Deciding the Satisfiability of Propositional Formulas in Finitely-Valued Signed Logics

Victor Chepoi,* Nadia Creignou†
LIF (CNRS, UMR 6166)
Université de la Méditerranée
13288 Marseille cedex 9, France

Miki Hermann‡
LIX (CNRS, UMR 7161)
École Polytechnique
91128 Palaiseau, France

Gernot Salzer§
Technische Universität Wien
Favoritenstraße 9-11
1040 Wien, Austria

Abstract

Signed logic is a way of expressing the semantics of many-valued connectives and quantifiers in a formalism that is well-suited for automated reasoning. In this paper we consider propositional, finitely-valued formulas in clausal normal form. We show that checking the satisfiability of formulas with three or more literals per clause is either NP*-complete or trivial, depending on whether the intersection of all signs is empty or not. The satisfiability of bijunctive formulas, i.e., formulas with at most two literals per clause, is decidable in linear time if the signs form a Helly family, and is* NP*-complete otherwise. We present a polynomial-time algorithm for deciding whether a given set of signs satisfies the Helly property. Our results unify and extend previous results obtained for particular sets of signs.*

1 Introduction

Signed logic [14, 16] is a general approach to deal with many-valued logics. Given the truth tables of arbitrary finitely-valued or certain infinitely-valued connectives and quantifiers, it is possible to construct systematically all kinds of sound and complete calculi – like natural deduction, tableaux systems, or sequent calculi – based on expressions of the form $S{:}\varphi$, where S is a set of truth values, called sign, and φ is a many-valued formula. Such an expression is true if the formula φ evaluates to a value in S, and false otherwise. In other words, these expressions are two-valued atoms that can be used as basic building blocks of classical two-valued formulas.

Signed formulas can be transformed into conjunctive normal form (CNF) using clause formation rules. In the case of propositional logic, these normal forms are ordinary two-valued CNFs, except that propositional variables are replaced by atoms of the form $S{:}x$, which are true if the variable x is interpreted by a truth-value in the sign S. Many-valued resolution can be applied to these clauses to perform automated deduction similar to classical logic [4].

It is natural to ask for the complexity of deciding the satisfiability of many-valued CNFs, like in the two-valued case. Several authors classified the complexity of CNFs depending on the structure of signs and clauses. In this paper we address this problem once more and give a complete classification with respect to the signs occurring in the CNF. We unify and extend previous results obtained for particular sets of signs, show that the polynomial cases can be decided in linear instead of quadratic time, and give a polynomial algorithm for classifying sets of signs.

In Section 2 we give a precise definition of the problem and state the main result, namely a theorem classifying the complexity of deciding the satisfiability of signed formulas based on the signs allowed to occur in the formula. The proof of the theorem is split into several parts that are covered in the following sections. As application of our theorem, we discuss some particular families of signs in Section 7. The final section reviews related work. Due to space reasons, several proofs are missing; they can be found in [11].

2 Problem and Main Result

Let D be a finite set of at least two truth values, and let V be a set of variables. For a variable $x \in V$ and a set $S \subseteq D$ (called sign), the expression $S{:}x$ is called a *signed literal*. A *signed clause* is a disjunction of signed literals and of the constant symbols $\top$ and $\bot$ (representing true and false, respectively). A *signed formula* is a conjunction of signed clauses. A clause is called *bijunctive* if it contains at most two literals; a formula is called bijunctive if all its

*email: chepoi@lif.univ-mrs.fr
†email: creignou@lif.univ-mrs.fr
‡email: hermann@lix.polytechnique.fr
§email: salzer@logic.at

0195-623X/08 $25.00 © 2008 IEEE
DOI 10.1109/ISMVL.2008.41

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clauses have this property.

An *interpretation* is a mapping $I\colon V \to D$ assigning a domain element $I(x)$ to each variable $x \in V$. It satisfies a literal $S{:}x$, if $I(x) \in S$. It satisfies a clause if it satisfies at least one of its literals, and it satisfies a formula if it satisfies each of its clauses. A formula is D'-satisfiable if it is satisfied by some interpretation over $D' \subseteq D$.

Given a set $\mathcal{S}$ of signs over the domain D, a formula using only signs from $\mathcal{S}$ is referred to as an *$\mathcal{S}$-formula*. In this paper, we study the complexity of deciding the satisfiability of $\mathcal{S}$-formulas depending on the set $\mathcal{S}$. More precisely we are interested in the following decision problem.

