INF421, Lecture 8
Graphs

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Course

- **Objective**: to teach you some data structures and associated algorithms
- **Evaluation**: TP noté en salle info le 16 septembre, Contrôle à la fin. Note: $\max(CC, \frac{3}{4}CC + \frac{1}{4}TP)$
- **Organization**: fri 26/8, 2/9, 9/9, 16/9, 23/9, 30/9, 7/10, 14/10, 21/10, amphi 1030-12 (Arago), TD 1330-1530, 1545-1745 (SI31,32,33,34)
- **Books**:
  1. Ph. Baptiste & L. Maranget, *Programmation et Algorithmique*, Ecole Polytechnique (Polycopié), 2009
  2. G. Dowek, *Les principes des langages de programmation*, Editions de l'X, 2008
  3. D. Knuth, *The Art of Computer Programming*, Addison-Wesley, 1997
  4. K. Mehlhorn & P. Sanders, *Algorithms and Data Structures*, Springer, 2008
- **Website**: `www.enseignement.polytechnique.fr/informatique/INF421`
- **Contact**: `liberti@lix.polytechnique.fr` (e-mail subject: INF421)

Lecture summary

- Graph definitions
- Operations on graphs
- Combinatorial problems on graphs
- Easy and hard problems
- Modelling problems for a generic solution method

The minimal knowledge

- **Operations on graphs**: complement, line graph, contraction
- **Decision/optimization problems**: finding subgraphs with given properties
- **Easy problems**: solvable in polynomial time (**P**), e.g. minimum cost spanning tree, shortest paths, maximum matching
- **Hard problems**: efficient method for solving one would solve all of them (**NP**-hard), e.g. maximum clique, maximum stable set, vertex colouring
- **Mathematical Programming**: a generic model-and-solve approach

Graph definitions

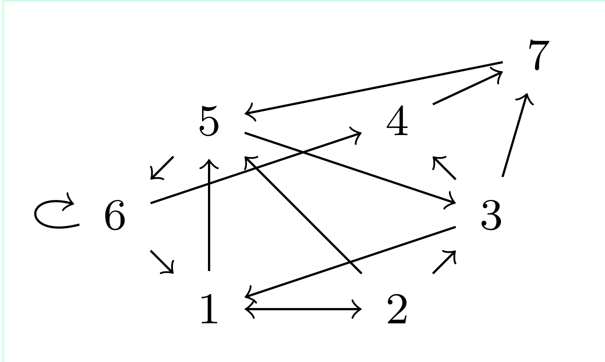
Motivation

The ultimate data structure

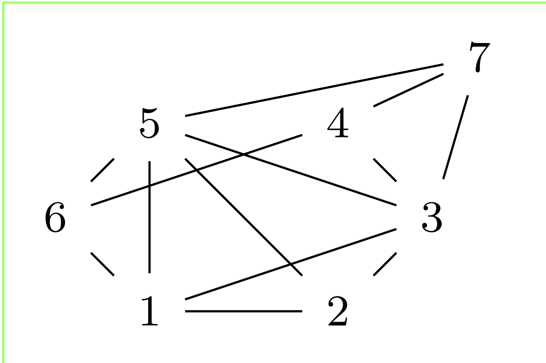
Every time you see arrow connecting boxes, circles or black dots in a computer science course, you can think of graphs and digraphs!

Graphs and digraphs

- Digraph $G = (V, A)$: relation A on set V
  - V: set of **nodes**
  - A: set of **arcs** (u, v) with $u, v \in V$



- Graph $G = (V, E)$: symmetric relation E on set V
  - V: set of **vertices**
  - E: set of **edges** $\{u, v\}$ with $u, v \in V$



- **Simple** (di)graphs: relation is *irreflexive*
  (I.e., v *not related to itself for all* $v \in V$)

Remarks

- I shall mainly present results for **undirected** graphs
- Most results extend trivially to **directed** graphs (*digraphs*)
- Detailing such extensions is a good exercise
- **Warning**: not all such extensions are trivial

Example

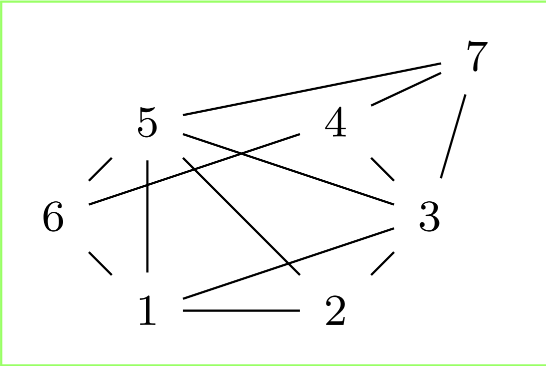
- If G is a graph, we indicate its set of vertices by $V(G)$ and its set of edges by $E(G)$
- **Example of extension to digraphs**:
  If G is a digraph, we indicate its set of nodes by $V(G)$ and its set of arcs by $A(G)$

Stars

Stars: sets of nodes/vertices or arcs/edges adjacent to a given node

$\forall v \in V(G)$,

- if G is undirected,



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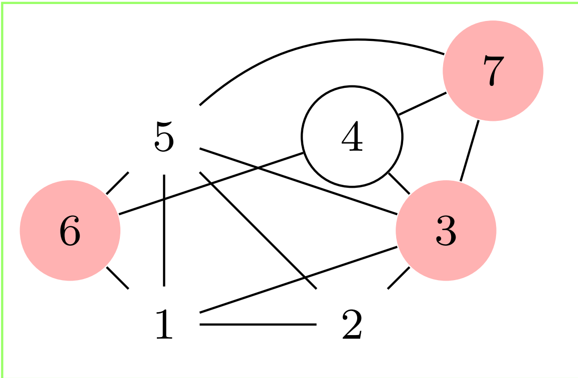

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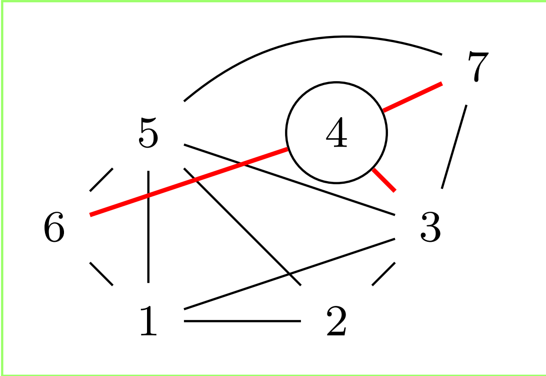

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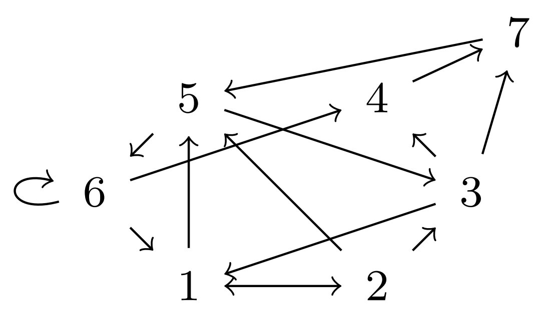

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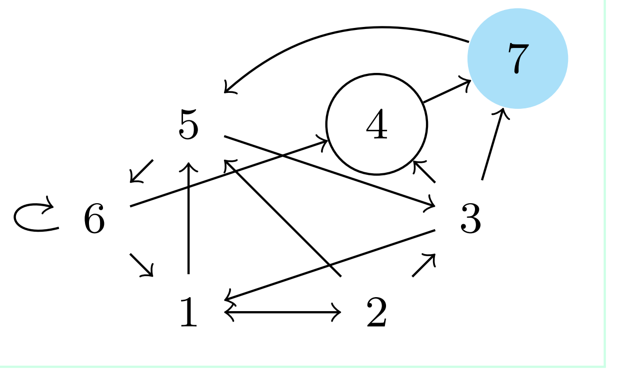

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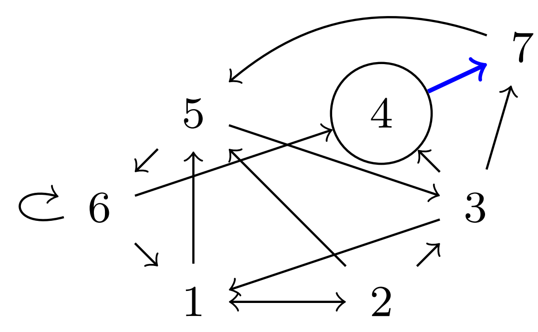

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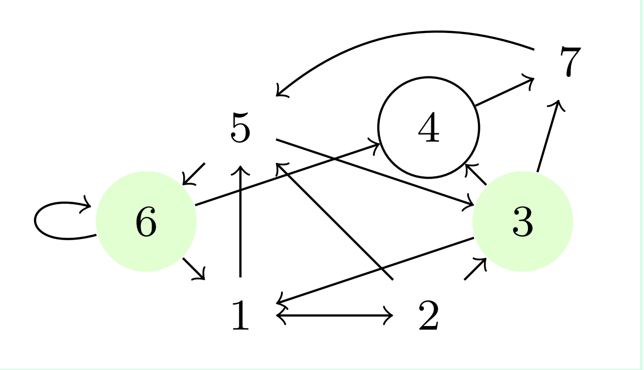

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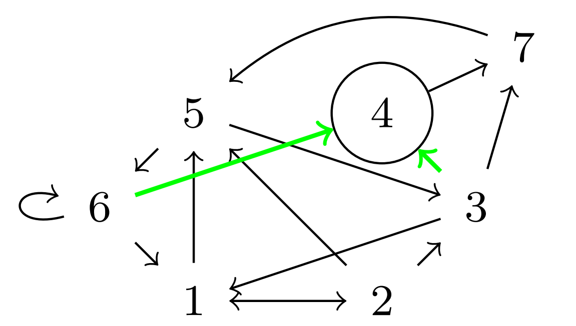

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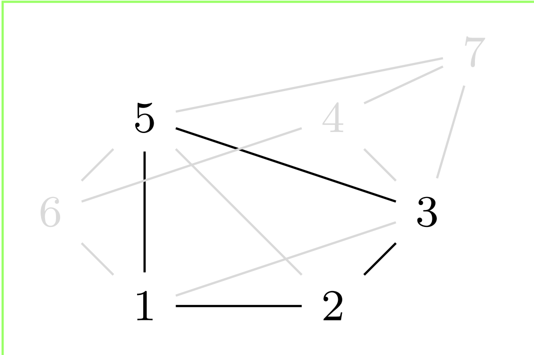
- $|N(v)| =$**degree**, $|N^+(v)| =$**outdegree**, $|N^-(v)| =$**indegree** of v
- If v belongs to two graphs G, H, write $N_G(v)$ and $N_H(v)$

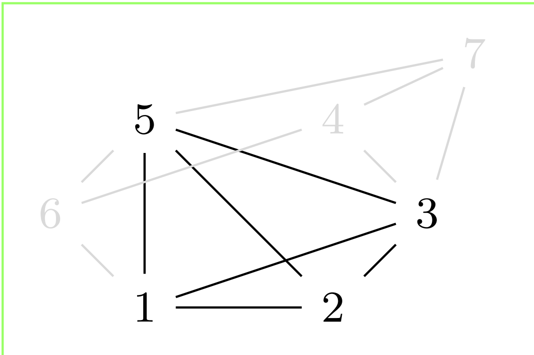
  (similarly for other star notation)

Subgraphs

- A graph $H = (U, F)$ is a **subgraph** of $G = (V, E)$ if $U \subseteq V$, $F \subseteq E$ and $\forall \{u, v\} \in F \ (u, v \in U)$



