Concurrency and Directed Algebraic Topology

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Chapter 1

Why and How

Directed algebraic topology is motivated by the idea that the execution traces of a program can be understood as the paths on a topological space whose points represent the states that might be reached during an execution. In some sense we topologize the space of states of the program. In particular there is infinitely many states along an execution trace.

In classial models of concurrency, the collection of states is seen as a combinatorial structure. An execution trace in such a model is then a (possibly infinite) sequence of "arrows[1]" which are the atomic steps of execution. The time in such models is therefore discrete.

However, the programming language semantics lay down constraints of "causality" between states. In **transitions systems** [Win95], these constraints are formalized by labelled graph whose vertices are the states, while in **Mazurkiewicz traces** [Win95] they are encoded by relations (upon the states space) submitted to axioms involving execution traces. In the framework of directed algebraic topology, each topological space will be provided with a structure carrying these constraints. According to the nature of the structure, we obtain **partially ordered spaces** [Nac65], **locally (partially) ordered spaces**, **streams** [Kri07]) or **d-spaces** [Gra03].

Each of these topological notions give rise to a category together with a functor which associates each object with its **fundamental category**. All the approaches mentioned above are in line with a common framework so they can be compared. In particular the fundamental category of a topological space is its **fundamental groupoid** [Hig71, Spa95, Bro06] : this elementary construction of algebraic topology is basically the only one that will be refered to in the sequel and no prerequisite of algebraic topology is assumed.

[1] One may also write "transitions" or "actions".

The necessary basic category theory as well as some is distilled along these notes when it is required. Speaking of category theory, any of the books [Awo06], [Bor94a] and [Mac98], cover all the needed concepts (and beyond), the one by *Steve Awodey* being more accessible.

1.1 Cartesian Product and Parallelism

1.1.1 Categories

A **category** $\mathcal{C}$ is defined by a collection of **objects** denoted by $\mathsf{Ob}(\mathcal{C})$, a collection of **morphisms** denoted $\mathsf{Mo}(\mathcal{C})$, three mappings id, s and t

$$\begin{tikzcd} \mathsf{Mo}(\mathcal{C}) \arrow[r, shift left=2, "\mathsf{s}"] \arrow[r, shift right=2, "\mathsf{t}"'] & \mathsf{Ob}(\mathcal{C}) \arrow[l, "\mathsf{id}" description] \end{tikzcd}$$

together with a mapping called the **composition (law)** of $\mathcal{C}$ whose domain of definition is the collection of ordered pairs (γ,δ) of morphisms of $\mathcal{C}$ such that $\mathsf{s}(\gamma)=\mathsf{t}(\delta)$. The image of (γ,δ) this mapping is denoted by $\gamma\circ\delta$ and called the **composite** of δ followed by γ. The objects $\mathsf{s}(\gamma)$ and $\mathsf{t}(\gamma)$ are the **source** and the **target** of the morphism γ. The morphism $\mathsf{id}(x)$, usually denoted by id_x, is the **identity** of the object x. These data form a category when they satisfy the following axioms:

The composition law is associative
For all morphisms γ one has $\mathsf{id}_{\mathsf{t}(\gamma)}\circ\gamma=\gamma=\gamma\circ\mathsf{id}_{\mathsf{s}(\gamma)}$
For all objects x one has $\mathsf{s}(\mathsf{id}_x)=x=\mathsf{t}(\mathsf{id}_x)$
For all morphisms γ and δ such that $\mathsf{s}(\gamma)=\mathsf{t}(\delta)$
one has $\mathsf{s}(\gamma\circ\delta)=\mathsf{s}(\delta)$ and $\mathsf{t}(\gamma\circ\delta)=\mathsf{t}(\gamma)$

A **sub-category** of $\mathcal{C}$ is the a sub-collection M of $\mathsf{Mo}(\mathcal{C})$ and a sub-collection O of $\mathsf{Ob}(\mathcal{C})$ such that
if x belongs to O, then id_x belongs to M
if γ is an element of M, then $\mathsf{s}(\gamma)$ and $\mathsf{t}(\gamma)$ belongs to O and
if δ is an element of M such that $\mathsf{s}(\gamma)=\mathsf{t}(\delta)$, then the composite $\delta\circ\gamma$ belongs to M.
The restrictions of the mappings id, s, t and $\circ$ to the collections O and M provide a structure of category.

A morphism γ from x to y is caled an **isomorphism** when there exists a morphism δ from y to x such that $\delta\circ\gamma=\mathsf{id}_x$ and $\gamma\circ\delta=\mathsf{id}_y$. In this case one says that δ (respectively γ) is the **inverse** of γ (respectively δ) and we usually write $\delta=\gamma^{-1}$ and $\gamma=\delta^{-1}$ (one easily checks that a morphism has at most one inverse). One also says that the objects x and y are **isomorphic**, which is denoted by $x\cong y$, when there exists an isomorphism from x to y.

The standard example of category is provided by the sets, it is often denoted by Set. It is a well-known fact that the collection of all sets is not a set, however the category theory would not be so interesting and useful if we had to refrain from handling such collections. While this issue is irrelevant to most branches of mathematics, it is crucial to category theory and none of the solutions that have been proposed to adress it has the agreement of all.

When the collection of morphisms of a category is a set, one says it is **small**. On the contrary, a category which is not small is sometimes said to be **large**. Anyway, the categories $\mathcal{C}$ we will have to deal with are **locally small**, in other words for all objects x and y of $\mathcal{C}$, the collection of morphisms whose source and target are x and y is a set denoted by $\mathcal{C}[x, y]$.

The **opposite** category of $\mathcal{C}$, denoted by $\mathcal{C}^{\mathsf{op}}$, is obtnained by swapping the roles of the mappings source and target. In other words $\mathcal{C}$ and $\mathcal{C}^{\mathsf{op}}$ have the same objects but γ is a morphism from x to y in $\mathcal{C}^{\mathsf{op}}$ if and only if it is a morphism from y to x in $\mathcal{C}$.

In these notes, any object of a large category can be understood as an abstraction of all the states of some program while morphisms can be thought of as "simulations".

1.1.2 Cartesian Product

The **Cartesian product** of two sets A and B is the set, denoted by $A \times B$, of ordered pairs (a, b) of elements of $A \cup B$ such that $a \in A$ and $b \in B$. It naturally comes with two **projections** π_A and π_B defined by $\pi_A(a, b) = a$ and $\pi_B(a, b) = b$. The mappings π_A and π_B respectively belong to the sets $\mathsf{Set}[A \times B, A]$ and $\mathsf{Set}[A \times B, B]$, moreover they satisfy the following property: for all sets X the following map is a bijection.

$$\begin{array}{ccc} \mathsf{Set}[X, A \times B] & \longrightarrow & \mathsf{Set}[X, A] \times \mathsf{Set}[X, B] \\ h & \longmapsto & (\ \pi_A \circ h\ ,\ \pi_B \circ h\ ) \end{array}$$

More generally, given two objects a and b of a category $\mathcal{C}$, a **Cartesian product** of a and b (in $\mathcal{C}$) is defined, when it exists, as an object c of $\mathcal{C}$ together with $(\pi_a, \pi_b) \in \mathcal{C}[c, a] \times \mathcal{C}[c, b]$ such that for all objects x of $\mathcal{C}$ the folowing map is a bijection.

$$\begin{array}{ccc} \mathcal{C}[x, c] & \longrightarrow & \mathcal{C}[x, a] \times \mathcal{C}[x, b] \\ h & \longmapsto & (\ \pi_a \circ h\ ,\ \pi_b \circ h\ ) \end{array}$$

One says (c, π_a, π_b) fulfill the **universal property** of the Cartesian product.

The general notion of Cartesian product is thus related to the specific notion of Cartesian product of sets. Yet, it is worth to notice that in general the Cartesian product is not uniquely defined since one may find (c, π_a, π_b) and (c', π'_a, π'_b) both satisfying the universal property. However in this case one has an isomorphism ϕ from c to c' such that $\pi_a \circ \phi = \pi'_a$ and $\pi_b \circ \phi = \pi'_b$. We give a proof:

Since (c, π_a, π_b) satisfy the universal property of the Cartesian product, we put $x := c'$ and it comes an element $\phi \in \mathcal{C}[c', c]$ such that

$$\pi_a \circ \phi = \pi'_a \quad \text{and} \quad \pi_b \circ \phi = \pi'_b$$

The same agrument applied to c' provides some $\phi' \in \mathcal{C}[c, c']$ such that

$$\pi'_a \circ \phi' = \pi_a \quad \text{and} \quad \pi'_b \circ \phi' = \pi_b$$

Hence we have

$$\pi_a \circ \phi \circ \phi' = \pi_a \quad \text{and} \quad \pi_b \circ \phi \circ \phi' = \pi_b$$

Still applying the universal property of the Cartesian product satisfied by (c, π_a, π_b), we put $x := c$ and obtains some $\xi \in \mathcal{C}[c, c]$ such that

$$\pi_a \circ \xi = \pi_a \quad \text{and} \quad \pi_b \circ \xi = \pi_b$$

Then $\xi = \mathsf{id}_c$ and $\xi = \phi \circ \phi'$ are two solutions to the preceding system of equations hence $\phi \circ \phi' = \mathsf{id}_c$ and one checks the same way that $\phi' \circ \phi = \mathsf{id}_{c'}$ which provides the expected isomorphism.

Conversely, if (c, π_a, π_b) fulfils the universal property of the Cartesian product and ϕ is an isomorphsm from c' to c, then $(c', \pi_a \circ \phi, \pi_b \circ \phi)$ also satisfies it and both are said to be isomorphic.

Provided one has proven the existence of some (c, π_a, π_b) satisfying the universal property of the Cartesian product, "the" Cartesian product of a and b is an implicitly reference to the isomorphism class of (c, π_a, π_b). Nevertheless, in most of the des categories, one can describe a representative denoted by $a \times b$ and for $f \in \mathcal{C}[x, a]$ and $g \in \mathcal{C}[x, b]$, we denote by $f \times g$ the unique element of $\mathcal{C}[x, a \times b]$ such that $\pi_a \circ (f \times g) = f$ and $\pi_b \circ (f \times g) = g$.

In particular, if $a \times b$ exists and if $a \cong a'$ and $b \cong b'$, then $a' \times b'$ also exists and $a \times b \cong a' \times b'$. The collection of all objects of $\mathcal{C}$ which are isomorphic to a given object x is called th isomorphism class of x. According to the preceding remarks, we define the Cartesian product of two isomorphism classes without any ambiguity.

<u>Exercice 1</u> : Prove that $a \times b$ exists if and only if $b \times a$ exists and that when it is the case, one has

$$a \times b \cong b \times a$$

For all non zero integers n, one defines the Cartesian product of n objects $a_1, \ldots, a_n$ by means of n-uples and we denote the i^{th} projection by π_i instead of π_{a_i} : the Cartesian product of $(a_1, \ldots, a_n)$ is an $(n+1)$-uple $(c, \pi_1, \ldots, \pi_n)$ where c is an object of $\mathcal{C}$, $\pi_i \in \mathcal{C}[c, a_i]$ and the following map is a bijection

$$\begin{array}{ccc} \mathcal{C}[x,c] & \longrightarrow & \mathcal{C}[x,a_1]\times\cdots\times\mathcal{C}[x,a_n] \\ h & \longmapsto & (\ \pi_1\circ h\ ,\cdots,\ \pi_n\circ h\ ) \end{array}$$

Exercice 2 : Given three objects a, b and c such that $a \times b$, $b \times c$ and $a\times b\times c$ exist, prove

$$(a\times b)\times c \ \cong\ a\times b\times c \ \cong\ a\times(b\times c)$$

In particular, the Cartesian product in Set defines a commutative and associative (up to isomorphism) binary operation. Furthermore the Cartesian product of a singleton and any other set is isomorphic to this set. This remark suggests that the singleton (strictly speaking the collection of all singletons) is the neutral element of the Cartesian product in Set. Hence, as a convention, one defines the Cartesian product of the 0-uple in Set as the singleton and choose $\{\emptyset\}$ for its representative. The universal property of the Cartesian product of the 0-uple in some category $\mathcal{C}$ asserts that for all objects x of $\mathcal{C}$, the following map is a bijection

$$\begin{array}{ccc} \mathcal{C}[x,c] & \longrightarrow & \{\emptyset\} \\ h & \longmapsto & \emptyset \end{array}$$

in other words for all objects x there is a unique morphism of $\mathcal{C}$ from x to c. Then c is called the **terminal** object of $\mathcal{C}$, once again, it is unique only up to isomorphism and we denote it by $\top$.

A category $\mathcal{C}$ is said to be **Cartesian** when all n-uples of objects of $\mathcal{C}$ have a Cartesian product. For example Set is Cartesian.

Exercice 3 : Suppose the category $\mathcal{C}$ admits a terminal objecty, prove that for all objects a of $\mathcal{C}$, the Cartesien produits $\top\times a$ and $a\times\top$ exist and satisfy

$$a \times \top \ \cong\ a \ \cong\ \top \times a$$

We come back to computer science and suppose we have associated each program $\overrightarrow{P}$ with an object $[\![\overrightarrow{P}]\!]$ of some fixed category $\mathcal{C}$. By definition $[\![\overrightarrow{P}]\!]$ is the model of $\overrightarrow{P}$. Also suppose we have a distinguished object X such that for all programs $\overrightarrow{P}$, the homset $\mathcal{C}\big[X,[\![\overrightarrow{P}]\!]\big]$ is (or at least contains) the set of all execution traces of the program. Then given $\overrightarrow{P}$ and $\overrightarrow{Q}$, if $[\![\overrightarrow{P}\,|\,\overrightarrow{Q}]\!] \cong [\![\overrightarrow{P}]\!]\times[\![\overrightarrow{Q}]\!]$, the universal property of Cartesian products asserts that each execution trace of the program $[\![\overrightarrow{P}\,|\,\overrightarrow{Q}]\!]$ is completely determined by two elements independently picked from the set of execution traces of $\overrightarrow{P}$ and from the one of $\overrightarrow{Q}$. When we meet the above-mentioned situation, we say that $\overrightarrow{P}$ and $\overrightarrow{Q}$ are executed **independently**. We extend this defintion to an n-uple of programs $\overrightarrow{P}^{(1)}, \dots, \overrightarrow{P}^{(n)}$ saying they are executed independently when

$$[\![\overrightarrow{P}^{(1)} \mid \dots \mid \overrightarrow{P}^{(n)}]\!] \cong [\![\overrightarrow{P}^{(1)}]\!] \times \dots \times [\![\overrightarrow{P}^{(n)}]\!]$$

1.1.3 The PV language syntax

The PV language has been introduced by *Edsger Wybe Dijkstra* as an example of a toy language allowing concurrent execution of sequential processes [Dij68]. The PV language offers only two instructions `P` and `V` as shortcuts for the dutch words "Prolaag" (short for "probeer te verlagen", literally "try to reduce") and "Verhogen" ("increase"). Let $\mathcal{S}$ be a set whose elements are called the **semaphores**. Each semaphore `s` is associated with an **arity** that is to say an integer $\alpha_{\mathtt{s}} \geq 2$. We suppose that for each integer $\alpha \geq 2$, there exist infinitely many semaphores whose arity is α. The only **instructions** are $\mathtt{P(}s\mathtt{)}$ and $\mathtt{V(}s\mathtt{)}$ where s is some semaphore. The **processes** of the langage are the finite sequences of instructions. When P is a process and j an integer less or equal to the length of P, we denote by $P(j)$ the j^{th} instruction of the process, in particular $P(1)$ is the first instruction. The syntaxic convention requires that the instructions of a process are separated by a dot, mostly in order to make them easier to read. For example we have the processes

```
P(a).V(a)
P(a).P(b).V(a).V(b)
```

Then a PV program is a finite sequence of processes separated by the operator `|` which should be read "run concurrently with". Thus

```
P(a).V(a) | P(a).V(a)
```

is an example of PV program made of two copies of a process while

```
P(a).P(b).V(a).V(b) | P(b).P(a).V(b).V(a)
```

is an example made of two distinct processes. Therefore a PV program can be seen as a matrix of instructons each line of which being a process.

1.1.4 A first set theoretic semantic of the PV language

A PV program can be thought of as a vector of processes denoted by $\overrightarrow{P}$ so $\overrightarrow{P}_i$ is the i^{th} process of the program and $\overrightarrow{P}_i(j)$ is the j^{th} instruction of the i^{th} process of the program.

If $\overrightarrow{P}$ is made of n processes and for each $i \in \{1, \dots, n\}$ we denote by l_i the number of instructions of the process $\overrightarrow{P}_i$ (indexed from 1 to l_i), then the expression $\overrightarrow{P}_i(j)$ makes sense only if $1 \leq i \leq n$ and $1 \leq j \leq l_i$ and we define [2]

$$\mathsf{dom}(\overrightarrow{P}) := \{0, \dots, l_1\} \times \dots \times \{0, \dots, l_n\}$$

[2] One has intentionally included 0 in the intervals of integers in the product that defines $\mathsf{dom}(\overrightarrow{P})$.

In analogy with the terminology of linear algebra, we call **canonical base** of $\mathbb{N}^n$ the set $\{e_1, \ldots, e_n\}$ of n-uples defined by

$$e_i(j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

The **occupancy function** of the process $\overrightarrow{P_i}$ is a mapping denoted by Φ_i whose domain of definition consists on the triples $(\mathtt{s}, \gamma, \tau)$ such that

1) $\mathtt{s}$ is a semaphore,

2) γ is a mapping from $\{0, \ldots, l\}$ (where $l \in \mathbb{N}$) to $\mathsf{dom}(\overrightarrow{P})$ such that

i) $\gamma(0) = \overrightarrow{0}$ that is to say the n-uple of zeroes

ii) for all $t < l$, the vector $\gamma(t+1) - \gamma(t)$ belongs to the canonical base and

3) $0 \leq \tau \leq l$.