Problem: k-MVSAT($\mathcal{S}$)
Input: An $\mathcal{S}$-formula φ with at most k literals per clause.
Question: Is φ satisfiable?

As to be expected, the complexity of the problem differs for $k = 2$ and $k \geq 3$. In the latter case, the problem is either trivial or NP-complete, whereas the classification for bijunctive signed formulas is related to a well-studied property in combinatorics and discrete mathematics, namely the Helly property. A set $\mathcal{S}$ of signs over D is called a *Helly family*, or $\mathcal{S}$ has the *Helly property* [8], if every subset $\mathcal{T} \subseteq \mathcal{S}$ satisfying $\bigcap \mathcal{T} = \emptyset$ contains two signs $S, S' \in \mathcal{T}$ such that $S \cap S' = \emptyset$. (The expression $\bigcap \mathcal{T}$ is an abbreviation for $\bigcap_{S \in \mathcal{T}} S$.) Now we can state the main result of this paper.

Theorem 1 *For $k \geq 3$, k-MVSAT($\mathcal{S}$) is polynomial (in fact trivial) if $\bigcap \mathcal{S} \neq \emptyset$, and NP-complete otherwise. 2-MVSAT($\mathcal{S}$) is polynomial (in fact linear) if $\mathcal{S}$ is a Helly family, and NP-complete otherwise. Checking whether $\mathcal{S}$ is a Helly family can be done in polynomial time.*

The proof of Theorem 1 is split into several parts. First, Section 3 presents the intractable cases. Note that $\bigcap \mathcal{S} \neq \emptyset$ implies that k-MVSAT($\mathcal{S}$) is trivially in P, because every $\mathcal{S}$-formula is satisfiable by an interpretation assigning to all variables a value from the intersection $\bigcap \mathcal{S}$. Section 4 links the completeness of binary resolution for signed formulas to the Helly property. Section 5 describes polynomial time algorithms for 2-MVSAT($\mathcal{S}$) when $\mathcal{S}$ is a Helly family. In particular, it presents a linear-time algorithm for evaluating 2-MVSAT($\mathcal{S}$)-formulas defined on Helly families, which is a generalization of the Aspvall-Plass-Tarjan algorithm [2] for 2-SAT-formulas. Section 6 shows that the distinction between tractability and intractability, i.e. the Helly property, is polynomially decidable.

3 Intractable Cases

The case $k \geq 3$. We show that k-MVSAT($\mathcal{S}$) is NP-complete if $\bigcap \mathcal{S} = \emptyset$. We encode 3-SAT as an instance of 3-MVSAT($\mathcal{S}$). Let $\varphi = C_1 \wedge \cdots \wedge C_k$ be a conjunction of clauses, where each clause is of the form $l_1 \vee l_2 \vee l_3$ and the literals l_i are Boolean variables or their negations. Let $\mathcal{T}$ be a minimal subset of $\mathcal{S}$ satisfying $\bigcap \mathcal{T} = \emptyset$, and let $\mathcal{T}_0$ and $\mathcal{T}_1$ be disjoint non-empty subsets of $\mathcal{T}$ such that $\mathcal{T}_0 \cup \mathcal{T}_1 = \mathcal{T}$. Let f be a function mapping Boolean literals to signed formulas in the following way:

$$f(l) \quad = \quad \begin{cases} \bigwedge_{T \in \mathcal{T}_0} T{:}x & \text{for } l = \neg x \\ \bigwedge_{T \in \mathcal{T}_1} T{:}x & \text{for } l = x \ . \end{cases}$$

For a clause C, let $f(C)$ be the conjunctive normal form of $f(l_1) \vee f(l_2) \vee f(l_3)$. For a formula φ, let $f(\varphi)$ be the conjunction $f(C_1) \wedge \cdots \wedge f(C_k)$. Since $f(l)$ consists of at most $|\mathcal{S}|$ conjuncts, $f(\varphi)$ is an $\mathcal{S}$-formula whose length is $O(|\varphi| \cdot |\mathcal{S}|^3)$, where $|\varphi|$ is the number of literals in φ. It is straight-forward to show that φ is $\{0,1\}$-satisfiable if and only if $f(\varphi)$ is D-satisfiable.