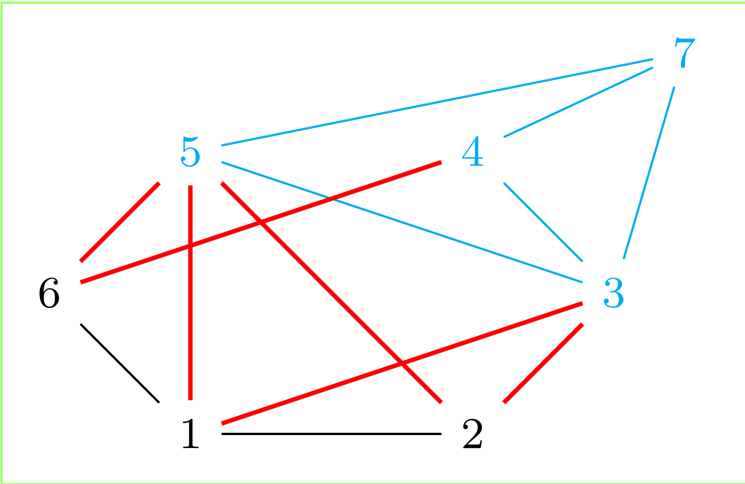
- A subgraph $H = (U, F)$ of $G = (V, E)$ is **spanning** if $U = V$
- A subgraph $H = (U, F)$ of $G = (V, E)$ is **induced** by U if $\forall u, v \in U \ (\{u, v\} \in E \rightarrow \{u, v\} \in F)$



Induced subgraph notation: $H = G[U]$

Cutsets

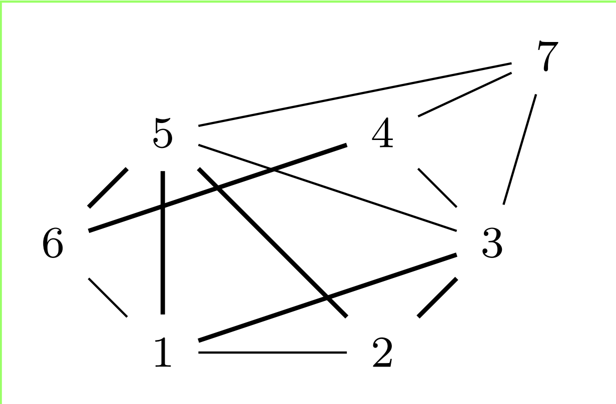
- Let $H = (U, F)$ be a subgraph of $G = (V, E)$ (i.e. $U \subsetneq V$)
- The **cutset** $\delta(H) = \left(\bigcup_{u \in U} \delta(u) \right) \smallsetminus F$
  is the edge set "separating" U and $V \smallsetminus U$
- E.g. let $U = \{1, 2, 6\}$ and $H = G[U]$, then $\delta(H)$ is shown by the red edges below



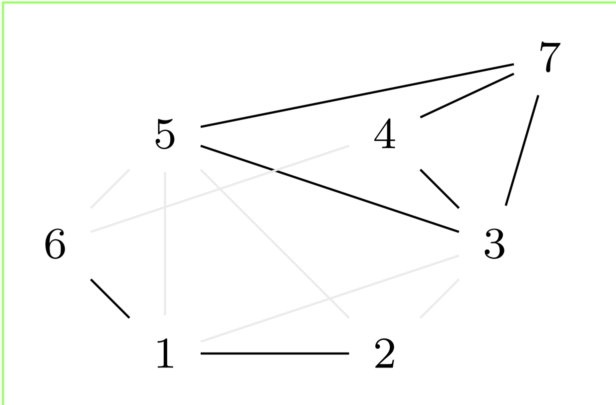
- Similar definitions hold for **directed cutsets**
- If G is undirected, $\delta(U) = \delta(V \smallsetminus U)$ for all $U \subseteq V(G)$

Connectedness

- A graph is **connected** if there are no empty nontrivial cutsets



Connected

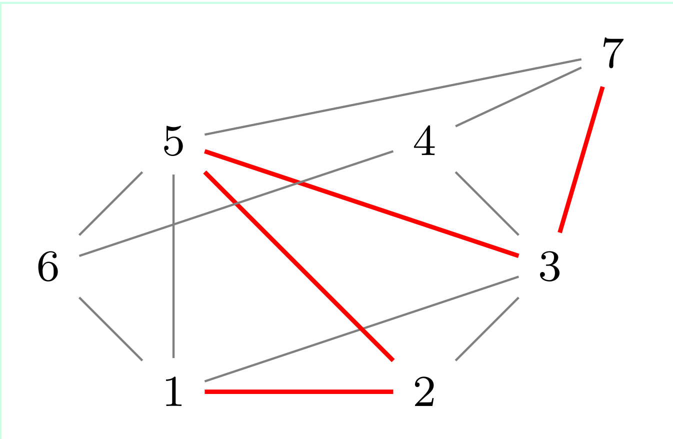


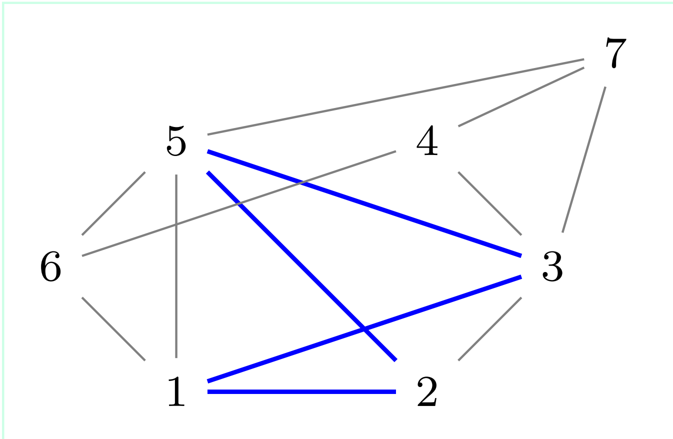
Not connected: $\delta(\{1,2,6\}) = \varnothing$

- Each maximal connected subgraph of a graph is a **connected component**
- Most graph algorithms assume the input graph to be connected: if not, just apply it to each connected component

Paths and cycles

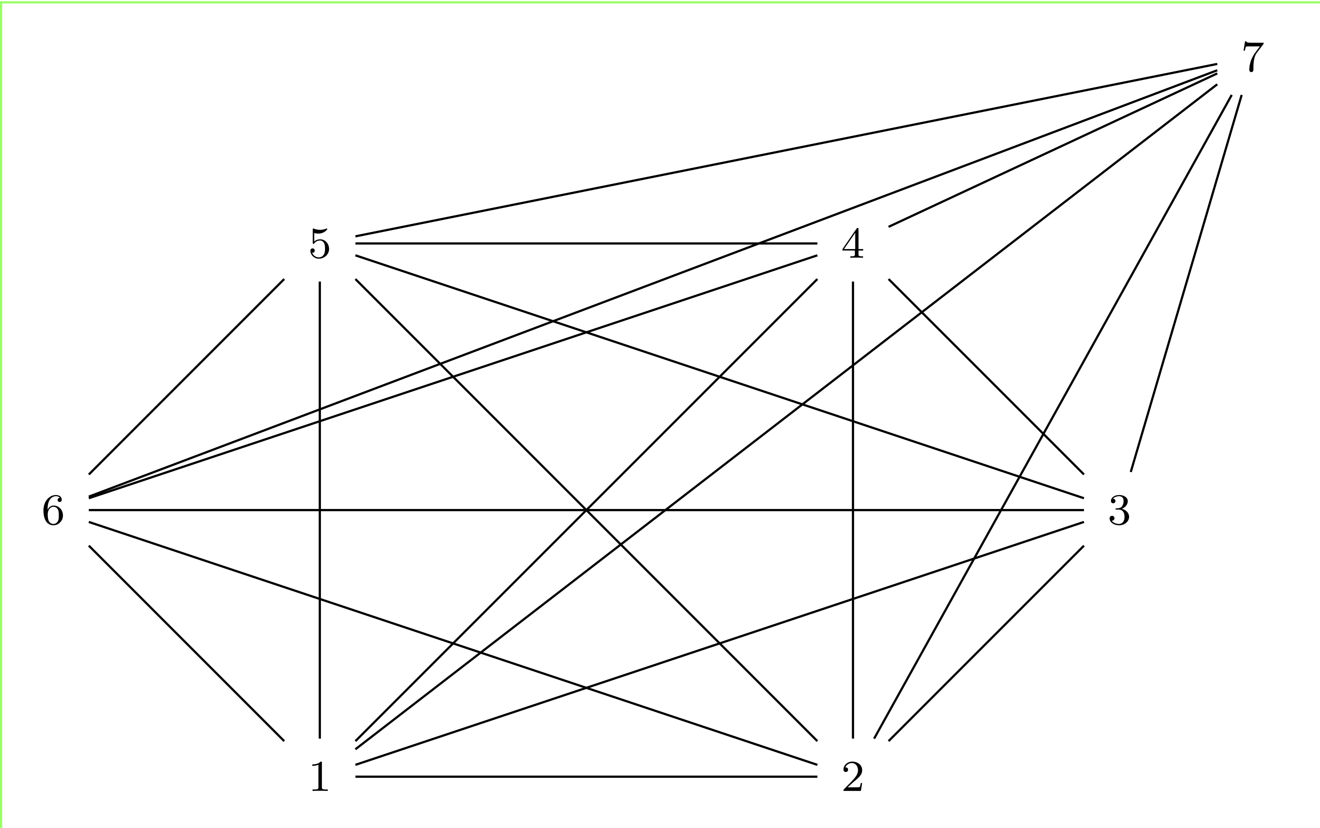
- Let G be a graph and $u, v \in V(G)$
- A **simple path** P from u to v in G is a connected subgraph of G s.t.:
  1. each vertex w in P different from u, v has $|N(w)| = 2$
  2. if $u \neq v$ then $|N(u)| = |N(v)| = 1$
  3. if $u = v$ then either $E(P) = \varnothing$ or $|N(u)| = |N(v)| = 2$
- We indicate a path from u to v by the notation $u \to v$
- If P is a path $u \to v$, then u, v are called the **endpoints** of the path
- A **simple cycle** is a simple path with equal endpoints
- Definitions in Lecture 6 equivalent but more general
- Will simply say paths/cycles to mean *simple* paths/cycles





Complete graph

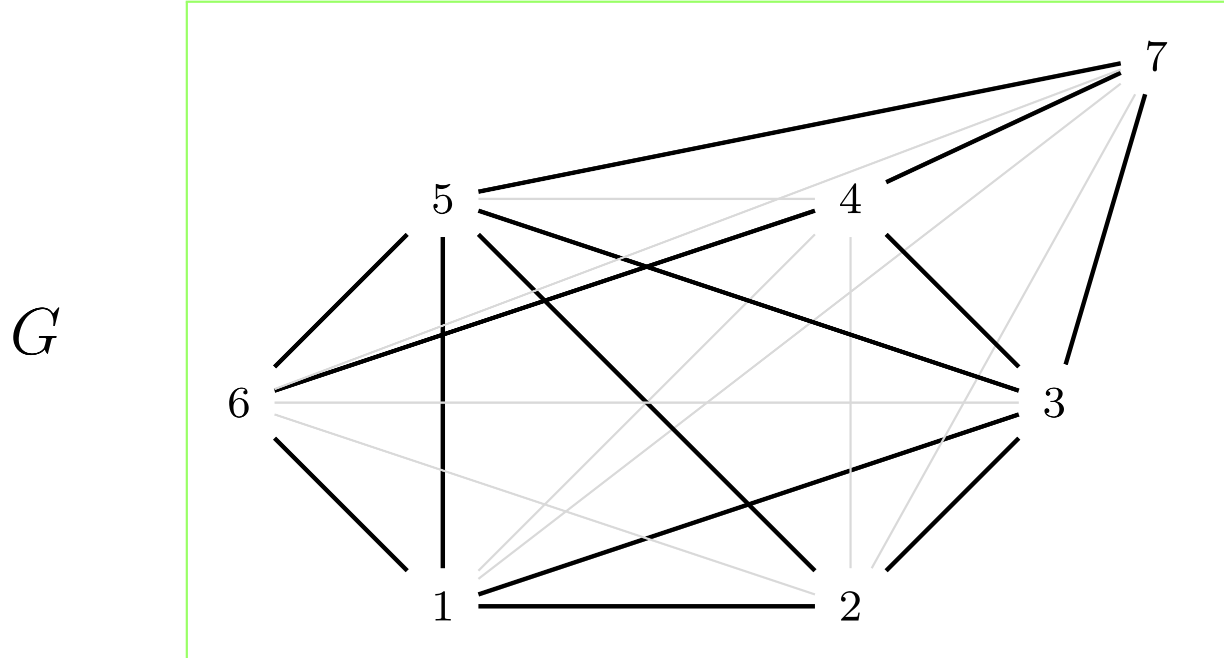
- The **complete graph** K_n on n vertices has all possible edges