The occupancy function is defined inductively as follows :
$\Phi_i(\mathtt{s}, \gamma, 0) = \mathtt{false}$ and for all integers $t > 0$, if $\gamma(t) - \gamma(t-1)$ is the i^{th} vector of the canonical base we define

$$\Phi_i(\mathtt{s}, \gamma, t) = \begin{cases} \mathtt{true} & \text{if } \overrightarrow{P_i}\big(\pi_i(\gamma(t))\big) = \mathtt{P(s)} \\ \mathtt{false} & \text{if } \overrightarrow{P_i}\big(\pi_i(\gamma(t))\big) = \mathtt{V(s)} \\ \Phi(\mathtt{s}, \gamma, t-1) & \text{otherwise} \end{cases}$$

Intuitively γ is a potential execution trace and the fact that $\gamma(t) - \gamma(t-1) = e_i$ means that on step t the i^{th} process of the progrm $\overrightarrow{P}$ executes an instruction. Hence the terms of the n-uple $\gamma(t)$ can be seen as the instruction pointers of the processes whose program $\overrightarrow{P}$ is made of, thus $\pi_i(\gamma(t))$ is the index of the next instruction that the process $\overrightarrow{P_i}$ has to execute. The boolean value $\Phi_i(\mathtt{s}, \gamma, t)$ is the answer to the question : "Is the semaphore $\mathtt{s}$ held by the process $\overrightarrow{P_i}$ on step t of the execution γ ?" The **ressource gauge** is the mapping Φ defined on the intersection of all domains of defintion $\mathsf{dom}(\Phi_i)$ by

$$\Phi(\mathtt{s}, \gamma, t) := \mathsf{Card}\big\{i \in \{1, \ldots, n\} \bigm| \Phi_i(\mathtt{s}, \gamma, t) = \mathtt{true}\big\}$$

The **execution traces** of this semantic are the mapping γ from $\{0, \ldots, l\}$ to $\mathsf{dom}(\overrightarrow{P})$ (where $l \in \mathbb{N}$) such that for all semaphore $\mathtt{s}$ and for all integer $t \in \{0, \ldots, l\}$, the triple $(\mathtt{s}, \gamma, t)$ lies in the domain of defintion of Φ and satisfies the following inequality.

$$\Phi(\mathtt{s}, \gamma, t) < \alpha_{\mathtt{s}}$$

Exercice 4 : Prove there exists an application Ψ defined over $\mathcal{S} \times \mathsf{dom}(\overrightarrow{P})$ such that for all triples $(\mathtt{s}, \gamma, t)$ taken from the domain of definition of Φ one has

$$\Phi(\mathtt{s}, \gamma, t) = \Psi(\mathtt{s}, \gamma(t))$$

We say the ressource gauge is memoryless (or forgetful but this terminology is widely used in category theory so we refrain from using it here). Then one defines the semantic of the program $\overrightarrow{P}$ as

$$[\![\overrightarrow{P}]\!] := \big\{\overrightarrow{x} \in \mathsf{dom}(\overrightarrow{P}) \bigm| \forall \mathtt{s} \in \mathcal{S} \;\; \Psi(\mathtt{s}, \overrightarrow{x}) < \alpha_{\mathtt{s}}\big\} \subseteq \mathbb{N}^n$$

By definition, we say that the programs $\overrightarrow{P}$ and $\overrightarrow{Q}$ run independently when

$$[\![\overrightarrow{P} \mid \overrightarrow{Q}]\!] = [\![\overrightarrow{P}]\!] \times [\![\overrightarrow{Q}]\!]$$

1.1.5 Another set theoretic semantic of the PV language

We provide another semantic to PV language based on sets. It let us characterize the fact that the processes of the program $P_1 \mid \ldots \mid P_n$ are executed independently by the fact that the semantic of the program $[\![P_1 \mid \ldots \mid P_n]\!]$ is the Cartesian product $[\![P_1]\!] \times \cdots \times [\![P_n]\!]$ of the semantics of each of its processes taken apart.

We denote the real positive half-line, that is to say $= [0, +\infty[$, by $\mathbb{R}_+$. For each process P, each semaphore $\mathtt{s}$ and each point $x \in \mathbb{R}_+$, we define

$$a_x := \max \left\{ k \in \mathbb{N} \;\middle|\; k \leq x \text{ and } P(k) = \mathtt{P(s)} \right\}$$

and

$$b_x := \min \left\{ k \in \mathbb{N} \;\middle|\; a_x \leq k \text{ and } P(k) = \mathtt{V(s)} \right\}$$

with the convention that $\max \emptyset = \min \emptyset = \infty$. Then we say the semaphore $\mathtt{s}$ is **occupied** or **held** by the process P at point x when $x \in [a_x, b_x[$. The **occupied/held part** (by the process P) of the semaphore $\mathtt{s}$ is defined as

$$B_{\mathtt{s}}(P) := \left\{ x \in \mathbb{R}_+ \;\middle|\; \mathtt{s} \text{ is held by } P \text{ at point } x \right\}$$

<u>Exercice 5</u> : Let $x, y \in \mathbb{R}_+$ prove the following facts.
1) if $a_x = a_y$, then $b_x = b_y$
2) if $a_x = \infty$ then $b_x = \infty$.
3) if $x \leq y$ and $a_y = \infty$, then $a_x = \infty$.

<u>Exercice 6</u> : Find $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ two finite sequences of elements of $\mathbb{N} \cup \{\infty\}$ such that

$$B_{\mathtt{s}}(P) = \bigcup_{n \in \mathbb{N}} [a_n, b_n[$$

Then we denote by $\chi^{\mathtt{s}}_P$ the indicator function of the set $B_{\mathtt{s}}(P)$ i.e.

$$\begin{array}{ccc} \mathbb{R}_+ & \longrightarrow & \{0,1\} \\ x & \longmapsto & \begin{cases} 1 & \text{if } x \in B_{\mathtt{s}}(P) \\ 0 & \text{otherwise} \end{cases} \end{array}$$

Each program $P_1 \mid \ldots \mid P_n$ is represented by a subset of $\mathbb{R}^n_+$ (the n-fold Cartesian product of $\mathbb{R}_+$). Intuitively, the number of processus which run concurrently is the "dimension" of the model. If $\overrightarrow{f} := (f_1, \ldots, f_n)$ is a n-uple of functions from $\mathbb{R}_+$ to $\mathbb{R}$ and $\overrightarrow{x} := (x_1, \ldots, x_n)$ is a n-uple of elements of $\mathbb{R}_+$, in other words a point of $\mathbb{R}^n_+$, we define

$$\overrightarrow{f} \cdot \overrightarrow{x} \;:=\; \sum_{i=1}^{n} f_i(x_i)$$

In particular, if $\overrightarrow{\chi}$ is the n-uple $(\chi^{\mathtt{s}}_{P_1}, \dots, \chi^{\mathtt{s}}_{P_n})$ of indicator fonctions of the sets $B_{\mathtt{s}}(P_1), \dots, B_{\mathtt{s}}(P_n)$, we define the **forbidden region** generated by the semaphore $\mathtt{s}$ in the model of the program $P_1 \,|\, ... \,|\, P_n$ as

$$F_{\mathtt{s}} := \left\{ \overrightarrow{x} \in \mathbb{R}^n_+ \;\middle|\; \overrightarrow{\chi} \cdot \overrightarrow{x} \geq \alpha \right\}$$

where α is the arity of the semaphore $\mathtt{s}$. Therefore the **forbidden area** of the program is

$$F := \bigcup_{\mathtt{s} \in \mathcal{S}} F_{\mathtt{s}}$$

The model of the program, denoted by $[\![P_1 \,|\, ... \,|\, P_n]\!]$, is then defined as the set theoretic complement (relatively to $\mathbb{R}^n_+$) of its forbidden area.

<u>Exercice 7</u> : Prove the model of a PV program with a single process is $\mathbb{R}_+$.

<u>Exercice 8</u> : Given a program $\overrightarrow{P}$ and a semaphore $\mathtt{s}$, prove for each $\overrightarrow{x}$ in the domain of defintion of the gauge Ψ (defined in the preceding section) one has $\Psi(\mathtt{s}, \overrightarrow{x}) = \overrightarrow{\chi} \cdot \overrightarrow{x}$.

Once again, two programs $P_1 \,|\, \dots \,|\, P_n$ and $Q_1 \,|\, \dots \,|\, Q_m$ are said to be run **independently** when

$$[\![(P_1 \,|\, ... \,|\, P_n) \,|\, (Q_1 \,|\, \dots \,|\, Q_m)]\!] = [\![P_1 \,|\, ... \,|\, P_n]\!] \times [\![Q_1 \,|\, ... \,|\, Q_m]\!]$$

This definition is obviously extended to the case where we have N programs $\overrightarrow{P_1}, \dots, \overrightarrow{P_N}$ which will be said to be run independently when

$$[\![\overrightarrow{P_1} \,|\, ... \,|\, \overrightarrow{P_N}]\!] = [\![\overrightarrow{P_1}]\!] \times \cdots \times [\![\overrightarrow{P_N}]\!]$$

The underlying idea is that the forbidden area portray the conflicts that might occur when several processes try to access a ressource which cannot satisfy all the requests. Rephrasing again, the forbidden area are holes in the model which represent the potential lack of ressources.

Let us see the case of the program `P(a).V(a) | P(a).V(a)` and write P for the process `P(a).V(a)`. For $x \in \mathbb{R}_+$, one has

x	a_x	b_x	$\chi^{\mathtt{s}}_P(x)$
$[0,1[$	∞	∞	0
$[1,2[$	1	2	1
$[2,3[$	1	2	0

The forbidden area of the program is then $[1,2[\times[1,2[$. In particular, the processes of this program, cannot be run independently.

<u>Exercice 9</u> : Find (draw some picture) the forbidden area of the program

```
P(a).P(b).V(a).V(b) | P(b).P(a).V(b).V(a)
```

Exercice 10 : Find three programs $\overrightarrow{P}_1$, $\overrightarrow{P}_2$ and $\overrightarrow{P}_3$ such that for each pair $\{i,j\} \subseteq \{1,2,3\}$ one has

$$[\![\overrightarrow{P}_i \mid \overrightarrow{P}_j]\!] = [\![\overrightarrow{P}_i]\!] \times [\![\overrightarrow{P}_j]\!]$$

although

$$[\![\overrightarrow{P}_1 \mid \overrightarrow{P}_2 \mid \overrightarrow{P}_3]\!] \neq [\![\overrightarrow{P}_1]\!] \times [\![\overrightarrow{P}_2]\!] \times [\![\overrightarrow{P}_3]\!]$$

Remark : In Probability theory, the independence of random variables is also characterized by means of Cartesian product. Indeed, a family $X_1, \ldots, X_n$ of random variables over a probability space (Ω, μ) is said to be independent when the law of probability of the random vector $\overrightarrow{X}$ is the product of the laws of probability of its components, that is to say when

$$\mu\big(\{\overrightarrow{X} \in U_1 \times \cdots \times U_n\}\big) = \mu(\{X_1 \in U_1\}) \cdots \mu(\{X_n \in U_n\})$$

for all n-uples $(U_1, \ldots, U_n)$ of measurable subsets of Ω. As an example, suppose a player simultaneously toss a coin and roll a dice then we have

$$\Omega := \{\mathtt{pile}, \mathtt{face}\} \times \{1,2,3,4,5,6\}$$

every subset of Ω is measurable and the probabilit that the event $\{(b,n)\}$ occurs is $\mu_{\text{pice}}(\{b\}) \cdot \mu_{\text{d}}(\{n\})$.
Actually the semantics we have given are not fully satisfactory. Indeed we would like the homset $\mathsf{Set}\big[[0,1], [\![\overrightarrow{P}]\!]\big]$ to contain only execution traces. Formally we would like for each of its elements γ, the restriction of γ to

$$\big\{t \in \mathbb{N} \bigm| \gamma(t) \in \mathsf{dom}(P)\big\}$$

be an execution trace in the sense of the preceding section. In order to achieve this, we look for models of PV programs in another category than Set.

Exercice 11 : Find a program $\overrightarrow{P}$ and an element of $\mathsf{Set}\big[[0,1], [\![\overrightarrow{P}]\!]\big]$ which is not an execution trace of $\overrightarrow{P}$.

1.2 Partially ordered spaces

The oldest article dedicated to partially ordered spaces (in the sequel we write pospace for short) I have found is due to *Samuel Eilenberg* [Eil41]. The pospaces have also widely been studied by *Leopoldo Nachbin* initially with a view toward functional analysis. Indeed one may find a series of articles [Nac48a, Nac48b, Nac48c] published in 1948 and then a book [Nac65] upon the subject. They contain in particular several results of pointset topology adapted to pospaces. From the computer science point of view, the works by *E.W. Dijkstra* [Dij68] and then by *Scott D. Carson* and *Paul F. Reynolds Jr* [CJ87] have shown that these mathematical objects provide very intuitive models to the PV language.

1.2.1 Topological Spaces

A topological space is a set X together with a collection $\mathcal{O}(X)$ of subsets of X which satisfy the following properties :

- $\emptyset$ and X belong to $\mathcal{O}(X)$
- if $(O_i)_{i\in I}$ is any family of elements of $\mathcal{O}(X)$, then

$$\bigcup_{i\in I} O_i \in \mathcal{O}(X)$$

- for all $n \in \mathbb{N}$, if $O_0, \ldots, O_n$ belong to $\mathcal{O}(\mathcal{X})$, then

$$\bigcap_{i=0}^{n} O_i \in \mathcal{O}(X)$$

The elements of $\mathcal{O}(X)$ are called the **open** subsets of X and the complement (in X) of an open subset of X is a **closed** subset of X. In the sequel, we write X both for the topological space and its underlying set. If X and Y are topological spaces, a **continuous** mapping from X to Y is an element f of $\mathsf{Set}[X, Y]$ such that for all open subset O of Y, the set

$$f^{-1}(O) := \big\{x \in X \;\big|\; f(x) \in O\big\}$$

is an open subset of X.

Exercice 12 : Check that the collection of subsets O of the real line $\mathbb{R}$ such that for all $x \in O$ there exists $\varepsilon > 0$ such that $]x - \varepsilon, x + \varepsilon[\subseteq O$, forms a topology over $\mathbb{R}$. It is called the **standard** topology of $\mathbb{R}$ and unless otherwise stated, it is always the one we refer to when we write $\mathbb{R}$.

Exercice 13 : Check that the collection of topological spaces form a Cartesian category, denoted by Top, whose morphisms are the continuous maps. Determine $\mathbb{R} \times \mathbb{R}$.

Exercice 14 : Let X be a topological space and A be a subset of X. Check that the collection $A \cap O$ where O runs through the collection of open subsets of X provides A with a structure of topological space. This structure is said to be **induced** by X over A. We also say that A is a subspace of X.

In particular, the unit segment $[0, 1]$ inherits from the standard topology of $\mathbb{R}$ and we define the **paths** over the topological space X as the elements of the homset

$$\mathsf{Top}\big[[0, 1], X\big]$$

1.2.2 The category of partially ordered spaces

A **partially ordered spaces**[3] is a topological spaces X equiped with a partial order relation $\sqsubseteq$ over its underlying set whose graph

$$\{(x,y) \in X \times X \mid x \sqsubseteq y\}$$

is a closed subset of $X \times X$. The generic notation for a pospace is $\overrightarrow{X}$.

A topological space X is said to be **separated** (in the sense of Hausdorff) when for all distinct points x and x' of X, there exist two open subsets O and O' such that $x \in O$, $x' \in O'$ and $O \cap O' = \emptyset$. A topological space is separated if and only if its diagonal $\{(x,x) \mid x \in X\}$ is a closed subset of $X \times X$.

Exercice 15 : Prove that the underlying topological space of a pospace is separated. Prove any separated space X equiped with the diagonal relation (i.e. $x \subseteq x'$ iff $x = x'$) over X is a pospace.

Exercice 16: Check that the real line $\mathbb{R}$ together with its standard topology and order forms a pospace that we denote by $\overrightarrow{\mathbb{R}}$.

Exercice 17 : Check that the collection of pospaces forms a Cartesian category, denoted by Po, whose morphisms are the increasing continuous maps, in other words a morphism from $\overrightarrow{X}$ to $\overrightarrow{Y}$ is an element of $\mathsf{Top}[X,Y] \cap \mathsf{Pos}[\sqsubseteq_X, \sqsubseteq_Y]$.

Exercice 18 : Given a pospace $\overrightarrow{X}$ and a subset A of the set X, prove that the topology and the order induced by $\overrightarrow{X}$ on A provide it with a structure of pospace.

In virtue of the preceding results, if $\overrightarrow{P}$ is a PV program then its model $[\![\overrightarrow{P}]\!]$ is a subset of $\mathbb{R}_+^n$ (for some $n \in \mathbb{N}$) hence it inherits a pospace structure induce by $\overrightarrow{\mathbb{R}}_+^n$. In particular the unit segment $[0,1]$ inherits a pospace structure from $\overrightarrow{\mathbb{R}}$: we denote it by $\overrightarrow{[0,1]}$. Then we define the **paths**[4] on a pospace $\overrightarrow{X}$ as the elements of

$$\mathsf{Po}\big[\overrightarrow{[0,1]}, \overrightarrow{X}\big]$$

that is to say the paths γ on the topological space X such that

$$t \leq t' \;\Rightarrow\; \gamma(t) \sqsubseteq_X \gamma(t')$$

Exercice 19 : Prove that a path γ on a pospace is constant (i.e. $\forall t, t' \in [0,1]\ \gamma(t) = \gamma(t')$) if and only if $\gamma(0) = \gamma(1)$.

Exercice 20 : Prove that an element θ of $\mathsf{Po}\big[\overrightarrow{[0,1]}, \overrightarrow{[0,1]}\big]$ is onto[5] if and only if $\gamma(0) = 0$ and $\gamma(1) = 1$.

[3] Also called *pospace* for short.
[4] Sometimes we write **directed path** or **dipath** to insist on the order preservation feature.
[5] i.e. $\forall y \in [0,1] \exists x \in [0,1], \theta(x) = y$ (surjective)

Theorem 1
Let $\overrightarrow{P}$ be a PV program, any path on $[\![\overrightarrow{P}]\!]$ induces an execution trace of $\overrightarrow{P}$ and conversely, any execution trace of $\overrightarrow{P}$ is induced by some path on $[\![\overrightarrow{P}]\!]$.

At the end of the preceding section, we have seen that the set $\mathsf{Set}\big[[0,1],[\![\overrightarrow{P}]\!]\big]$ contains many "parasites" in addition to all the execution traces of $\overrightarrow{P}$. By contrast, the Theorem 1 asserts that $\mathsf{Po}\big[\overrightarrow{[0,1]},[\![\overrightarrow{P}]\!]\big]$ contains exactly all the execution traces of $\overrightarrow{P}$. From this point of view one may say that no path on the model is pathological. The next result formalises this idea in a wider context. A path on $\overrightarrow{X}$ is in particular an element of $\mathsf{Set}\big[[0,1],[\![\overrightarrow{P}]\!]\big]$. With a bit of set theory we prove there exists a bijection (isomorphism in Set) between the sets $[0,1]$ and $[0,1]\times[0,1]$. Furthermore, the *Peano* curve provides a continuous map from $[0,1]$ *onto* $[0,1]\times[0,1]$. In terms of execution traces, such maps are pathological.
We define the **image** of an element f of $\mathsf{Po}[\overrightarrow{X},\overrightarrow{Y}]$ as the following subset of Y

$$f(X) := \{f(x) \mid x \in X\}$$

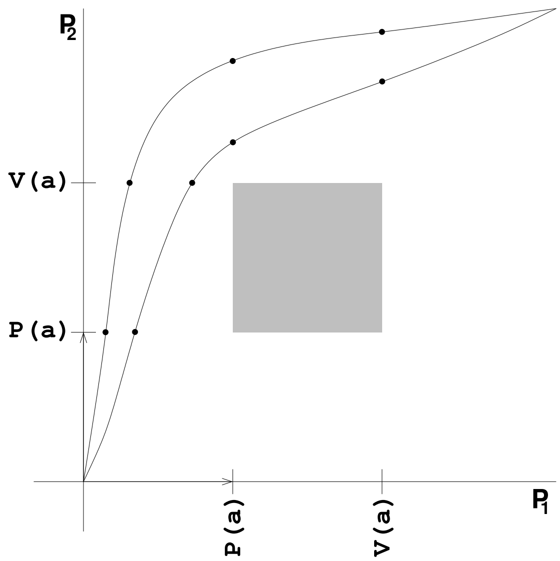
seen as a sub-pospace of $\overrightarrow{Y}$. If γ is a constant path, its image is reduced to a singleton, otherwise:

Theorem 2 The image of a non constant path on a pospace is isomorphic to $\overrightarrow{[0,1]}$.