The case $k = 2$. We show by a reduction from the coloring problem that 2-MVSAT($\mathcal{S}$) is NP-complete if $\mathcal{S}$ is not a Helly family. An *r-coloring* of a graph $G = (V, E)$ is a mapping $c\colon V \to C$ such that $|C| = r$ and $c(v) \neq c(w)$ whenever v and w are adjacent in G. The elements of the set C are called the available *colors*. The r-coloring problem r-COL asks whether a graph G admits an r-coloring. It is known to be NP-complete for any $r \geq 3$.

Note that $\mathcal{S}$ is not a Helly family if and only if there exists a subset $\mathcal{T} \subseteq \mathcal{S}$ of cardinality at least 3 such that $\bigcap \mathcal{T} = \emptyset$ and $\bigcap(\mathcal{T} - \{T\}) \neq \emptyset$ for all signs $T \in \mathcal{T}$. We use $\mathcal{T}$ as the set of colors, i.e., $r = |\mathcal{T}|$. Consider the bijunctive $\mathcal{S}$-formula

$$\varphi_{G,\mathcal{T}} := \bigwedge_{(x,y) \in E} \bigwedge_{T \in \mathcal{T}} (T{:}x \vee T{:}y).$$

over the variables V and the signs in $\mathcal{T}$. Then the following result holds.

Proposition 2 *A graph $G = (V, E)$ admits an r-coloring if and only if the bijunctive $\mathcal{S}$-formula $\varphi_{G,\mathcal{T}}$ is satisfiable.*

4 Binary Resolution and the Helly Property

Let $C_1 = S_1{:}x \vee D_1$ and $C_2 = S_2{:}x \vee D_2$ be clauses such that $S_1 \cap S_2 = \emptyset$. Then $C = D_1 \vee D_2$ is called the *binary resolvent* of the parent clauses C_1 and C_2. If the binary resolvent contains literals $S_1{:}x$ and $S_2{:}x$ such that $S_1 \subseteq S_2$, then they are merged to $S_2{:}x$. If C_1 or C_2 contains just one literal, we assume $D_1 = \bot$ or $D_2 = \bot$, respectively. A *proof* of a clause C from a formula φ is a sequence of clauses $C_1, \ldots, C_n$ such that $C_n = C$ and for each k, either C_k is a clause of φ, or C_k is a binary resolvent of C_i and C_j for $i, j < k$. A *refutation* of φ is a proof of $\perp$ from φ. Binary signed resolution is sound and, in the case of a Helly family, also refutationally complete. The base case of the induction proof of the latter result provides the link to the Helly property.

Proposition 3 *Let $\mathcal{S}$ be a Helly family. Then binary resolution is refutationally complete for $\mathcal{S}$-formulas, i.e., every unsatisfiable $\mathcal{S}$-formula admits a refutation.*

Proof: Let $e(\varphi)$ denote the number of excess literals of the $\mathcal{S}$-formula φ, i.e., the total number of literals in φ minus the number of clauses in φ. We prove the proposition by induction on $e(\varphi)$.

Base case: $e(\varphi) = 0$. All clauses in φ are unit clauses, since the number of literals equals the number of clauses. For a variable x, we denote by φ_x the unit clauses involving x. The unsatisfiability of φ implies that for some x the intersection of all signs in φ_x is empty. Since $\mathcal{S}$ is a Helly family, there must be two signs S_1 and S_2 in φ_x such that their intersection is empty. By resolving the corresponding literals we obtain a refutation of φ.

Induction step: see e.g. the proof in [6, Section 5.3]. □

5 Tractable Case

In this section we prove that 2-MvSAT($\mathcal{S}$) is in P if $\mathcal{S}$ is a Helly family. Since binary resolution is sound and complete in this case, we immediately obtain a polynomial algorithm: Compute the quadratic number of all binary resolvents and check for a contradiction (see also [6, Section 5.3]).

We can do better, however, by generalizing the linear algorithm of Aspvall *et al.* [2] for 2-SAT. Given a 2-SAT-formula φ over the variables V and the clauses C, this algorithm constructs a directed graph $G(\varphi)$ with $2\,|V|$ vertices $v, \neg v$ and $2\,|C|$ arcs $\neg u \rightarrow v$ and $\neg v \rightarrow u$ for each clause $u \vee v$. The formula φ is satisfiable if and only if each pair of vertices $u, \neg u$ belong to different strongly connected components of the graph $G(\varphi)$. The satisfying assignment for φ can be computed by traversing the strongly connected components of $G(\varphi)$ in reverse topological order.