- K_n is also called n**-clique**; a complete graph on a vertex set U is denoted by $K(U)$

Complement graph

- Given $G = (V, F)$ with $|V| = n$, the **complement** of G is $\bar{G} = (V, E(K_n) \setminus F)$



- The complement of K_n is the **empty graph** $(V, \varnothing)$ on n vertices

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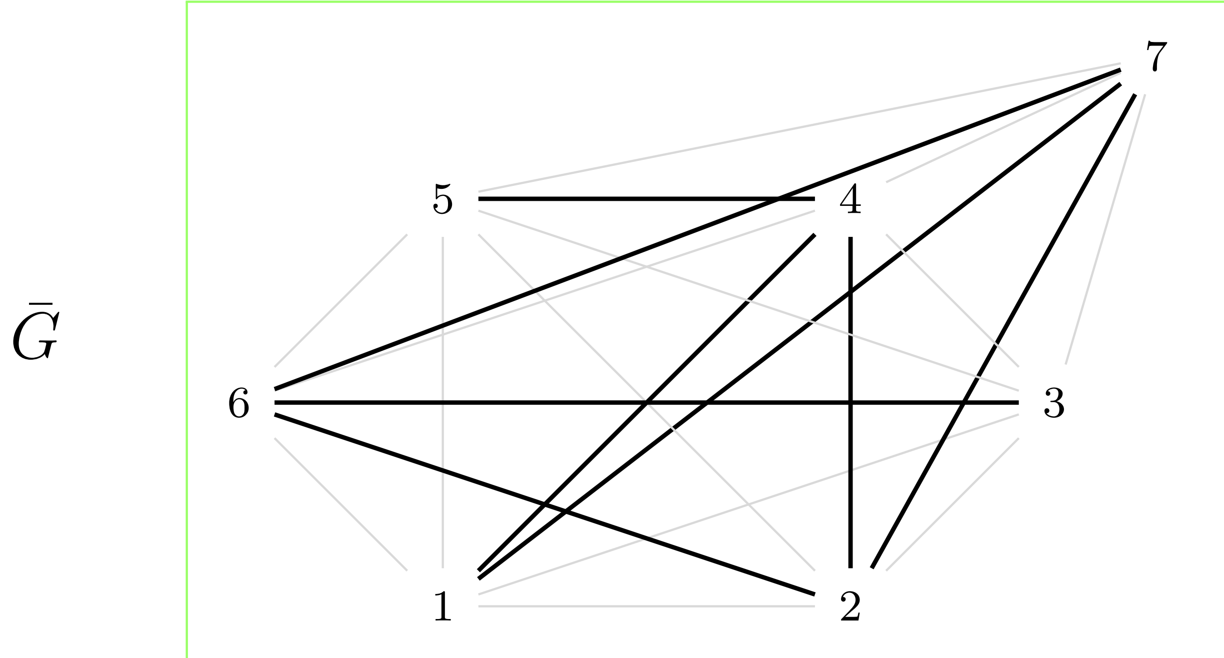

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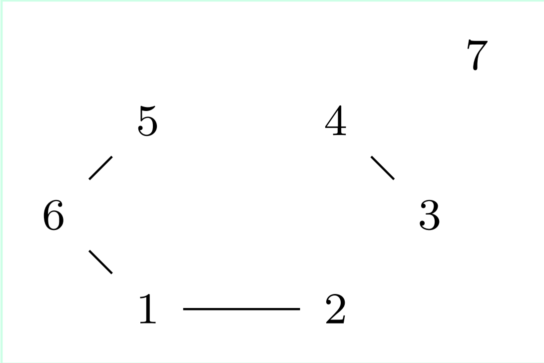
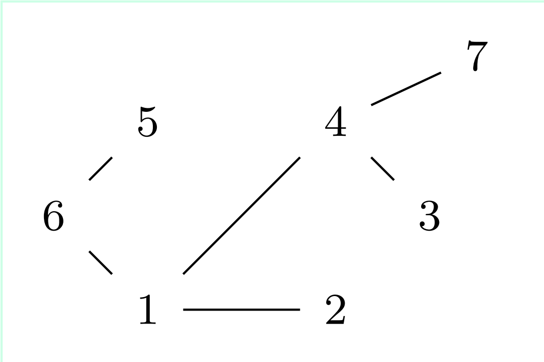
Forests and trees

- A **forest** is a graph with no cycles



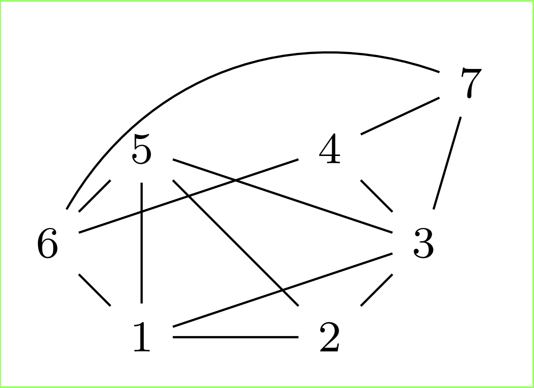
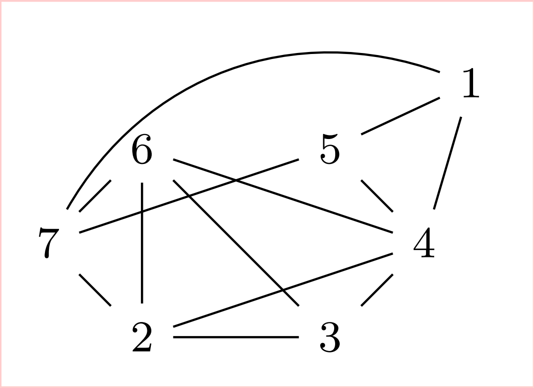
- A **tree** is a connected forest



- If a tree is a subgraph of another graph G, we also call it a **spanning tree**

Graph isomorphism

- Let $|V| = n$ and S_n be the symmetric group of order n
  $\pi \in S_n$ acts on V, get new graph πG

$G =$  $\xrightarrow{\pi=(1,2,3,4,5,6,7)}$  $= H$

- $\exists \pi \in S_n \; (G = \pi H) \Rightarrow G, H$ **isomorphic,** π **graph isomorphism**

  Take $G = (1, 7, 6, 5, 4, 3, 2)H$

- If $(\pi G = G)$, then π is an **automorphism** of G

  Automorphism group of G is Aut$(G) = \langle (1, 5), (4, 7) \rangle \cong C_2 \times C_2$

$N(1) = \{2, 3, 5, 6\}$, $N(2) = \{1, 3, 5\}$		$N(5) = \{2, 3, 1, 6\}$, $N(2) = \{5, 3, 1\}$
$N(3) = \{1, 2, 4, 5, 7\}$, $N(4) = \{3, 6, 7\}$	$=$	$N(3) = \{5, 2, 7, 1, 4\}$, $N(7) = \{3, 6, 4\}$
$N(5) = \{1, 2, 3, 6\}$, $N(6) = \{1, 4, 5, 7\}$		$N(1) = \{5, 2, 3, 6\}$, $N(6) = \{5, 7, 1, 4\}$
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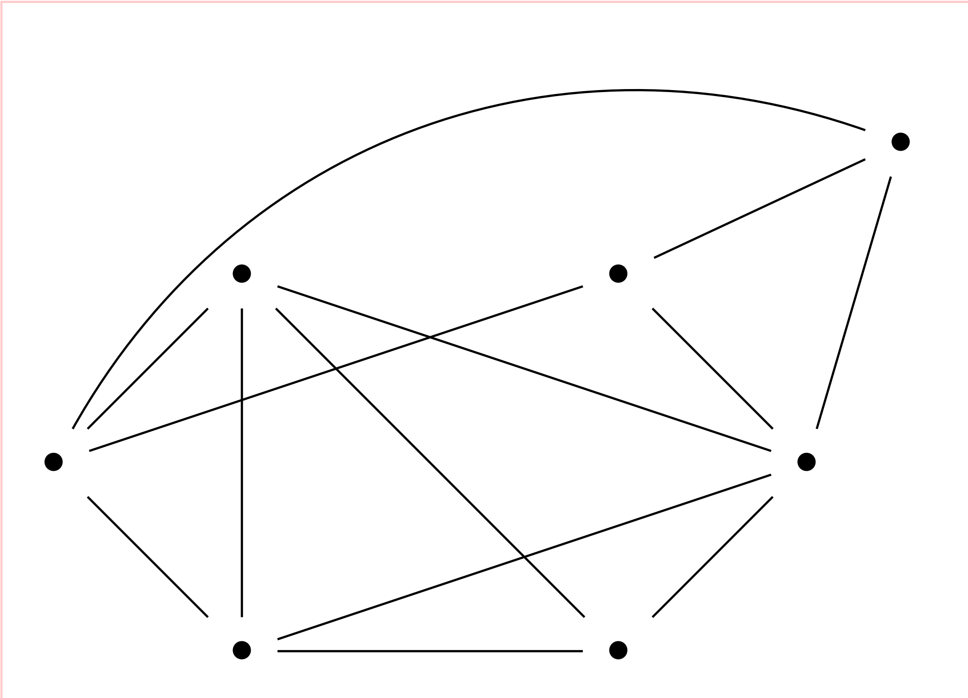
Graphs modulo symmetry

- Symmetries act on vertex labels — can represent equivalence classes of graphs modulo symmetry by simply ignoring labels



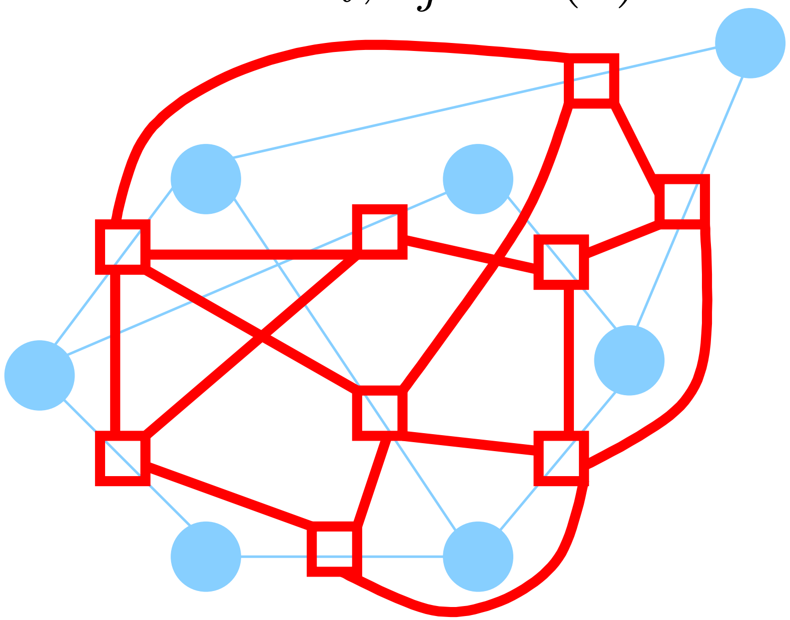
- Not easy to treat mathematically, though. . .

