INF421, Lecture 7
Balanced Trees

Leo Liberti

LIX, École Polytechnique, France

Course

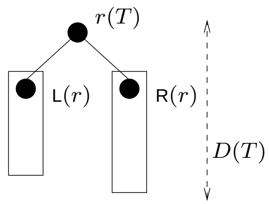
- **Objective**: to teach you some data structures and associated algorithms
- **Evaluation**: TP noté en salle info le 16 septembre, Contrôle à la fin. Note: $\max(CC, \frac{3}{4}CC + \frac{1}{4}TP)$
- **Organization**: fri 26/8, 2/9, 9/9, 16/9, 23/9, 30/9, 7/10, 14/10, 21/10, amphi 1030-12 (Arago), TD 1330-1530, 1545-1745 (SI31,32,33,34)
- **Books**:
  1. Ph. Baptiste & L. Maranget, *Programmation et Algorithmique*, Ecole Polytechnique (Polycopié), 2009
  2. G. Dowek, *Les principes des langages de programmation*, Editions de l'X, 2008
  3. D. Knuth, *The Art of Computer Programming*, Addison-Wesley, 1997
  4. K. Mehlhorn & P. Sanders, *Algorithms and Data Structures*, Springer, 2008
- **Website**: `www.enseignement.polytechnique.fr/informatique/INF421`
- **Contact**: `liberti@lix.polytechnique.fr` (e-mail subject: INF421)

Lecture summary

- Binary search trees
- AVL trees
- Heaps and priority queues
- Tries

Notation

- Tree T
- Node set of T: $\mathsf{V}(T)$ (with $|\mathsf{V}(T)| = n$)
- Root: $r(T)$
- Tree rooted at v: $\mathsf{T}(v)$
- Node: $v \in \mathsf{V}(T)$
- Root node of left subtree of v: $\mathsf{L}(v)$
- Root node of right subtree of v: $\mathsf{R}(v)$
- If $\mathsf{L}(v) = \mathsf{R}(v) = \varnothing$, v is a **leaf node**
- Parent node of v: $P(v)$
- $\Rightarrow T = \langle \mathsf{L}(r(T)), r(T), \mathsf{R}(r(T)) \rangle$
- For all $v \in \mathsf{V}(T)$: $p(v)$ =unique path $r(T) \to v$
- Path length: $\lambda(T) = \sum_{v \in \mathsf{V}(T)} |p(v)|$
- Depth (or height): $D(T) = \max_{v \in \mathsf{V}(T)} |p(v)|$



The minimal knowledge

- Let $(V, <)$ be a totally ordered set
- V stored as a binary tree T:

  $$\mathsf{L}(v) = u \Rightarrow u \le v \qquad \mathsf{R}(v) = u \Rightarrow u > v \quad (\dagger)$$

- `find, insert, delete, min, max`:
  $O(\log n)$ on average, $O(n)$ worst case
- **AVL trees**: balance

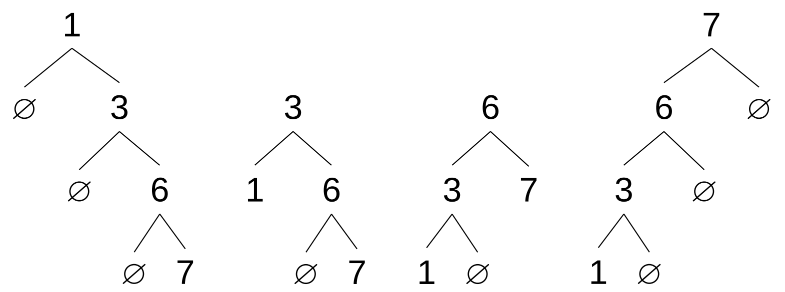
  $$B(T) = D(\mathsf{T}(\mathsf{L}(r(T)))) - D(\mathsf{T}(\mathsf{R}(r(T)))) \in \{-1, 0, 1\}$$

- If an operation unbalances, use a **rebalancing** operation
- $\Rightarrow$ all operations are $O(\log n)$ in the worst case
- Can use a special balanced tree (a **heap**) to implement a **priority queue** (`min/max, insert, delete`)
- **Tries** are k-ary trees that encode words prefix-wise

Binary search trees (BST)

Sorted sequences

- Used to store a set V as a **sorted sequences**
- Makes it efficient to answer the question $v \in V$
- Each node v in the tree is such that $\mathsf{L}(v) \le v < \mathsf{R}(v)$
- Example: $V = \{1, 3, 6, 7\}$