Now, let $\mathcal{S}$ be a Helly family defined over a finite domain D and let φ be a bijunctive $\mathcal{S}$-formula over the variables V. In order to capture the Helly property of $\mathcal{S}$ and the satisfiability of φ, we define the following directed graph $G(\varphi)$: (a) For each literal $S{:}x$, we add two vertices $S{:}x$ and $\neg S{:}x$ to $G(\varphi)$ to be interpreted as "$S{:}x$ is true" and "$S{:}x$ is false", respectively; (b) for each clause $S{:}x \vee T{:}y$ of φ, add the arcs $\neg S{:}x \rightarrow T{:}y$ and $\neg T{:}y \rightarrow S{:}x$ to $G(\varphi)$; (c) for each pair of literals of φ of the form $S{:}x$ and $T{:}x$ such that $S \cap T = \emptyset$ we add the arcs $S{:}x \rightarrow \neg T{:}x$ and $T{:}x \rightarrow \neg S{:}x$ to $G(\varphi)$.

As in the case of the 2-SAT problem, the graph $G(\varphi)$ has the following *duality property:* $G(\varphi)$ is isomorphic to the graph obtained by reversing all arcs and all nodes of $G(\varphi)$. By this property, every strongly connected component H of $G(\varphi)$ has a dual component $\overline{H}$ induced by the complements of the vertices in H (two vertices u, v belongs to the same strongly connected component if there exist directed paths from u to v and from v to u).

Suppose that φ is satisfied by an interpretation I. We say that the vertex $S{:}x$ of $G(\varphi)$ is *satisfied* by I if $I(x) \in S$; then $\neg S{:}x$ is said to be *unsatisfied*. Otherwise, if $I(x) \notin S$, then we say that $\neg S{:}x$ is *satisfied* and $S{:}x$ is *unsatisfied*. Note that (1) exactly one of the vertices $S{:}x$ and $\neg S{:}x$ is satisfied by I, and (2) no arc $u \rightarrow v$ of $G(\varphi)$ has u satisfied and v unsatisfied, or equivalently, no directed path leads from a satisfied vertex to an unsatisfied vertex.

Vice versa, if we partition all vertices of $G(\varphi)$ into *satisfied* and *unsatisfied* vertices and this assignment obeys the conditions (1) and (2), then we can define an interpretation I of φ compatible with this assignment. Indeed, for each variable x, let $\mathcal{S}_x$ denote the subset of $\mathcal{S}$ consisting of all signs S such that the vertex $S{:}x$ is satisfied. We assert that $\mathcal{S}_x$ being non-empty implies $\bigcap \mathcal{S}_x \neq \emptyset$. In view of the Helly property, it suffices to show that the sets of $\mathcal{S}_x$ pairwise intersect. Indeed, if $\mathcal{S}_x$ contains two disjoint signs S and T, since $S{:}x \rightarrow \neg T{:}x$ is an arc of $G(\varphi)$ and the vertex $S{:}x$ is satisfied, condition (1) implies that $\neg T{:}x$ must be satisfied as well, yielding that $S{:}x$ is not satisfied. This contradicts the choice of T. Thus $\bigcap \mathcal{S}_x$ is indeed non-empty. Now define an interpretation I of φ by letting $I(x) \in \bigcap \mathcal{S}_x$ for all variables x with nonempty $\mathcal{S}_x$. We assert that the $\mathcal{S}$-formula is satisfied by I. Pick an arbitrary clause $S{:}x \vee T{:}y$ of φ. If $S \in \mathcal{S}_x$, then the first literal of this clause is satisfied, and we are done. Otherwise, if $S \notin \mathcal{S}_x$ then the vertex $\neg S{:}x$ is satisfied. Since $\neg S{:}x \rightarrow \neg T{:}y$ is an arc of $G(\varphi)$, condition (2) yields that the vertex $T{:}y$ must be satisfied, thus $T \in \mathcal{S}_y$ establishing our assertion.

Proposition 4 *Given a Helly family $\mathcal{S}$ over D, a bijunctive $\mathcal{S}$-formula φ is satisfiable if and only if no vertex $S{:}x$ is in the same strong component as its complement $\neg S{:}x$. Deciding whether a bijunctive $\mathcal{S}$-formula is satisfiable can be done in time $O(|\varphi| \cdot |\mathcal{S}|^2)$. Computing a satisfying interpretation requires $O(|\varphi| \cdot |D|)$ extra time.*

When we consider 2-MvSAT($\mathcal{S}$), the family $\mathcal{S}$ and the domain D are not part of the input, but $\mathcal{S}$ parameterizes the problem. We obtain the following complexity result.