By definition, any subset A of a pospace $\overrightarrow{X}$ which is isomorphic to $\overrightarrow{[0,1]}$ is the image of a path and one can easily check there actually exist uncountably many paths whose image is A. Indeed, for all elements θ of $\mathsf{Po}\big[\overrightarrow{[0,1]},\overrightarrow{[0,1]}\big]$ the paths γ and $\gamma\circ\theta$ share the same image as soon as θ in onto. The Theorem 2 thus allows to reduce the problem of classification of paths on a pospace to the classifiaction of its sub-pospaces isomorphic to $\overrightarrow{[0,1]}$. Nevertheless, given a program $\overrightarrow{P}$ and two points x and y of its model $[\![\overrightarrow{P}]\!]$, there still exists uncountably many paths γ on $[\![\overrightarrow{P}]\!]$ such that $\gamma(0) = x$, $\gamma(1) = y$, whose images are pairwise distinct though they all represent the same execution trace.

Let us go back to the program `P(a).V(a) | P(a).V(a)` then observe the distinct images of two paths sharing the same extremities and both representing the following execution trace

$$P_2 : \mathtt{P(a)} \;\to\; P_2 : \mathtt{V(a)} \;\to\; P_1 : \mathtt{P(a)} \;\to\; P_1 : \mathtt{V(a)}$$



The next part of the course describe an invariant (namely the fundamental category) which removes these redundancies at least when the extremities of the paths are fixed.

Chapter 2

The Fundamental Category

Samuel Eilenberg and *Saunders MacLane* laid the bases of category theory in order to, among other things, clarify the formalisation of algebraic topology. Indeed, notions like homotopy and homology induce functors from the category of topology spaces to the category of groups. The notion of fundamental category is related to the notion of homotopy and it induces a functor that we describe after we have given some complements of category theory.

2.1 Functors and Natural Transformations

The functors are the morphisms of categories. Provided we are cautious with the notion of collection, we can collect all the functors from a category $\mathcal{C}$ to a category $\mathcal{D}$ to form a category whose morphisms are the natural transformations[1].

A **functor** f from a category $\mathcal{C}$ to a category $\mathcal{D}$ is defined by two "mappings" denoted by $\mathsf{Ob}(f)$ and $\mathsf{Mo}(f)$ such that the law of composition of $\mathcal{C}$ is preserved and the following diagram commutes

$$\begin{tikzcd}
\mathsf{Mo}(\mathcal{C}) \arrow[r, shift left=2, "\mathsf{s}"] \arrow[r, shift right=2, "\mathsf{t}"'] \arrow[d, "\mathsf{Mo}(f)"'] & \mathsf{Ob}(\mathcal{C}) \arrow[l, "\mathsf{id}" description] \arrow[d, "\mathsf{Ob}(f)"] \\
\mathsf{Mo}(\mathcal{D}) \arrow[r, shift left=2, "\mathsf{s}'"] \arrow[r, shift right=2, "\mathsf{t}'"'] & \mathsf{Ob}(\mathcal{D}) \arrow[l, "\mathsf{id}'" description]
\end{tikzcd}$$

in other words for each object x and each morphism γ of $\mathcal{C}$:

the source of $\mathsf{Mo}(f)(\gamma)$ is $\mathsf{Ob}(f)(\mathsf{s}(\gamma))$

the target of $\mathsf{Mo}(f)(\gamma)$ is $\mathsf{Ob}(f)(\mathsf{t}(\gamma))$

the identity of $\mathsf{Ob}(f)(x)$ is $\mathsf{Mo}(f)(\mathsf{id}_x)$

[1] The relations satisfied by these structures give rise to the axioms of the notion of 2-category.

and for each morphism δ of $\mathcal{C}$ such that $\mathsf{s}(\gamma) = \mathsf{t}(\delta)$ the composite of $\mathsf{Mo}(f)(\delta)$ followed by $\mathsf{Mo}(f)(\gamma)$ is $\mathsf{Mo}(f)(\gamma \circ \delta)$.
In the sequel we will write f to denote $\mathsf{Ob}(f)$ as well as $\mathsf{Mo}(f)$. So we can summarize the preceding axioms as follows $\mathsf{s}'(f(\gamma)) = f(\mathsf{s}(\gamma)$, $\mathsf{t}'(f(\gamma)) = f(\mathsf{t}(\gamma)$, $\mathsf{id}'_{f(x)} = f(\mathsf{id}_x)$ and $f(\gamma \circ \delta) = f(\gamma) \circ' f(\delta)$ where $\circ$ and $\circ'$ denote the laws of composition of $\mathcal{C}$ and $\mathcal{D}$. We can also encompass all of them in the following diagram

$$\begin{tikzcd} x \arrow[r, "\delta"'] \arrow[rr, bend left, "\gamma\circ\delta"] & y \arrow[r, "\gamma"'] & z \end{tikzcd}$$

$$\begin{tikzcd} f(x) \arrow[r, "f(\delta)"'] \arrow[rr, bend left, "f(\gamma\circ\delta)"] & f(y) \arrow[r, "f(\gamma)"'] & f(z) \end{tikzcd}$$

Given two objects x and y of $\mathcal{C}$ the restriction of $\mathsf{Mo}(f)$ to $\mathcal{C}[x, y]$ induces a mapping to the set $\mathcal{D}[\mathsf{Ob}(f)(x), \mathsf{Ob}(f)(y)]$. When for all objects x and y of $\mathcal{C}$ this restriction is one-to-one (respectively onto), one says that the functor is **faithful** (respectively **full**). A full and faithful functor is said to be **fully faithful**. A faithful functor fidle such that for all objects x and y one has

$$\mathsf{Ob}(f)(x) = \mathsf{Ob}(f)(y) \;\Rightarrow\; x = y$$

is called an **embeding**. The following exercise is a "light" version of the (famous and ubiquitous) *Yoneda* lemma.

Exercice 21 : Prove that any small category can be embedded in Set.

When $\mathcal{C}$ is a sub-category of $\mathcal{D}$, the inclusions of $\mathsf{Mo}(\mathcal{C})$ and $\mathsf{Ob}(\mathcal{C})$ in $\mathsf{Mo}(\mathcal{D})$ and $\mathsf{Ob}(\mathcal{D})$ induce an **inclusion** functor of $\mathcal{C}$ in $\mathcal{D}$. When it is full, one says that $\mathcal{C}$ is a full sub-category of $\mathcal{D}$.

Exercice 22 : Check that the collection of small categories form, together with the functors, a locally small Cartesian category denoted by Cat. In particular check that the Cartesian product

$$\left(\left(\mathsf{Mo}(\mathcal{C}) \underset{\mathsf{t}}{\overset{\mathsf{s}}{\underset{\longleftarrow \mathsf{id}}{\rightrightarrows}}} \mathsf{Ob}(\mathcal{C}) \right), \circ \right) \times \left(\left(\mathsf{Mo}(\mathcal{D}) \underset{\mathsf{t}'}{\overset{\mathsf{s}'}{\underset{\longleftarrow \mathsf{id}'}{\rightrightarrows}}} \mathsf{Ob}(\mathcal{D}) \right), \circ' \right)$$

is equal to

$$\left(\left(\mathsf{Mo}(\mathcal{C}) \times \mathsf{Mo}(\mathcal{D}) \underset{\mathsf{t}\times\mathsf{t}'}{\overset{\mathsf{s}\times\mathsf{s}'}{\underset{\longleftarrow \mathsf{id}\times\mathsf{id}'}{\rightrightarrows}}} \mathsf{Ob}(\mathcal{C}) \times \mathsf{Ob}(\mathcal{D}) \right), \circ \times \circ' \right)$$

A functor f from C to D is said to **preserve Cartesian products** when for all ordered pairs (x, x') of objects of C, if $x \times x'$ exists in C, then $f(x) \times f(x')$

exists in D and it is isomorphic to $f(x \times x')$.

Exercice 23 : Check that the collection of monoids together with their morphisms form a locally small Cartesian category denoted by Mon. Prove that one can define a monoid as a small category with a single object and describe an embeding of Mon in Cat which preserves the Cartesian product.

Exercice 24 : Check that the collection of groups together with their morphisms form a locally small Cartesian category denoted by Gr. Describe an embeding of Gr in Mon which preserves the Cartesian product.

Exercice 25 : Check that the collection of Abelian groups together with their morphisms form a locally small Cartesian category denoted by Ab and describe an embeding of Ab in Gr which preserves the Cartesian product.

Exercice 26 : Check that the collection of partially ordered sets (posets for short) together with their morphisms form a locally small Cartesian category denoted by Pos. Given a poset $(X, \sqsubseteq)$, prove that

$$\big\{(x, y) \in X \times X \;\big|\; x \sqsubseteq y\big\}$$

can be taken as the set of morphisms of some category such that the source and the target of (x, y) are x and y. Describe an embeding of Pos in Cat which preserves the Cartesian product.

Given two functors f and g from $\mathcal{C}$ to $\mathcal{D}$, a **natural transformation** from f to g is a collection $(\eta_x)_{x \in \mathsf{Ob}(\mathcal{C})}$ such that $\eta_x \in \mathcal{D}[fx, gx]$ and for all morphisms α of $\mathcal{C}[x, y]$ we have the following commutative diagram:

$$\begin{tikzcd}
fx \arrow[r, "f\alpha"] \arrow[d, dotted, "\eta_x"'] & fy \arrow[d, dotted, "\eta_y"] \\
gx \arrow[r, "g\alpha"'] & gy
\end{tikzcd}$$

in other words $g\alpha \circ \eta_x = \eta_y \circ f\alpha$. This situation is depicted by the following diagram.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D}
\end{tikzcd}$$

Given two natural transformations η and θ repsectively from f to g and from g to h, where f, g and h are functors from $\mathcal{C}$ to $\mathcal{D}$, we define the composite $\theta \circ \eta$ by $(\theta_x \circ \eta_x)_{x \in Ob(\mathcal{C})}$.

Diagramatically we have

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "f"{name=F}] \arrow[r, "g"{name=G, description}] \arrow[r, bend right=60, "h"'{name=H}] & \mathcal{D}
\arrow[from=F, to=G, Rightarrow, "\eta"]
\arrow[from=G, to=H, Rightarrow, "\theta"]
\end{tikzcd}
\quad \text{whence} \quad
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "f"{name=F}] \arrow[r, bend right=60, "h"'{name=H}] & \mathcal{D}
\arrow[from=F, to=H, Rightarrow, "\theta\circ\eta"]
\end{tikzcd}
$$

The composition is defined componentwise so the associativity derives from the associativity of the composition law of $\mathcal{B}$. Moreover, the collection $(\mathsf{id}_{fx})_{x\in\mathsf{Ob}(\mathcal{C})}$ is a natural transformation from f to f which is denoted by id_f. The notation is made sound by the following equality.

$$\eta \circ \mathsf{id}_f = \eta = \mathsf{id}_g \circ \eta$$

<u>Exercice 27</u> : Check that the collection of functors from $\mathcal{C}$ to $\mathcal{D}$ forms a category $\mathsf{Fun}[\mathcal{C},\mathcal{D}]$ whose morphisms are the natural transformations. In particular, if $\mathcal{C}$ and $\mathcal{D}$ are small categories, the set of objects of $\mathsf{Fun}[\mathcal{C},\mathcal{D}]$, which is also denoted by $\mathcal{D}^{\mathcal{C}}$, is the homset $\mathsf{Cat}[\mathcal{C},\mathcal{D}]$.

Actually, the functors whose target is $\mathcal{C}$ and those whose target is $\mathcal{D}$ act on the right and on the left over the morphisms of $\mathsf{Fun}[\mathcal{C},\mathcal{D}]$. Formally, given two functors f and g from $\mathcal{C}$ to $\mathcal{D}$, a natural transformation η from f to g, a functor h from $\mathcal{C}'$ to $\mathcal{C}$ and a functor k from $\mathcal{D}$ to $\mathcal{D}'$, the collection $(\eta_{hx'})_{x'\in Ob(\mathcal{C}')}$ is a natural transformation from $f\circ h$ to $g\circ h$ while $(k\eta_x)_{x\in Ob(\mathcal{C})}$ is a natural transformation from $k\circ f$ to $k\circ g$. These natural transformations are respectively denoted by $\eta\cdot h$ and $k\cdot\eta$. We have the following diagrams

$$
\begin{tikzcd}
\mathcal{C}' \arrow[r, "h"] & \mathcal{C} \arrow[r, bend left=50, "f"{name=F}] \arrow[r, bend right=50, "g"'{name=G}] & \mathcal{D} \arrow[r, "k"] & \mathcal{D}'
\arrow[from=F, to=G, Rightarrow, "\eta"]
\end{tikzcd}
$$

whence

$$
\begin{tikzcd}
\mathcal{C}' \arrow[r, bend left=50, "f\circ h"{name=F}] \arrow[r, bend right=50, "g\circ h"'{name=G}] & \mathcal{D}
\arrow[from=F, to=G, Rightarrow, "\eta\cdot h"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "k\circ f"{name=F}] \arrow[r, bend right=50, "k\circ g"'{name=G}] & \mathcal{D}'
\arrow[from=F, to=G, Rightarrow, "k\cdot\eta"]
\end{tikzcd}
$$

Besides, there are rules of associativity and distributivity involving the composition of functors denoted by $\circ$, the one of natural transformations also denoted by $\circ$ and the operator $\cdot$ previously defined.

$$\left.\begin{array}{l} h'\cdot(h\cdot\eta) = (h'\circ h)\cdot\eta \\ (\eta\cdot k)\cdot k' = \eta\cdot(k\circ k') \\ (h\cdot\eta)\cdot k = h\cdot(\eta\cdot k) \end{array}\right\} \text{heterogeneous associativity}$$

$$\left.\begin{array}{l} h\cdot(\eta'\circ\eta) = (h\cdot\eta')\circ(h\cdot\eta) \\ (\eta'\circ\eta)\cdot k = (\eta'\cdot k)\circ(\eta\cdot k) \end{array}\right\} \text{distributivity of } \cdot \text{ over } \circ$$

The operator $\cdot$ satisfies

$$\mathsf{id}_{\mathcal{D}} \cdot \eta = \eta = \eta \cdot \mathsf{id}_{\mathcal{C}}$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathsf{id}_{\mathcal{C}}"] & \mathcal{C} \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D} \arrow[r, "\mathsf{id}_{\mathcal{D}}"] & \mathcal{D} \end{tikzcd}$$

and

$$\mathsf{id}_f \cdot h = \mathsf{id}_{f \circ h} \text{ and } k \cdot \mathsf{id}_f = \mathsf{id}_{k \circ f}$$

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "h"] & \mathcal{C} \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow\mathsf{id}_f"] & \mathcal{D} \arrow[r, "k"] & \mathcal{D}' \end{tikzcd}$$

It worth to notice that $h \cdot (\eta' \circ \eta)$ and $(h \cdot \eta') \circ \eta$, as well as $(\eta' \circ \eta) \cdot k$ and $\eta' \circ (\eta \cdot k)$, may not be simultaneously defined.

We are about to define another operator $*$ over the natural transformations. Let f and g be functors from $\mathcal{A}$ to $\mathcal{B}$, f' and g' be functors from $\mathcal{B}$ to $\mathcal{C}$ and η and η' by natural transformations respectively from f and f' to g and g' i.e.

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{B} \arrow[r, bend left, "f'"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\eta'"] & \mathcal{C} \end{tikzcd} .$$

Thus we have

$$\begin{tikzcd} f' \circ f \arrow[r, Rightarrow, "f' \cdot \eta"] & f' \circ g \\ g' \circ f \arrow[r, Rightarrow, "g' \cdot \eta"'] & g' \circ g \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} f' \circ f \arrow[d, Rightarrow, "\eta' \cdot f"'] & f' \circ g \arrow[d, Rightarrow, "\eta' \cdot g"] \\ g' \circ f & g' \circ g \end{tikzcd}$$

<u>Exercice 28</u> : Prove that we have

$$\big(\eta' \cdot g\big) \circ \big(f' \cdot \eta\big) = \big(g' \cdot \eta\big) \circ \big(\eta' \cdot f\big)$$

$$\begin{tikzcd} f' \circ f \arrow[r, Rightarrow, "f' \cdot \eta"] \arrow[d, Rightarrow, "\eta' \cdot f"'] \arrow[dr, Rightarrow, "\eta' * \eta"] & f' \circ g \arrow[d, Rightarrow, "\eta' \cdot g"] \\ g' \circ f \arrow[r, Rightarrow, "g' \cdot \eta"'] & g' \circ g \end{tikzcd}$$

Then we define $\eta' * \eta$ as the aforementioned natural transformation from $f' \circ f$ to $g' \circ g$. The operator $*$ is called the ***Godement* product** or **juxtaposition**. We have some additional rules

$$(\eta'' * \eta') * \eta = \eta'' * (\eta' * \eta) \ \Big\} \text{ associativit de } *$$

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{B} \arrow[r, bend left=50, "f'"] \arrow[r, bend right=50, "g'"'] \arrow[r, phantom, "\Downarrow\eta'"] & \mathcal{C} \arrow[r, bend left=50, "f''"] \arrow[r, bend right=50, "g''"'] \arrow[r, phantom, "\Downarrow\eta''"] & \mathcal{D}
\end{tikzcd}$$

$$\left.\begin{array}{l}(\eta' * \eta) \cdot h = \eta' * (\eta \cdot h) \\ k \cdot (\eta' * \eta) = (k \cdot \eta') * \eta\end{array}\right\} \text{heterogeneous associativity of } \cdot \text{ and } *$$

$$\begin{tikzcd}
\mathcal{A}' \arrow[r, "h"] & \mathcal{A} \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{B} \arrow[r, bend left=50, "f'"] \arrow[r, bend right=50, "g'"'] \arrow[r, phantom, "\Downarrow\eta'"] & \mathcal{C} \arrow[r, "k"] & \mathcal{C}'
\end{tikzcd}$$

$$\big(\theta' \circ \eta'\big) * \big(\theta \circ \eta\big) = \big(\theta' * \theta\big) \circ \big(\eta' * \eta\big) \;\; \Big\} \;\; \textbf{exchange}$$

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\eta"] \arrow[r, phantom, shift right=3ex, "\Downarrow\theta"] & \mathcal{B} \arrow[r, bend left=60, "f'"] \arrow[r, "g'" description] \arrow[r, bend right=60, "h'"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\eta'"] \arrow[r, phantom, shift right=3ex, "\Downarrow\theta'"] & \mathcal{C}
\end{tikzcd}$$

$$\left.\begin{array}{l}\mathsf{id}_k * \eta = k \cdot \eta \\ \eta * \mathsf{id}_h = \eta \cdot h\end{array}\right\} \text{behaviour with respect to identities}$$

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left=50, "h"] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, "\Downarrow\mathsf{id}_h"] & \mathcal{B} \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{C} \arrow[r, bend left=50, "k"] \arrow[r, bend right=50, "k"'] \arrow[r, phantom, "\Downarrow\mathsf{id}_k"] & \mathcal{D}
\end{tikzcd},$$

The rules involving $\circ$, $*$ and $\cdot$ are called the *Godement*'s rules. The operator $*$ can be seen as an extension of the operator $\cdot$.