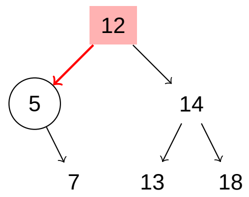


- Several possibilities

BST `min/max`

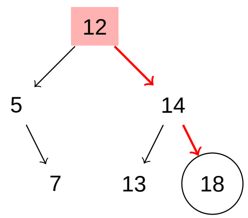
- `min`(v):

  if $\mathsf{L}(v) = \varnothing$ **then**
      return v;
  else
      return $\min(\mathsf{L}(v))$;
  end if



- `max`(v):

  if $\mathsf{R}(v) = \varnothing$ **then**
      return v;
  else
      return $\max(\mathsf{R}(v))$;
  end if



Base cases for recursion

All other BST functions $\texttt{f}(k, v)$ are assumed to be implemented so that $\texttt{f}(k, \varnothing)$ returns without doing anything (base case of recursion)

BST `find`

- `find`(k, v):

```
ret = not_found;
if v = k then
  ret = v;
else if k < v then
  ret = find(k, L(v));
else
  ret = find(k, R(v));
end if
return ret;
```

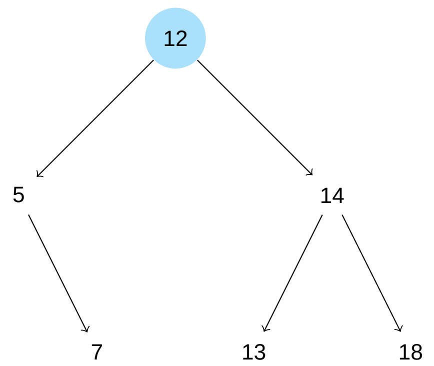

BST `insert`

- `insert`(k, v):

```
if k = v then
  return already_in_set; // if multiset:  add new node
else if k < v then
  if L(v) = ∅ then
    L(v) = k;
  else
    insert(k, L(v));
  end if
else
  if R(v) = ∅ then
    R(v) = k;
  else
    insert(k, R(v));
  end if
end if
```


Insert example 1/3

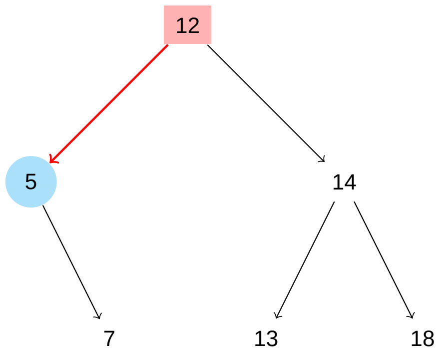
`insert`$(1, r(T))$



$1 < 12$, take left branch

Insert example 2/3

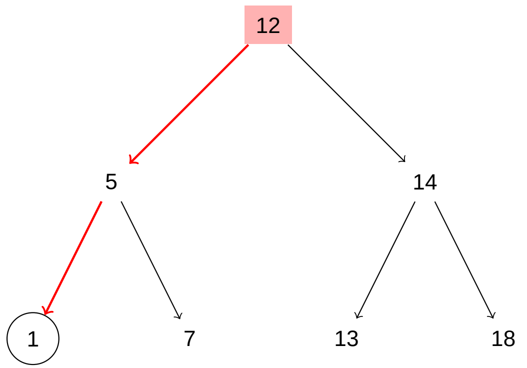
$\texttt{insert}(1, r(T))$



$1 < 5$, should take left branch but $\text{L}(5) = \varnothing$

Insert example 3/3

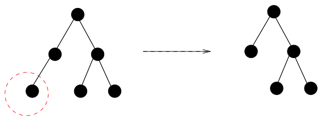
$\texttt{insert}(1, r(T))$



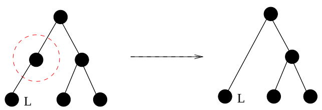
Add $k = 1$ as $\text{L}(5)$

Deletion is not so easy

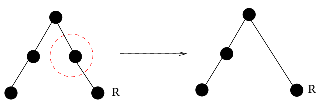
- If node v to delete is a leaf, easy: "cut" it (`unlink`)



- If $\text{R}(v) = \varnothing$ and $\text{L}(v) \neq \varnothing$, replace with $\text{L}(v)$



