A MATROID VIEW OF KEY THEOREMS FOR EDGE-SWAPPING ALGORITHMS

JON LEE AND LEO LIBERTI

Abstract. We demonstrate that two key theorems of Amaldi, Liberti, Maffiolo and Maculan (2009), which they presented with rather complicated proofs, can be more easily and cleanly established using a simple and classical property of binary matroids. Besides a simpler proof, we see that both of these key results are manifestations of the same essential property.

Our goal is to demonstrate that two graph-theoretic theorems from [1] are direct manifestations of a simple and classical property of binary matroids. This follows the line of matroids sometimes serving as a means to simplify and unify combinatorial theorems involving graphs and coordinatized vector spaces.

We assume some very basic familiarity with matroid terminology and theory (see [2]), and we rely on elementary notions from graph theory. For a matroid M, we denote its dual matroid by M^* (i.e., the matroid on the same ground set as M but having its set of bases to be the set of complements of bases of M). For $S \subset E$, we let S^c denote the complement of S in E. For $S, U \subset E$, we denote the symmetric difference $(S \setminus U) \cup (U \setminus S)$ by $S \Delta U$.

Let G be a finite connected graph with edge set E. Let $T \subset E$ be a spanning tree of G. For an edge $f \notin T$, let $C(T, f)$ be the unique cycle of G contained in $T \cup \{f\}$. Similarly, for an edge $e \in T$, let $D(T, e)$ be the unique cocycle of G contained in $T^c \cup \{e\}$. From the point of view of matroid theory, T is a base of the graphic matroid of G and T^c is a base of the cographic matroid of G, so $C(T, f)$ and $D(T, e)$ are fundamental circuits (in the matroid sense).

For an edge $f \notin T$ and an edge $e \in C(T, f) \setminus \{f\}$, $T \cup \{f\} \setminus \{e\}$ is also a spanning tree of G, and hence a base of the graphic matroid of G. Of course then, $(T \cup \{f\} \setminus \{e\})^c$ is also the complement of a spanning tree of G, so $(T \cup \{f\} \setminus \{e\})^c$ is a base of the cographic matroid of G.

With our notation, Theorem 1 of [1] is as follows

Theorem 1. Let $e \in T^c$, $f \in T^c$, $e \neq f$, $g \in C(T, e) \cap C(T, f)$. Then

$$C(T \cup \{e\} \setminus \{g\}, f) = C(T, e) \ \Delta \ C(T, f).$$

The proof of Theorem 1 presented in [1] is not egregiously lengthy, but it is a bit ad hoc. However, the authors also present a Theorem 9, which they demonstrate with a long (three page) proof that is relegated to an appendix. In the way that it is stated, it is not immediately clear that the result is dual to Theorem 1. Here, we state it in a way that makes it easier to directly compare.

Theorem 9. Let $e \in T$, $f \in T$, $e \neq f$, $g \notin D(T, e) \cap D(T, f)$. Then

$$D(T \cup \{e\} \setminus \{g\}, f) = D(T, e) \ \Delta \ D(T, f).$$

Date: January 20, 2012. Revised March 27, 2012.

We can easily see, now, that these theorem are just manifestations of the same phenomenon, which we state now in matroid language.

Let $B \subset E$ be a base of a matroid M on ground set E. For $e \in B^c$, $\mathcal{C}_M(B,e)$ denotes the unique circuit of M contained in $B \cup \{e\}$. The matroid M is *binary* if it can be represented by a matrix over the two element field $GF(2)$.

Theorem P. Suppose that B is a base of a binary matroid M. Let $e \in B^c$, $f \in B^c$, $e \neq f$, $g \in \mathcal{C}_M(B,e) \cap \mathcal{C}_M(B,f)$. Then

$$\mathcal{C}_M(B \cup \{e\} \setminus \{g\}, f) = \mathcal{C}_M(B,e) \ \Delta \ \mathcal{C}_M(B,f).$$

Theorem 1 is just Theorem P, when M is the graphic matroid of G. Theorem 9 is just Theorem P, when M is the cographic matroid of G. Note that graphic and cographic matroids are binary.

It can also be useful to think of the equivalent dual re-statement of Theorem 9:

Theorem P*. Suppose that B is a base of a binary matroid M. Let $e \in B$, $f \in B$, $e \neq f$, $g \in \mathcal{C}_{M^*}(B^c,e) \cap \mathcal{C}_{M^*}(B^c,f)$. Then

$$\mathcal{C}_{M^*}(B^c \cup \{g\} \setminus \{e\}, f) = \mathcal{C}_{M^*}(B^c,e) \ \Delta \ \mathcal{C}_{M^*}(B^c,f).$$

Theorems P is classical and can be found, for example, as a special case of Theorem 9.1.2, part (vii) in [2, p. 304]. In the interest of making this note self contained and to demonstrate how truly elementary Theorem P is, we present a very short and direct proof of it. Before proceeding to the proof of Theorem P, we need to recall basic facts concerning representations of a matroid. If B is a base of a matroid M that is representable over a field $\mathbb{F}$, then M has a representation over $\mathbb{F}$ as a *standard representative matrix*

$$A = \begin{array}{c} \\ B \end{array} \begin{array}{c} \begin{array}{cc} B & B^c \end{array} \\ \left[\begin{array}{c|c} I & N \end{array} \right] \end{array},$$

where here we think of B and B^c as *ordered* sets. Furthermore, for $e \in B$, the support of the row of A indexed by e is precisely $\mathcal{C}_{M^*}(B^c, e)$.

Proof. (Theorem P). We proceed by establishing the equivalent Theorem P*. The matrix A has a pair of rows indexed by $e, f \in B$, and $g \in B^c$ is in the support of each of these two rows:

$$\begin{array}{c} e \\ f \end{array} \left[\begin{array}{cccccccc|ccccccc} 0 & \ldots & 0 & \overset{e}{1} & \overset{f}{0} & 0 & \ldots & 0 & \times & \ldots & \times & \overset{g}{1} & \times & \ldots & \times \\ 0 & \ldots & 0 & 0 & 1 & 0 & \ldots & 0 & \times & \ldots & \times & 1 & \times & \ldots & \times \end{array} \right].$$

Pivoting on the entry in the top row and column g leads to a standard representative matrix for M with respect to the base $B \cup \{e\} \setminus \{g\}$, the complement of which is $B^c \cup \{g\} \setminus \{e\}$. So, after the pivot, the support of the second row is precisely $\mathcal{C}_{M^*}(B^c \cup \{g\} \setminus \{e\}, f)$. Moreover, the result in the second row is the $GF(2)$ sum of the two rows, which is precisely $\mathcal{C}_{M^*}(B^c,e) \ \Delta \ \mathcal{C}_{M^*}(B^c,f)$. □

References

1. E. Amaldi, L. Liberti, F. Maffioli and N. Maculan, Edge-swapping algorithms for the minimum fundamental cycle basis problem, *Mathematical Methods of Operations Research*, Vol. 69 (2009) 205–223.
2. J. Oxley, "Matroid Theory,", Oxford University Press, New York, 1992.

Jon Lee: IOE Department, University of Michigan, Ann Arbor, MI 48109-2117 USA
E-mail address: jonxlee@umich.edu

Leo Liberti: LIX, École Polytechnique, 91128 Palaiseau, France.
E-mail address: liberti@lix.polytechnique.fr