Line graphs

- Given a graph $G = (V, E)$ where $E = \{e_1, \ldots, e_m\}$
- The **line graph** of G is

$$L(G) = (E, \{\{e_i, e_j\} \mid e_i \cap e_j \neq \varnothing\})$$

- Every vertex of $L(G)$ is an edge of G
- Two vertices e_i, e_j of $L(G)$ are adjacent if there is $v \in V$ such that $e_i, e_j \in \delta(v)$



Property: the degree $|N_{L(G)}(e)|$ of a vertex $e = \{u, v\}$ of $L(G)$ is $|N_G(u)| + |N_G(v)| - 2$.

$L(G)$ can be constructed from G in polynomial time (how?)

Operations on graphs

Addition and removal

- Add a vertex v:

  update $V \leftarrow V \cup \{v\}$

- Add an edge $e = \{u, v\}$:

  add vertices u, v, update $E \leftarrow E \cup \{e\}$

- Remove an edge $e = \{u, v\}$:

  update $E \leftarrow E \smallsetminus \{e\}$

- Remove a vertex v:

  update $V \leftarrow V \smallsetminus \{v\}$ and $E \leftarrow E \smallsetminus \delta(v)$

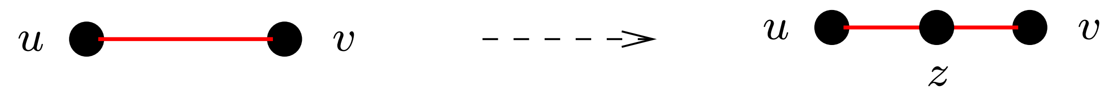
- Operations on sets of vertices/edges:

  Apply operation to each set element

Subdivision and contraction

- Subdivide an edge $e = \{u, v\}$:

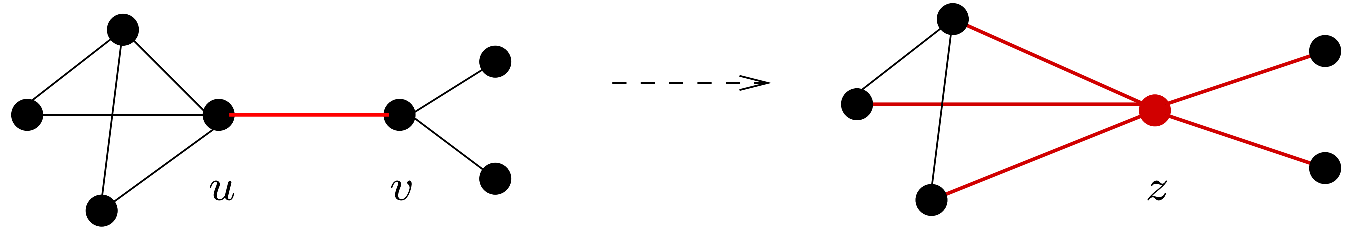
remove e, let $z \notin V$, add edges $\{u, z\}$ and $\{z, v\}$



- Contract an edge $e = \{u, v\}$:

`contract`(G,e):

Let $N(e) = (N(u) \cup N(v)) \smallsetminus \{u, v\}$
Let z be a vertex $\notin V$;
Add vertex z;
for $v \in N(e)$ **do**
    Add edge $\{v, z\}$;
end for
Remove edge e;



Subgraph contraction

- Let $G=(V,E)$, $U \subseteq V$ and $H = G[U]$
- The **contraction** G/U is a sort of "G modulo H"

```
contract(G,U):
  Let z be a new vertex not in V;
  Add vertex z;
  for {u,v} ∈ δ(H) (assume WLOG u ∈ U, v ∈ V ∖ U) do
     add edge {v,z};
     remove edge {u,v};
  end for
  remove G[U];
  return G;
```

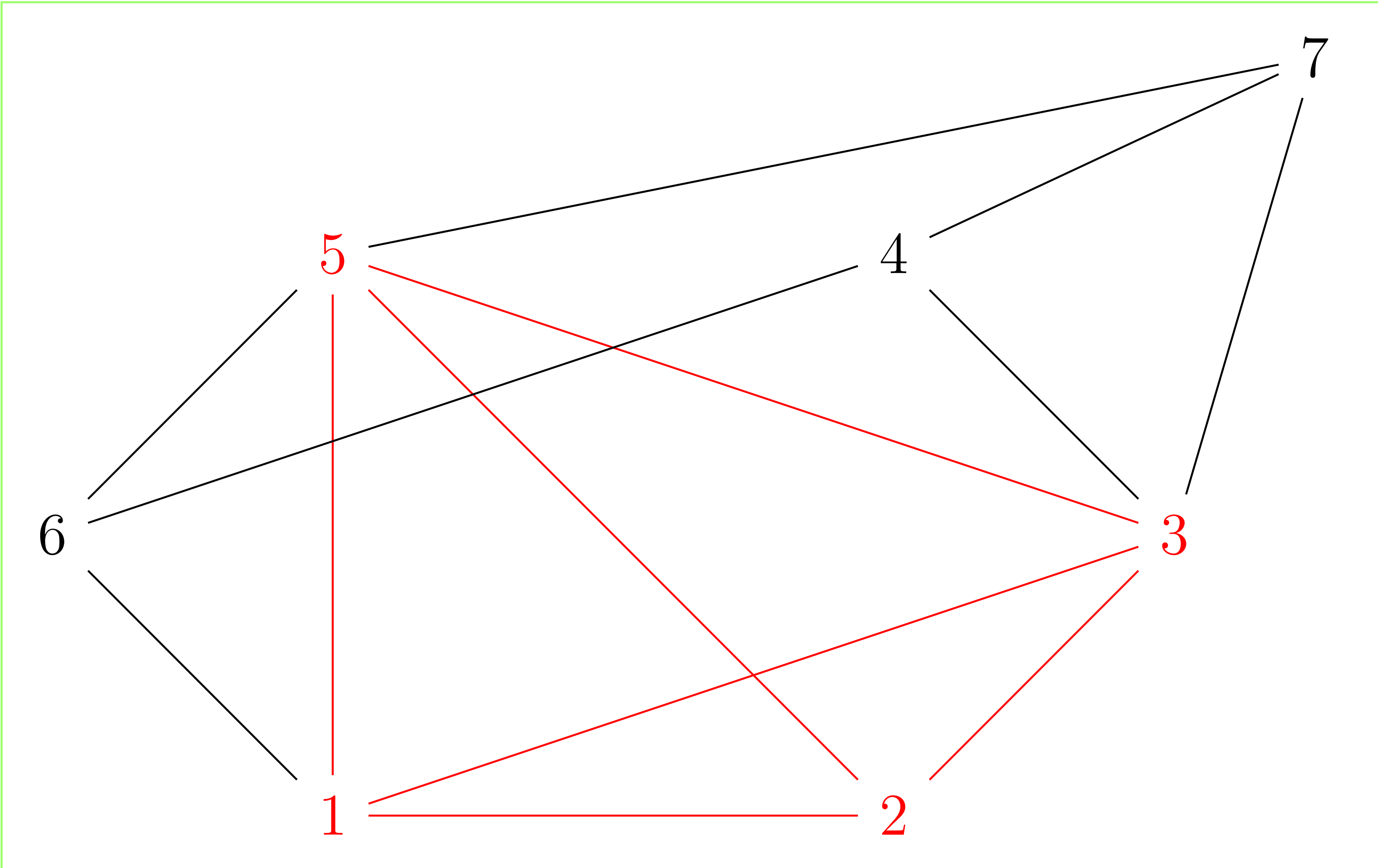
- At the end of the contraction algorithm, the whole subgraph H is "replaced" by a single vertex z
- G/U is formally defined to be `contract`(G,U)

Thm.