- If $\text{L}(v) = \varnothing$ and $\text{R}(v) \neq \varnothing$, replace with $\text{R}(v)$



- If v has both subtrees, not evident

Replacing a node

Replace link $\{P(v), v\}$ with $\{P(v), u\}$, then unlink v

- $\texttt{replace}(u, v)$

  if $\text{R}(P(v)) = v$ **then**
  $\quad \text{R}(P(v)) \leftarrow u$; `// u is a right subnode`
  else
  $\quad \text{L}(P(v)) \leftarrow u$; `// u is a left subnode`
  end if
  if $u \neq \varnothing$ **then**
  $\quad P(u) \leftarrow P(v)$;
  end if
  unlink v;

- `unlink`: set $\text{L}(v) = \text{R}(v) = P(v) = \varnothing$

Deleting v : $\mathsf{L}(v) \neq \varnothing \wedge \mathsf{R}(v) \neq \varnothing$

Idea: swap v with $u = \texttt{min}(\mathsf{R}(v))$ then delete it

- The minimum u of a BST is always the leftmost node without a left subtree
- Hence we know how to delete u (case $\mathsf{L}(\cdot) = \varnothing$ in previous slide)
- We replace the value of v by that of u then delete u
- Because $u = \min \mathsf{T}(\mathsf{R}(v))$, we have $u < w$ for all $w \in \mathsf{T}(\mathsf{R}(v))$
- Since the value of v is now the value of u, v is now the minimum over all nodes in $\mathsf{T}(\mathsf{R}(v))$; hence $v < r(\mathsf{R}(v))$
- Moreover, since the value of v used to be u, a node in $\mathsf{R}(v)$, we have $v > r(\mathsf{L}(v))$, satisfying the BST defn. (†)

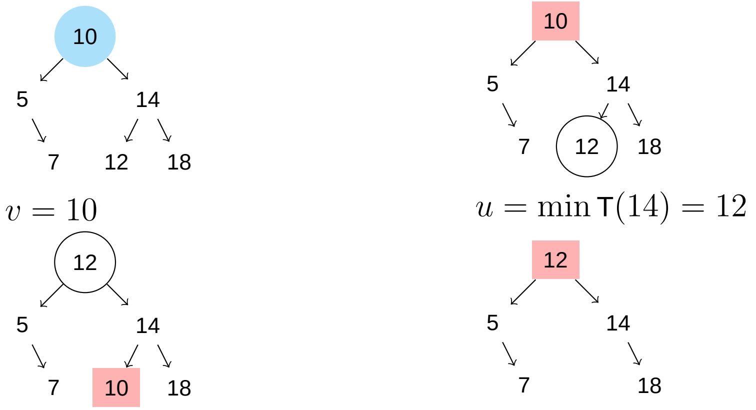
BST delete

- $\texttt{delete}(k, v)$:

```
if k < v then
  delete(k, L(v));
else if k > v then
  delete(k, R(v));
else
  if L(v) = ∅ ∨ R(v) = ∅ then
    delete v; // one of the easy cases
  else
    u = min(R(v));
    swap_values(u, v);
    delete u; // an easy case, as L(u)=null
  end if
end if
```


Delete example

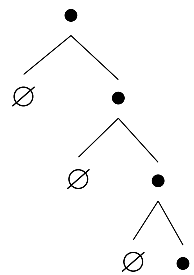
$\texttt{delete}(10, r(T))$



swap values of 10 and 12 delete 10

Complexity

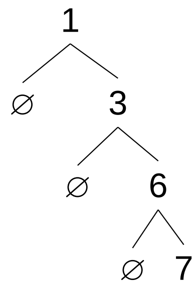
- Each IF case involves at most one recursive call
- Recurse along <u>one branch only</u>
- Worst-case complexity proportional to depth $D(T)$
- If tree is balanced, $D(T)$ is $O(\log n)$ (see INF311)
- In the worst case, $D(T)$ is $O(n)$



Adelson-Velskii & Landis (AVL) trees

AVL Trees

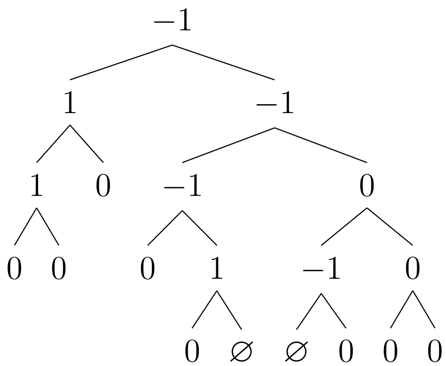
- Try inserting $1, 3, 6, 7$ in this order: get **unbalanced tree**



