Formulation Space Search for Circle Packing

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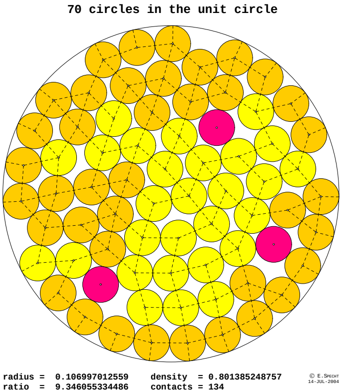
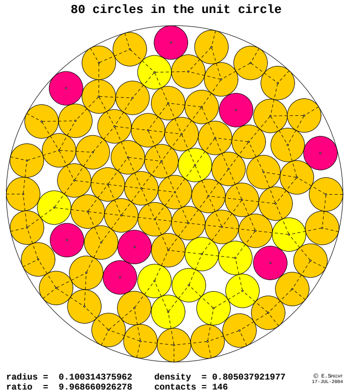
Packing circles into a circle

What is the smallest circle in which N unit circles may be packed ?

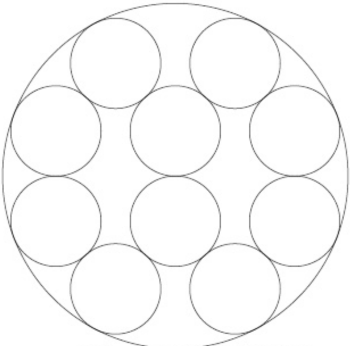




For much more see E.Specht website (updated 25 oct 2008)
`http://hydra.nat.uni-magdeburg.de/packing/cci/cci.html`

Optimality not so simple.

E.G. $n = 10$

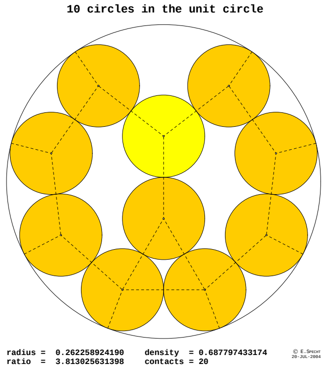


$n = 10$, box radius : 3.828427125

Published by Martin Gardner 1992, following S.Kravitz 1967.

First-order stationary solution of cartesian formulation.

Better solution :



Obtained by a 'linear' move in a polar coordinate formulation.

Proven to be optimal by Pirl 1969
according to E. Friedman, website 'Erich's Packing Center'
`http://www.stetson.edu/~efriedma/packing.html`

RD: Reformulation Descent

Observation:

Local search result depends on formulation

- Extension of local search
- Needed: at least two formulations of a same problem
- Method
  - Use sequentially local search for each formulation in turn
  - until full cycle without improvement occurs

Mladenović, Plastria, Urošević,
Reformulation Descent applied to circle packing problems,
Computers & Operations Research 32 (9), 2005, 2419-2434

RD for Circle packing

Use two formulations:

- in cartesian coordinates
- in polar coordinates (w.r.t. enclosing circle's center)

Not linearly related. So first order optimisation procedure behaves differently

Tests:

- MINOS for local search
- Multistart RD (10 times)
- $N = 10, 11, \ldots, 100$
- obtaining the best known results from the literature in 40% of the cases, and otherwise with an error of less than 1%
- similar results than more sophisticated NLP methods but 150 times faster.

Reduced RD for Circle packing

- Problems with size of full formulation
  $O(n^2)$ ‘non-overlap’ constraints for all pairs of disks
- In local search only non-overlap is needed for disks which are ‘close’
- Reduced RD deletes all non-overlap constraints for centers at distance $> 4r$
- number of constraints is now $O(n)$.
- Usually results remain the same

Formulation space

- Optimisation problem (P)
- Several "different" formulations $\phi \in \Phi$
- Clear correspondence of solutions between formulations
- Easy translation of solutions to other formulations.
- Formulations not related
  i.e. corresponding local search not equivalent
- Formulation space

= Set of pairs (ϕ, x) with
x solution for (P)
ϕ formulation for (P).

FSS : Formulation Space Search

apply VNS in Formulation space

- Local search method for each formulation
- Nested neighbourhood structure in formulation search:

  For each $\phi \in \Phi$
  "neighbourhoods" $\phi \in N_1(\phi) \subset \ldots \subset N_m(\phi) \subset \Phi$

VNS

- Obtain initial pair (ϕ, x) and set $k = 1$
- while $k <= m$

  1. Choose a formulation $\phi' \in N_k(phi)$
  2. Apply local search for ϕ' starting from x, yielding x'
  3. **If** x' better than x

     Then set $x \leftarrow x'$, $\phi \leftarrow \phi'$
     and restart loop with $k = 1$