<u>Exercice 29</u> : Given a category $\mathcal{C}$ and f an **endofunctor** of $\mathcal{C}$ (i.e. a functor from $\mathcal{C}$ to $\mathcal{C}$). Let δ and μ be natural transformations from f^n to f^{n+1} and from f^{n+1} to f^n where $n \in \mathbb{N}$. By calculating $\mu * \mu$ and $\delta * \delta$, prove that

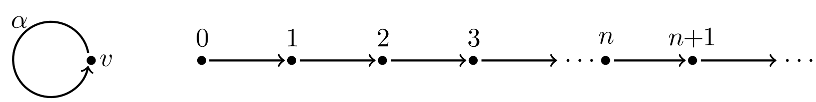
$$(\mu \cdot f^n) \circ (f^{n+1} \cdot \mu) = (f^n \cdot \mu) \circ (\mu \cdot f^{n+1}) \text{ et } (f^{n+1} \cdot \delta) \circ (\delta \cdot f^n) = (\delta \cdot f^{n+1}) \circ (f^n \cdot \delta)$$

2.2 The category of Graphs

A **graph**[2] is defined by two mappings as follow. The elements of A are called the **arrows** while the elements of V are called the **vertices**.

$$A \underset{\mathsf{t}}{\overset{\mathsf{s}}{\rightrightarrows}} V$$

For each arrow α, we write $\mathsf{s}(\alpha)$ and $\mathsf{t}(\alpha)$ to denote the **source** and the **target** of α. For each integer $n \in \mathbb{N}$, the graph **segment** of **length** n, denoted by I_n, is the graph whose sets of vertices and arrows are respectively $\{0, \ldots, n\}$ and $\{(0,1), \ldots, (n-1, n)\}$. For $k \in \{1, \ldots, n\}$, the source and the target of $(k-1, k)$ are respectively $k-1$ and k. By extension, **segment** graph of **infinite length** is the one whose set of vertices is $\mathbb{N}$ and whose arrows are the ordered pairs $(n, n+1)$ for $n \in \mathbb{N}$, it is denoted by I_∞.



A morphism of graphs is an ordered pair of mappings (ϕ_0, ϕ_1) such that the following diagram commutes

$$\begin{tikzcd}
A \arrow[r, shift left, "\mathsf{s}"] \arrow[r, shift right, "\mathsf{t}"'] \arrow[d, "\phi_1"'] & V \arrow[d, "\phi_0"] \\
A' \arrow[r, shift left, "\mathsf{s}'"] \arrow[r, shift right, "\mathsf{t}'"'] & V'
\end{tikzcd}$$

$$\text{i.e.}\quad \mathsf{s}'(\phi_1(\alpha)) = \phi_0(\mathsf{s}(\alpha)) \quad \text{and} \quad \mathsf{t}'(\phi_1(\alpha)) = \phi_0(\mathsf{t}(\alpha))$$

<u>Exercice 30</u> : Check there exists a unique graph with a single vertex and a single arrow. Denoting it by $\mathcal{Q}_1$ prove that for all graph $\mathcal{Q}$ there exists a unique morphism of graph from $\mathcal{Q}$ to $\mathcal{Q}_1$.

<u>Exercice 31</u> : Check that the graphs and their morphisms form a locally small Cartesian category denoted by Gph. In particular, check the following equality

$$\left(A \underset{\mathsf{t}}{\overset{\mathsf{s}}{\rightrightarrows}} V\right) \times \left(A' \underset{\mathsf{t}'}{\overset{\mathsf{s}'}{\rightrightarrows}} V'\right) \cong \left(A \times A' \underset{\mathsf{t}\times\mathsf{t}'}{\overset{\mathsf{s}\times\mathsf{s}'}{\rightrightarrows}} V \times V'\right)$$

Though the category Gph is Cartesian, the product does not behave as expected. Indeed, the product $I_1 \times I_1$ is the left hand side graph while we would expect the right one.

[2] We can also define a "graph" as a 1-dimensional presimplicial set.

$$
\begin{tikzcd}
\bullet & \bullet \\
\bullet \arrow[ur] & \bullet
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bullet \arrow[r] & \bullet \\
\bullet \arrow[u] \arrow[r] & \bullet \arrow[u]
\end{tikzcd}
$$

From the concurrency point of view, we would like to consider I_1 as the atomic step from a state to the next one through the execution of an instruction along the unique arrow of the graph. The parallel execution of two instructions is thus represented by the right hand side graph : independance is formalized by the fact that we can execution each of the instructions before the other. This is the **interleaving** semantic. This approach is implemented by the classical notion of **systems of transitions** [Win95].

<u>Exercice 32</u> : Given a small category $\mathcal{C}$, check that the sets $\mathsf{Ob}(\mathcal{C})$, $\mathsf{Mo}(\mathcal{C})$ and the mappings source and target form a graph which is called the **underlying** graph of $\mathcal{C}$.

$$
\left(\left(\begin{tikzcd} \mathsf{Mo}(\mathcal{C}) \arrow[r, shift left=2, "\mathsf{s}"] \arrow[r, shift right=2, "\mathsf{t}"'] & \mathsf{Ob}(\mathcal{C}) \arrow[l, "\mathsf{id}" description] \end{tikzcd} \right), \circ \right) \xmapsto{\mathsf{Oubli}} \left(\begin{tikzcd} \mathsf{Mo}(\mathcal{C}) \arrow[r, shift left, "\mathsf{s}"] \arrow[r, shift right, "\mathsf{t}"'] & \mathsf{Ob}(\mathcal{C}) \end{tikzcd} \right)
$$

Moreover, if f is a functor from $\mathcal{C}$ to another small category $\mathcal{D}$, prove that the ordered pair $(\mathsf{Ob}(f), \mathsf{Mo}(f))$ is a morphism from the underlying graph of $\mathcal{C}$ to the underlying graph of $\mathcal{D}$. Check we have defined a functor U from Cat to Gph which is said to be **forgetful**. Prove that the graphs $U(\mathcal{C}) \times U(\mathcal{D})$ and $U(\mathcal{C} \times \mathcal{D})$ are isomorphic. Consider the poset $\{0 < 1\}$ as a category, then compare $U(\{0 < 1\})$ and I_1.

We have seen, through the forgetful functor, that we can think of graphs as small categories from which the composition and the identities have been removed. The ill-behaviour [3] of the Cartesian product in Gph is due to the lack of "identities". Indeed, if we consider the poset $\{0 < 1\}$ as a small category, then the underlying graph of $\{0 < 1\} \times \{0 < 1\}$ is

$$
\begin{tikzcd}
\bullet \arrow[loop left] \arrow[r] & \bullet \arrow[loop right] \\
\bullet \arrow[loop left] \arrow[u] \arrow[r] \arrow[ur] & \bullet \arrow[loop right] \arrow[u]
\end{tikzcd}
$$

where the "circular" arrows represent the identities while the "diagonal" one represents the unique morphism of the homset it belongs to. From the concurrency point of view, this arrow is the simultaneous execution of both instructions. The Cartesian product in Cat thus better fits to the study of concurrency. Actually, though the definition of a transitions system lays on the notion of graph, the identities implicitly appear via the fact that "partial" functions are allowed as morphisms. The other way consists on explicitly adding *idle transitions* which

[3] From the concurrency point of view.

play the role of identities [Win95]. Our approach is based on the notion of category.

By definition the elements of $\mathsf{Gph}[I_n, \mathcal{Q}]$ are the **paths** of **lentgh** n on the graph $\mathcal{Q}$. Then, a path γ is defined by two mappings (ϕ_0, ϕ_1), however we choose the following notation $\gamma_k := \phi_0(k)$ for all $k \in \{0, \ldots, n\}$ and $\gamma_{k-1}^k := \phi_1(k-1, k)$ for all $k \in \{1, \ldots, n\}$. In particular we have $\mathsf{s}(\gamma_{k-1}^k) = \gamma_{k-1}$ and $\mathsf{t}(\gamma_{k-1}^k) = \gamma_k$. Thus we define the **source** and the **target** of γ as $\mathsf{s}(\gamma) := \gamma_0$ and $\mathsf{t}(\gamma) := \gamma_n$. Then given a path δ of length $m \in \mathbb{N}$ over $\mathcal{Q}$ such that $\mathsf{s}(\gamma) = \mathsf{t}(\delta)$ we can define the **concatenation** of δ followed by γ, denoted by $\gamma \cdot \delta$, as below

$$(\gamma \cdot \delta)_k = \begin{cases} \delta_k & \text{if } k \in \{0, \ldots, m\} \\ \gamma_{k-m} & \text{if } k \in \{m, \ldots, m+n\} \end{cases}$$

and

$$(\gamma \cdot \delta)_{k-1}^k = \begin{cases} \delta_{k-1}^k & \text{if } k \in \{1, \ldots, m\} \\ \\ \gamma_{k-m-1}^{k-m} & \text{if } k \in \{m+1, \ldots, m+n\} \end{cases}$$

The concatenation law formalizes the idea that one runs along the path γ once δ has been run. Thus $\gamma \cdot \delta$ is a path of length $n+m$ from $\mathsf{s}(\delta) = \delta_0$ to $\mathsf{t}(\gamma) = \gamma_n$ over $\mathcal{Q}$. Besides, the paths of null length let the other paths unchanged under concatenation, that is to say if α and β are two such paths such that $\alpha_0 = \mathsf{s}(\gamma)$ and $\beta_0 = \mathsf{t}(\gamma)$, then $\beta \cdot \gamma = \gamma = \gamma \cdot \alpha$. It follows that the vertices of the graph $\mathcal{Q}$ are the objects of a small category whose morphisms are the paths on $\mathcal{Q}$, the composition law being given by the concatenation while the identities are the paths of null length. This category is denoted by $F(\mathcal{Q})$ and called the **free category** generated by $\mathcal{Q}$. In addition, a morphism of graph f from $\mathcal{Q}$ to $\mathcal{Q}'$ induces a functor of $F(f)$ from $F(\mathcal{Q})$ to $F(\mathcal{Q}')$ since for all paths γ on $\mathcal{Q}$, the composite (in Gph) $f \circ \gamma$ is a path over $\mathcal{Q}'$ and that we have $f \circ (\gamma \cdot \delta) = (f \circ \gamma) \cdot (f \circ \delta)$. We have thus defnied the **free functor** F from Gph to Cat.

Exercice 33 : Determine the category freely generated by the graph I_∞ as well as the category freely generated by the graph with a single vertex and a single arrow.

Exercice 34 : The poset $(\mathbb{R}, \leq)$ is seen as a small category denoted by $\mathcal{R}$, given two real numbers x and y such that $x < y$, compare the homsets $\mathcal{R}[x, y]$ and $\big(F \circ U(\mathcal{R})\big)[x, y]$.

In general, when u and f are two functors between the categories $\mathcal{C}$ and $\mathcal{D}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "u"] & \mathcal{D} \arrow[l, shift left, "f"] \end{tikzcd}$$

we say that f is **left adjoint** to u (or equivalently that u is **right adjoint** to f) when there exists a natural transformation η from $\mathsf{id}_{\mathcal{D}}$ to $u \circ f$ and a natural transformation ε from $f \circ u$ to $\mathsf{id}_{\mathcal{C}}$ such that $(u \cdot \varepsilon) \circ (\eta \cdot u) = \mathsf{id}_u$ et $(\varepsilon \cdot f) \circ (f \cdot \eta) = \mathsf{id}_f$. These facts are summarized by the notation $f \dashv u$.

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "u"'] \arrow[rr, bend left=50, "\mathsf{id}_{\mathcal{C}}"] \arrow[rr, phantom, shift left=3ex, "\Uparrow\varepsilon"] & \mathcal{D} \arrow[r, "f"] \arrow[rr, bend right=50, "\mathsf{id}_{\mathcal{D}}"'] \arrow[rr, phantom, shift right=3ex, "\Uparrow\eta"] & \mathcal{C} \arrow[r, "u"] & \mathcal{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{D} \arrow[r, "f"'] \arrow[rr, bend left=50, "\mathsf{id}_{\mathcal{D}}"] \arrow[rr, phantom, shift left=3ex, "\Downarrow\eta"] & \mathcal{C} \arrow[r, "u"] \arrow[rr, bend right=50, "\mathsf{id}_{\mathcal{C}}"'] \arrow[rr, phantom, shift right=3ex, "\Downarrow\varepsilon"] & \mathcal{D} \arrow[r, "f"] & \mathcal{C}
\end{tikzcd}
$$

The composite $u' \circ u$ of two functors u and u' which have f and f' as left (respectively right) adjoint functors, admits $f \circ f'$ as left (respectively right) adjoint functor.

Up to isomorphism in the category $\mathsf{Fun}[\mathsf{D}, \mathsf{C}]$, a functor from C to D admits at most one left (respectively right) adjoint. In other words, if F and F' are left (respectively right) adjoint to U, then there exist two natural transformations from F to F' and from F' to F whose composite are id_F and $\mathsf{id}_{F'}$.

<u>Exercice 35</u>: In the case where f if left adjoint to u and with the preceding notations prove that teh following mappings

$$
\begin{array}{ccc}
\mathcal{C}[f(d), c] & \longrightarrow & \mathcal{D}[d, u(c)] \\
\gamma & \longmapsto & u(\gamma) \circ \eta_d
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathcal{D}[d, u(c)] & \longrightarrow & \mathcal{C}[f(d), c] \\
\delta & \longmapsto & \varepsilon_c \circ f(\delta)
\end{array}
$$

are inverse of each other.

The terminology "adjoint functor" can be understood by a formal analogy with the notion of adjoint matrix (in linear algebra) : given a square matrix M of size n with entires in $\mathbb{R}$ (for example), the adjoint matrix $M^{\perp}$ is defined as the unique one which satisfies

$$< Mx | y > = < x | M^{\perp} y >$$

for all vectors x and y of $\mathbb{R}^n$ where $< \_\_ | \_\_ >$ denotes the scalar product of $\mathbb{R}^n$. The analogy then comes from the preceding bijection by considering as the scalar product the mapping which sends an ordered pair (c, c') of objects of C to the homset $\mathsf{C}[c, c']$. Precisely the sets, small categories, functors, objects and homsets play the roles of scalars, vector spaces, linear mappings, vectors and scalar products.

The exercises aim at proving that the free functor from Gph to Cat is the left adjoint of the forgetful functor U from Cat to Gph. Depending on the context, the symbol $\circ$ both denote the composition law $\mathcal{C}$, Cat, Gph as well as the composition between functors and natural transformations.

Exercice 36 : Given a small category $\mathcal{C}$, each path γ of length n over the underlying graph $U(\mathcal{C})$ is associated with the following morphism of $\mathcal{C}$

$$\mathsf{id}_{\gamma_n} \circ \gamma_{n-1}^{n} \circ \cdots \circ \gamma_0^1 \circ \mathsf{id}_{\gamma_0}$$

Prove that we thus define a functor $\varepsilon_{\mathcal{C}}$ from $F(U(\mathcal{C}))$ to $\mathcal{C}$ and that the collection $(\varepsilon_{\mathcal{C}})_{\mathcal{C}\in\mathsf{Cat}}$ is a natural transformation from $F \circ U$ to $\mathsf{id}_{\mathsf{Cat}}$.

Exercice 37 : Given a graph $\mathcal{Q}$, each arrow α of $\mathcal{Q}$ is associated with γ, a path de longueur 1 on $\mathcal{Q}$ defined by $\gamma_0^1 = \alpha$. Prove that we thus define a morphism of graphs $\eta_{\mathcal{Q}}$ from $\mathcal{Q}$ to $U(F(\mathcal{Q}))$ and that the collection $(\eta_{\mathcal{Q}})_{\mathcal{Q}\in\mathsf{Gph}}$ is a natural transformation from $\mathsf{id}_{\mathsf{Gph}}$ to $U \circ F$.

Exercice 38 : Prove the following equalities $(U\cdot\varepsilon)\circ(\eta\cdot U) = \mathsf{id}_U$ et $(\varepsilon\cdot F)\circ(F\cdot\eta) = \mathsf{id}_F$. Then find two bijections, inverse of each other, between the homsets $\mathsf{Gph}[\mathcal{Q}, U(\mathcal{C})]$ and $\mathsf{Cat}[F(\mathcal{Q}), \mathcal{C}]$.

The following exercises provide several examples of forgetful functors. We recall that the underlying topological space of a partially ordered space is separated.

Exercice 39 : Denote by Hsd for the full sub-category of Top whose objects are the separated topological spaces. In virtue of the exercice 15, we define the forgetful functor U from Po to Hsd associating a pospace with its underlying topological space and each morphism of pospaces with its underlying continuous map. We also define a functor F from Hsd to Po associating each separated topological space with to pospace obtained by equipping X with the diagonal relation (i.e. $x \sqsubseteq x'$ if and only if $x = x'$). Prove that F is left adjoint to U.

Exercice 40 : Associate each set X with the monoid $F(X)$ the set of finite sequences of elements of X, the composition is the concatenation and the empty sequence is neutral[4]. Given a mapping f from the set X to the set Y, check that the mapping $F(f)$ with sends each sequence $(x_n, \ldots, x_1)$ to the sequence $(f(x_n), \ldots, f(x_1))$ is a morphism of monoids[5]. Prove that the functor F admits a right adjoint.

The following construction is the first step to the definition of the fundamental category of a pospace.

Let $\overrightarrow{X}$ be a pospace, the arrows of the **graph of paths** on $\overrightarrow{X}$, denoted by $Q(\overrightarrow{X})$, are the paths on $\overrightarrow{X}$. The set of vertices of $Q(\overrightarrow{X})$ is X, thus the source and the target of a path γ on $\overrightarrow{X}$ are respectively $\gamma(0)$ and $\gamma(1)$. Besides, if f is a morphism of pospaces from $\overrightarrow{X}$ to $\overrightarrow{Y}$, then $f \circ \gamma$ is a path on $\overrightarrow{Y}$ whose source

[4] This monoid is also called the monoid of words over the alphabet X.

[5] In order to stick to the standard convention in category theory, the concatenation of the word w followed by the word w' is denoted by $w'w$, in other words the words are read from right to left.

and but are respectively $f(\gamma(0))$ and $f(\gamma(1))$. So we have defined a functor Q from Po to Gph.

<u>Exercice 41</u> : Given two points x and y of a pospace $\overrightarrow{X}$, prove that the following are equivalent :

i) there exists a path on X from x to y

ii) there exists a path on $Q(X)$ from x to y

iii) there exists a path of length 1 on $Q(X)$ from x to y

Moreover, prove that if there exists a path on $\overrightarrow{X}$ from x to y and another from y to x, then $x = y$ and these two paths are the following constant one $t \in \overrightarrow{[0,1]} \mapsto x \in \overrightarrow{X}$.