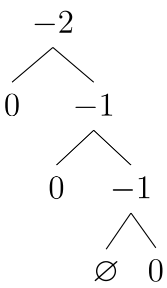
- Worst case find (i.e., find the key 7) is $O(n)$
- Need to *rebalance* the tree to be more efficient
- **AVL trees**: at any node, $B(T) =$depth difference between left and right subtrees $\in \{-1, 0, 1\}$

Examples

AVL tree:



Non-AVL tree:

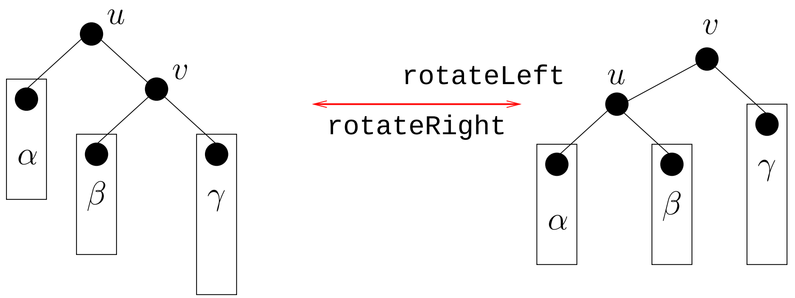


Nodes indicate $B(\mathsf{T}(v))$

In general

- We can decompose balanced trees operations into:
  - the operation itself
  - a sequence of *rebalancing operations* (when required), called **rotations**
- The operations `min/max, find, insert, delete` are as in BST (with one simple modification)
- Unbalancing can occur on insertion and deletion
- Since we insert/delete only one node at a time, unbalance offset is at most 1 unit
- I.e., $B(T) =$depth difference between left and right subtrees, could be $\{-2, 2\}$

Left and right rotation



Algebraic interpretation

- Let α, β, γ be trees, u, v be nodes not in α, β, γ
- Define:
  - $\texttt{rotateLeft}(\langle \alpha, u, \langle \beta, v, \gamma \rangle\rangle) = \langle\langle \alpha, u, \beta \rangle, v, \gamma\rangle$
  - $\texttt{rotateRight}(\langle\langle \alpha, u, \beta \rangle, v, \gamma\rangle) = \langle \alpha, u, \langle \beta, v, \gamma \rangle\rangle$
- A sort of "associativity of trees"
- Remark: `rotateLeft`, `rotateRight` are inverses

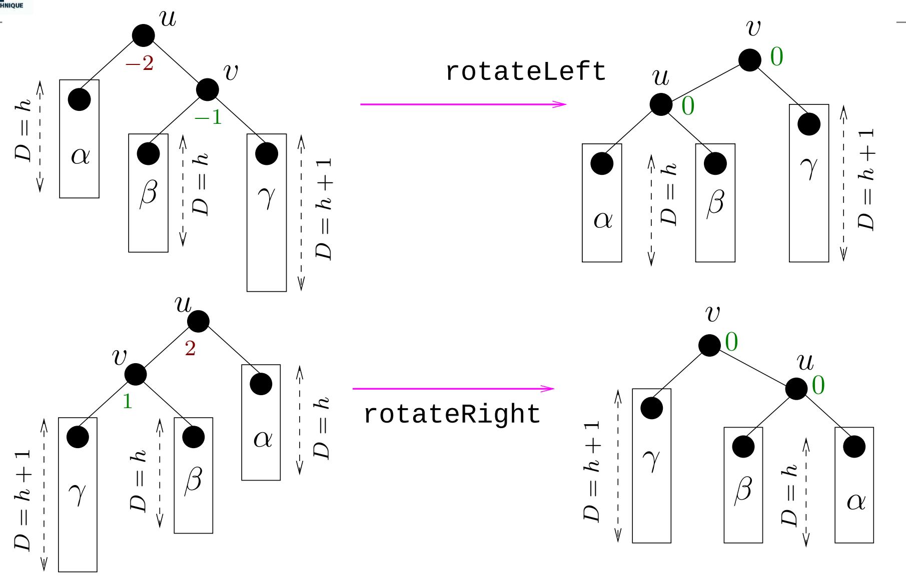
Thm.

