & (A \supset \Diamond A) \quad \forall w. wRw \\
    b: \quad & (A \supset \Box A) \quad \forall w. \forall v. wRv \supset vRw \\
    4: \quad & (\Diamond A \supset \Diamond A) \quad \forall w. \forall v. \forall u. wRu \wedge vRu \supset wRu \\
    5: \quad & (\Diamond A \supset \Box A) \quad \forall w. \forall v. \forall u. wRu \wedge wRu \supset vRu
    \end{align*}

- Context: $\Gamma\{\}$ is a sequent with a hole (an occurrence of $\{\}$ in place of a formula)

- A nested sequent is valid iff the corresponding formula is valid.

Nested sequents

- Recall sequents: $\Gamma = A_1, \ldots, A_n$
  - corresponding formula: $fm(\Gamma) = A_1 \vee \cdots \vee A_n$

- Nested sequents: $\Gamma = A_1, \ldots, A_n, [\Gamma_1], \ldots, [\Gamma_m]$
  - (considered as multiset)
  - corresponding formula: $fm(\Gamma) = A_1 \vee \cdots \vee A_n \vee \Box fm(\Gamma_1) \vee \cdots \vee \Box fm(\Gamma_m)$
The inference rules

- axiom and logical rules for system KN:

- Note: no structural rules here (they are admissible as we see later.)

- The cut rule in nested sequents:

- Inference rules for the other modal axioms:

  - For $X \subseteq \{d, t, b, 4, 5\}$, we write $X^\circ$ for the corresponding set of inference rules, and we write $\text{KN} + X^\circ$ for the system KN extended with $X^\circ$, and we write $\text{KN} + X^\circ + \text{cut}$ for the system that also includes cut.

Proposition 1. The axiom

\[
\text{id} \quad \frac{\Gamma \{A, A\}}{\Gamma \{\{\}}
\]

is derivable in KN.

Proof. By induction on $A$. [Exercise: Complete this proof.]

Soundness and Completeness

We now see that for arbitrary $X \subseteq \{d, t, b, 4, 5\}$, the system $\text{KN} + X^\circ + \text{cut}$ is sound and complete for the modal logic $K + X$.

Theorem 2 (Soundness). If a sequent $\Gamma$ is provable in $\text{KN} + X^\circ + \text{cut}$, then it is $X$-valid.

Proof. By induction on the height of the derivation $\mathcal{D}$ for $\Gamma$.

We show for each rule instance: if all premises are valid, then so is the conclusion.

Theorem 3 (Completeness). If a formula $A$ is $X$-valid, then it is provable in $\text{KN} + X^\circ + \text{cut}$.

Proof. By completeness of the Hilbert system, $A$ is provable in the Hilbert system. For showing that it is provable in $\text{KN} + X^\circ + \text{cut}$, we first show that every axiom of the Hilbert system can be proved in $\text{KN} + X^\circ$, and then we show how modus ponens and necessitation can be simulated in $\text{KN} + \text{cut}$. [Exercise: Complete this proof.]

Cut elimination

This time it is a bit more involved than for the plain sequent calculus.

- A set of axioms $X \subseteq \{d, t, b, 4, 5\}$ is 45-closed, if and only if the following two conditions hold:
  
  - whenever 4 is derivable in $K + X$, then $4 \in X$, and
  
  - whenever 5 is derivable in $K + X$, then $5 \in X$.

- Note: all logics in the cube can be described by a 45-closed set of axioms.

Theorem 4. Let $X \subseteq \{d, t, b, 4, 5\}$ be 45-closed. If a sequent $\Gamma$ is derivable in $\text{KN} + X^\circ + \text{cut}$ then it is also derivable in $\text{KN} + X^\circ$. 
To prove this, we need a lot of lemmas, and again a new variant of the cut:

- For \( Y \subseteq \{4, 5\} \) and \( n \geq 0 \) we define the rule
  \[
  \text{cut}_{Y} \quad \frac{\Gamma \{ \square A \} \{ \emptyset \}^{n} \quad \Gamma \{ \lozenge A \} \{ \lozenge A \}^{n}}{\Gamma \{ \emptyset \} \{ \emptyset \}^{n}}
  \]
  
- We write \( \text{cut}_{r} \) or \( \text{cut}_{Y} \) to indicate that the depth of the cut formula (=the rank of the cut) is \( r \). The\( \) cut-rank\( \) of a derivation \( D \) is the maximal rank of a cut occurring in \( D \).

- Structural rules for modal axioms:

  \[
  \begin{align*}
  \text{d}_1 \quad & \frac{\Gamma \{ [ ] \} \quad \Gamma \{ [ ] \}}{\Gamma \{ [ ] \}} \\
  \text{t}_1 \quad & \frac{\Gamma \{ [ \Delta ] \} \quad \Gamma \{ [ ] \}}{\Gamma \{ [ ] \}} \\
  \text{b}_1 \quad & \frac{\Gamma \{ [ \Lambda, [ \Delta ] ] \} \quad \Gamma \{ [ ] \}}{\Gamma \{ [ ] \}} \\
  \text{d}_5 \quad & \frac{\Gamma \{ [ \Delta ] \} \{ [ ] \}}{\Gamma \{ [ ] \}} \\
  \text{t}_5 \quad & \frac{\Gamma \{ [ ] \} \{ [ ] \}}{\Gamma \{ [ ] \}} \\
  \end{align*}
  \]

  \( \text{d}_5 \) \quad \text{t}_5 \quad \text{b}_1 \quad \text{d}_1 \quad \text{t}_1 \quad \text{b}_1 \quad \text{d}_5 \quad \text{t}_5 \)

- For \( X \subseteq \{d, t, b, 4, 5\} \), we write \( X^{\dagger} \) for the corresponding set of structural rules.