<u>Exercice 42</u> : Prove that we can substitute topologial spaces to pospaces in the description of the Q, thus giving rise to the functor Q' from Top to Gph. Given two points x and y of a topological space X, prove that the following are equivalent :

i) there exists a path on X from x to y

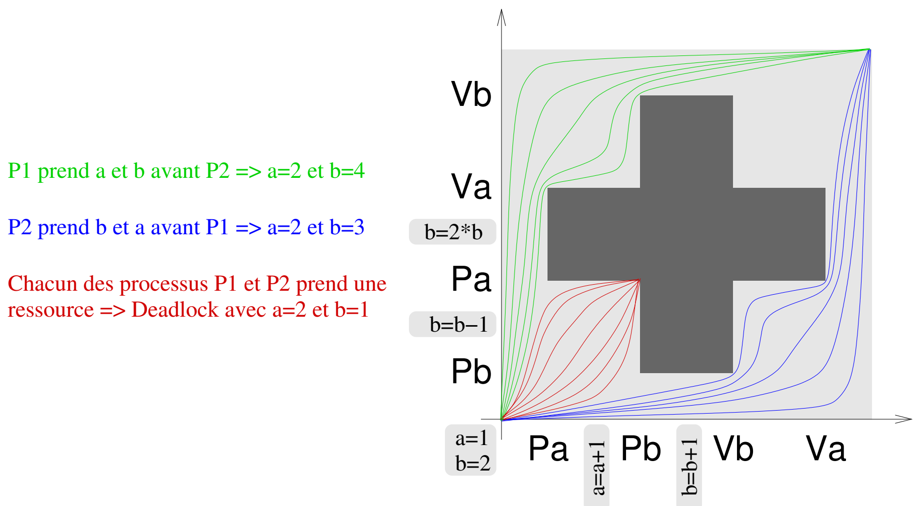
ii) there exists a path on $Q(X)$ from x to y

iii) there exists a path of length 1 on $Q(X)$ from x to y

iv) there exists a path on $Q(X)$ from y to x

2.3 Homotopy of paths

The processes P_1 and P_2 are respectively represented on the horizontal and vertical axes of the following picture. By the way we have introduced variables and affected their content though this kind of instruction were not formally allowed in the syntax we have described. The paths sharing the same color are intuitively "close" to each other, furthermore all of them induce the same execution trace. The notion of homotopy formalizes the idea of nearness.



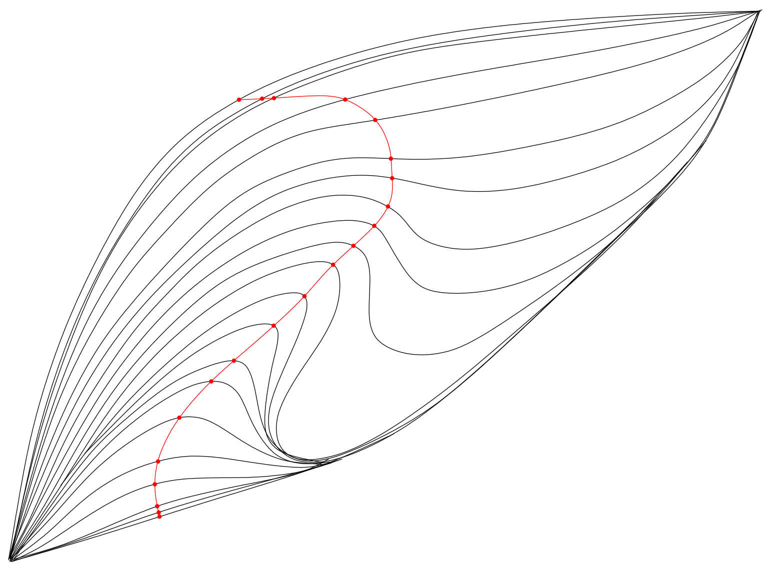
Given two paths γ and δ on a topological space X, a **homotopy** from γ to δ is an element h of de $\mathsf{Top}\big[[0,1]\times[0,1],X\big]$ such that :

i) for all $t\in[0,1]$ we have $h(t,0)=\gamma(t)$ and $h(t,1)=\delta(t)$

ii) for all $s\in[0,1]$ we have $h(0,s)=\gamma(0)$ and $h(1,s)=\gamma(1)$

<u>Exercice 43</u> : Check that if there exists a homotopy from γ to δ then γ and δ share the same extremities that is to say $\gamma(0)=\delta(0)$ and $\gamma(1)=\delta(1)$.

On the following picture, each black curve represent the image of a path some homotopy goes by to go from γ to δ, in other words they are the images of mappings $t\in[0,1]\mapsto h(t,s)\in X$ for several chosen values of s. Besides, the red curve describes the image of the mapping $s\in[0,1]\mapsto h(t,s)\in X$ for some chosen values $t\in[0,1]$. One can think of t as the "time variable" and s as the "deformation parameter".



The paths γ and δ on X are both elements of the homset $\mathsf{Top}\big[[0,1],X\big]$ while h can be seen as a mapping sending each $s \in [0,1]$ to the mapping $(t \in [0,1] \mapsto h(t,s) \in X)$ which is an element of the homset $\mathsf{Top}\big[[0,1],X\big]$. Equiping the homset $\mathsf{Top}\big[[0,1],X\big]$ with the suitable topology, we would like to say that a homotopy is a path on $\mathsf{Top}\big[[0,1],X\big]$. We are about to formalize this idea.

<u>Exercice 44</u> : Let e be an object of some Cartesian category $\mathcal{C}$. Send each object x of $\mathcal{C}$ to $e \times x$ and each morphism f from x to x' to the morphism $\mathsf{id}_e \times f$. The check we have thus define an **endofunctor**[6] of $\mathcal{C}$ which is denoted by $e \times (-)$.

An object e of a category $\mathcal{C}$ is said to be **exponentiable** in $\mathcal{C}$ when the endofunctor $e \times (-)$ has a right adjoint which is then denoted by $(-)^e$. When e is exponentiable, we have for all objects a and x of $\mathcal{C}$ the following isomorphism

$$\mathcal{C}[e \times a, x] \cong \mathcal{C}[a, x^e]$$

<u>Exercice 45</u> : Prove that any set is exponentiable.

It happens that the compact unit segment $[0,1]$ is exponentiable in Top which provides the expected bijection

$$\mathsf{Top}\big[[0,1]\times[0,1],X\big] \cong \mathsf{Top}\big[[0,1],X^{[0,1]}\big]$$

where $X^{[0,1]}$ is the homset $\mathsf{Top}\big[[0,1],X\big]$ provided with the **compact-open** topology : in other words a subset O of $\mathsf{Top}\big[[0,1],X\big]$ is open in this topology when for all $\gamma \in O$, there exists some compact subset [7] K of $[0,1]$ and some

[6] An endofunctor of $\mathcal{C}$ is a functor from $\mathcal{C}$ to $\mathcal{C}$.

[7] The closed unit segment $[0,1]$ is a compact space, its closed subsets are exactly its closed subsets.

open subset O' of X such that

$$\gamma \in \big\{\delta \in \mathsf{Top}\big[[0,1], X\big] \;\big|\; \delta(K) \subseteq O'\big\} \subseteq O$$

Given two points x and x' of X, the set of paths on X whose source and target are respectively x and x' inherits from the topology of $X^{[0,1]}$: we denote by $X^{[0,1]}_{x,x'}$ the resulting topological space. Henceforth one can define a **homotopy** from γ to δ as a path on $X^{[0,1]}_{x,x'}$ from γ to δ.

<u>Exercice 46</u> : The set $\mathsf{Top}\big[[0,1], X\big]$ is provided with the **homotopy** relation $\underset{X}{\sim}$ defined by

$$\big\{(\gamma,\delta) \;\big|\; \text{there exists a homotopy from } \gamma \text{ to } \delta\big\}$$

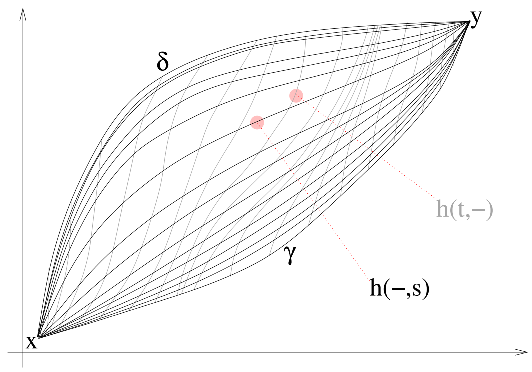
Given γ and δ two paths on X, prove that :
i) $\underset{X}{\sim}$ is an equivalenece relation
ii) if f is a continuous mapping from X to Y and $\gamma \underset{X}{\sim} \delta$, then $f \circ \gamma \underset{X}{\sim} f \circ \delta$
iii) if θ is a continuous mapping from $[0,1]$ onto $[0,1]$ such that $\theta(0) = 0$ and $\theta(1) = 1$, then $(\gamma \circ \theta) \underset{X}{\sim} \gamma$ (Put $h(t,s) := \gamma((1-s)t + s\theta(t))$).
The paths γ and δ are said to be **homotopic** when $\gamma \underset{X}{\sim} \delta$.

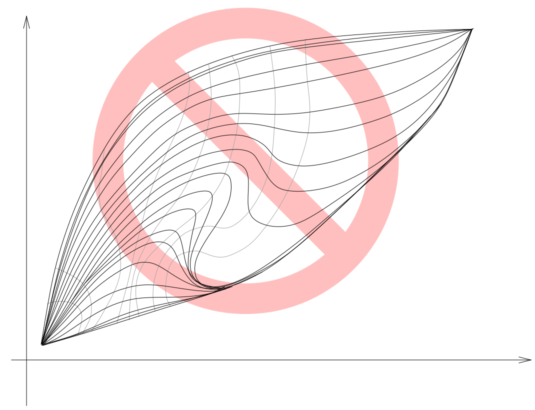
<u>Exercice 47</u> : Given two paths γ and γ' on X such that $\gamma'(0) = \gamma(1)$, we can define the **gluing** of γ followed by γ' as the path which associates each $t \in [0,1]$ with $\gamma(2t)$ when $t \in \left[0, \frac{1}{2}\right]$ and $\gamma'(2t-1)$ when $t \in \left[\frac{1}{2}, 1\right]$. This path is denoted by $\gamma' \cdot \gamma$. Prove that if $\delta \sim \gamma$ and $\delta' \sim \gamma'$ then $\gamma' \cdot \gamma \sim \delta' \cdot \delta$.

<u>Exercice 48</u> : Assuming the fact that the closed unit segment $[0,1]$ is exponentiable in Top, prove that $\overrightarrow{[0,1]}$ is exponentiable in Po.

The notion of homotopy between paths on a pospace straigtforwardly derives from the one wa have just defined. Indeed, given two paths γ and δ on a pospace $\overrightarrow{X}$, a homotopy from γ to δ is an element h of $\mathsf{Po}\big[\overrightarrow{[0,1]} \times \overrightarrow{[0,1]}, \overrightarrow{X}\big]$ such that $U(h)$ is a homotopy from $U(\gamma)$ to $U(\delta)$ where U is the forgetful functor from Po to Top.

Consider the ordered plane $\overrightarrow{\mathbb{R}} \times \overrightarrow{\mathbb{R}}$, the left hand side picture displays two paths γ and δ on $\overrightarrow{\mathbb{R}} \times \overrightarrow{\mathbb{R}}$ and a homotopy h from the first one to the second one. The right hand side picture displays a homotopy h' from $U(\gamma)$ to $U(\delta)$ which does not induce a homotopy from γ to δ. Indeed, there are values $s \in [0,1]$ such that the mapping $t \mapsto h'(t,s)$ is not increasing. Actually we also have values $t \in [0,1]$ such that the mapping $s \mapsto h'(t,s)$ neither is increasing.





In fact one easily checks that a homotopy h' from $U(\gamma)$ to $U(\delta)$ induces a homotopy h from γ to δ if and only if for all $s, t \in [0,1]$ the mappings $s \mapsto h'(t,s)$ and $t \mapsto h'(t,s)$ are increasing.

<u>Exercice 49</u> : The relation $\sqsubseteq$ on the homset $\mathsf{Po}\big[\overrightarrow{[0,1]}, \overrightarrow{X}\big]$ is defined by

$$\big\{(\gamma,\delta)\ \big|\ \text{there exists a homotopy from } \gamma \text{ to } \delta\big\}$$

Let γ and δ be two paths on $\overrightarrow{X}$, prove the follwing facts :

i) $\sqsubseteq$ is an order relation

ii) if f is a morphism of pospaces from X to Y and $\gamma \sqsubseteq \delta$, then $f \circ \gamma \sqsubseteq f \circ \delta$

iii) given θ and θ' two morphisme from $\overrightarrow{[0,1]}$ onto $\overrightarrow{[0,1]}$, prove the mapping $\theta''(t) = \max(\theta(t), \theta'(t))$ for $t \in [0,1]$ is still a morphism from $\overrightarrow{[0,1]}$ onto $\overrightarrow{[0,1]}$ and also $\gamma \circ \theta \sqsubseteq \gamma \circ \theta''$ (Define h by $h(t,s) := \gamma((1-s)\cdot\theta(t) + s\cdot\theta''(t))$).

We define the homotopy relation $\sim_{\overrightarrow{X}}$ as the least equivalence relation over $\mathsf{Po}\big[\overrightarrow{[0,1]}, \overrightarrow{X}\big]$ which contains $\sqsubseteq$. One says the paths γ and δ are **homotopic** when $\gamma \sim_{\overrightarrow{X}} \delta$.

iv) given some un morphisme θ from $\overrightarrow{[0,1]}$ onto $\overrightarrow{[0,1]}$, prove that γ and $\gamma\circ\theta$ are homotopic.

<u>Exercice 50</u> : Given two paths γ and γ' on $\overrightarrow{X}$ such that $\gamma'(0) = \gamma(1)$, we can define the **gluing** of γ followed by γ' as the path which associates $t \in [0,1]$ with $\gamma(2t)$ when $t \in \left[0, \frac{1}{2}\right]$ and $\gamma'(2t-1)$ when $t \in \left[\frac{1}{2}, 1\right]$. We denote this path by $\gamma' \cdot \gamma$. Prove that if $\delta \sqsubseteq \gamma$ and $\delta' \sqsubseteq \gamma'$ then $\gamma' \cdot \gamma \sqsubseteq \delta' \cdot \delta$.

The following result is specific to the directed algebraic topology of pospaces. It is obviously false if we replace “pospaces” by “topological spaces” in its statement.

Theorem 3 Two paths sharing the same image are homotopic.

<u>Preuve</u> : If the common image of γ and γ' is a singleton, then $\gamma = \gamma'$. Otherwise define $\delta(t) := \max(\gamma(t), \gamma'(t))$ for all $t \in [0,1]$ thus defining a path on $\overrightarrow{X}$. By the Theorem 2 we have an isomorphism ϕ from $\overrightarrow{[0,1]}$ to the common image of γ and γ', then we define

$$\begin{cases} h(s,t) := \phi\Big(s \cdot \phi^{-1}\big(\delta(t)\big) + (1-s) \cdot \phi^{-1}\big(\gamma(t)\big)\Big) \\ h'(s,t) := \phi\Big(s \cdot \phi^{-1}\big(\delta(t)\big) + (1-s) \cdot \phi^{-1}\big(\gamma'(t)\big)\Big) \end{cases}$$

which define two homotopies h and h' from γ and γ' to δ thus completing the proof.

Before defining the fundamental category of a pospace we need another notion from basic category theory. A **congruence** over a small category $\mathcal{C}$ is an equivalence relation $\sim$ over the set of all morphisms of $\mathcal{C}$ such that for all morphisms γ, δ, γ' and δ' :

i) if $\gamma \sim \delta$, then $\mathsf{s}(\gamma) = \mathsf{s}(\delta)$ and $\mathsf{t}(\gamma) = \mathsf{t}(\delta)$

ii) if $\gamma \sim \delta$, $\gamma' \sim \delta'$ and $\mathsf{s}(\gamma') = \mathsf{t}(\gamma)$, then $(\gamma' \circ \gamma) \sim (\delta' \circ \delta)$

The preceding axioms can be expressed by means of diagrams that should remind the reader about those ones seen to represent natural transformations.

$$\text{if } \begin{tikzcd} x \arrow[r, bend left=60, "\gamma"] \arrow[r] \arrow[r, bend right=60, "\delta"'] & y \end{tikzcd} \text{ then } \begin{tikzcd} x \arrow[r, bend left, "\gamma"] \arrow[r, bend right, "\delta"'] & y \end{tikzcd} \text{ and if } \begin{tikzcd} x \arrow[r, bend left, "\gamma"] \arrow[r, bend right, "\delta"'] & y \arrow[r, bend left, "\gamma'"] \arrow[r, bend right, "\delta'"'] & z \end{tikzcd} \text{ then } \begin{tikzcd} x \arrow[r, bend left, "\gamma' \circ \gamma"] \arrow[r, bend right, "\delta' \circ \delta"'] & z \end{tikzcd}$$

In this case, the $\sim$-equivalence classes are the morphisms of a category denoted by $\mathcal{C}/_\sim$, called the **quotient** of $\mathcal{C}$ by $\sim$, and whose objects are the objects of $\mathcal{C}$. Acoording to the axioms i) and ii) the source, the target and the composition law are defined from representatives of the equivalence classes. The identities $\mathcal{C}/_\sim$ are therefore the equivalence classes of the identities. The mapping which sends each morphism of $\mathcal{C}$ to its $\sim$-equivalence class induces a functor q from $\mathcal{C}$ to $\mathcal{C}/_\sim$, called the **quotient functor**, it satisfies the following property : for all functor f from $\mathcal{C}$ to $\mathcal{D}$ such that

$$\gamma \sim \delta \Rightarrow f(\gamma) = f(\delta)$$

there exists a unique functor g from $\mathcal{C}/_\sim$ to $\mathcal{D}$ such that $f = g \circ q$.

$$\begin{tikzcd} & \mathcal{C}/_\sim \arrow[d, "g"] \\ \mathcal{C} \arrow[ur, "q"] \arrow[r, "f"'] & \mathcal{D} \end{tikzcd}$$

<u>Exercice 51</u> : Let $\mathcal{C}$ be a small category. Prove there exists a congruence over $\mathcal{C}$ which contains all the other ones and a congruence over $\mathcal{C}$ which is contained

in all the other ones. Moreover if $(\sim_i)_{i\in I}$ is a non empty family of congruences over $\mathcal{C}$ $\mathcal{C}$, then prove that

$$\bigcap_{i\in I}\sim_i$$

is still a congruence over $\mathcal{C}$. Then deduce that for all binary relation R over the set of morphisms of $\mathcal{C}$ such that

$$\big(\gamma R\delta\big) \Longrightarrow \big(\mathsf{s}(\gamma)=\mathsf{s}(\delta) \text{ and } \mathsf{t}(\gamma)=\mathsf{t}(\delta)\big)$$

the collection of congruences containing R admits a least element which is called the **congruence generated** by R.

The notion of homotopy induces an equivalence relation over the set of paths over a pospace $\overrightarrow{X}$, that is to say the arrows of $Q(\overrightarrow{X})$. We would like to extend this relation to the collection of all paths over the graph $Q(\overrightarrow{X})$, in other words to a congruence over $F(Q(\overrightarrow{X}))$. To do so we extend the notion of gluing to any non empty finite sequence $(p_n,\dots,p_0)$ of mappings from $[0,1]$ to X that satisfies $p_k(0)=p_{k-1}(1)$ for all $k\in\{1,\dots,n\}$. The gluing of such a sequence is the mapping from $[0,1]$ to X which sends each t to $p_k((n+1)t-k)$ when $t\in\left[\frac{k}{n+1},\frac{k+1}{n+1}\right]$. In particular, if each term of the sequence is a path, whether on a topological space or a pospace, then its gluing is so.

<u>Exercice 52</u> : Prove that if f is a morphism from $\overrightarrow{X}$ to $\overrightarrow{Y}$ and γ is a path on $\overrightarrow{X}$, then $f\circ(r(\gamma)) = r\big((Q(f))(\gamma)\big)$.