$\texttt{rotateRight}(\texttt{rotateLeft}(T)) =$
$\texttt{rotateLeft}(\texttt{rotateRight}(T)) = T$

Proof

Directly from the definition

Rotating and rebalancing



Properties of rotation

Thm.

$\forall T$, $\texttt{rotateLeft}(T)$, $\texttt{rotateRight}(T')$ are BSTs

Proof

(Sketch): The tree order only changes locally for u, v. In T, $\texttt{T}(v) = \texttt{R}(u)$, which implies $u < v$. In $\texttt{rotateLeft}(T)$, $\texttt{T}(u) = \texttt{L}(u)$, which is consistent with $u < v$. Similarly for T'.

- Suppose $D(\alpha) = D(\beta) = h$ and $D(\gamma) = h + 1$
- Let $T = \langle \alpha, u, \langle \beta, v, \gamma \rangle\rangle$: then $B(T) = -2$
- Let $T' = \langle\langle \gamma, u, \beta \rangle, v, \alpha\rangle$: then $B(T') = 2$

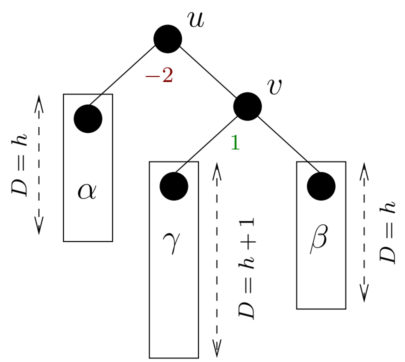
Thm.

T, T' as above $\Rightarrow B(\texttt{rotateLeft}(T)) = 0$, $B(\texttt{rotateRight}(T')) = 0$

Proof

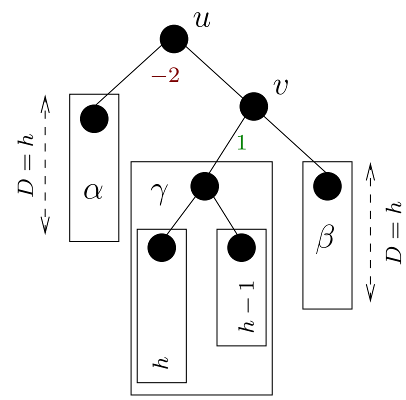
(Sketch): since subtrees α, γ are swapped, tree depth is $D = h$ for all subtrees

Is this enough?



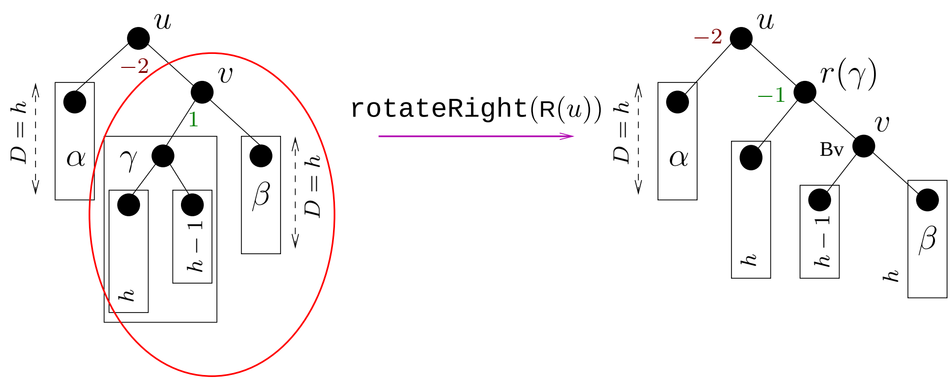
Rotating leaves γ at its place, doesn't work

Break γ up into subtrees



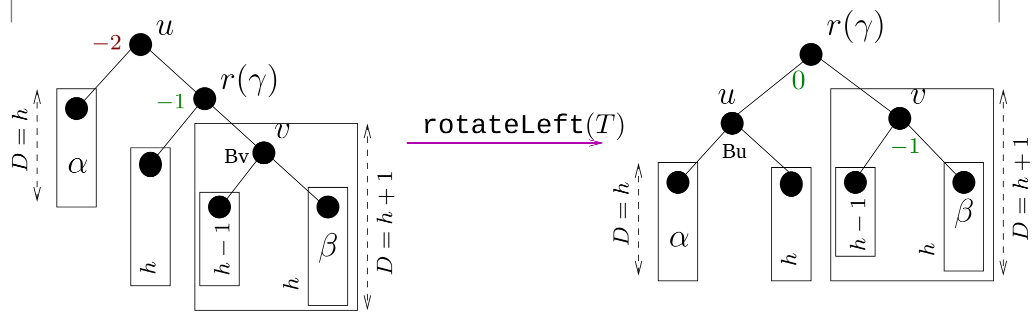
Now we can rotate $\mathsf{T}(v) = \mathsf{R}(u)$