Subgraph contraction is equivalent to a sequence of edge contractions

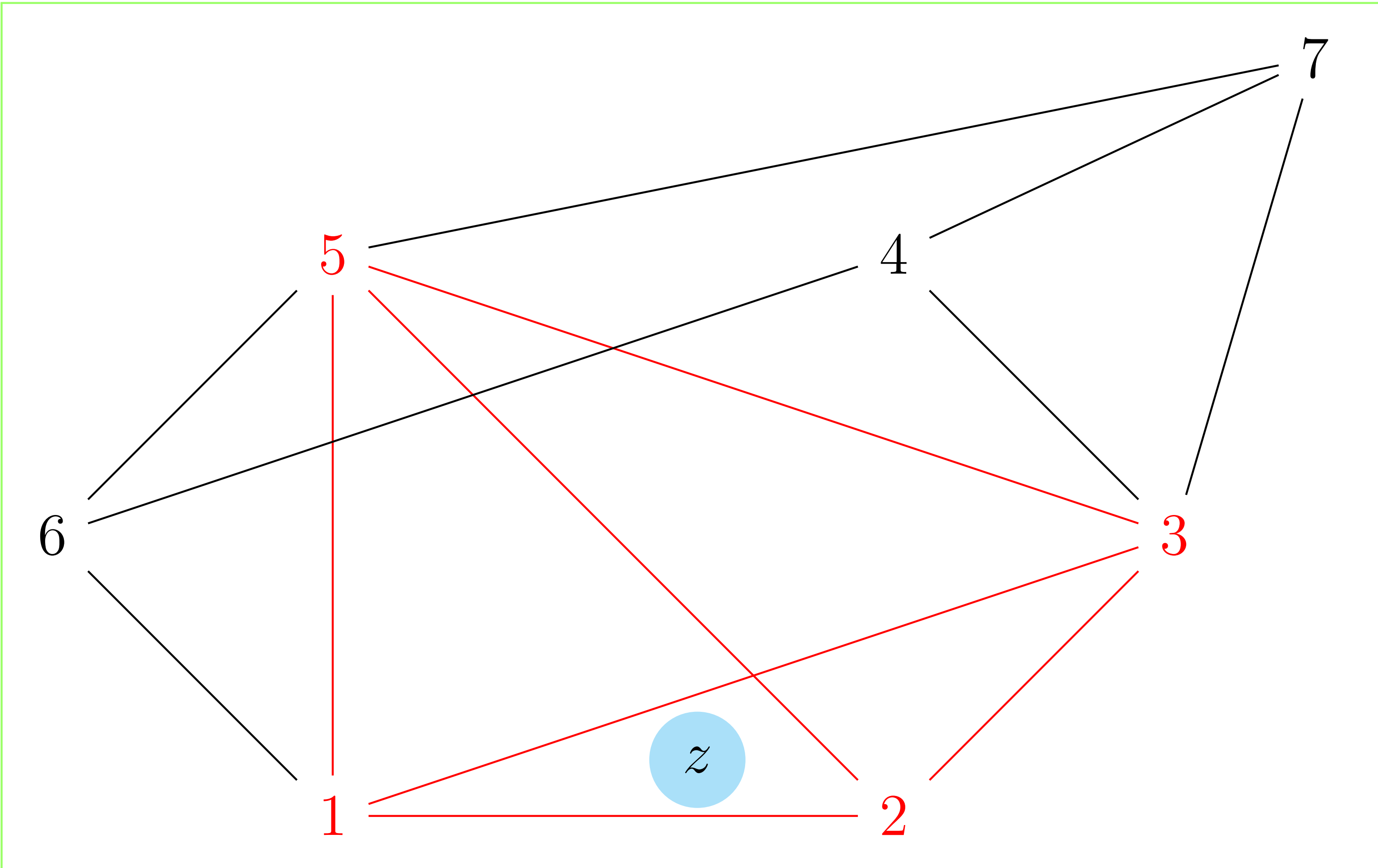
Subgraph contraction algorithm

$U = \{1, 2, 3, 5\}$, $G[U]$ **in red**



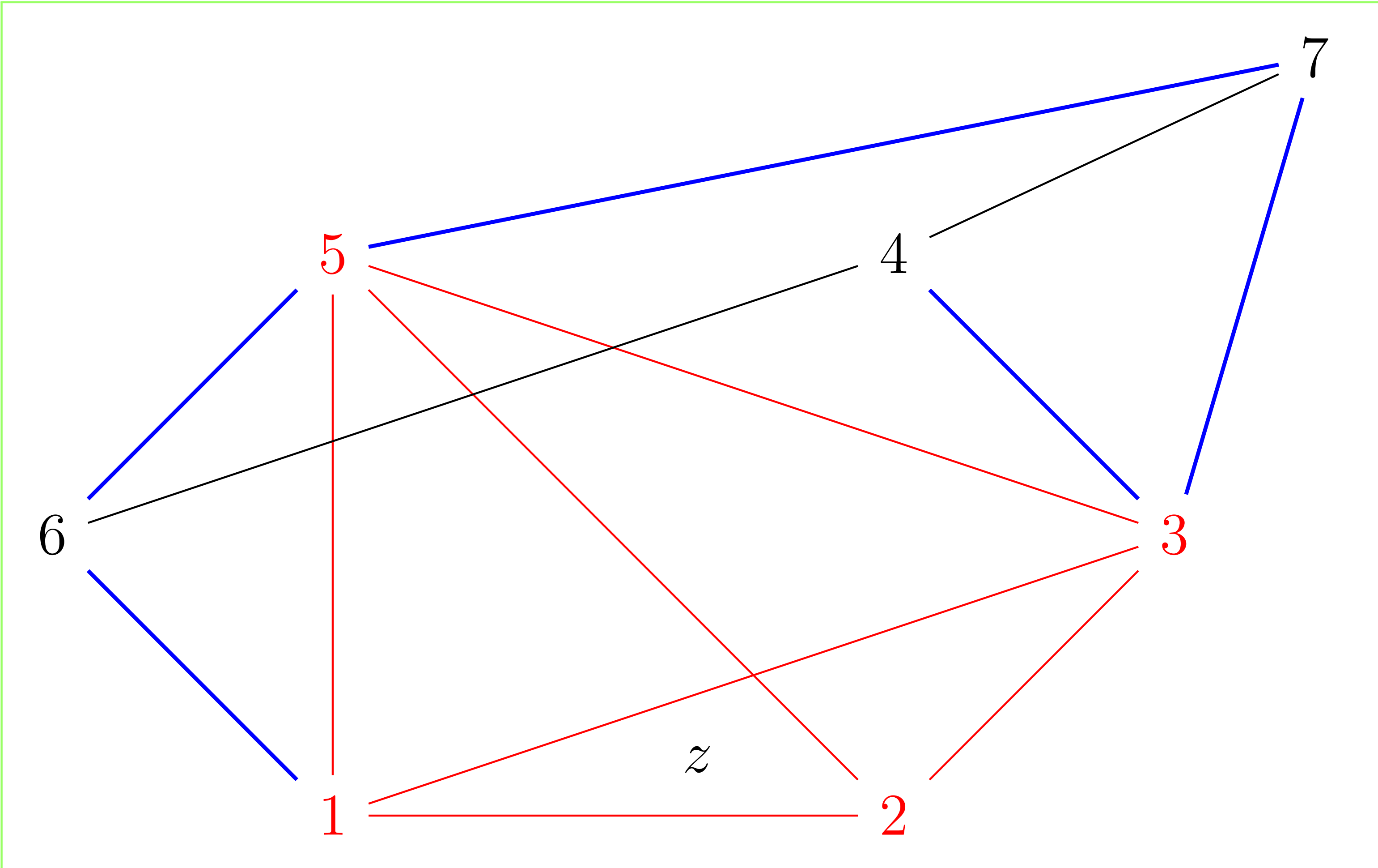
Subgraph contraction algorithm

Add z



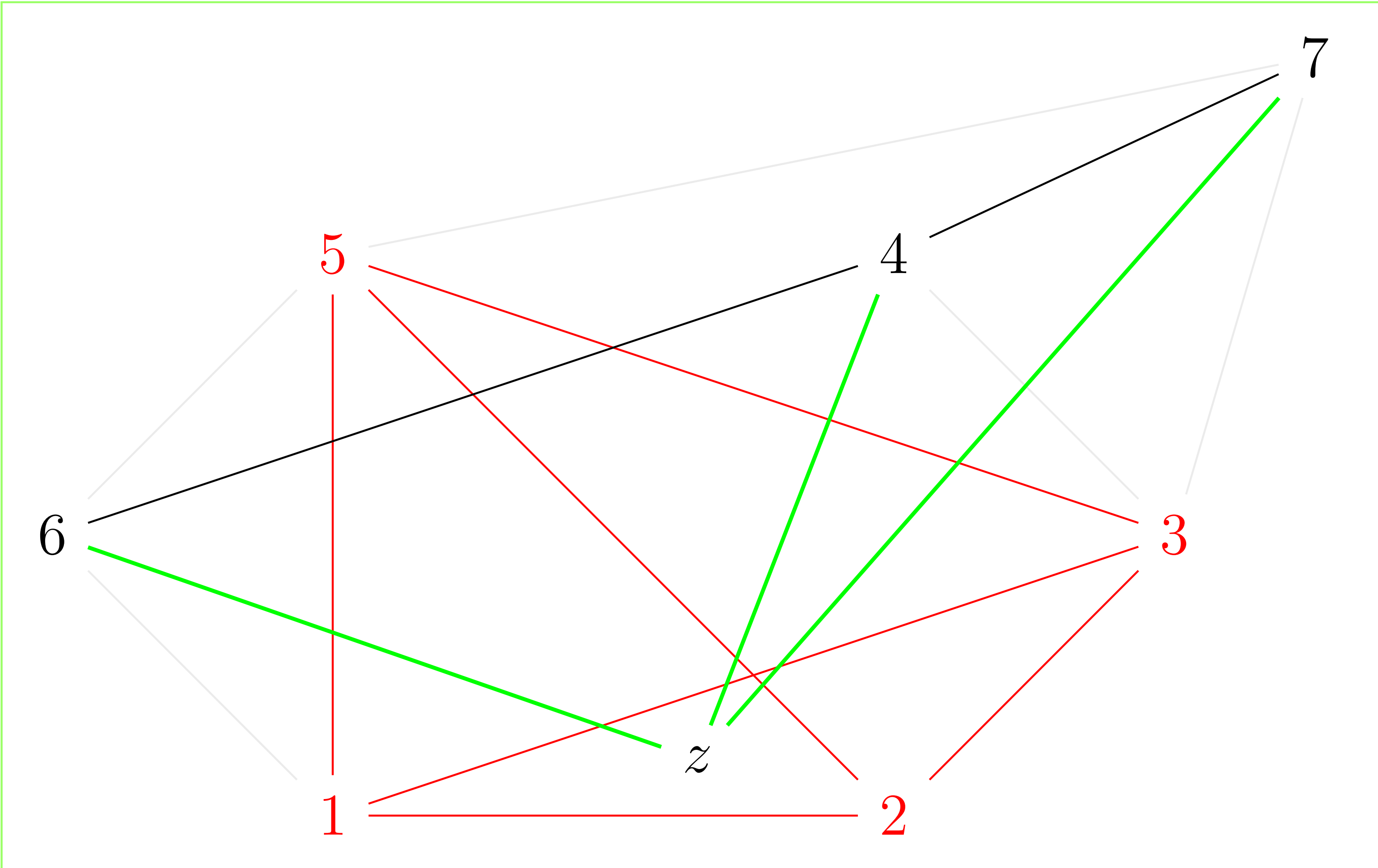
Subgraph contraction algorithm

$\delta(G[U])$ **in blue** *(edges with just one endpoint in U)*



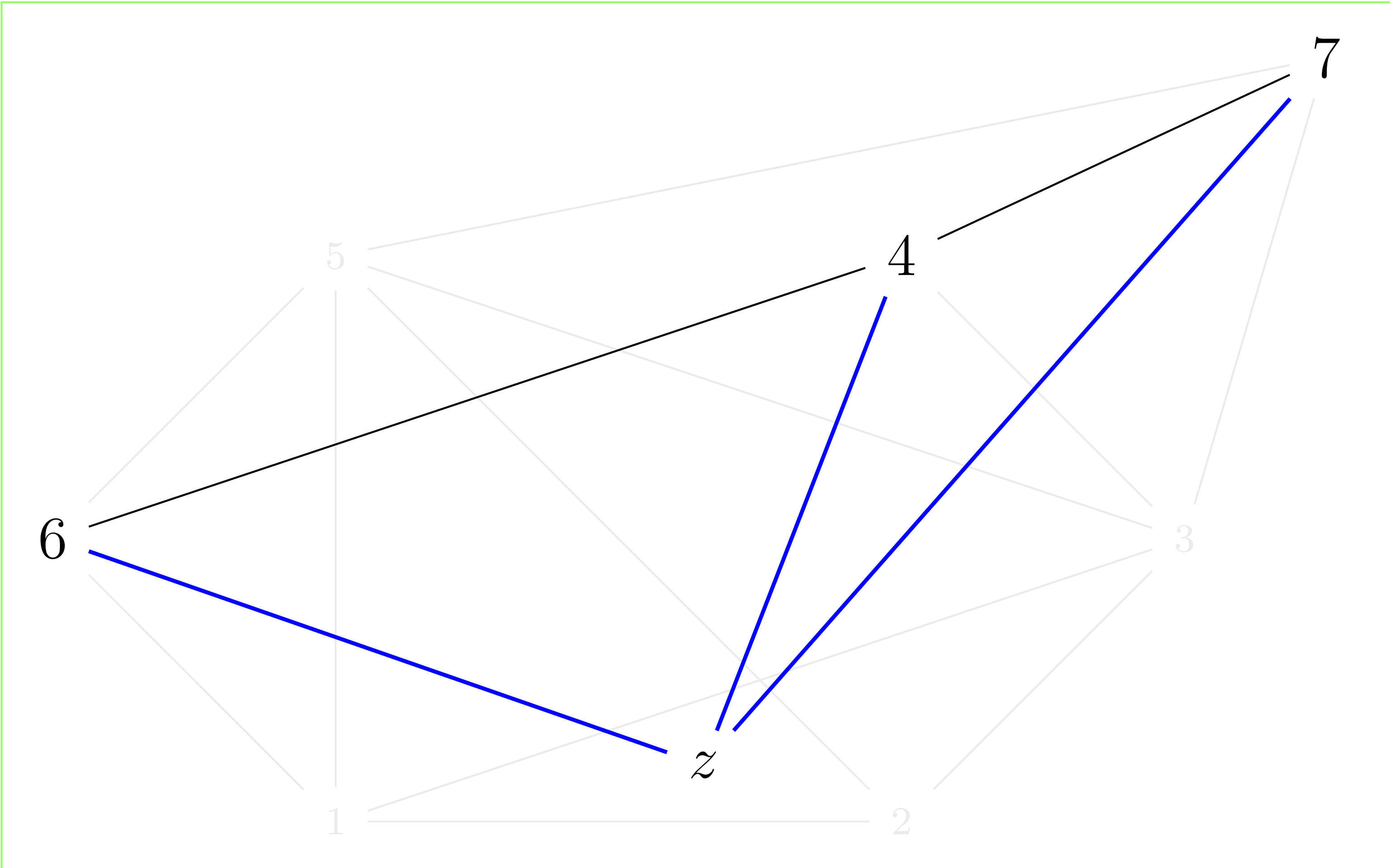
Subgraph contraction algorithm

Add $\{v, z\}$ and remove $\{u, v\}$



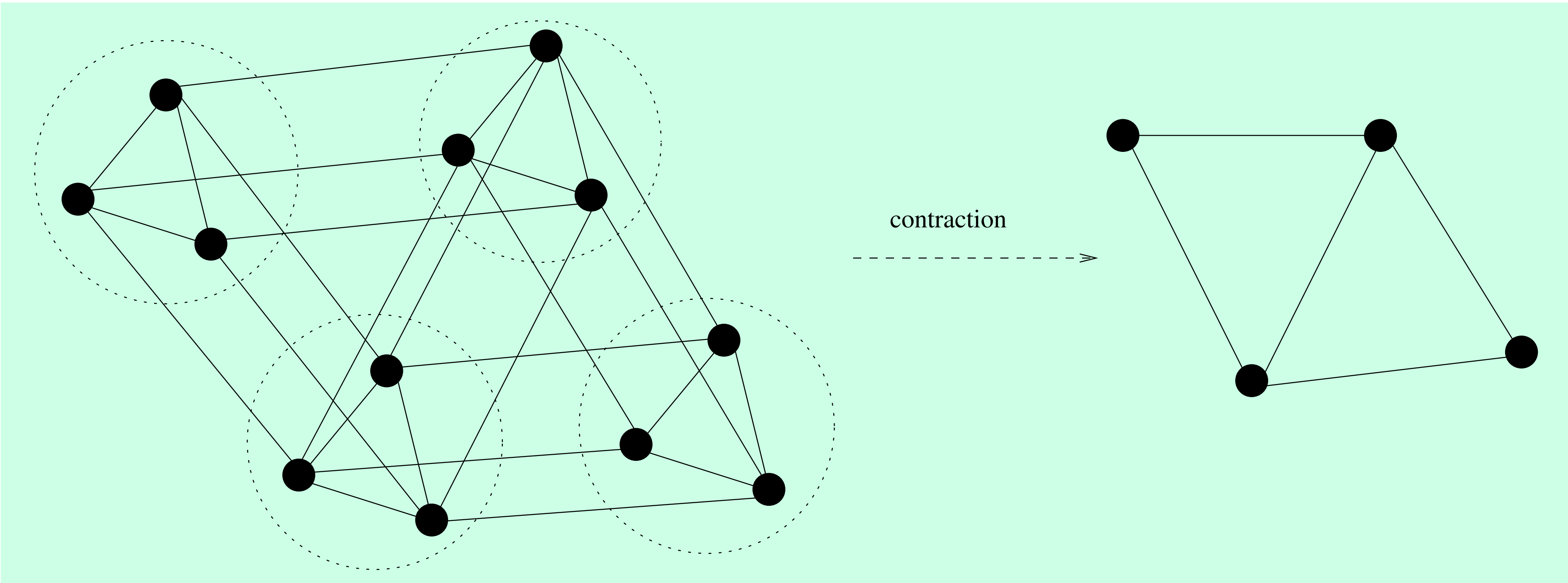
Subgraph contraction algorithm

Remove $G[U]$



Minors

- The graph H is a **minor** of the graph G if it is isomorphic to the graph obtained by a sequence of contractions
- Useful to underline "essential structure" of a complicated graph



Contract some triangles

Combinatorial problems on graphs

The subgraph problem

- Let $\mathbb{G}$ be the class of all graphs
- For a set of valid propositions $P(G)$ (for $G \in \mathbb{G}$), a typical decision problem in graph theory is the following:

  SUBGRAPH PROBLEM (SP$_P$). Given a graph G, does it have a subgraph H with property P?
- **Decision problem**: YES/NO question parametrized over symbols representing the **instance** (i.e. the input)
- Formally, a decision problem is the set of all possible inputs
- Require decision problems to always provide a **certificate** (a proof that certifies the answer)
- E.g. if $P(H) \equiv (H$ is a cycle$)$ the certificate is the cycle
- **NP** is the class of decision problems whose certificates for YES instances can be verified in polynomial time w.r.t. the instance size

Graph optimization problems

- To most decision problems on graphs there is a corresponding *optimization problem*
- Let $\mu : \mathbb{G} \to \mathbb{R}$ be a "measure" function for graphs
- E.g. μ could be the number of vertices, or of edges

SUBGRAPH PROBLEM, *optimization version* ($\text{SP}_{P,\mu}$). Given a graph G, does it have a subgraph H with property P and having minimum/maximum measure μ?

- Given a property P and a graph measure μ, the set of instances of $\text{SP}_{P,\mu}$ is an **optimization problem**

Easy problems

- We let **P** be the class of decision or optimization problems which can be solved in polynomial time
- We call problems in **P** "easy"
  - MINIMUM SPANNING TREE (MST)

    Seen in Lecture 6
  - SHORTEST PATH PROBLEM (SPP) from a vertex v to all other vertices