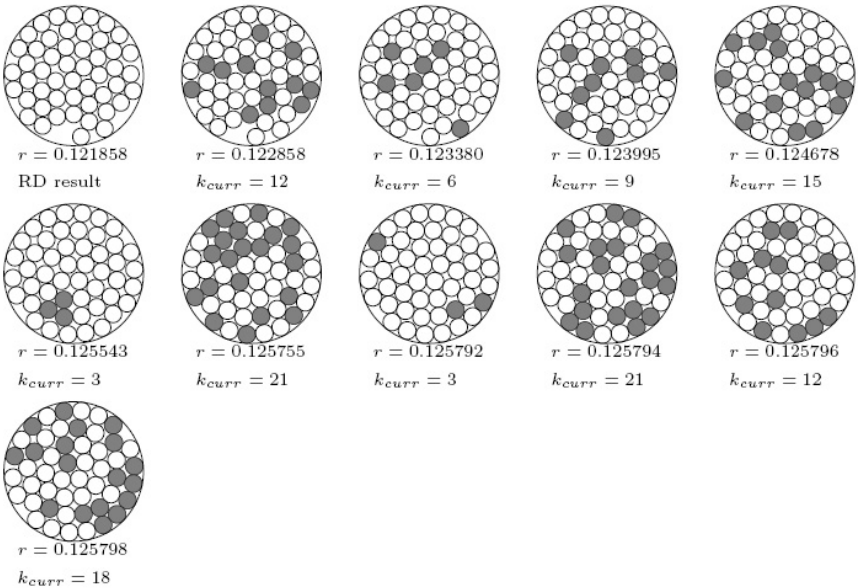
     Else increment k

FSS for circle packing - version 1

N.Mladenovic, F.Plastria & D.Urosevic,
Formulation space search for circle packing problems,
in Engineering Stochastic Local Search Algorithms
Proceedings, International Workshop SLS2007, Brussels, Belgium Sept 2007,
Edited by T.Stützle, M.Birattari,H.H.Hoos
LNCS 4638, Springer, 2007, 212-216

- Consider all formulations with some unit circle centres in cartesian coordinates, and the others in polar coordinates
- $\|\Phi\| = 2^N$
- k-Neighbourhood of $\phi \in \Phi$:
  switch coordinates of any k centres. $(k = 1, \ldots, N)$
- Starting solution and formulation obtained by random choice and RD
- Local search was reduced RD

Example


r = 0.121858
RD result
r = 0.122858
kcurr = 12
r = 0.123380
kcurr = 6
r = 0.123995
kcurr = 9
r = 0.124678
kcurr = 15
r = 0.125543
kcurr = 3
r = 0.125755
kcurr = 21
r = 0.125792
kcurr = 3
r = 0.125794
kcurr = 21
r = 0.125796
kcurr = 12
r = 0.125798
kcurr = 18

Some computational results

RD results from COR paper

FSS: 40 runs each
Fortran code, on a Pentium 3, 900 MHz computer.

		RD			FSS		
n	Best known	%Best	Avg.	Time	%Best	Avg.	Time
50	7.947515	0.06	0.79	3.19	0.00	0.24	80.54
55	8.211102	0.00	2.09	3.37	0.00	0.60	72.81
60	8.646220	0.03	1.40	4.71	0.00	0.95	84.39
65	9.017397	0.00	1.33	16.24	0.00	0.21	108.25
70	9.346660	0.10	0.99	19.56	0.01	0.27	151.64
75	9.678344	0.10	0.77	26.46	0.02	0.20	164.51
80	9.970588	0.10	0.93	39.15	0.04	0.23	229.49
85	10.163112	0.72	1.75	38.79	0.18	0.72	256.17
90	10.546069	0.02	1.27	96.82	0.02	0.56	294.77
95	10.840205	0.18	0.93	147.35	0.07	0.39	308.34
100	11.082528	0.30	1.01	180.32	0.12	0.68	326.67

FSS for circle packing - version 2

- Observation: polar coordinates were centered at center of enclosing circular box
- Dis-advantageous: polar coordinates "linearize" circular movements around this center.
  OK for those touching the outer boundary.
  But small circles should rather move around a neighbouring circle.
- So consider polar coordinates centered at some neighbouring (touching) circle
- Formulations: some unit circle centres in cartesian coordinates, and the others in polar coordinate, *with different origins*
- Formulation Space becomes much larger
- Many strategies possible according to the choice of origins.

Currently under investigation.

1 Some other packing problems that might benefit from FSS

1.1 Packing circles in boxes of other shapes

Any shape of box may be used.

- Square (well-studied)
- Equilateral triangle (well-studied)
- ellipses
- polygonal regions
- etc.

Much harder question
given n unit circles, what is the smallest area ellipse in which they may be packed ?

1.2 Packing unequal circles

- Applications in the electrical machine industry :
  minimise the holes through which a given bundle of wires of unequal size have to pass.
- Holes of circular, square, rectangular, elliptic shape
- Simplest instance only two sizes available.
  Variants according to numbers and relative sizes.

1.3 Packing unequal spheres

- Three-dimensional variant of previous problem.
- How to pack given unequal radius spheres into a given polytope or sphere.
- Applications in medicine :
  in radiosurgery and internal radio treatment of cancer.

1.4 Packing of other shapes

Packing of squares, triangles etc.
See e.g. Erich's packing center and links therein.

1.5 Packing on surfaces

Packing spheres on a sphere

'Kissing number'

Consider $r(n, d)$ = maximum radius of n identical nonoverlapping d-dimensional balls which touch the unit d-ball.

etc. etc.