Rotate a subtree right



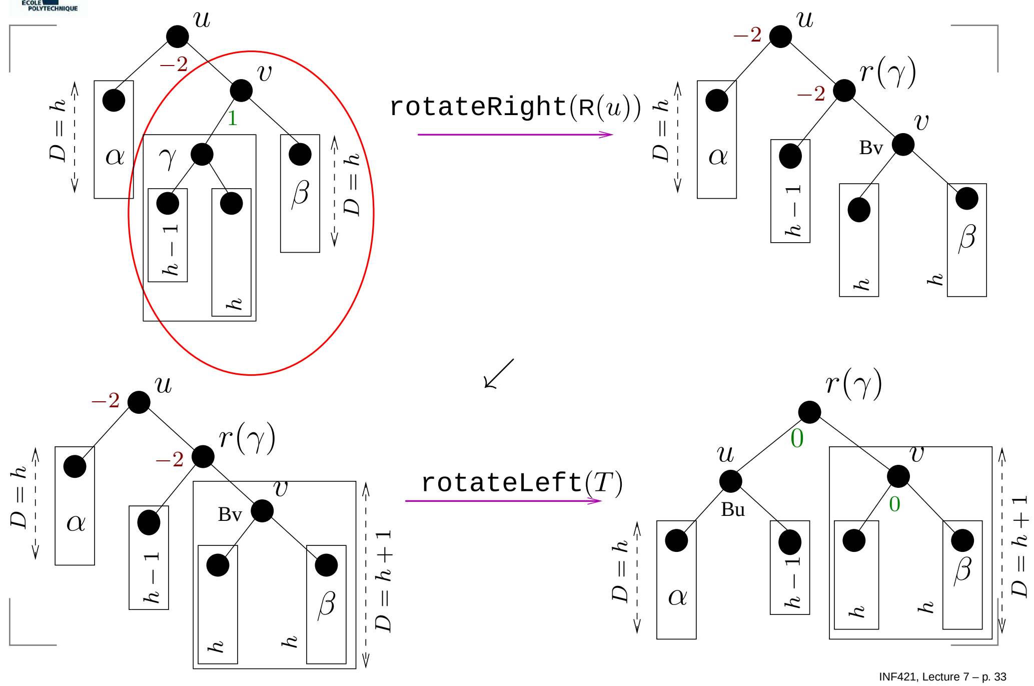
Rotate $\mathsf{R}(u)$ right

Finally, rotate left

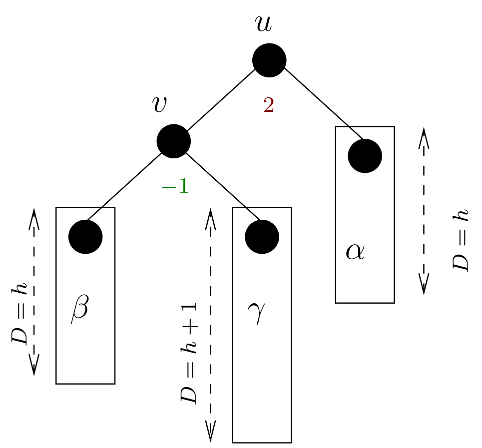


Rotate T left

Symmetric cases I



Symmetric cases II



Rebalance: $\texttt{rotateLeft}(\mathsf{L}(u))$, $\texttt{rotateRight}(T)$

Implementation of AVL trees

It took me TEN bloody hours to code a decent Java implementatation!

