Decomposition theorems for classes of graphs defined by constraints on connectivity

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Danish Graph Theory Meeting
April – May 2011

Outline

1. Minimally 2-connected graphs
2. Critically 2-connected graphs
3. Open questions

Definitions

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- A graph is **chordless** if every cycle is chordless.

Links between "chordless" and "minimally 2-connected"

Plummer's observations [1968]:

- A graph G is chordless if and only if for every subgraph H, either H has connectivity at most 1, or H is minimally 2-connected.
- A 2-connected graph is chordless if and only if it is minimally 2-connected.

Decomposing chordless graphs

Theorem (Lévêque, Maffray, NT 2009)

Let G be a chordless graph. Then either G is basic or G has a decomposition.

Basic class

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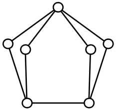
- The converse is false: 

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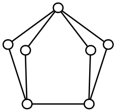
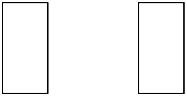
- The converse is false: 
- From any graph, one can obtain a sparse graph by subdividing several edges.

Decompositions

- **0-cutset**: 

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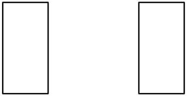
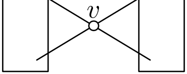
Decomposition
Connectivity

Minimally 2-connected graphs

Critically 2-connected graphs

Open questions

- **0-cutset**: 
- **1-cutset**: 

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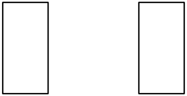
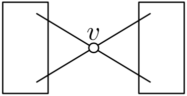
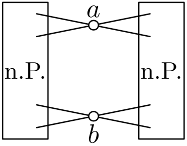
- **0-cutset**: 
- **1-cutset**: 

- **proper 2-cutset**: 

Applications (1): reproving Plummer's Theorem

Theorem (Plummer 1968)

Let G be a 2-connected graph. Then G is minimally 2-connected if and only if either

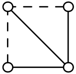
- *G is a cycle; or*
- *if S denotes the set of nodes of degree 2 in G, then there are at least two components in $G \setminus S$, each component of $G \setminus S$ is a tree and if C is any cycle in G and T is any component of $G \setminus S$, then the graph $(V(C) \cap V(T), E(C) \cap E(T))$ is empty or connected.*

Applications (2): edge- and total-colouring

Theorem (Machado, de Figueiredo, NT 2010)

If G is a chordless graph such that $\Delta(G) \geq 3$, then G is $\Delta(G)$-edge-colourable and $(\Delta(G) + 1)$-total-colourable.

Generalization: unichord-free graphs

- A graph is chordless if it does not contain  as a subgraph.

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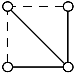
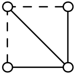
- A graph is chordless if it does not contain  as a subgraph.
- What about not containing  as an **induced** subgraph?

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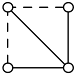
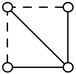
- A graph is chordless if it does not contain  as a subgraph.

- What about not containing  as an **induced** subgraph?

- A graph is **unichord-free** if it does not contain a cycle with a unique chord.

Unichord-free graphs defined by connectivity constraints

Theorem (McKee 2007)

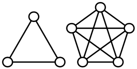
A graph G is unichord-free if and only if every minimal cutset of G is a stable set.

Decomposing unichord-free graphs

Theorem (NT, Vušković 2008)

Let G be a unichord-free graph. Then either G is basic or G has a decomposition.

Basic classes

- **cliques**: 

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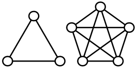
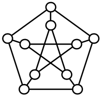
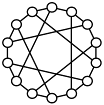
- **cliques**: 

- graphs obtained from **Petersen**  or **Heawood**  by deleting vertices and subdividing edges incident to at least one vertex of degree 2

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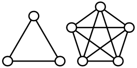
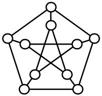
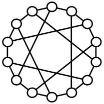
- **cliques**: 
- graphs obtained from **Petersen**  or **Heawood**  by deleting vertices and subdividing edges incident to at least one vertex of degree 2
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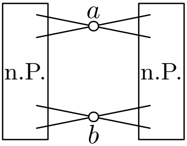
- **proper 2-cutset**: 

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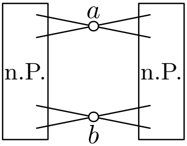
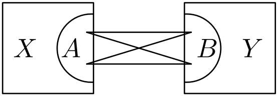
- **proper 2-cutset**: 

- **1-join**: 

Applications

- Detections of cycle with a unique chord: in time $O(nm)$ [NT, Vušković]
- Vertex colouring: a unichord-free graph G is either 3-colourable or $\omega(G)$-colourable [NT, Vušković].

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- *If squares are excluded and $\Delta \geq 4$, then both problems become polynomial.*

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- A **propeller** is a cycle together with a vertex (called the center), not in the cycle, that has at least two neighbors in the cycle.
- A k-**propeller** is a propeler such that the center has k neighbors in the cycle.