Corollary 5 *For a Helly family $\mathcal{S}$,* 2-MvSAT*($\mathcal{S}$) can be decided and solved in linear time $O(|\varphi|)$.*

6 Complexity of Classification

In this section we discuss the complexity of deciding for a given set $\mathcal{S}$ of signs, whether the problem 2-MvSAT($\mathcal{S}$)

is in P or is NP-complete. According to Theorem 1 this is equivalent to recognizing if the set $\mathcal{S}$ has the Helly property. We present two algorithms for this task that are polynomial in $|D|$ and $|\mathcal{S}|$. They follow from two classical characterizations of Helly families by Berge and Duchet in [8, pp. 22-23] and [9].

Proposition 6 *A set $\mathcal{S}$ of signs over D has the Helly property if and only if for any three elements $a, b, c \in D$, the subset $\mathcal{S}(a, b, c)$ of all signs $S \in \mathcal{S}$ containing at least two of the elements a, b, c has a non-empty intersection.*

Hence it suffices to generate the set $\mathcal{S}(a, b, c)$ for each triplet $a, b, c \in D$ and to test if $\bigcap \mathcal{S}(a, b, c) \neq \emptyset$. A straightforward way is to construct for all pairs of elements $a, b \in D$ the sets $\mathcal{S}(a, b)$ consisting of all signs $S \in \mathcal{S}$ which contain both a and b. This can be done in time $O(|D|^2 \cdot |\mathcal{S}|)$. For a fixed pair a, b, we find the intersection $\bigcap \mathcal{S}(a, b)$ in time $O(|D| \cdot |\mathcal{S}(a, b)|)$. All such intersections taken over all pairs of D can be computed in time $O(|D|^3 \cdot |\mathcal{S}|)$. Now having the intersections $\bigcap \mathcal{S}(a, b)$, $\bigcap \mathcal{S}(b, c)$, and $\bigcap \mathcal{S}(c, a)$ at hand, it takes $O(|D|)$ time to find $\bigcap \mathcal{S}(a, b, c)$, requiring time $O(|D|^4)$ to compute all such intersections. According to Proposition 6, the algorithm returns "NO" if a set $\mathcal{S}(a, b, c)$ is found where $\bigcap \mathcal{S}(a, b, c) = \emptyset$. Summarizing we obtain the following result.

Proposition 7 *Given a set $\mathcal{S}$ of signs over a domain D, we can decide in $O(|D|^4 + |D|^3 |\mathcal{S}|)$ time whether $\mathcal{S}$ is a Helly family.*

Our second algorithm is of a better complexity than the first one in the case when the size of D is significantly larger than the size of $\mathcal{S}$. First we need a few notions from hypergraph theory.

A domain element d dominates another domain element d' if for all $S \in \mathcal{S}$, $d' \in S$ implies $d \in S$; in this case, the element d' is called *redundant*. A set of signs $\mathcal{S}$ is called *reduced* if it contains no redundant domain elements. By the following lemma we may assume that the set of signs $\mathcal{S}$ is reduced.

Lemma 8 *Let d and d' be distinct domain elements such that d dominates d'. Let h be a homomorphism defined by $h(d') = d$ and $h(x) = x$ for $x \neq d'$. An $\mathcal{S}$-formula φ is satisfiable if and only if the corresponding $h(\mathcal{S})$-formula $h(\varphi)$ is satisfiable.*

For a set of signs $\mathcal{S}$ over D, let $\overline{\mathcal{S}} = \{D - S \mid S \in \mathcal{S}\}$. The *dual* of $\mathcal{S}$, denoted by $\mathcal{S}^*$, is the family of sets $\{S \in \mathcal{S} \mid d \in S\}$ for all $d \in D$. A set $T \subseteq D$ is a *transversal* of $\mathcal{S}$ if it intersects all sets of $\mathcal{S}$, i.e., $T \cap S \neq \emptyset$ for all $S \in \mathcal{S}$. The family of all transversals of $\mathcal{S}$ that are minimal with respect to inclusion is denoted by $Tr(\mathcal{S})$. Then the second characterization of Helly families by Berge and Duchet can be rephrased in the following way:

Proposition 9 *A set $\mathcal{S}$ of non-empty signs over D has the Helly property if and only if all minimal transversals of the set family $\overline{\mathcal{S}^*} = \{\{S \in \mathcal{S} \mid d \notin S\} \mid d \in D\}$ have size 2.*