Let $\overrightarrow{X}$ be a pospace (respectively a topological space X), given two morphisms γ and δ picked from the category freely generated by the graph $Q(\overrightarrow{X})$ (respectively $Q(X)$), we write $\gamma \sim_{\overrightarrow{X}} \delta$ (respectively $\gamma \sim_X \delta$) when the gluings of the following sequences are homotopic

$$(c_y,\gamma^n_{n-1},\dots,\gamma^1_0,c_x) \quad \text{and} \quad (c_y,\delta^m_{m-1},\dots,\delta^1_0,c_x)$$

In the previous notation, n and m are the lengths of the paths γ and δ on the graph $Q(\overrightarrow{X})$ (respectively $Q(X)$). We have put $x=\gamma^1_0(0)$ and $y=\gamma^n_{n-1}(1)$ while c_x and c_y denotes the constant mappings from $[0,1]$ to $\overrightarrow{X}$ whose single values are respectively x and y.

Proposition 1 Given a topological space X, the relation $\sim_X$ is a congruence over the category freely generated by the graph $Q(X)$. Moreover, if f is a continuous map from X to Y and $\gamma\sim_X\delta$, then $f\circ\gamma\sim_X f\circ\delta$.

<u>Proof</u> : Use the exercise 46 taking care of the fact that the relation $\sim_X$ over the paths of the graph $Q(X)$ is in some sense an extension of the homotopy relation (also denoted by $\sim_X$ in the exercise 46) over the collection of paths over the topological space X.

Proposition 2 Given a pospace $\overrightarrow{X}$, the relation $\sim_{\overrightarrow{X}}$ is a congruence over the category freely generated by the graph $Q(\overrightarrow{X})$. Moreover, if f is a morphism of posaces from $\overrightarrow{X}$ to $\overrightarrow{Y}$ and $\gamma \sim_{\overrightarrow{X}} \delta$, then $f \circ \gamma \sim_{\overrightarrow{X}} f \circ \delta$.

<u>Proof</u> : Use the exercice 49 taking care of tha fact that the relation $\sim_{\overrightarrow{X}}$ over the paths of the graph $Q(\overrightarrow{X})$ is in some sense an extension of the homotopy relation (also denoted by $\sim_{\overrightarrow{X}}$ in the exercice 49) over the collection of paths over the pospace $\overrightarrow{X}$.

For all topological spaces X ans all pospaces $\overrightarrow{X}$, the relations $\sim_X$ and $\sim_{\overrightarrow{X}}$ are called the **homotopy relations** while their equivalence classes are the **homotopy classes**.

Then we define the **fundamental category** of a pospace $\overrightarrow{X}$ as the quotient of the category freely generated by the graphs of paths on $\overrightarrow{X}$ by the congruence $\sim_{\overrightarrow{X}}$. Then we deduce from the Proposition 2 that for all morphisms f from $\overrightarrow{X}$ to $\overrightarrow{Y}$, there exists a unique functor $\overrightarrow{\pi_1}(f)$ which makes the following diagram commute

$$\begin{tikzcd} F(Q(\overrightarrow{X})) \arrow[r, "F(Q(f))"] \arrow[d, "q_{\overrightarrow{X}}"'] & F(Q(\overrightarrow{Y})) \arrow[d, "q_{\overrightarrow{Y}}"] \\ \overrightarrow{\pi_1}(\overrightarrow{X}) \arrow[r, "\overrightarrow{\pi_1}(f)"'] & \overrightarrow{\pi_1}(\overrightarrow{Y}) \end{tikzcd}$$

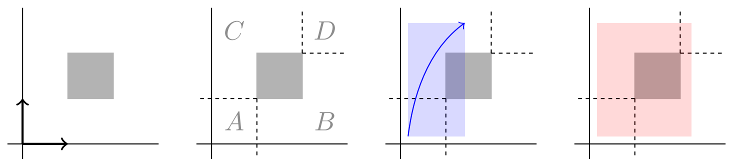
where $q_{\overrightarrow{X}}$ and $q_{\overrightarrow{Y}}$ are the quotient functors associated with the homotopy relations $\sim_{\overrightarrow{X}}$ and $\sim_{\overrightarrow{Y}}$. Whence the functor **fundamental category** from Po to Cat.

We determine the fundamental category of the pospace $\overrightarrow{X}$ obtained by digging up an open square $]1,2[^2$ from the ordered real plane $\overrightarrow{\mathbb{R}} \times \overrightarrow{\mathbb{R}}$. We split the collection of objects of $\overrightarrow{\pi_1}(\overrightarrow{X})$ into four parts $A :=]-\infty,1]^2$, $B :=]1,+\infty[\times]-\infty,1] \cup [2,+\infty[\times]-\infty,2[$, $C :=]-\infty,1]\times]1,+\infty[\cup]-\infty,2[\times[2,+\infty[$ and $D := [2,+\infty[^2$. Then given two points x and y of $\mathbb{R}^2$, the homset $\overrightarrow{\pi_1}(\overrightarrow{X})[x,y]$ is given by

$$\overrightarrow{\pi_1}(\overrightarrow{X})[x,y] := \left\{ \begin{array}{ll} \text{the empty set } \emptyset & \text{if } x \not\sqsubseteq y \\ \text{the singleton } \{(x,y)\} & \text{if } x \sqsubseteq y \text{ and } (x,y) \notin A \times D \\ \text{the pair } \{(x,B,y),(x,C,y)\} & \text{if } x \sqsubseteq y \text{ and } (x,y) \in A \times D \end{array} \right.$$

Let x, y, z be objects of $\overrightarrow{\pi_1}(\overrightarrow{X})$ and two morphisms γ and δ respectively picked from $\overrightarrow{\pi_1}(\overrightarrow{X})[y,z]$ and $\overrightarrow{\pi_1}(\overrightarrow{X})[x,y]$ the only case that should be considered is when we calculate the composite $\gamma \circ \delta$ with x and z respectively picked from A and D : the result is entirely determined by the element of the partition where y lies, indeed

$$\gamma \circ \delta := \left\{ \begin{array}{ll} (x,B,z) & \text{if } y \in B \\ (x,C,z) & \text{if } y \in C \end{array} \right.$$



The formalism of the description of $\mathcal{C}$ hides a very simple idea : each continuous increasing path on $\mathbb{R}^2 \backslash]1,2[^2$ starting in A and stopping in D have either to go under or over the open square $]1,2[^2$ and therefore either go through B or C.

Exercice 53 : A **groupoid** is a category all the morphisms of which are isomorphisms. Denote by Grd the full sub-category of Cat whose objects are the groupoids. Given a topological space X, prove that quotient of the category freely generated by the graph of paths X by the congruence $\sim_X$ is a **groupoid** called the **fondamental groupoid** of X.

Exercice 54 : Prove the identities of the fundamental groupoid of a topological space as well as those of the fundamental category of a pospace are the homotopy classes of the constant paths.

Exercice 55 : Prove the inclusion functor from Grd to Cat admits a left adjoint. The idea consists on adding an arrow $\overline{\gamma}$ for each arrow γ (with $\mathsf{s}(\overline{\gamma}) = \mathsf{t}(\gamma)$ and $\mathsf{t}(\overline{\gamma}) = \mathsf{s}(\gamma)$) of the underlying graph of a small category $\mathcal{C}$. Then a suitable choive of a congurence over the category freely generated by the resulting graph should provide the expected groupoid. This construction is a special case of **category of fractions** also called **localisation**. See the 5th chapter of [Bor94a] for the general approach. One meets lhe localizations in several branches of mathematics : algebraic topology [PGG99, GZ67], model categories [Hov99, Hir03] and homological algebra [SIG03] for example.

By the Proposition 1 and the exercice 53, one actually has a functor π_1 from Top to Grd which associates a topological space with its fundamental groupoid, Grd being a full sub-category of Cat whose corresponding inclusion admits a left adjoint. The idea of using algrebra to study topology is in part based on the fact that $\pi_1(X)$ is a groupoid.

The next part will reveal that we have an analogous situation considering the functor $\overrightarrow{\pi_1}$ provided we replace the category of groupoids by the category of loop-free categories.

Chapter 3

Loop-Free Categories

3.1 Origin

The concept of loop-free categories has been introduced by *Andr Haefliger* [BH99, Hae91, Hae92] in a context far away from the one we presently are interested in. He calls *scwols* for *small category without loops* what we call loop-free categories. As we shall see, they play, in the study of pospaces, the role that groupoids play in the study of topological spaces.

A category $\mathcal{C}$ is said to be **loop-free** when for all objects x and y of $\mathcal{C}$, if $\mathcal{C}[x,y] \neq \emptyset$ and $\mathcal{C}[y,x] \neq \emptyset$, then $x = y$ and $\mathcal{C}[x,x] = \{\mathsf{id}_x\}$. This property can be seen as a generalisation of the antisymetry property. We denote by Lf for the full sub-category of Cat whose objects are the (small) loop-free categories.

<u>Exercice 56</u> : Prove the quotient of a loop-free category by a congruence is still loop-free.

The next result justifies our interest in loop-free categories.

Proposition 3 The fundamental category of a pospace is loop-free.

<u>Preuve</u> : If $\overrightarrow{\pi_1}(\overrightarrow{X})[x,y] \neq \emptyset$ then there exists a path on $\overrightarrow{X}$ from x to y, denote such a path by γ. In particular γ is increasing hence $\gamma(0) \sqsubseteq \gamma(1)$ i.e. $x \sqsubseteq y$. If we also have $\overrightarrow{\pi_1}(\overrightarrow{X})[y,x] \neq \emptyset$ the the antisymetry of the relation $\sqsubseteq$ leads us to conclude that $x = y$. The elements of $\overrightarrow{\pi_1}(\overrightarrow{X})[x,x]$ are homotopy classes, one of their representative γ is in particular a path on $\overrightarrow{X}$ such that $\gamma(0) = \gamma(1)$: according to the exercise 19 it is therefore constant and by the exercise 54 the homotopy class it represents is an identity of $\overrightarrow{\pi_1}(\overrightarrow{X})$.

We provide a theorem which allows one to calculate the fundamental category of certain pospaces. The statement of this result requires some additional category theory. Given two morphisms $\alpha, \beta \in \mathcal{C}[x, y]$ we define, when it exists, the **coequalizer** of α and β in $\mathcal{C}$ as an object z together with a morphism $\gamma \in \mathcal{C}[y, z]$ such that

$$\gamma \circ \alpha = \gamma \circ \beta$$

and for all objects z' and all morphisms $\gamma' \in \mathcal{C}[y, z']$ satisfying

$$\gamma' \circ \alpha = \gamma' \circ \beta$$

one has a unique morphism $\zeta \in \mathcal{C}[z, z']$ such that $\gamma' = \zeta \circ \gamma$

$$\begin{tikzcd} x \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & y \arrow[r, "\gamma"] \arrow[rr, bend left, "\gamma'"] & z \arrow[r, "\zeta"] & z' \end{tikzcd}$$

Exercice 57 : Given two sets X and Y, find the coequalizer $\alpha, \beta \in \mathsf{Set}[X, Y]$. Same question replacing sets and applications by topological spaces and continous maps.

The coequalizers can be seen as an abstraction of the notion of quotient. In general, they are rather difficult to calculate and often use to exhibit "pathological" objects. In the sequel, we will need the coequalizers in Cat which are related to the notion of genralised congruences.
Let $\mathcal{C}$ ne a small category and an equivalence relation $\sim_0$ over $\mathsf{Ob}(\mathcal{C})$. The set of vertices and the set of arrows of the graph $U(\mathcal{C})/\sim_0$ are respectively $\mathsf{Mo}(\mathcal{C})$ and $\mathsf{Ob}(\mathcal{C})/\sim_0$ by composing q_0 on the left side of the applications source and target

$$\begin{tikzcd} \mathsf{Mo}(\mathcal{C}) \arrow[r, shift left, "\mathsf{s}"] \arrow[r, shift right, "\mathsf{t}"'] & \mathsf{Ob}(\mathcal{C}) \arrow[r, "q_0"] & \mathsf{Ob}(\mathcal{C})/\sim_0 \end{tikzcd}$$

where q_0 associates each element of $\mathsf{Ob}(\mathcal{C})$ with its $\sim_0$-equivalence class. In particular $U(\mathcal{C})/\sim_0$ has the same set of arrows than $U(\mathcal{C})$. So the mapping q_0 induces a morphism of graph from $U(\mathcal{C})$ to $U(\mathcal{C})/\sim_0$ which is still denoted by q_0

$$\begin{tikzcd} \mathsf{Mo}(\mathcal{C}) \arrow[r, shift left, "\mathsf{s}"] \arrow[r, shift right, "\mathsf{t}"'] \arrow[d, "\mathsf{id}_{\mathsf{Mo}(\mathcal{C})}"'] & \mathsf{Ob}(\mathcal{C}) \arrow[d, "q_0"] \\ \mathsf{Mo}(\mathcal{C}) \arrow[r, shift left, "q_0 \circ \mathsf{s}"] \arrow[r, shift right, "q_0 \circ \mathsf{t}"'] & \mathsf{Ob}(\mathcal{C})/\sim_0 \end{tikzcd}$$

from which we deduce the functor $F(q_0)$

$$F\big(U(\mathcal{C})\big) \xrightarrow{F(q_0)} F\big(U(\mathcal{C})/\sim_0\big)$$

A **generalised congruence** over $\mathcal{C}$ is then an ordered pair $(\sim_0, \sim_1)$ where $\sim_0$ is an equivalence relation on the set of vertices of the graph $U(\mathcal{C})$ and $\sim_1$ a congruence over $F\big(U(\mathcal{C})/\sim_{0}\big)$ such that there exists a unique functor q which satisfies $q_{1} \circ F(q_{0}) = q \circ \varepsilon_{\mathcal{C}}$

$$\begin{tikzcd}
F\big(U(\mathcal{C})\big) \arrow[r, "F(q_0)"] \arrow[d, "\varepsilon_{\mathcal{C}}"'] & F\big(U(\mathcal{C})/\sim_0\big) \arrow[d, "q_1"] \\
\mathcal{C} \arrow[r, "q"'] & F\big(U(\mathcal{C})/\sim_0\big)/\sim_1
\end{tikzcd}$$

In the preceding statement q_{1} is the quotient functor which comes along the congruence $\sim_{1}$ and for each morphism γ of $F(U(\mathcal{C}))$ one has

$$\varepsilon_{\mathcal{C}}(\gamma) := \mathsf{id}_{\gamma_n} \circ \gamma^n_{n-1} \circ \cdots \circ \gamma^1_0 \circ \mathsf{id}_{\gamma_0}$$

We check that such a functor exists if and only if each morphism γ of $F(U(\mathcal{C}))$ is $\sim_{1}$-equivalent to the path of length 1 over $U(\mathcal{C})$ which sends the unique arrow of the graph I_1 to the arrow $\varepsilon_{\mathcal{C}}(\gamma)$. The functor q and the category $F\big(U(\mathcal{C})/\sim_{0}\big)/\sim_{1}$ are called the **quotient functor** and the **quotient category**.

In general, for all categories $\mathcal{D}$ and all functors f from $\mathcal{C}$ to $\mathcal{D}$ such that
i) $x \sim_{0} y \;\Rightarrow\; f(x) = f(y)$ and
ii) $\gamma \sim_{1} \delta \;\Rightarrow\; f(\varepsilon_{\mathcal{C}}(\gamma)) = f(\varepsilon_{\mathcal{C}}(\delta))$
there exists a unique functors g from the quotient categories to $\mathcal{D}$ such that $f = g \circ q$. Furthermore this property characterizes the quotient in the sense that if q' if a functor whose domain is $\mathcal{C}$ and such that any functor f satisfying i) and ii) can be factorized in a unique way through q' as $f = g \circ q'$, then there exists an isomorphism ϕ from the codomain of q to te domain of q' such that $q' = \phi \circ q$.

In particular, when Σ is a set of morphisms of $\mathcal{C}$, we denote by $\sim_{0}$ the equivalence relation over $\mathsf{Ob}(\mathcal{C})$ generated by

$$\big\{(\mathsf{s}(\sigma), \mathsf{t}(\sigma)) \;\big|\; \sigma \in \Sigma\big\}$$

end we denote by $\sim_{1}$ the congruence over $F\big(U(\mathcal{C})/\sim_{0}\big)$ generated by

$$\big\{(\mathsf{id}_{\mathsf{t}(\sigma)}, \sigma), (\sigma, \mathsf{id}_{\mathsf{s}(\sigma)}) \;\big|\; \sigma \in \Sigma\big\}$$

thus defining a generalised congruence over $\mathcal{C}$. The quotient category is then denoted by $\mathcal{C}/\Sigma$. In this case, the quotient functor q is characterized by the property that any functor f whose domain is $\mathcal{C}$ and which sends each element of Σ to an identity can be factorized in a unique way as $f = g \circ q$.

<u>Exercice 58</u> : Prove the inclusion functor from Lf to Cat admits a left adjoint.

We construct the coequalizer of $f, g \in \mathsf{Cat}[\mathcal{D}, \mathcal{C}]$ by defining $\sim_{0}$ as the equivalence relation over $\mathsf{Ob}(\mathcal{C})$ generated by

$$\big\{(f(x), g(x)) \;\big|\; x \in \mathsf{Ob}(\mathcal{C})\big\}$$

and $\sim_1$ as the congruence over $F\big(U(\mathcal{C})/{\sim_0}\big)$ generated by

$$\big\{(f(\gamma), g(\gamma)) \;\big|\; \gamma \in \mathsf{Mo}(\mathcal{C})\big\}$$

the coequalizer is then provided by the quotient functor.

3.2 The *Seifert -Van Kampen* Theorem

It is a classical result about the fundamental groupoid of a topological space. As we shal see, it can be adapted so as to permit the calculation of the fundamental category of a pospace espace "piecewisely" provided we considered a covering enjoying some reasonable properties. A detailed proof in the case of the fundamental groupoid can be found in [Hig71].

The statement of the *Seifert - Van Kampen* theorem requires the notion of **pushout** (also called **amalgamated sum**).