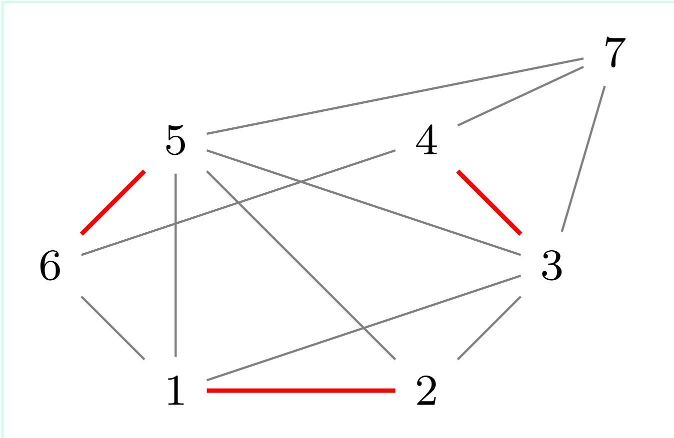
    To be seen in Lecture 9
  - MAXIMUM MATCHING problem (MATCHING)

    Discussed in INF550

Matching: subgraph given by set of mutually non-adjacent edges

A maximum matching M,

$\mu(M) = |E(M)|$

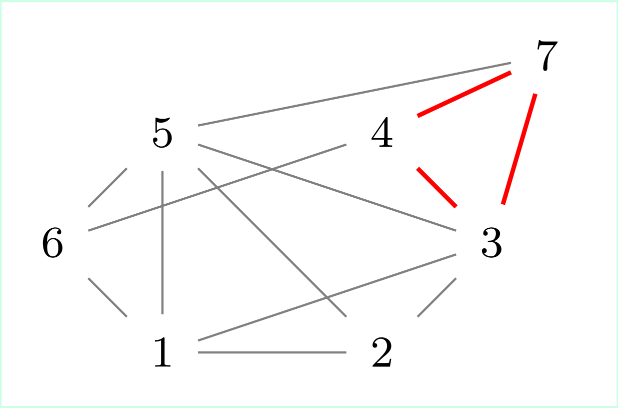


Hard(er) problems

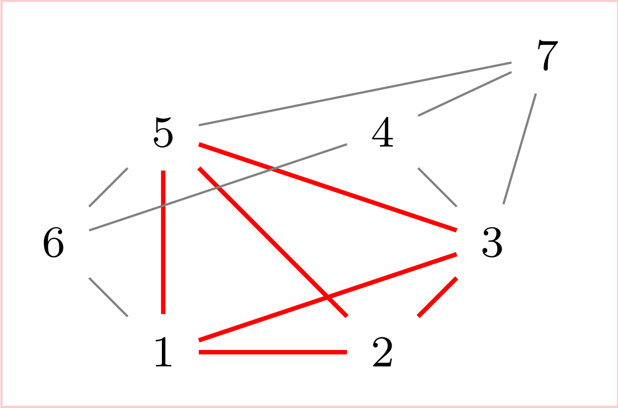
Maximum clique

CLIQUE PROBLEM (CLIQUE). Given a graph G, what is the largest n such that G has K_n as a subgraph?

- In CLIQUE, $\mu(H) = |V(H)|$ and $P(H) \equiv H = K(V(H))$



A clique in G



The largest clique in G

- Several applications to social networks and bioinformatics

Clique and NP-completeness

- The decision version of CLIQUE is:

  k-CLIQUE PROBLEM (k-CLIQUE). Given a graph G and an integer $k > 0$, does G have K_k as a subgraph?

- Consider the following result (which we won't prove) Thm.

  [Karp 1972] If CLIQUE $\in$ **P** then **P** $=$ **NP**

- Any decision problem for which such a result holds is called **NP**-complete
- It is not known whether **NP**-complete problems can be solved in polynomial time; the current guess is NO

Solving **NP**-complete problems

- Essentially, proving **NP**-completeness of a problem amounts to say "it's really hard"

  If it were easy, every problem in **NP** would be easy, which is unlikely: so it's likely to be hard

- Solution methods for **NP**-complete problems include:
  - **exact** but **exponential** worst-case complexity algorithms
  - **heuristic** algorithms

    whenever they find a YES answer they provide a certificate, but there is no guarantee that they are able to determine whether an answer is NO in a finite amount of time

- Some optimization problems are also such that "solvable in polynomial time" implies **P** $=$ **NP**: they are called **NP**-hard

- With **NP**-hard optimization problems, can use **approximation** algorithms

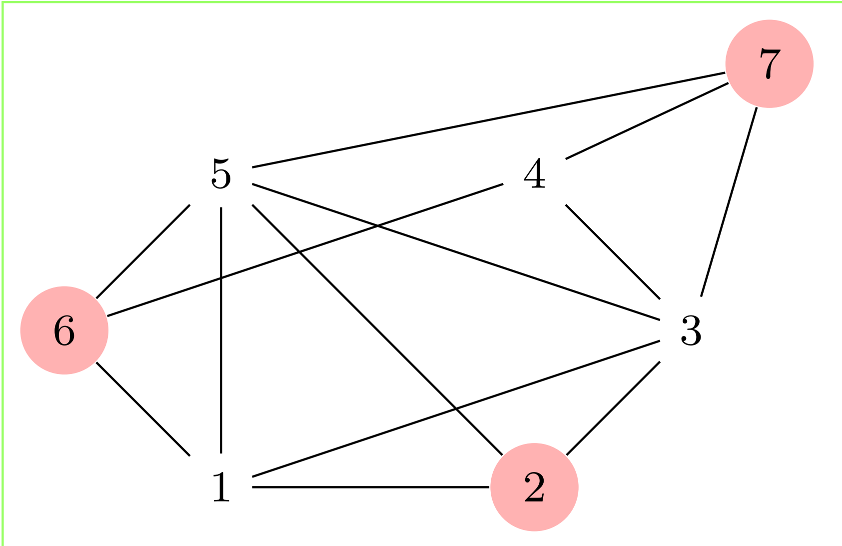
  their solutions have a μ-value which is no more than $f(|G|)$-fold worse than the optimal one *(e.g. $\mu \leq f(|G|)\mu^*$, where μ^* is the cost of an optimal solution)*

Stables

- A **stable** (or **independent set**) of a graph $G = (V, E)$ is a subset $U \subseteq V$ such that $\forall u, v \in U \ (\{u, v\} \notin E)$
  Thm.

  U is a stable in G if and only if $\bar{G}[U]$ is a clique

  a stable in G



- Decision problem: k-STABLE

  Given G and $k \in \mathbb{N}$, is there a stable $U \subseteq V(G)$ of size k?