For the sake of notational simplicity, we set $n = |D|$ and $m = |\mathcal{S}|$. Then $\mathcal{S}^*$ and $\overline{\mathcal{S}^*}$ contain n sets each and are defined on the domain $\mathcal{S}$ of size m. The set family $\overline{\mathcal{S}^*}$ can be constructed in time $O(n \cdot m)$ by first transposing the $(0, 1)$ incidence matrix of $\mathcal{S}$ (this defines the dual family $\mathcal{S}^*$) and then switching the 0 and the 1 values of the resulting matrix. Next we compute in time $O(m^2 \cdot n)$ the set E of all minimal transversals of size 2 of $\overline{\mathcal{S}^*}$. Let $G = (\mathcal{S}, E)$ be the non-oriented simple graph defined by the set E. According to Proposition 9, $\mathcal{S}$ is a Helly family if and only if $Tr(\overline{\mathcal{S}^*}) = E$ holds. The following result shows that instead of checking $Tr(\overline{\mathcal{S}^*}) = E$ it suffices to check if $Tr(E) = \overline{\mathcal{S}^*}$. Its proof relies on the result $Tr(Tr(\overline{\mathcal{S}^*})) = \overline{\mathcal{S}^*}$ by Edmonds and Fulkerson [13].

Lemma 10 *$Tr(\overline{\mathcal{S}^*}) = E$ if and only if $Tr(E) = \overline{\mathcal{S}^*}$.*

Note that $Tr(E)$ consists of all subsets of the vertices of the graph G that are minimal w.r.t. inclusion and that meet all edges of E (i.e., $Tr(E)$ is the set of all minimal vertex covers of G). The complements of minimal vertex covers are the stable sets of the graph G that are maximal w.r.t. inclusion. Johnson, Yannakakis, and Papadimitriou [15] developed an algorithm which enumerates all maximal independent sets of a graph with m vertices with delay $O(m^3)$ between two subsequent maximal independent sets. We run this algorithm on the graph G until it has computed the first $n + 1 = \left|\overline{\mathcal{S}^*}\right| + 1$ maximal independent sets of G; this can be done in time $O(m^3 \cdot n)$. Let $\mathcal{I}$ be the collection of these independent sets. If $\overline{\mathcal{I}} = \overline{\mathcal{S}^*}$ (or, more simply, if $\mathcal{I} = \mathcal{S}^*$), then $Tr(\overline{\mathcal{S}^*}) = E$ and $\mathcal{S}$ is a Helly family. Otherwise, by the considerations above, $\overline{\mathcal{S}^*}$ has a minimal transversal of size at least 3 and therefore $\mathcal{S}$ is not Helly. The last test can be performed in $O(n^2 \cdot m)$ time. Hence the total complexity of the algorithm is $O(m^3 \cdot n + n^2 \cdot m)$.

Summarizing, we obtain the following algorithm for testing $\mathcal{S}$ for the Helly property. First, construct the dual family $\mathcal{S}^*$ and its complement $\overline{\mathcal{S}^*}$, and compute the set E of minimal transversals of size 2 of $\overline{\mathcal{S}^*}$. Then, using the algorithm of Johnson *et al.* [15], compute $\left|\overline{\mathcal{S}^*}\right| + 1$ maximal independent sets of the graph $G = (\mathcal{S}, E)$. If the returned family of independent sets coincides with $\mathcal{S}^*$, then return the answer "$\mathcal{S}$ is Helly", otherwise return the answer "$\mathcal{S}$ is not Helly." We obtain the following result.

Proposition 11 *Given a set $\mathcal{S}$ of signs over a domain D, we can decide in time $O(|\mathcal{S}|^3 \cdot |D| + |D|^2 \cdot |\mathcal{S}|)$ whether $\mathcal{S}$ is a Helly family.*

7 Examples

We first need some basic definitions from lattice theory. Let $L = \langle D; \wedge, \vee \rangle$ be a finite lattice with the induced ordering $\leq$ defined by $a \leq b$ if $a \vee b = b$. For each element $a \in D$, the *up-set* (or principal filter) of a is given by $\uparrow a = \{d \in D \mid d \geq a\}$ and the *down-set* (or principal ideal) of a by $\downarrow a = \{d \in D \mid d \leq a\}$. The complements of the up- and down-set with respect to D are denoted by $\overline{\uparrow a}$ and $\overline{\downarrow a}$, respectively. For a pair of elements $a, b \in D$, the *interval* $[a, b]$ is the set $\{d \in D \mid a \leq d \leq b\}$. A lattice L is called *distributive* if it satisfies the distributive identity $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all elements $a, b, c \in D$. L is called *modular* if it satisfies the modular condition that $a \geq c$ implies the identity $a \wedge (b \vee c) = (a \wedge b) \vee c$.