- We define \( \tilde{X}^{\circ} = X^{\circ} \) if \( d \notin X \) and \( \tilde{X}^{\circ} = X^{\circ} \setminus \{d^{\circ}\} \cup \{d^{\dagger}\} \) if \( d \in X \).

- We also define a \( Y^{\dagger} \)-rule for \( Y \subseteq \{4, 5\} \):

  \[
  \text{Y}_{Y} \quad \frac{\Gamma \{ [ ] \} \{ [ ] \}}{\Gamma \{ [ ] \}}
  \]

  with the side condition that

  - if \( Y = 0 \) then \( \Gamma \{ \{ \} \} = \Gamma' \{ \{ \} \} \) for some \( \Gamma' \{ \} \),
  - if \( Y = \{4\} \) then \( \Gamma \{ \{ \} \} = \Gamma' \{ \} \), \( \Gamma'' \{ \} \) for some \( \Gamma' \{ \} \) and \( \Gamma'' \{ \} \).
  - if \( Y = \{5\} \) then \( \text{depth}(\Gamma' \{ \} \{ \emptyset \}) > 0 \),
  - if \( Y = \{4, 5\} \) then there is no side condition.

- Note: if \( Y = X \subseteq \{4, 5\} \) then the rule \( Y^{\dagger} \) is derivable using the rules in \( X^{\dagger} \). [Exercise: Prove that.]

Lemma 5. Let \( X \subseteq \{d, t, b, 4, 5\} \). Then the rules

  \[
  \begin{align*}
  \text{weak} \quad & \frac{\Gamma \{ \emptyset \}}{\Gamma \{ [ ] \}} \\
  \text{contr} \quad & \frac{\Gamma \{ [ ] \} \{ [ ] \}}{\Gamma \{ [ ] \}} \\
  \end{align*}
  \]

are height-preserving and cut-rank preserving admissible for \( KN + \tilde{X}^{\circ} + \text{cut} \).

[Exercise: Prove this.]

Lemma 6. Let \( X \subseteq \{d, t, b, 4, 5\} \). Then all rules in \( KN + \tilde{X}^{\circ} \) are height-preserving and cut-rank preserving invertible.

[Exercise: Prove this.]

Lemma 7. Let \( X \subseteq \{d, t, b, 4, 5\} \) be \( 45 \)-closed and let \( x \in X \). If \( x \neq d \) then \( x^{\dagger} \) is cut-rank preserving admissible for \( KN + \tilde{X}^{\circ} + \text{cut} \). If \( x = d \) then \( x^{\dagger} \) is admissible for \( KN + \tilde{X}^{\circ} \).

[Exercise: Prove this.]

Lemma 8. Let \( X \subseteq \{d, t, b, 4, 5\} \) be \( 45 \)-closed and let \( Y = \{4, 5\} \) such that \( Y \subseteq X \). Then \( Y^{\dagger} \) is cut-rank preserving admissible for \( KN + \tilde{X}^{\circ} + \text{cut} \).

[Exercise: Prove this.]

Lemma 9 (Reduction Lemma). Let \( X \subseteq \{d, t, b, 4, 5\} \) be \( 45 \)-closed and let \( Y \subseteq X \cap \{4, 5\} \) and let \( r > 0 \) and \( n \geq 0 \).
• If there is a derivation in $\text{KN} + \overline{\text{X}}^0 + \text{cut}$ of the form

$$
\begin{array}{c}
\text{cut}_{r+1} \\
D_1 \Gamma \{ \overline{A} \} \\
D_2 \Gamma \{ \overline{A} \} \\
\hline
\Gamma \{ \emptyset \}
\end{array}
$$

where $D_1$ and $D_2$ have both cut-rank $\leq r$, then there is a derivation

$$
\begin{array}{c}
\text{cut}_{r+1} \\
D' \Gamma \{ \emptyset \}
\end{array}
$$

in $\text{KN} + \overline{\text{X}}^0 + \text{cut}$ with cut-rank $\leq r$.

• If there is a derivation

$$
\begin{array}{c}
\text{Y-cut}_{r+1} \\
D_1 \Gamma \{ \square \overline{A} \} \{ \emptyset \}^n \\
D_2 \Gamma \{ \Diamond \overline{A} \} \{ \Diamond \overline{A} \}^n \\
\hline
\Gamma \{ \emptyset \} \{ \emptyset \}^n
\end{array}
$$

where $D_1$ and $D_2$ are in $\text{KN} + \overline{\text{X}}^0 + \text{cut}$ and have both cut-rank $\leq r$, then there is a derivation

$$
\begin{array}{c}
\text{Y-cut}_{r+1} \\
D' \Gamma \{ \emptyset \} \{ \emptyset \}^n
\end{array}
$$

in $\text{KN} + \overline{\text{X}}^0 + \text{cut}$ with cut-rank $\leq r$.

[Exercise: Prove this.]

References