Given three objects a, b, c of a category $\mathcal{C}$ and an ordered pair of morphisms $(i, i') \in \mathcal{C}[c, a] \times \mathcal{C}[c, b]$, the **pushout** of (i, i') is an object d together with an ordered pair of morphisms $(j, j') \in \mathcal{C}[a, d] \times \mathcal{C}[b, d]$ such that :

i) $j \circ i = j' \circ i'$

ii) for all objects x and all morphisms $(k, k') \in \mathcal{C}[a, x] \times \mathcal{C}[b, x]$ such that $k \circ i = k' \circ i'$, there exists a unique morphism $\alpha \in \mathcal{C}[d, x]$ such that $k = \alpha \circ j$ and $k' = \alpha \circ j'$

$$\begin{tikzcd}
 & x & \\
 & d \arrow[u, "\alpha"] & \\
a \arrow[uur, "k"] \arrow[ur, "j"'] & & b \arrow[uul, "k'"'] \arrow[ul, "j'"] \\
 & c \arrow[ul, "i"] \arrow[ur, "i'"'] &
\end{tikzcd}$$

By a slight abuse of language, the object d is also called **pushout**, thus omitting any reference to j and j'. The notion of pushout is the abstract and categorical way to express the idea of a "gluing". It is unique only up to isomorphisms. First we treat an example in the category of sets. We define $A := \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$, $B := \{\mathtt{d}, \mathtt{e}, \mathtt{f}\}$ and $C := \{\mathtt{g}, \mathtt{h}\}$. Let $i \in \mathsf{Set}[C, A]$ and $i' \in \mathsf{Set}[C, B]$ defined by

$$\begin{array}{ll} i(\mathtt{g}) = \mathtt{b} & i'(\mathtt{g}) = \mathtt{d} \\ i(\mathtt{h}) = \mathtt{c} & i'(\mathtt{h}) = \mathtt{e} \end{array}$$

The pushout can then be described as follows : $D := \{\mathtt{a}, \mathtt{g}, \mathtt{h}, \mathtt{f}\}$ while the mappings $j \in \mathsf{Set}[A, D]$ and $j' \in \mathsf{Set}[B, D]$ are defined by

$$\begin{array}{ll} j(\mathtt{a}) = \mathtt{a} & \\ j(\mathtt{b}) = \mathtt{g} & j'(\mathtt{d}) = \mathtt{g} \\ j(\mathtt{c}) = \mathtt{h} & j'(\mathtt{e}) = \mathtt{h} \\ & j'(\mathtt{f}) = \mathtt{f} \end{array}$$

We have actually "glued" the sets A and B by identifying some of their elements. More generally, one can construct the pushout of $i \in \mathsf{Set}[C, A]$ and $i' \in \mathsf{Set}[C, B]$ as follows : first define the **sum** of A and B as the set

$$A \sqcup B := \big(A \times \{0\}\big) \cup \big(B \times \{1\}\big)$$

and the two "canonical" inclusions k and k' by

$$\begin{array}{ccc} A \longrightarrow A \sqcup B & \qquad & B \longrightarrow A \sqcup B \\ a \longmapsto (a, 0) & & b \longmapsto (b, 1) \end{array}$$

Then equip the set $A \sqcup B$ with the equivalence relation $\sim$ generated by

$$\Big\{ \big(k \circ i(c), k' \circ i'(c)\big) \;\big|\; c \in C \Big\}$$

and finally $j := k \circ i$ and $j' := k' \circ i'$. The pushout of (i, i') is thus given by (j, j').

<u>Exercice 59</u> : With the preceding notation, prove that for all sets X, there is a bijection

$$\begin{array}{ccc} \mathsf{Set}[A \sqcup B, X] & \longrightarrow & \mathsf{Set}[A, X] \times \mathsf{Set}[B, X] \\ h & \longmapsto & (\ h \circ k\ ,\ h \circ k'\ ) \end{array}$$

then mimic the abstract presentation of the Cartesian product to define the **sum** of two objects of a given category. In particular, describe the sum in the case of objects of the categories Gph and Cat.

<u>Exercice 60</u> : Let i and i' be two morphisms of $\mathcal{C}$ having the same source c. Then suppose the sum (k, k') of the objects $a := \mathsf{t}(i)$ and $b := \mathsf{t}(i')$ exist. Also suppose the coequalizer ζ of the morphisms $k \circ i$ and $k' \circ i'$ exist and denote its target by d. Then prove that $(\zeta \circ k, \zeta' \circ k')$ is the pushout of (i, i').

$$\begin{tikzcd}
b \arrow[rr, "\zeta \circ k'"] \arrow[dr, "k'"] & & d \\
& a \sqcup b \arrow[ur, "\zeta"] & \\
c \arrow[uu, "i'"] \arrow[rr, "i"'] & & a \arrow[uu, "\zeta \circ k"'] \arrow[ul, "k"']
\end{tikzcd}$$

The exercise 60 proves in particular than a category which admits all the sums and all the coequalizers actually admit all the pushouts. Indeed, the calculation of a pushout amounts to the calculation of a sum and then the calculation of a coequalizer.

<u>Exercice 61</u> : We work in the category Top. Let C be the discrete topological space over the pair $\{0,1\}$. Define $A = B = [0,1]$ the compact unit segment. Denote by i and i' the inclusion maps from $\{0,1\}$ to $[0,1]$. Prove the pushout of i and i' is the circle. Then denote by i'' the mapping from $\{0,1\}$ to $[0,1]$ which sends t to $1-t$. Then prove the the pushout of i and i' is still a circle. In general describe the sum of two topological spaces.

<u>Exercice 62</u> : We come back to the preceding exercise replacing Top by Po. The pospace $\overrightarrow{C}$ is the discrete topological space $\{0,1\}$ equiped with the diagonal relation. Define $\overrightarrow{A} = \overrightarrow{B} = \overrightarrow{[0,1]}$. Then denote by i and i' the inclusions morphisms from $\{0,1\}$ to $[0,1]$. Describe the pushout of i and i'. Then denote by i'' the morphism from $\{0,1\}$ to $[0,1]$ which sends t to $1-t$. Does the pushout of i and i'' exist in Po ?

The *Seifert-Van Kampen* theorem expresses the fact that one can calculate the fundamental category of a pospace by calculating the fundamental categories of some of its open subsets provided they cover its underlying topological space. From exercise 59 we know that the category Cat admits all the sums and we have seen that, by means of the generalised congruences, it admits all the coequalizers : therefore in virtue of the Exercise 60 the category Cat admits all the pushouts and we even have a method to construct them.

Moreover, it is konwn from basic category theory that a left adjoint preserves the pushouts and the Exercise 58 show that the inclusion functor from Lf to Cat admits a left djoint R. The pushouts in Lf are obtained by calculating them first in Cat and then applying the functor R.

We finally come to the statement of the theorem : given a pospace $\overrightarrow{X}$ and an open covering $\{Y, Y'\}$ of its underlying topological space de son, denote the pospace structures they inherit from $\overrightarrow{X}$ by $\overrightarrow{Y}$ and $\overrightarrow{Y'}$. Also denote by $\overrightarrow{Y} \cap \overrightarrow{Y'}$ the pospace structure induced by $\overrightarrow{X}$ on the intersection $Y \cap Y'$. Then we have four morphisms of pospaces induced by the corresponding inclusions.

$$i : \overrightarrow{Y} \cap \overrightarrow{Y'} \subseteq \overrightarrow{Y} \quad i' : \overrightarrow{Y} \cap \overrightarrow{Y'} \subseteq \overrightarrow{Y'} \quad j : \overrightarrow{Y} \subseteq \overrightarrow{X} \quad \text{and} \quad j' : \overrightarrow{Y'} \subseteq \overrightarrow{X}$$

Theorem 4 (Seifert - Van Kampen)
Under the preceding hypothses, $(\overrightarrow{\pi_1}(j), \overrightarrow{\pi_1}(j'))$ is the pushout of $(\overrightarrow{\pi_1}(i), \overrightarrow{\pi_1}(i'))$ in Lf.

$$
\begin{tikzcd}
\overrightarrow{Y'} \arrow[r, "\subseteq"] & \overrightarrow{X} \\
\overrightarrow{Y} \cap \overrightarrow{Y'} \arrow[u, "\subseteq"] \arrow[r, "\subseteq"'] & \overrightarrow{Y} \arrow[u, "\subseteq"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\overrightarrow{\pi}_1(\overrightarrow{Y'}) \arrow[r, "\subseteq"] & \overrightarrow{\pi}_1(\overrightarrow{X}) \\
\overrightarrow{\pi}_1(\overrightarrow{Y} \cap \overrightarrow{Y'}) \arrow[u, "\subseteq"] \arrow[r, "\subseteq"'] & \overrightarrow{\pi}_1(\overrightarrow{Y}) \arrow[u, "\subseteq"']
\end{tikzcd}
$$

3.3 The monoid of non-empty finite connected loop-free categories

A category $\mathcal{C}$ is said to be **finite** when $\mathsf{Mo}(\mathcal{C})$ is a finite set. It's worth to notice that if $\mathsf{Mo}(\mathcal{C})$ is finite, then so is $\mathsf{Ob}(\mathcal{C})$ since there is exactly one identity per object. Also remark that two finite isomorphic categories have the same number of morphisms. So we can associate the isomorphism class of a finite category $\mathcal{C}$ with the cardinal of $\mathsf{Mo}(\mathcal{C})$ which is also called the **size** of $\mathcal{C}$. Hence for each integer $n \in \mathbb{N}$ there are, up to isomorphism, a finite number of categories of size n.

Exercice 63 : Prove that, up to isomorphism, there is a unique category of size 0, we denote it by the bold zero $\mathbf{0}$. Prove for all categories $\mathcal{C}$, there is a unique functor from $\mathbf{0}$ to $\mathcal{C}$. The category $\mathbf{0}$ is said to be the **initial** object of Cat.

Prove that, up to isomorphism, there is a unique category of size 1, we denote it by the bold one $\mathbf{1}$. Prove for all categories $\mathcal{C}$, there is a unique functor from $\mathcal{C}$ to $\mathbf{1}$. The category $\mathbf{1}$ is said to be the **terminal** object of Cat. Give the list of all categories whose size is less or equal 3.

A morphism of equivalence relation from $(X, \sim)$ to $(Y, \sim')$ is a mapping f from X to Y such that for all $x, y \in X$, if $x \sim y$, then $f(x) \sim f(y)$. We denote the categories of equivalence relations by Eq.

Exercice 64 : One associates a set X with two equivalence relations :

$$\big\{(x,x) \mid x \in X\big\} \quad \text{and} \quad X \times X$$

prove we have thus defined two functors G and D from Set to Eq and

$$G \dashv U \dashv D$$

in other words G and D are respectively the left and the right adjoint to the forgetful functor U from Eq to Set. The relations $G(X)$ are $D(X)$ are respectively said to be **discrete** and **chaotic** on X.

Given a small category $\mathcal{C}$ the set of objects $\mathsf{Ob}(\mathcal{C})$ is equiped with the equivalence relation $\sim$ generated[1] by

$$\big\{(x,y) \mid \mathcal{C}[x,y] \neq \emptyset\big\}$$

[1] That i to say the least equivalence relation containing the given binary relation.

Then two objects x and y of $\mathcal{C}$ are said to be **connected** when $x \sim y$. A category is said to be **connected** when any couple of its objects are connected. In particular if f is a functor from $\mathcal{C}$ to $\mathcal{D}$ and x and y are two connected objects of $\mathcal{C}$, then $f(x)$ and $f(y)$ are two connected objects of $\mathcal{D}$. Thus we have a functor from Cat to Eq.

Exercice 65 : Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, prove that
$\mathcal{C} \times \mathcal{D} \cong \mathbf{0}$ if and only if $\mathcal{C} \cong \mathbf{0}$ or $\mathcal{D} \cong \mathbf{0}$.
From now on we suppose that $\mathcal{C} \ncong \mathbf{0}$ and $\mathcal{D} \ncong \mathbf{0}$, then prove that
$\mathcal{C} \times \mathcal{D}$ is finite if and only if $\mathcal{C}$ are $\mathcal{D}$ finite,
$\mathcal{C} \times \mathcal{D}$ is connected if and only if $\mathcal{C}$ and $\mathcal{D}$ are connected,
$\mathcal{C} \times \mathcal{D}$ are loop-free if and only if $\mathcal{C}$ and $\mathcal{D}$ are loop-free,
Finally, in the case where $\mathcal{C}$ is finite, prove that $\mathcal{C} \times \mathcal{D} \cong \mathcal{C}$ if and only if $\mathcal{D} \cong \mathbf{1}$. Find a counter-example in the case where $\mathcal{C}$ is not finite.

Thus the collection of isomorphism classes of finite categories is countably infinite as a countable (and disjoint) union of finite sets. We denote the set of isomorphism classes of non-empty finite connected loop-free categories by $\mathcal{M}$.

In virtue of the Exercise 65 we provide $\mathcal{M}$ with a structure of commutative monoid whose composition law is the Cartesian product of isomorphism classes [2], denoted by $\times$. The neutral element of $\mathcal{M}$ is (the isomorphism class of) $\mathbf{1}$. In the sequel, we often write "category" to mean "isomorphism class of category".

The next important result of these lecture notes is stated in terms of commutative monoids so we now give some elementary facts about them. Let $(M, *, e)$ be a commutative monoid. Given two elements x and y of M, we say that x **divide** y and we write $x|y$ when there exists some element z of M such that $y = x * z$. We also say that x is a **factor** of y and that it is **non trivial** if it is not inversible, that is to say when there is no element a in M such that $x * a = e$.

Exercice 66 : Let a be an element of a commutative monoid $(M, *, e)$, prove the following are equivalent :
1) a is inversible
2) the mapping $x \in M \mapsto a * x \in M$ is onto
3) the mapping $x \in M \mapsto a * x \in M$ is a bijection

An element i of M is said to be **irreducible** when for all a and b in M, if $ab = i$ then either a or b is inversible (but not both of them). An element p of M is said to be **prime** when for all a and b in M, if p divides ab then p divides a or p divides b (possibly both of them). Let us see some examples.

In the commutative monoid $(\mathbb{N}\backslash\{0\}, \times, 1)$ an element is prime if and only if it is irreducible. In fact, the prime numbers are often defined as the irreducible elements of $(\mathbb{N}\backslash\{0\}, \times, 1)$. However an irreducible element may not be prime,

[2] See section 1.1.2.

as it happens in following counter-example due to *Tadasi Nakayama* and *Junji Hashimoto* [NH50]. In the multiplicative monoid $(\mathbb{N}[X], \times, 1)$ of polynomials with coefficients in $\mathbb{N}$ we have

$$(1+X)(1+X^2+X^4) = 1+X+X^2+X^3+X^4+X^5 = (1+X^3)(1+X+X^2)$$

The monomial $1+X$ is clearly irreducible though it is not prime since it neither divides $(1+X^3)$ nor $(1+X+X^2)$. Conversely, if we consider the semi-lattive $(\{0,1\}, \vee, 0)$ then we see that 1 is prime though it is not irreducible because $1 \vee 1 = 1$. Finally the commutative monoid $(\mathbb{R}_+, +, 0)$ provides an example in which there are neither irreducible nor prime elements.

We denote the category of commutative monoids by CM. Given two mappings ϕ and ψ from X to $\mathbb{N}$, we define the sum $\phi + \psi$ as the mapping $x \in X \mapsto \phi(x) + \psi(x) \in \mathbb{N}$. For all mappings ϕ from some set X to $\mathbb{N}$ we define the **support** of ϕ as the set of elements of X whose image by ϕ is not null.

$$\mathsf{Supp}(\phi) := \{x \in X \mid \phi(x) \neq 0\}$$

If the supports of ϕ and ψ are finite then so is the support of their sum $\phi + \psi$. The collection of mappings from X to $\mathbb{N}$ with finite support turns out to be a commutative monoid $F(X)$ whose neutral element is the null mapping $x \in X \mapsto 0 \in \mathbb{N}$. These remarks lead to call **linear combination** of elements of X with **coefficients** in $\mathbb{N}$ any mapping from X to $\mathbb{N}$ with finite support. The coefficient of some $x \in X$ in the linear combination ϕ is by definition $\phi(x)$. Whence the notation

$$\phi := \sum_{x \in X} \phi(x) \cdot x$$

which can be given a formal meaning provided one identifies each element $x \in X$ with the following map

$$\begin{array}{ccl} X & \longrightarrow & \mathbb{N} \\ x' & \longmapsto & \begin{cases} 1 & \text{if } x' = x \\ 0 & \text{otherwise} \end{cases} \end{array}$$

Besides, if f is a function from X to Y and ϕ a mapping from X to $\mathbb{N}$ with finite support, then we define the function $\big(F(f)\big)(\phi)$ from Y to $\mathbb{N}$ as below :

$$\big(F(f)\big)(\phi) := \begin{cases} Y & \longrightarrow \quad \mathbb{N} \\ y & \longmapsto \quad \displaystyle\sum_{\substack{x \in X \\ f(x) = y}} \phi(x) \end{cases}$$

which can also be written as

$$\big(F(f)\big)(\phi) := \sum_{x \in X} \phi(x) f(x)$$

The right hand side expression being sound since the support of ϕ is finite. We have thus described a functor F from Set to CM and one can easily check it admits a right adjoint. The commutative monoid $F(X)$ is said to be **freely generated** by X. It must be noticed that the notion of "freeness" implicitly depends of the domain of the forgetful functor of an adjunction. For example both forgetful functors

$$\mathsf{Mon} \xrightarrow{U} \mathsf{Set} \qquad \text{and} \qquad \mathsf{CM} \xrightarrow{U} \mathsf{Set}$$

admit a left adjoint though they differ. Thus, given a set X, the free monoid generated by X is the monoid of words on (the alphabet) X (it is commutative if and only if X is empty or reduced to a singleton) while the free commutative monoid generated by X consists on the collection of linear combinations of elements of X with coefficients in $\mathbb{N}$.

The freeness of a commutative monoid is tightly bound to the comparison between its prime elements and its irreducible elements.

Proposition 4 A commutative monoid is free if and only if
1) all its elements can be written as a product of irreducible elements and
2) its prime elements are its irreducible elements.
Moreover the commutative monoids $F(X)$ and $F(Y)$ are isomorphic if and only if there is a bijection between X and Y ie

$$F(X) \cong F(Y) \text{ in } \mathsf{CM} \text{ if and only if } X \cong Y \text{ in } \mathsf{Set}$$

In particular the commutative monoid $(\mathbb{N}\backslash\{0\}, \times, 1)$ is freely generated by the prime numbers . It is thus isomorphic to $F(X)$ as soon as X is countably infinite.

The commutative monoid $(\mathbb{N}[X], \times, 1)$ satisfies the first axiom while the semilattice $(\{0,1\}, \vee, 0)$ satisfies the second one.

The following result generalises a theorem by *Junji Hashimoto* [Has51], it has been proven by *Thibaut Balabonski*.

Theorem 5 An element of the monoid $(\mathcal{M}, \times, \mathbf{1})$ is prime if and only if it is irreducible.

We have already seen that the monoid $(\mathcal{M}, \times, \mathbf{1})$ is at most countable and it is easy to see that the finite totally ordered sets are irreducible.

$$\{0 < \ldots < n\}$$

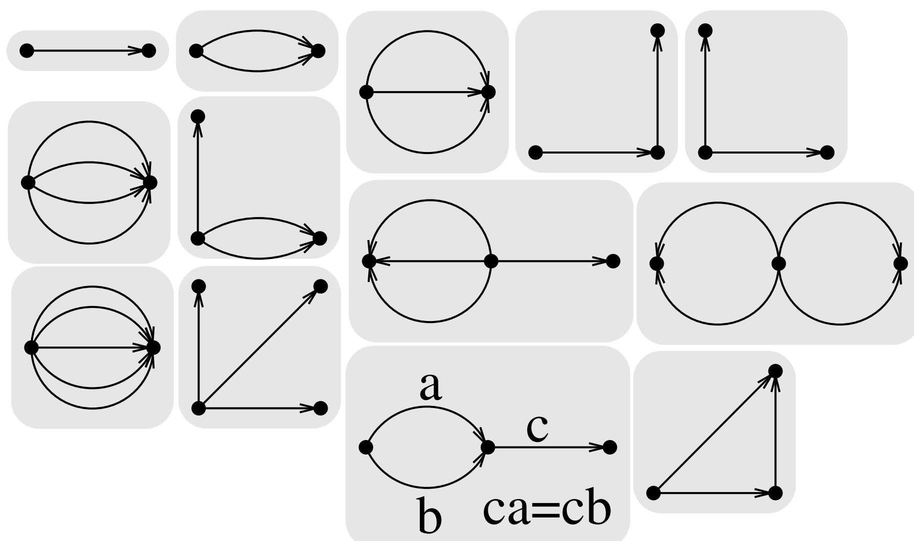
Furthermore, a straightforward argument of cardinality proves that any element of $(\mathcal{M}, \times, \mathbf{1})$ is a finite product of irreducible elements of $(\mathcal{M}, \times, \mathbf{1})$. The Proposition 4 then provides the following result.