- **Definition of "decent implementation"**:
- Recursive implementation for didactical value
- Methods act on `this` node, for consistency with other lectures
- Efficient update of $B(v)$ after insertions and rotations

In view of my coding odyssey, in retrospect these were poor choices

- **Advice**:
- Consider iterative implementations using stacks or three threading
- Declare static methods and pass the relevant nodes as arguments

  this frees you from several constraints, e.g. you can't set `this` *to* `null`
- If you have trouble keeping balances updated in an efficient manner, you can always re-compute them recursively at each node, using depth

  yields a slower code but worst-case complexity is the same
- Look at my (online) code and INF421 Polycopié's

Balanced vs. random BST

- Balanced binary trees have $O(\log n)$ `insert`, `delete`, `query` ops
- What about an average (not necessarily balanced) BST?
- Given a sequence $\sigma \in \{1, \dots, n\}^n$, we insert it in a BST T
- Nodes to the left of $r(T)$ are $\leq r(T)$, nodes to the right of are $> r(T)$
- Let K be the number of nodes in $\mathsf{L}(T)$, so that $|\mathsf{R}(T)| = n - 1 - K$
- Uniform distribution on K i.e. $P(K = k) = \frac{1}{n}$ for all $k \in \{0, \dots, n-1\}$

σ	(1,2,3)	(1,3,2)	(2,1,3)	(2,3,1)	(3,1,2)	(3,2,1)
T	1 ↘ 2 ↘ 3	1 ↘ 3 ↙ 2	2 ↙ 1, ↘ 3	2 ↙ 1, ↘ 3	3 ↙ 1 ↘ 2	2 ↙ 1 ↙ 3
type	A	B	C	C	D	E

Type C (balanced) twice as likely as any other type!

Average depth and path length

- Average depth for BFSs: $O(\log n)$ [Devroye, 1986]
- Average path length for BFSs: $O(n \log n)$ [Vitter & Flajolet, 1990]
- This shows that BFSs are pretty balanced on average

Heaps and priority queues

Queues reminder

- A **queue** is a data structure with main operations:
  - `pushBack`(v): inserts v at the end of the queue
  - `popFront`$()$: returns and removes an element at the beginning of the queue
- Queues implement the Last-In-First-Out principle
- Definitions in Lecture 2
- Used by BFS to compute paths with fewest arcs
- If arcs are prioritized (e.g. travelling times for route segments), we want the queue to return the *element of highest priority*

This may not be at the beginning of the queue

Priority queues

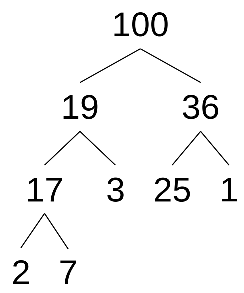
- Let V be a set and $(S, <)$ be a totally ordered set
- A **priority queue** on V, S is a set Q of pairs (v, p_v) s.t. $v \in V$ and $p_v \in S$
- Usually, p_v is a number
- E.g., if p_v is the rank of entrance of v in Q, then Q is a standard queue
- Supports three main operations:
  - `insert`(v, p_v): inserts v in Q with priority p_v
  - `max`$()$: returns the element of Q with maximum priority
  - `popMax`$()$: returns and removes `max`$()$
- Implemented as **heaps**

Heap

- A (binary) **heap** is an abstract, tree-like data structure which offers:
  - $O(\log |Q|)$ `insert`
  - $O(1)$ `max`
  - $O(\log |Q|)$ `popMax`
- The $O(1)$ is obtained by storing the maximum priority element as the root of a binary tree
- Distinguishing properties
  - *shape property*: all levels except perhaps the last are fully filled; the last level is filled left-to-right
  - *heap property*: every node stores an element of higher priority than its subnodes

Example

Let $V = \mathbb{N}$, and for all $v \in V$ we let $p_v = v$



A balanced tree

Thm.

If Q is a binary heap, $B(Q) \in \{0, 1\}$

Proof

This follows trivially from the shape property. Since all levels are filled completely apart perhaps from the last, $B(Q) \in \{-1, 0, 1\}$. Since the last is filled left-to-right, $B(Q) \neq -1$

Cor.

A binary heap is a balanced binary tree

Warning: *NOT a BST/AVL*: **heap property not compatible with BST definition** $\mathsf{L}(v) \leq V \mathsf{R}(v)$

Keep the heap balanced: need $O(\log |Q|)$ work to insert/remove

Insert

- Add new element (v, p_v) at the bottom of the heap (last level, leftmost free "slot")
- Compare with its (unique) parent (u, p_u); if $p_u < p_v$, swap u and v's positions in the heap
- Repeat comparison/swap until heap property is attained

Insertion maintains the heap

- Worst case: `insert` takes time proportional to tree depth: $O(\log n)$
- The shape property is maintained:
  - on adding a new element at last level, leftmost free slot
  - on swapping node values along a path to the root
- The heap property is not maintained after adding a new element
- However, it is restored after the sequence of swaps

Thm.

