Advanced Mathematical Programming

Formulations & Applications

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INF580 — 2017

Practicalities

- **URL:**
http://www.lix.polytechnique.fr/~liberti/teaching/dix/inf580-17
- **Dates:** wed-fri
4-6, 11-13, 18, 25-27 jan
1-3, 8-10, 22-24 feb
1-3, 8-10, 15 mar
- **Place:** PC 37 (lectures & tutorials)
bring your laptops! (Linux/MacOSX/Windows)
- **Exam:** either a project (max 2 people) or oral

Section 1

Introduction

What is *Mathematical Programming*?

- Formal declarative language for describing optimization problems
- As expressive as any imperative language
- Interpreter = solver
- Shifts focus from *algorithmics* to *modelling*

Syntax

- **A valid sentence:**

$$\left.\begin{array}{ll} \min & x_1 + 2x_2 - \log(x_1 x_2) \\ & x_1 x_2^2 \geq 1 \\ & 0 \leq x_1 \leq 1 \\ & x_2 \in \mathbb{N}. \end{array}\right\} \qquad [P]$$

- **An invalid one:**

$$\left.\begin{array}{ll} \min & \frac{\cdot}{x_2} + x_1 + + \sin\cos \\ & x_{x_2} \geq x_{x_1} \\ & \sum\limits_{i \leq x_1} x_i = 0 \\ & x_1 \neq x_2 \\ & x_1 < x_2. \end{array}\right\}$$

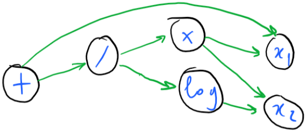
MINLP Formulation

Given functions $f, g_1, \ldots, g_m : \mathbb{Q}^n \to \mathbb{Q}$ **and** $Z \subseteq \{1, \ldots, n\}$

$$\left.\begin{array}{rrcl} \min & f(x) & & \\ \forall i \leq m & g_i(x) & \leq & 0 \\ \forall j \in Z & x_j & \in & \mathbb{Z} \end{array}\right\}$$

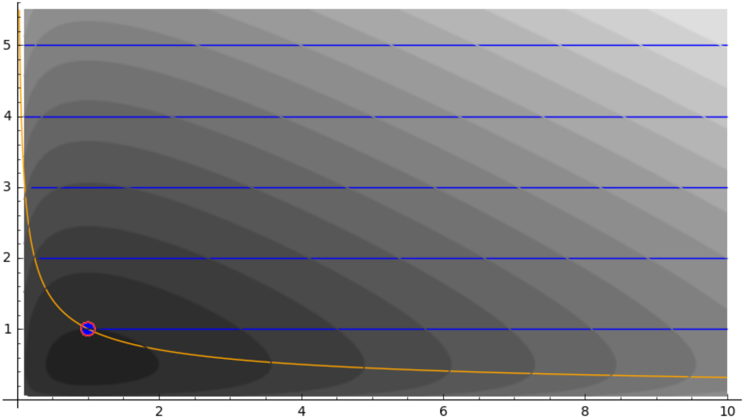
- $\phi(x) = 0 \quad \Leftrightarrow \quad (\phi(x) \leq 0 \wedge -\phi(x) \leq 0)$
- $L \leq x \leq U \quad \Leftrightarrow \quad (L - x \leq 0 \wedge x - U \leq 0)$
- f, g_i **represented by** ***expression DAGs***

$$x_1 + \frac{x_1 x_2}{\log(x_2)}$$



Semantics

$$P \equiv \min\{x_1 + 2x_2 - \log(x_1 x_2) \mid x_1 x_2^2 \geq 1 \wedge 0 \leq x_1 \leq 1 \wedge x_2 \in \mathbb{N}\}$$



$[\![P]\!] = (\mathsf{opt}(P), \mathsf{val}(P))$ $\mathsf{opt}(P) = (1, 1)$ $\mathsf{val}(P) = 3$

What is a solution of an MP?

- Given an MP P, there are three possibilities:
  1. $[\![P]\!]$ exists
  2. P is unbounded
  3. P is infeasible
- P *has a feasible solution* iff $[\![P]\!]$ exists or is unbounded
  otherwise it is infeasible
- P *has an optimum* iff $[\![P]\!]$ exists
  otherwise it is infeasible or unbounded
- Asymmetry between optimization and feasibility
- Feasibility prob. $g(x) \leq 0$ can be written as MP

$$\min\{0 \mid g(x) \leq 0\}$$

Solvers (or "interpreters")

- Take formulation P as input
- Output $[\![P]\!]$ *and possibly other information*
- Trade-off between generality and efficiency
  (i) Linear Programming (LP)
  f, g_i linear, $Z = \varnothing$
  (ii) Mixed-Integer Linear Programming (MILP)
  f, g_i linear, $Z \neq \varnothing$
  (iii) Nonlinear Programming (NLP)
  some nonlinearity in f, g_i, $Z = \varnothing$
  (iv) Mixed-Integer Nonlinear Programming (MINLP)
  some nonlinearity in f, g_i, $Z \neq \varnothing$

  (way more classes than these!)
- Each solver targets a given class

Why should you care?