Proposition 12 *Let $\langle D; \vee, \wedge \rangle$ be a finite lattice with the induced ordering $\leq$. Then the set of non-empty intervals over D is a Helly family.*

Corollary 13 2-MVSAT*($\mathcal{S}$) can be decided in linear time if $\mathcal{S}$ is a set of intervals with respect to some lattice.*

This corollary subsumes two previous results. Beckert *et al.* [6] showed that the satisfiability of bijunctive clause sets is polynomially decidable, if the signs are up- and down-sets in a lattice. Ansótegui and Manyà [1] showed an analogous result for signs that are intervals in a totally ordered domain.

Problem: 2-MVSAT-CUD(L)
Input: A bijunctive $\mathcal{S}$-formula φ, where $\mathcal{S}$ is the set of complements of up- and down-sets wrt. lattice L.
Question: Is φ satisfiable?

Proposition 14 2-MVSAT-CUD(L) *is* NP*-complete if the lattice L contains at least two incomparable elements, otherwise it is in* P.

Proof: Let E be a set of pairwise incomparable elements (containing at least two elements) that is maximal in the following sense: (1) every other element in the domain is comparable to some element in E, and (2) every domain element greater than some element in E is in fact greater than at least two elements in E. Such a maximal set always exists if the lattice contains at least two incomparable elements. Consider the set of signs $\mathcal{S} = \{\overline{\downarrow e} \mid e \in E\} \cup \{\overline{\uparrow(e \vee e')} \mid e, e' \in E, e \neq e'\}$; it contains at least three signs since E has at least two elements. We have $E - \{e\} \subseteq \overline{\downarrow e}$, $E \subseteq \overline{\uparrow(e \vee e')}$, $\top \in \overline{\downarrow e}$, and $\bot \in \overline{\uparrow(e \vee e')}$ for all $e \neq e'$.[1] Hence the pairwise intersection of any two signs in $\mathcal{S}$ is non-empty. The intersection of all signs, however, is empty: every domain element is either less than or equal to some $e \in E$ and therefore does not occur in $\overline{\downarrow e}$, or it is greater than some $e \in E$ and therefore does not occur in $\overline{\uparrow(e \vee e')}$ for some $e' \in E$. Hence $\mathcal{S}$ is not a Helly family, and by Theorem 1 we conclude that 2-MVSAT$(\mathcal{S})$ and therefore 2-MVSAT-CUD(L) is NP-complete. Otherwise, if every two elements are comparable, the domain is linearly ordered and the complements of up- and down-sets can be regarded as intervals. Hence, by Corollary 13, 2-MVSAT-CUD(L) is in P. □

[1] $\bot$ and $\top$ denote the bottom and top element, respectively.

Signs $\mathcal{S}$	ordering	complexity	why?
$\uparrow, \overline{\downarrow}$	any	P	triv. sat. by $\top$
$\overline{\uparrow}, \downarrow$	any	P	triv. sat. by $\bot$
$\uparrow, \downarrow$	any	P	Cor. 13
$\uparrow, \downarrow, \overline{\uparrow}, \overline{\downarrow}$	linear	P	Cor. 13
$\uparrow, \overline{\uparrow}$	non-linear	NP	Prop. 15
$\downarrow, \overline{\downarrow}$	non-linear	NP	Prop. 15, dual
$\overline{\uparrow}, \overline{\downarrow}$	non-linear	NP	Prop. 14

Figure 1. Classification of 2-MVSAT($\mathcal{S}$)

As a corollary we obtain that 2-MVSAT-CUD(L) is NP-complete for arbitrary modular non-distributive lattices L: By the M_3-N_5 theorem (see e.g. [12, Theorem 6.10]) a modular non-distributive lattice contains the sublattice M_3, i.e., it contains at least three incomparable elements. Proposition 14 subsumes the result in [6] that shows the NP-completeness of 2-MVSAT-CUD(M_3).

Problem: 2-MVSAT-UCU(L)
Input: A bijunctive $\mathcal{S}$-formula φ, where $\mathcal{S}$ is the set of up-sets in lattice L and of their complements.
Question: Is φ satisfiable?