The insertion operation maintains the heap

Max

- *Easy*: return the root of the heap tree
- Evidently $O(1)$

Removal of max

- Let $\mathtt{last}(Q)$ be the rightmost non-empty element of the last heap level
- Move node $\mathtt{last}(Q)$ to the root $r(Q)$
- Compare v with its children u, w: if $p_v \geq p_u, p_v \geq p_w$, heap is in correct order
- Otherwise, swap v with $\max_p(u, v)$ *(use $\min_p$ if min-heap)* and repeat comparison/swap until termination

Efficient construction

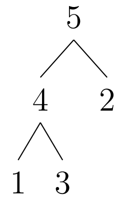
- Suppose we have n elements of V to insert in an empty heap
- Trivially: each insert takes $O(\log n)$, get $O(n \log n)$ to construct the whole heap
- Instead:
  1. arbitrarily put the element in a binary tree with the shape property (can do this in $O(n)$)
  2. lower level first, move nodes down using the same swapping procedure as for `popMax`
- At level ℓ, moving a node down costs $O(\ell)$ (worst-case)
- There's $\leq \lceil \frac{n}{2^{\ell+1}} \rceil$ nodes at level ℓ and $O(\log n)$ possible levels

$$\sum_{\ell=0}^{\lceil \log n \rceil} \frac{n}{2^{\ell+1}} O(\ell) = O(n \sum_{\ell=0}^{\lceil \log n \rceil} \frac{1}{2^\ell}) \leq O(n \sum_{\ell=0}^{\infty} \frac{1}{2^\ell}) = O(2n) = O(n)$$

Implementation

- A priority queue is implemented as a heap
- But we didn't say how a heap is to be implemented
- It *behaves* like a tree
- We're going to use an array instead (practically very efficient)

Binary trees in arrays



Node	5	4	2	1	3
Index	0	1	2	3	4
		i		$2i+1$	$2i+2$

- Heap Q of n elements stored in an array q of length n
- $q_0 = r(Q)$
- **Subnodes**
  If $q_i = v$, then $q_{2i+1} = r(\mathsf{L}(v))$ and $q_{2i+2} = r(\mathsf{R}(v))$ (whenever $2i+1, 2i+2 < n$)
- **Parent**
  If $q_i = v \neq r(Q)$, $q_j = P(v)$ where $j = \lfloor \frac{i-1}{2} \rfloor$

We now have all the elements: start implementing!

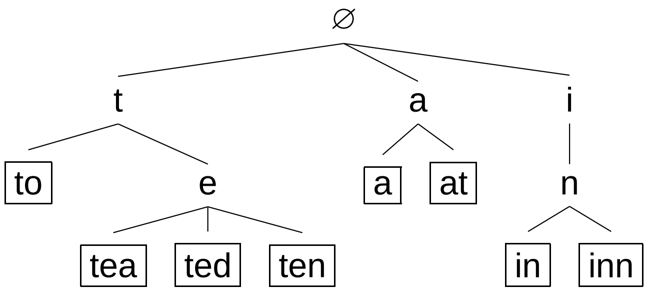
k-ary Search Trees

Tries

- Recall SEARCH problem: given a set V and a key v, determine whether $v \in V$
- Hash functions: $O(1)$ in the average case
- Let V be a set of words from same alphabet L
- We can organize keys in a k**-ary tree** for answering SEARCH
- In a k-ary tree, each node has at most k subnodes

Trie example

$V = \{$a,at,to,tea,ted,ten,in,inn$\}$



- Each key is stored at a leaf node ℓ
- Each non-leaf node v represents a prefix of all keys stored in the tree rooted at v
- The trie root node is $\varnothing$, the empty string

Trie properties

- The path of the trie corresponding to a key k is given by the key itself

  Compare with hash functions: the hash value is specified by the key
- This path has the same length m as the key
- `find`, `insert` and `delete` take worst-case $O(m)$
- If $m, |L|$ are bounded by a constant w.r.t. $n = |V|$, then methods are $O(1)$ in the worst case (w.r.t. set size)
- Comparison to hash functions
  - With respect to hashing, tries support “ordered iteration”
  - Hash tables need re-hashing (expensive) as they become full; tries adjust to size gracefully
  - No need to construct good hash functions

Warning: there are several trie variants

End of Lecture 7