- Optimization problem: STABLE

  Given G, find the stable of G of maximum size

Stables

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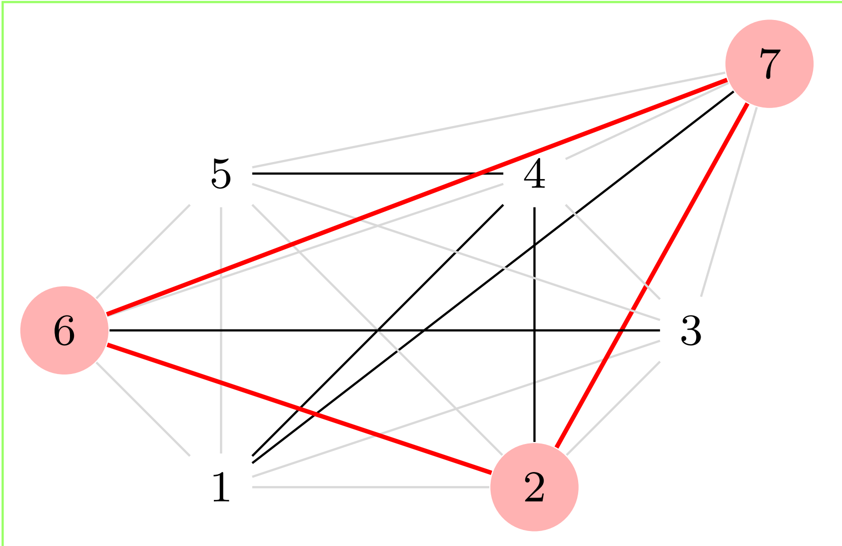
$\bar{G}[U]$ *is a clique*



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  Given G, find the stable of G of maximum size

NP-completeness of k-STABLE

Thm.

k-STABLE is **NP**-complete

Proof

Consider an instance (G, k) of k-CLIQUE

The complement graph $\bar{G}$ can be obtained in polynomial time $(*)$

It is easy to show that $\bar{\bar{G}} = G \quad (**)$

By $(**)$ and previous thm.,

(G, k) is a YES instance of k-CLIQUE iff $(\bar{G}, k)$ is a YES instance of k-STABLE

By $(*)$, if k-STABLE $\in$ **P** then k-CLIQUE $\in$ **P**

By **NP**-completeness of k-CLIQUE, k-STABLE $\in$ **P** implies **P** $=$ **NP**

Hence k-STABLE is **NP**-complete

- How to show that a problem $\mathcal{P}$ is **NP**-complete:
  - Take another **NP**-complete problem $\mathcal{Q}$ "similar" to $\mathcal{P}$
  - Transform (with a poly. alg.) an instance of $\mathcal{Q}$ to an instance of $\mathcal{P}$
  - Show that transformation preserves the YES/NO property

Stable heuristic

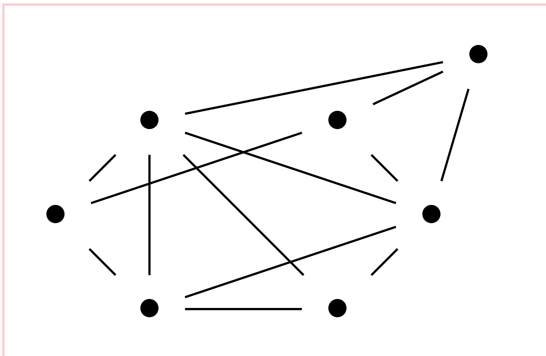
- It suffices to give an algorithm for STABLE, the one for CLIQUE will follow trivially (why?)
- The following **greedy** method will find a *maximal* stable

$U = \varnothing$;
order V by increasing values of $|N(v)|$;
while $V \neq \varnothing$ **do**
    $v = \min V$;
    $U \leftarrow U \cup \{v\}$;
    $V \leftarrow V \smallsetminus (\{v\} \cup N(v))$
end while

- Worst-case: $O(n)$ (given by an empty graph)

degree sequence

$(3, 3, 3, 3, 4, 5, 5)$



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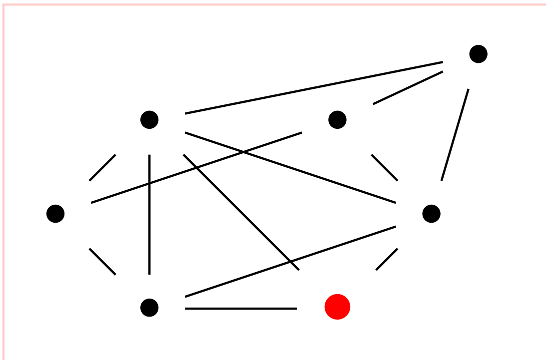
```

- Worst-case: $O(n)$ (given by an empty graph)

select $\min V$

put it in U



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```

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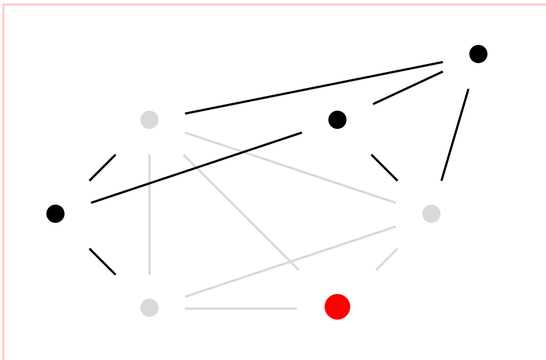
remove v and its star from V



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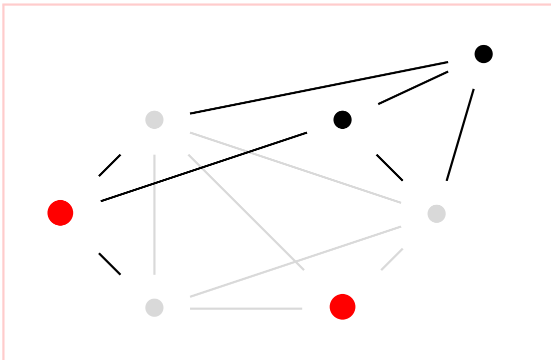

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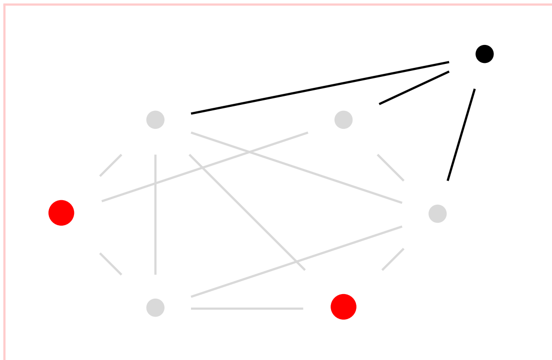

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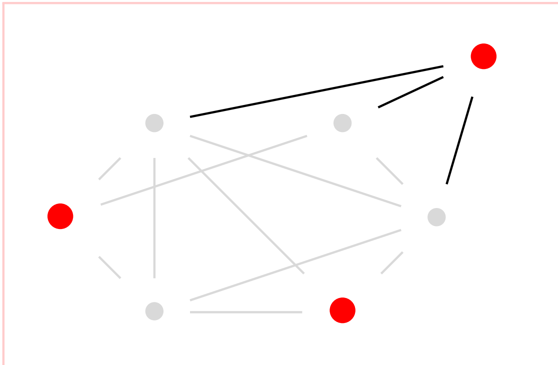
- The following **greedy** method will find a *maximal* stable

1: $U = \varnothing$;
2: order V by increasing values of $|N(v)|$;
3: **while** $V \neq \varnothing$ **do**
4: $\quad v = \min V$;
5: $\quad U \leftarrow U \cup \{v\}$;
6: $\quad V \leftarrow V \smallsetminus (\{v\} \cup N(v))$
7: **end while**

- Worst-case: $O(n)$ (given by an empty graph)

select $\min V$

put it in U



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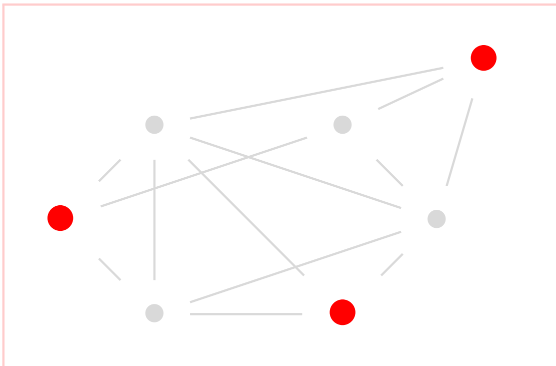
end while
```

- Worst-case: $O(n)$ (given by an empty graph)

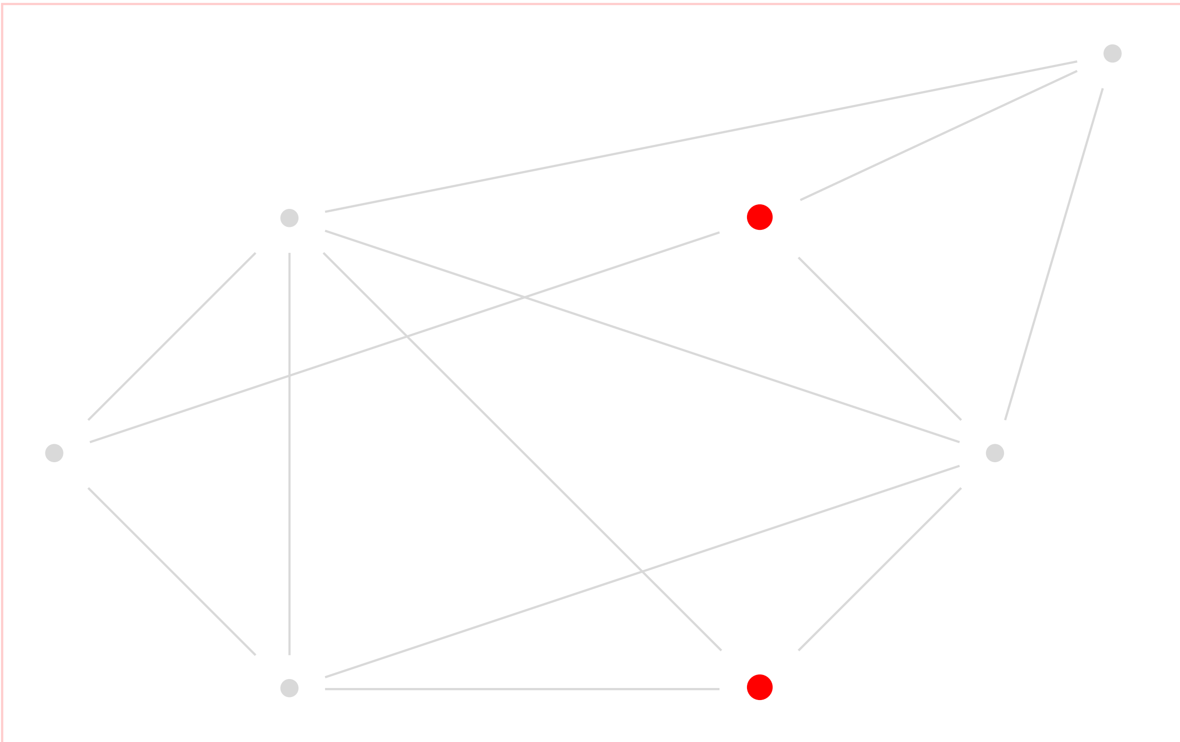
remove v and its star from V

stop: maximal stable



Heuristic fails

- The above algorithm may fail to find a maximum stable
- When choosing second element of U, instead of choosing leftmost vertex, choose:



- Then algorithm stops immediately with a stable of cardinality 2

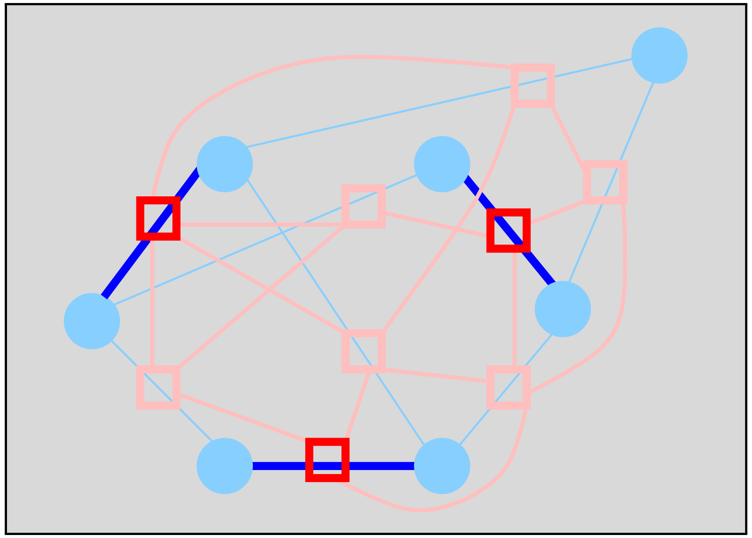
A polynomial case

- Not all instances of an **NP**-complete problem are hard
- Let P be a decision problem and C⊆P be an infinite set of instances for which there exists a polynomial algorithm
- Then C∈**P**, and C is a **polynomial case** of P
- For example, let $\mathcal{L} = \{H \in \mathbb{G} \mid \exists G \in \mathbb{G}\ (H = L(G))\}$ be the class of graphs that are line graphs of another graph

Thm.