Links between "propeller" and "critically 2-connected"

- A graph G contains no propeller (as a subgraph) if and only if for every subgraph H, either H has connectivity at most 1, or H is critically 2-connected.
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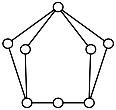
- The converse is false: 

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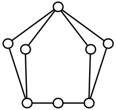
- The converse is false: 
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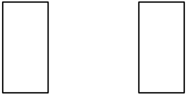
- **0-cutset**: 

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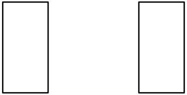
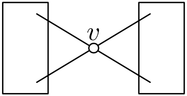
- **0-cutset**: 
- **1-cutset**: 

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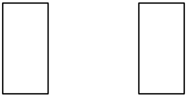
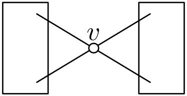
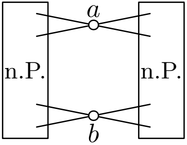
- **0-cutset**: 
- **1-cutset**: 

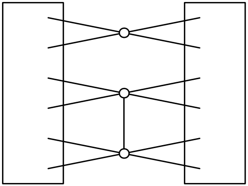
- **proper 2-cutset**: 

- **K_2-cutset**

Decomposing graphs with no propeller as an induced subgraph

- To decompose graphs with no propeller as an **induced subgraph**, one more decomposition is needed.
- An ***I*-cutset** in a graph G is cutset S made of three vertices, with only one edge linking them, and such that $G \setminus I$ has at least two components containing neighbors of all vertices of S.

An I-cutset



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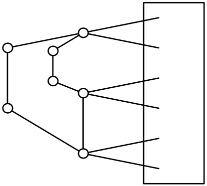

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- Consider the following algorithm:
  Test for all paths a-b-c whether in $G \setminus b$, an induced cycle goes through a, c.
- Difficult to implement in polytime because Bienstock proved that testing for an induced cycle through 2 given vertice is NPC.
- Detecting a k-propeller such that $k \geq 4$ is NP-complete.

Application (2): edge-colouring

Theorem (Aboulker, Radovanović, NT, Vusković 2011)

A 2-connected graph with no induced propeller contains an edge with both ends of degree at most 2.
Hence, every graph G with no induced propeller, and which is not an odd cycle, is $\Delta(G)$-edge colourable.

Outline

Higher connectivity?

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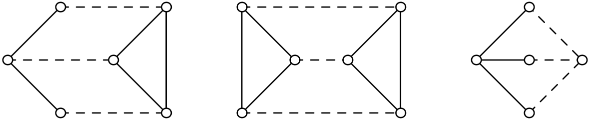
Minimally 3-connected graphs?

Truemper configurations

- 3 paths configurations



pyramid prism theta

- Wheels

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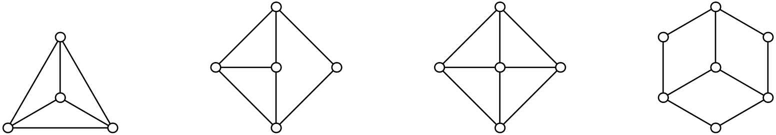

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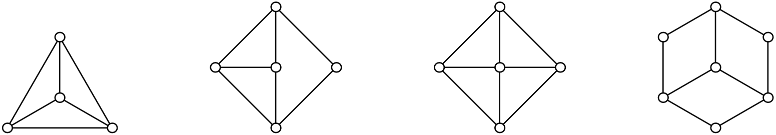

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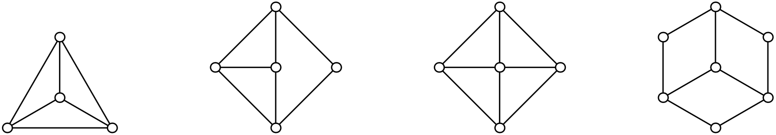

Open question: detecting a wheel as an induced subgraph in polytime.
It is **NPC** to detect k-propeller with $k \geq 4$.

Bounding the chromatic number of wheel-free graphs

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- Do they have a structure?
- Yes if induced subdivisions of K_4 are also excluded [Lévêque, Maffray, NT].
- Yes in the particular case of unichord-free graphs.

Excuding 2-propellers and colouring

Let G be a graph with no triangle, no cube and no 2-propeller (as induced subgraphs). Is it true that G contains a vertex of degree 2?

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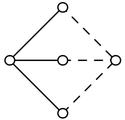
Theorem (Radovanović, Vusković 2010)

If a graph contains no triangle, no cube and no theta (as induced subgraphs), then it has a vertex of degree at most 2.

Remind that a theta is: 

Detecting 2-propellers

Is there a polytime algorithm to detect 2-propellers (as induced subgraphs)?