Proposition 15 2-MVSAT-UCU(L) *is* NP*-complete if the lattice L contains at least two incomparable elements, otherwise it is in* P.

Proof: Let a and b be incomparable elements, and consider the signs $\mathcal{S} = \{\uparrow a, \uparrow b, \overline{\uparrow(a \vee b)}\}$. The pairwise intersections are non-empty, since $a \in \uparrow a \cap \overline{\uparrow(a \vee b)}$, $b \in \uparrow b \cap \overline{\uparrow(a \vee b)}$ and $a \vee b \in \uparrow a \cap \uparrow b$. On the other hand, the intersection of all signs in $\mathcal{S}$ is empty since $\uparrow a \cap \uparrow b = \uparrow(a \vee b)$. Hence $\mathcal{S}$ is not a Helly family, and by Theorem 1 we conclude that 2-MVSAT$(\mathcal{S})$ and therefore 2-MVSAT-UCU(L) is NP-complete. Otherwise, if every two elements are comparable, the domain is linearly ordered and the up-sets and their complements can be regarded as intervals. Hence, by Corollary 13, 2-MVSAT-UCU(L) is in P. □

Proposition 15 subsumes the result in [6] that shows the NP-completeness of 2-MVSAT-UCU(M_2).

The above results lead to a complete classification of 2-MvSAT($\mathcal{S}$), when $\mathcal{S}$ consists of up-sets, down-sets, and/or complements thereof (Fig. 1). Note that not all sets of signs possessing the Helly property can be viewed as sets of intervals. E.g., the set $\mathcal{S} = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,6\},\{1,6\},\{2,5\},\{3,6\}\}$ is a Helly family but there exists no lattice such that the signs in $\mathcal{S}$ can be interpreted as intervals with respect to this lattice. In fact, hypergraph theory knows many instances of Helly families that are not related to lattices; see e.g. [8] for several examples. This shows that the complexity of bijunctive formulas depends on a combinatorial property of the signs rather than on the algebraic structure of the set of truth values motivated by the logical interpretation.

8 Related Work

The study of k-MvSAT($\mathcal{S}$) was started in [17] and further continued in [1, 5–7, 10]. Manyà [17] established that 2-MvSAT is NP-complete using a reduction from the 3-coloring problem. He also established that 2-MvSAT($\mathcal{S}$) is polynomially solvable if $\mathcal{S}$ consists of regular signs of the form $\uparrow a$ and $\downarrow a$ of a totally ordered domain D. Béjar, Hähnle, and Manyà [7] reduced the problem of satisfiability of regular signed formulas on totally ordered domains to satisfiability of classical formulas. In particular, a regular 2-MvSAT($\mathcal{S}$) formula φ is reduced to a 2-CNF formula of size $O(|\varphi| \log |\varphi|)$ [7], which leads to an algorithm of complexity $O(|\varphi| \log |\varphi|)$ to test the satisfiability of a regular 2-MvSAT($\mathcal{S}$) formula φ in using the linear time algorithm of Aspvall *et al.* for 2-SAT [2]. Baaz and Fermüller [3] established that the 2-MvSAT($\mathcal{S}$) problem for monosigned CNF formulas φ ($\mathcal{S}$ consisting of signs of the form $\{d\}, d \in D$) is polynomially solvable and Manyà [17] presented a $O(|\varphi| \cdot |D|)$ time algorithm for this problem. Using the binary resolution method, Beckert, Hähnle and Manyà [6] showed that the problem 2-MvSAT($\mathcal{S}$) is polynomially solvable if D is a lattice and $\mathcal{S}$ consists of regular signs $\uparrow a$ and $\downarrow a$ of D. More recently, Charatonik and Wrona [10] showed that this problem can be solved in quadratic time and in linear time in the size of the formula, if the lattice is fixed. For this, they used a reduction of a many-valued satisfiability problem on a lattice to a classical one. Extending the intractability result of [17], Beckert *et al.* [6] showed that 2-MvSAT($\mathcal{S}$) is NP-complete (1) if the domain D is a modular lattice and $\mathcal{S}$ consists of complements of regular signs $\uparrow a$ and $\downarrow a$ of D or (2) if the domain D is a distributive lattice and $\mathcal{S}$ consists of regular signs of D and their complements.

Acknowledgment. The second algorithm for testing the Helly property results from discussions of the authors with Gérard Cornuejols and Yann Vaxès.

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