A maximum matching in G is a stable in $L(G)$

Proof



- Since MATCHING∈**P** and finding $L(G)$ can be done in polynomial time, STABLE$_{\mathcal{L}}$ ∈**P**

Vertex colouring

- Decision problem

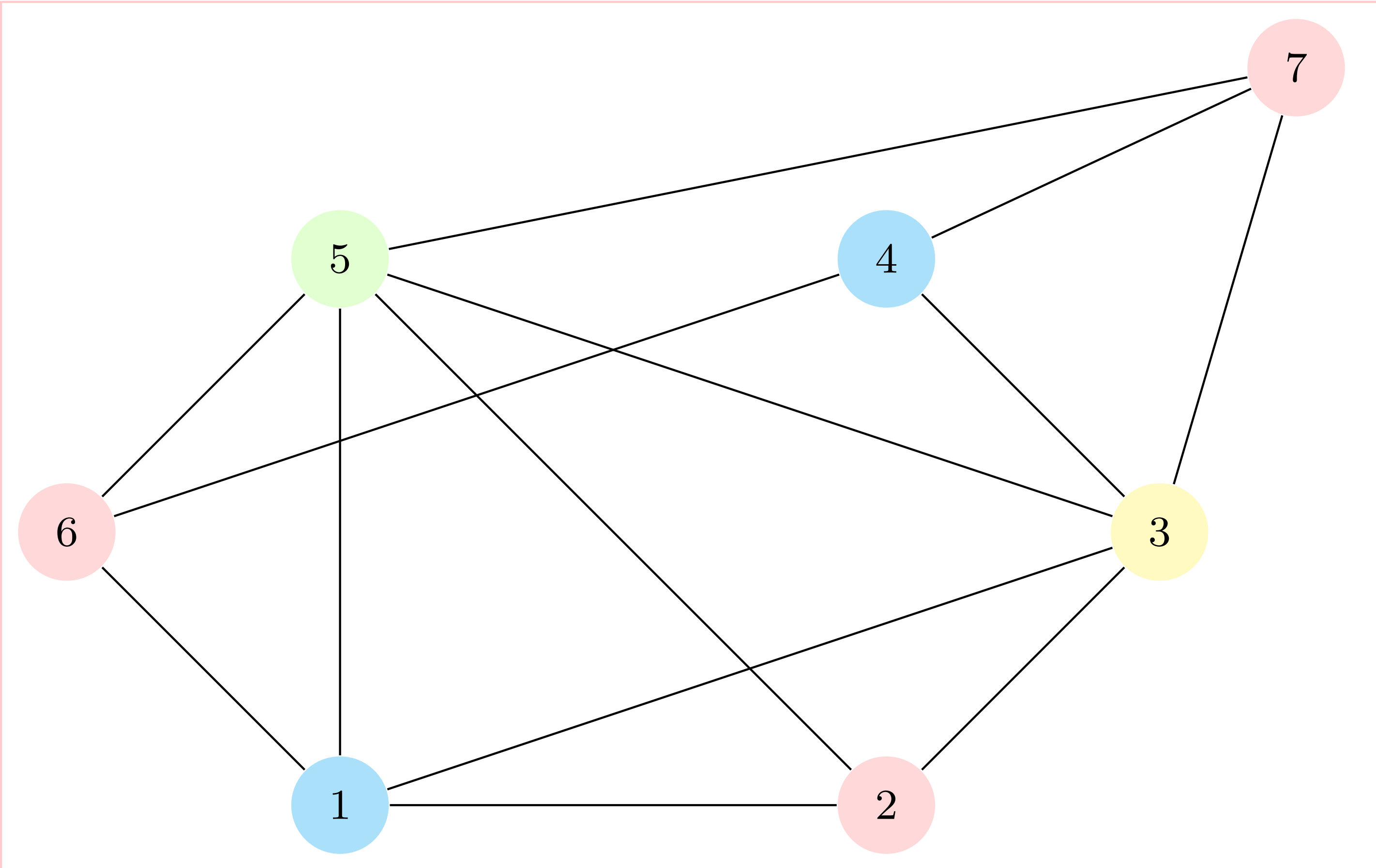
  VERTEX k-COLOURING PROBLEM (k-VCP). Given a graph $G = (V, E)$ and an integer $k > 0$, find a function $c : V \to \{1, \ldots, k\}$ such that $\forall\{u, v\} \in E \; (c(u) \neq c(v))$

- Optimization problem

  VERTEX COLOURING PROBLEM (VCP). Given a graph $G = (V, E)$, find the minimum $k \in \mathbb{N}$ such that there is a function $c : V \to \{1, \ldots, k\}$ with $\forall\{u, v\} \in E \; (c(u) \neq c(v))$

- Applications to scheduling and wireless networks
- In general, find how to allocate network resources to a few capacities such that there is no conflict

Vertex colouring example



Vertex colouring heuristic

Thm.

Each color set $C_k = \{v \in V \mid c(v) = k\}$ is a stable

- Use stable set heuristic as a sub-step

```
k = 1;
U = V;
while U ≠ ∅ do
    C_k = maximalStable(G[U]);
    U ← U ∖ C_k;
    k ← k + 1;
end while
```

- Worst-case: $O(n)$ (given by an empty or complete graph)

Model-and-solve

Mathematical programming

- Take e.g. the STABLE problem
- Input (also called **parameters**):
  - set of vertices V
  - set of edges E
- Output: $x : V \to \{0,1\}$

$$\forall v \in V \quad x(v) = \begin{cases} 1 & \text{if } v \in \text{maximum stable} \\ 0 & \text{otherwise} \end{cases}$$

- We also write $x_v = x(v)$
- We'd like $x = (x_v \mid v \in V) \in \{0,1\}^{|V|}$ to be the **characteristic vector** of the maximum stable S^*
- $x_1, \ldots, x_{|V|}$ are also called **decision variables**

Objective function

- If we take $x = (0, 0, 0, 0, 0, 0, 0)$, $S^* = \varnothing$ and $|S^*| = 0$ (minimum possible value)
- If we take $x = (1, 1, 1, 1, 1, 1, 1) = \mathbf{1}$, $|S^*| = |V| = 7$ has the maximum possible value
- Characteristic vector x should satisfy the **objective function**

$$\max_x \sum_{v \in V} x_v$$

Constraints

- Consider the solution $x = \mathbf{1}$
- x certainly maximizes the objective
- ...but $S^* = V$ is not a stable!

  $x = 1$ is an **infeasible solution**

- The **feasible set** is the set of all vectors in $\{0,1\}^{|V|}$ which encode stable sets
- Defining property of a stable:

  Two adjacent vertices cannot both belong to the stable

- In other words,

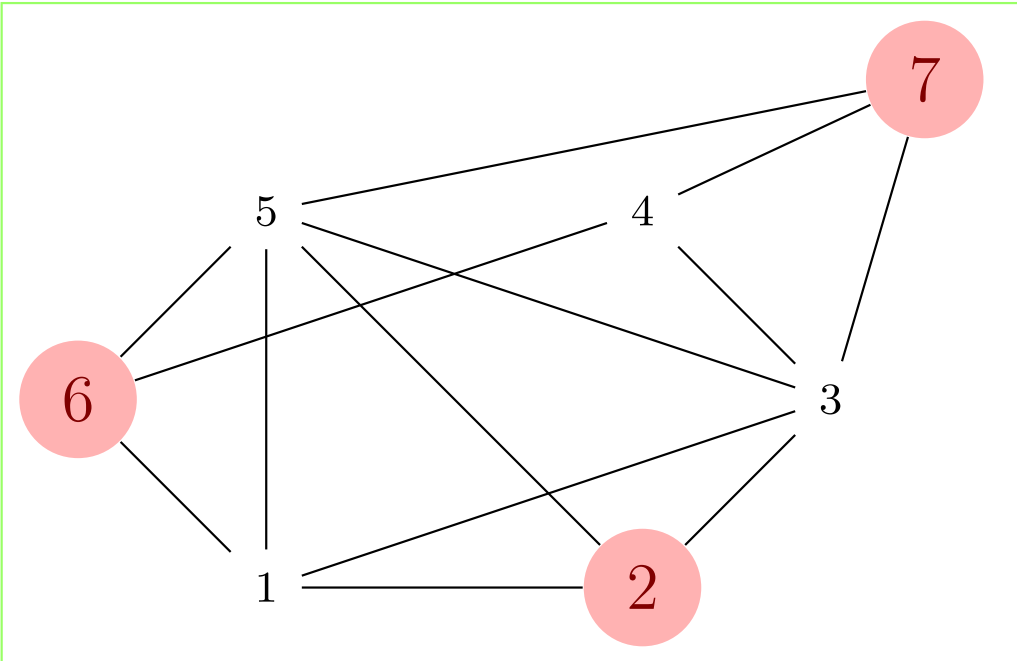
  choose at most one vertex adjacent to each edge

- Written formally,

$$\forall \{u,v\} \in E \quad x_u + x_v \leq 1$$

Verify the constraints

- $x = (0,1,0,0,0,0,1,1)$ encodes $S^* = \{2,6,7\}$
- $x_u + x_v = 2$ only for $\{u,v\} \in F = \{\{2,6\},\{2,7\},\{6,7\}\}$
- Notice $F \cap E = \varnothing$
- Hence, $x_u + x_v \leq 1$ for all $\{u,v\} \in E$



So what?

- OK, so the **Mathematical Programming** (MP) **formulation**

$$\begin{array}{rrcl} \max_x & \sum_{v \in V} x_v & & \\ \forall \{u,v\} \in E & x_u + x_v & \le & 1 \\ & x & \in & \{0,1\}^{|V|} \end{array}$$

  describes STABLE correctly
- As long as we can't solve it, why should we care?

The magical method

- But WE CAN!
- Use generic MP solvers
- These algorithms can solve *ANY* MP formulation expressed with linear forms, or *prove* that there is no solution
- Based on Branch-and-Bound (BB)
- The YES certificate is the characteristic vector of a feasible solution
- The NO certificate is the whole BB tree, which implicitly (and intelligently) enumerates the feasible set
- YES certificate lengths are polynomial, NO certificates may have exponential length

CLIQUE and MATCHING

- Clique (use complement graph):

$$\begin{array}{rrcl} & \max_x \quad \sum_{v\in V} x_v & & \\ \forall \{u,v\} \not\in E, u \neq v & x_u + x_v & \le & 1 \\ & x & \in & \{0,1\}^{|V|} \end{array}$$

- Matching:

$$\begin{array}{rrcl} \max_x & \sum_{\{u,v\}\in E} x_{uv} & & \\ \forall u \in V & \sum_{v\in N(u)} x_{uv} & \le & 1 \\ & x & \in & \{0,1\}^{|E|} \end{array}$$

Warning: although MATCHING∈**P**, solving the MP formulation with BB is exponential-time

How to

- Come see me, I'll give you a personal demo
- Go to `www.ampl.com` and download the AMPL software, student version
- AMPL is for modelling, i.e. writing MP formulations
- Still from `www.ampl.com`, you can download a student version of the ILOG CPLEX BB implementation

And tomorrow?

If you're interested in modelling problems as MPs

- M1:
  - INF572 (Optimization: Modelling and Software)
  - MAP557 (Optimization: Theory and Applications)
- M2:
  - MPRO (Master Parisien en Recherche Operationnelle)
    `http://uma.ensta-paristech.fr/mpro/`

End of Lecture 8