- Production industry
  planning, scheduling, allocation, ...
- Transportation & logistics
  facility location, routing, rostering, ...
- Service industry
  pricing, strategy, product placement, ...
- Energy industry (*all of the above*)
- Machine Learning & Artificial Intelligence
  clustering, approximation error minimization
- Biochemistry & medicine
  protein structure, blending, tomography, ...
- Mathematics
  Kissing number, packing of geometrical objects,...

Section 2

Decidability

Formal systems (FS)

- A *formal system* consists of:
  - an *alphabet*
  - a *formal grammar*
    allowing the determination of *formulæ* and *sentences*
  - a set A of *axioms* (given sentences)
  - a set R of *inference rules*
    allowing the derivation of new sentences from old ones
- A *theory* T is the smallest set of sentences that is obtained by recursively applying R to A
- Example 1 (PA1): $+, \times, \wedge, \vee, \forall, \exists, =$ and variable names; 1st order sentences about $\mathbb{N}$; Peano's Axioms; *modus ponens* and generalization
- Example 2 (Reals): $+, \times, \wedge, \vee, =, >$, variables, real constants; polynomials over $\mathbb{R}$; field and order axioms for $\mathbb{R}$, "basic operations on polynomials"

What is decidability?

Given a FS $\mathcal{F}$,

- **a *decision problem* P in $\mathcal{F}$ is a set of sentences in $\mathcal{F}$**
- **Decide whether a given sentence f in $\mathcal{F}$ belongs to P or not**
- PA1: **decide whether a sentence f about $\mathbb{N}$ has a proof or not**

  a *proof* of f is a sequence of sentences that begins with axioms and ends with f, each other sentence in the sequence being derived from applying inference rules to previous sentences
- Reals: **decide whether a given system of polynomials p on $\mathbb{R}$ has a solution or not**

Decision and proof in PA1

- **Given a decision problem, is there an algorithm with input f, output YES/NO?**

  YES: "f has proof in $\mathcal{F}$"

  NO: "f does not have a proof in $\mathcal{F}$"

- **[Turing 1936]: an encoding of HALTING PROBLEM in PA1 is undecidable in PA1**

- **A FS $\mathcal{F}$ is *complete* if, for every f in $\mathcal{F}$ either f or $\neg f$ is provable in $\mathcal{F}$**

  Gödel's first incompleteness theorem $\Rightarrow$ PA1 is incomplete $\exists f$ s.t. f and $\neg f$ are unprovable in $\mathcal{F}$ *(such f are called independent in $\mathcal{F}$)*

- **PA1 is undecidable and incomplete**

Decision and proof in Reals

- **Given poly system $p(x) \geq 0$, is there alg. deciding YES/NO?**

  YES: "$p(x) \geq 0$ has a solution in $\mathbb{R}$"

  NO: "$p(x) \geq 0$ has no solution in $\mathbb{R}$"
- **[Tarski 1948]:** Reals **is decidable**
- **Tarski's algorithm:**

  constructs solution sets (YES) or derives contradictions (NO)

  Best kind of decision algorithm: also provides proofs!

  $\Rightarrow$ Reals is also complete
- Reals **is decidable and complete**

A stupid FS

- NoInference:
  Any FS with $<\infty$ axiom schemata and no inference rules
- Only possible proofs: sequences of axioms
- Only provable sentences: axioms
- For any other sentence f: no proof of f or $\neg f$
- Trivial decision algorithm:
  given f, output YES if f is an axiom, NO otherwise
- NoInference is decidable and incomplete

Undecidability & Incompleteness

- **[Nonexistence of a proof for f] $\not\equiv$ [Proof of $\neg f$]**
  In a decidable and incomplete FS, a decision algorithm answers NO to both f and $\neg f$ if f is independent
- ***Information complexity*:**
  decision = 1 bit, proof = many bits
- **Undecidability and incompleteness are different!**

Decidability, computability, solvability

- Decidability: applies to decision problems
- Computability: applies to function evaluation
  - Is the function f, mapping i to the i-th prime integer, computable?
  - Is the function g, mapping Cantor's CH to 1 if provable in ZFC axiom system and to 0 otherwise, computable?
- Solvability: applies to other problems
  E.g. to optimization problems!

Is MP solvable?

- Hilbert's 10th problem: is there an algorithm for solving polynomial Diophantine equations?
- *Modern formulation*:
  are polynomial systems over $\mathbb{Z}$ solvable?
- [Matiyasevich 1970]: NO
  can encode universal TMs in them
- Let $p(\alpha, x) = 0$ be a Univ. Dioph. Eq. (UDE)
- $\min\{0 \mid p(\alpha, x) = 0\}$ is an undecidable (feasibility) MP
- $\min (p(\alpha, x))^2$ is an unsolvable (optimization) MP