Theorem 6 The commutative monoid $(\mathcal{M}, \times, \mathbf{1})$ is freely generated by its set of irreducible elements, which is countably infinite.

Exercice 67 : Prove the commutative monoid of non-empty finite loop-free categories is not free. In other words the connectedness hypothesis cannot be dropped from the statement of the Theorem 6.

Exercice 68 : Prove that all the elements of the commutative monoid $(\mathcal{M}, \times, \mathbf{1})$ whose size is at most 8 are prime and that there exists a unique non-prime category of size 9.

So the commutative monoid $(\mathcal{M}, \times, \mathbf{1})$ is isomorphic to the commutative monoid $(\mathbb{N}, \times, 1)$. In particular we may try to determine all its prime elements by means of an algorithm. We give the list of all[3] the prime elements whose size is at most 7 and notice in particular that $\mathcal{M}$ has no element of size 2.



The proofs of theorems 5 and 6 by *Thibaut Balabonski* provide an algorithm which decomposes any element of $\mathcal{M}$ as a product of prime elements of $\mathcal{M}$, however this algorithm is (extremely) unefficient.

The existence of a unique decomposition actually has a practical interest. Indeed the notion of **category of components** "reduces" the fundamental category of the model $\llbracket \overrightarrow{P} \rrbracket$ of a PV program to an element of $\mathcal{M}$. Then its decomposition indicates how we can "optimally gather" processes into subprograms that can be executed independently. Let us see an example : suppose `a` and `b` are semaphore of arity 2 and `c` is a semaphore of arity 3. We focus on the program below

[3] This list is exhaustive only up to opposite.

```
P(a).P(c).V(c).V(a) |
P(a).P(c).V(c).V(a) |
P(b).P(c).V(c).V(b) |
P(b).P(c).V(c).V(b)
```

We remark that the semaphore `a` prevents the two first processes to hold simultaneously an occurence of `c`. The same remark, involving the semaphore `c`, can be applied to the two last processes. It follows that the forbidden area generated by the semaphore `c` is empty. Hence the semaphore `c` does not interfer in the execution of the program which is then in some sense "equivalent" to

```
P(a).V(a) |
P(a).V(a) |
P(b).V(b) |
P(b).V(b)
```

which can be "separated" into two sub-programs

```
P(a).V(a) |             P(b).V(b) |
P(a).V(a)       and     P(b).V(b)
```

that can be executed independently. This result can also been obtained by calculating the decomposition of the category of components of the model of the program. The last part of the course is dedicted to the notion of category of components.

Chapter 4

The categories of Components

4.1 What is good for ?

We have associated each PV program $\overrightarrow{P}$ with a pospace $\overrightarrow{X} := [\![\overrightarrow{P}]\!]$ and seen that the paths on $\overrightarrow{X}$ correspond to the execution traces of $\overrightarrow{P}$. Whence our interest in $F(Q(\overrightarrow{X}))$, the category freely generated by the graph of paths on $\overrightarrow{X}$. Yet, each homset of $F(Q(\overrightarrow{X}))$ is either empty or contains uncountably many elements. Nevertheless, the examination of some concrete examples give the (right) intuition that most of the morphisms of $F(Q(\overrightarrow{X}))$ should be identified according to a suitable notion of deformation coming from the topological structure of $\overrightarrow{X}$. This idea is formalised in the concept of homotopic paths which leads to the definition of the fundamantal category of $\overrightarrow{X}$ as a quotient. Then we check that the homsets of the fundamental category of (the model of) a PV program are finite. Unfortunately it still has uncountably many objects though in practice we observe that the local composition laws, that is to say the applications defined by

$$\begin{array}{ccc} \overrightarrow{\pi}_1(\overrightarrow{X})[y,z] \times \overrightarrow{\pi}_1(\overrightarrow{X})[x,y] & \xrightarrow{\Phi_{x,y,z}} & \overrightarrow{\pi}_1(\overrightarrow{X})[x,z] \\ (\gamma,\delta) & \longmapsto & \gamma \circ \delta \end{array}$$

are "isomorphic" when we get x, y and z running through some well chosen subsets C_x, C_y and C_z. Thus we expect to find the "best" (in other word the coarsest) partition of the collection of objects of $\overrightarrow{\pi}_1(\overrightarrow{X})$ such that the local composition law $\Phi_{x,y,z}$ only depends on the elements of the partition x, y and z belong to. The elements of this partition form the collection of objects of what we call the category of components. The soundness of the construction we are about to detail does not formally require any restriction about the category it is applied to, nevertheless it provides the expected result only under loop-freeness hypothesis. This constraint is not impeding since the fundamental category of a pospace is loop-free.

From a technical point of view, we try to stick to the well-known situation met in classical algebraic topology. The **skeleton** of a category $\mathcal{C}$ is a full subcategory having exactly one object per isomorphism class. One easily proves that two skeleta of a given category are isomorphic hence we can write "the" skeleton meaning it is unique up to isomorphism. The interest of this notion lies in the fact that any category is equivalent to its skeleton in the following sense : an **equivalence** is a fully faithful functor E such that all objects of the codomain of E is isomorphic to come objet of its image. One can prove that a functor E from $\mathcal{C}$ to $\mathcal{D}$ is an equivalence if and only if there exists a functor E' from $\mathcal{D}$ to $\mathcal{C}$ such that $E' \circ E$ and $E \circ E'$ are respectively isomorphic to $\mathsf{id}_{\mathcal{C}}$ and $\mathsf{id}_{\mathcal{D}}$. Then two categories are said to be **equivalent** when there exists an equivalence between them and we can prove that two categories are equivalent if and only if they share the same skeleta. In many cases the skeleton of a category is "smaller" than the original. In classical algebraic topology the fundamental groupoid $\pi_1(X)$ of a connected topological space is equivalent to the group $\pi_1(X)[x,x]$ for all points x : it is the **fondamental group** of X. By considering the fundamental group instead of the fundamental groupoid we reduce the set of objects to a singleton yet preserving the "essential" information.

We are looking for a similar situation in the case of the fundamental categories of pospaces. First we notice that the usual notion of skeleton is vain in this context since any loop-free category[1] is its own skeleton : this is due to the fact that the only isomorphisms of a loop-free category are its identities. In the sequel we outline the axiomatic presentation of a collection of morphisms which contains slightly more than the identities of a loop-free category. Then we prove that the morphisms of this collection can be formally turned into isomorphisms keeping the information we are concerned about safe.

4.2 The *Yoneda* Morphisms

Let x be an object of a small category $\mathcal{C}$, the **category over** x is denoted by $\mathcal{C}{\downarrow}x$, its objects are the morphisms of $\mathcal{C}$ whose target is x while

$$\mathcal{C}{\downarrow}x[\delta,\delta'] := \left\{ \alpha \in \mathcal{C}[\mathsf{s}(\delta'),\mathsf{s}(\delta)] \;\middle|\; \delta' = \delta \circ \alpha \right\}$$

$$\begin{tikzcd} \cdot \arrow[dr, "\delta"'] & & \cdot \arrow[ll, "\alpha"'] \arrow[dl, "\delta'"] \\ & x & \end{tikzcd}$$

[1] Recall the fundamental category of a pospace is loop-free.

Then we denote by $\mathcal{C}{\downarrow}y, x$ for the full sub-category of $\mathcal{C}{\downarrow}y$ whose objects dont les objets δ satisfy the following property

$$\mathcal{C}[\mathsf{s}(\delta), x] \neq \emptyset$$

In a analogous way we denote by $y{\downarrow}\mathcal{C}$ for the **category below** y, its objects are the morphisms of $\mathcal{C}$ whose source is y while

$$y{\downarrow}\mathcal{C}[\gamma, \gamma'] := \left\{\beta \in \mathcal{C}[\mathsf{t}(\gamma), \mathsf{t}(\gamma')] \;\middle|\; \gamma' = \beta \circ \gamma\right\}$$

$$\begin{tikzcd} & y \arrow[dl, "\gamma"'] \arrow[dr, "\gamma'"] & \\ {} \arrow[rr, "\beta"'] & & {} \end{tikzcd}$$

We denote by $y, x{\downarrow}\mathcal{C}$ for the full sub-category of $x{\downarrow}\mathcal{C}$ whose objects γ satisfy the following property

$$\mathcal{C}[y, \mathsf{t}(\gamma)] \neq \emptyset$$

Given a morphism $\sigma \in \mathcal{C}[x, y]$ we can define the functors F_σ ("future") and P_σ ("past") as follows

$$y{\downarrow}\mathcal{C} \xrightarrow{F_\sigma} y, x{\downarrow}\mathcal{C} \qquad \mathcal{C}{\downarrow}x \xrightarrow{P_\sigma} \mathcal{C}{\downarrow}y, x$$

$$\begin{tikzcd} \gamma \arrow[r, mapsto] \arrow[d, "\beta"] & \gamma \circ \sigma \arrow[d, "\beta"'] & & \delta \arrow[r, mapsto] \arrow[d, "\alpha"] & \sigma \circ \delta \arrow[d, "\alpha"'] \\ \gamma' \arrow[r, mapsto] & \gamma' \circ \sigma & & \delta' \arrow[r, mapsto] & \sigma \circ \delta' \end{tikzcd}$$

Indeed, if $\gamma \in \mathcal{C}[y, z]$, then $\gamma \circ \sigma \in \mathcal{C}[x, z]$ and the homset $\mathcal{C}[y, z]$ is not empty since it contains γ, it is therefore an object of $y, x{\downarrow}\mathcal{C}$. Also if $\delta \in \mathcal{C}[z, x]$, then $\sigma \circ \delta \in \mathcal{C}[z, y]$ and the homset $\mathcal{C}[z, x]$ is not empty since it contains δ, it is therefore an object of $\mathcal{C}{\downarrow}y, x$.
When σ is an isomorphism of $\mathcal{C}$ the functors F_σ and P_σ are isomorphism of Cat. We say that σ is a ***Yoneda* morphism** when both F_σ and P_σ are isomorphisms. In particular we can prove that σ is a *Yoneda* morphism when for all morphisms ζ of $\mathcal{C}$:
if $\mathsf{s}(\zeta) = x$ and $\mathcal{C}[y, \mathsf{t}(\zeta)] \neq \emptyset$, then there exists a unique γ in $\mathcal{C}[y, \mathsf{t}(\zeta)]$ such that

$$\zeta = \gamma \circ \sigma$$

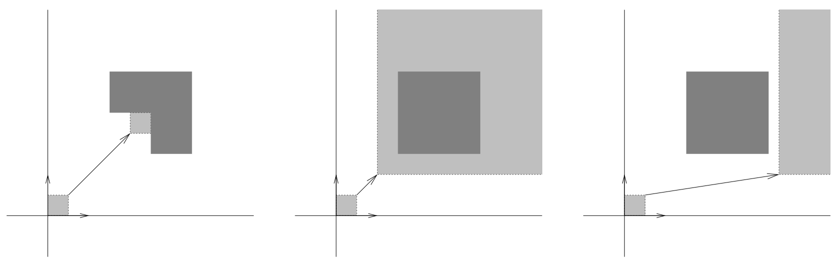
if $\mathsf{t}(\zeta) = y$ and $\mathcal{C}[\mathsf{s}(\zeta), x] \neq \emptyset$, then there exists a unique δ in $\mathcal{C}[\mathsf{s}(\zeta), x]$ such that

$$\zeta = \sigma \circ \delta$$

$$\begin{tikzcd} x \arrow[r, "\sigma"] & y & & \mathsf{t}(\zeta) \\ \mathsf{s}(\zeta) \arrow[u, "\delta"] \arrow[ur, "\zeta"'] & & x \arrow[ur, "\zeta"] \arrow[r, "\sigma"'] & y \arrow[u, "\gamma"'] \end{tikzcd}$$

The restrictions $\mathcal{C}[\mathsf{s}(\zeta),x] \neq \emptyset$ and $\mathcal{C}[y,\mathsf{t}(\zeta)] \neq \emptyset$ come from the fact that we consider the categories $y,x{\downarrow}\mathcal{C}$ and $\mathcal{C}{\downarrow}y,x$ as respective codomains of the functors F_σ and P_σ instead of $x{\downarrow}\mathcal{C}$ and $\mathcal{C}{\downarrow}y$. If we omit these restrictions, then the category $x{\downarrow}\mathcal{C}$ contains the object id_x and the first condition furnishes a morphism $\gamma \in \mathcal{C}[y,x]$ such that $\gamma \circ \sigma = \mathsf{id}_x$ while the category $\mathcal{C}{\downarrow}y$ contains the object id_y and the second condition gives rise to a morphism $\delta \in \mathcal{C}[y,x]$ such that $\sigma \circ \delta = \mathsf{id}_y$. Then σ is an isomorphism and we have $\gamma = \delta = \sigma^{-1}$. In particular, if $\mathcal{C}[x,y] \neq \emptyset$ and $\mathcal{C}[y,x] \neq \emptyset$, then an element of $\mathcal{C}[x,y]$ is a *Yoneda* morphism if and only if it is an isomorphism. Thus the notion of *Yoneda* morphism is preferably used when $\mathcal{C}$ is loop-free.

However we easily find *Yoneda* morphisms which are not isomorphisms. It suffices to consider the poset $\{0 < 1\}$ as a category, whose unique element from 0 to 1 happens to be a *Yoneda* morphism.
We give three examples of pospaces, on the the two first ones we have drawn a path inducing a *Yoneda* morphism on the fundamental category while the path pictured on the third one does not.



From the concurrency point of view the first path is pathological. On one hand, denoting by x and y for its source and target, we can find two paths from x to the same target, z say, inducing two distinct morphisms of the fundamental category. On the other hand, in the fundamental category, the homset from y to some z' is either empty or reduce to a singleton. In some sense, one has "chosen" to go to a dead end (which will corrspond to a deadlock in the sequel) along the path.

Technically, if we formally add an inverse to any *Yoneda* morphism of (the fundamental category of) the first example then we would obtain a category which is isomorphic to the one that we would obtain if we formally add an inverse to any *Yoneda* morphism of (the fundamental category of) the second example.

So the notion of *Yoneda* morphism should be strengthened.

4.3 Systmes de *Yoneda*

The *Yoneda* morphisms are designed to generalize the isomorphisms. Hence we will only consider a subcollection of *Yoneda* morphisms which is stable under pushout and pullback[2].

Formally we can define the notion of pullback (or fibered product) in a category $\mathcal{C}$ as the pushout in the opposite catyegory $\mathcal{C}^{\mathsf{op}}$. In other words, given three objects a, b, c of a category $\mathcal{C}$ and an ordered pair of morphisms $(p, p') \in \mathcal{C}[a, c] \times \mathcal{C}[b, c]$, the **pullback** (or **fibered product**) of (p, p') is an object d and an ordered pair of morphisms $(q, q') \in \mathcal{C}[d, a] \times \mathcal{C}[d, b]$ such that :

i) $p \circ q = p' \circ q'$

ii) for all object x and all ordered pair of morphisms $(k, k') \in \mathcal{C}[x, a] \times \mathcal{C}[x, b]$ tels que $p \circ k = p' \circ k'$, there exists a unique morphism $\alpha \in \mathcal{C}[x, d]$ such that $k = q \circ \alpha$ and $k' = q' \circ \alpha$

$$\begin{tikzcd}
 & x \arrow[d, "\alpha"'] \arrow[ddl, "k"'] \arrow[ddr, "k'"] & \\
 & d \arrow[dl, "q"] \arrow[dr, "q'"'] & \\
a \arrow[dr, "p"'] & & b \arrow[dl, "p'"] \\
 & c &
\end{tikzcd}$$

By the same abuse of language, we also call **pushout** (or **fibered product**) the object l'objet d alone, omitting the reference to q and q'.

A collection Σ of morphisms of $\mathcal{C}$ is said to be stable under pullback and pushout when for all squares

$$\begin{tikzcd}
 & \cdot & \\
\cdot \arrow[ur, "f'"] & & \cdot \arrow[ul, "\sigma'"'] \\
 & \cdot \arrow[ul, "\sigma"] \arrow[ur, "f"'] &
\end{tikzcd}$$

if $\sigma' \in \Sigma$ then the pullback of (f', σ') exists
if $\sigma \in \Sigma$ then the pushout of (f, σ) exists
if $\sigma' \in \Sigma$ and (f, σ) is a pullback of (f', σ'), then $\sigma \in \Sigma$ and
if $\sigma \in \Sigma$ and (f', σ') is a pushout of (f, σ), then $\sigma' \in \Sigma$

We say that Σ is **stable under composition** when $\sigma' \circ \sigma \in \Sigma$ as soon as σ and σ' are two elements Σ such that $\mathsf{s}(\sigma') = \mathsf{t}(\sigma)$. We also say that Σ is **pure** when two morphisms γ and δ such that $\mathsf{s}(\gamma) = \mathsf{t}(\delta)$ belong to Σ as soon as their composite $\gamma \circ \delta$ does.

[2] One also says the collcetion is stabel under **change of base** and **cochange of base**

A collection Σ of *Yoneda* morphisms of $\mathcal{C}$ is a ***Yoneda* system** when it is stable under pullback, pushout, composition, and contains all the isomorphisms.

Proposition 5 All the *Yoneda* systems of a loop-free category are pure.

We will see that the collection of all *Yoneda* systems of a loop-free category has a structure which can be seen as an abstraction of a topological space. A poset is said to be a **complete lattice** when any of its subset (even the empty one) admits a least upper bound and a greatest lower bound. In particular we denote by $\perp$ (respectively $\top$) for the least (respectively greatest) element of a complete lattice and we have

$$\bigvee \emptyset = \perp \quad \text{and} \quad \bigwedge \emptyset = \top$$

A complete lattice is a **locale** when for all elements x and all families: $(y_i)_{i\in I}$ of elements of this set one has

$$x \wedge \Big(\bigvee_{i\in i} y_i \Big) = \bigvee_{i\in i} (x \wedge y_i)$$

where $\vee$ and $\wedge$ respectivement represent the least upper bound and the greatest lower bound. A morphism of locale[3] from $(X', \sqsubseteq')$ to $(X, \sqsubseteq)$ is a mapping from X to X' such that $f(\perp) = \perp'$, $f(\top) = \top'$, $f(x \wedge y) = f(x) \wedge f(y)$ and

$$f\Big(\bigvee_{i\in i} x_i \Big) = \bigvee_{i\in i} f(x_i)$$

In particular the collection of open subsets of a topological space forms a locale. Furthermore, if f is a continuous mapping from X' to X, then the mapping f^* associating each open subset U of X with its inverse image by f i.e.

$$\big\{ x' \in X' \bigm| f(x') \in U \big\}$$

is a morphism from the locale of open subsets of X' to the one of open subsets of X. Denoting the category of locale by Loc, we have just described a functor from Top to Loc. This functor actually admits a left adjoint described in [Bor94b], [Joh82] and [PPT04].

Let us go back to the *Yoneda* systems. The collection of all morphisms of a grouod $\mathcal{G}$ is its unique *Yoneda* system. Indeed, the collection of all isomorphisms of a category $\mathcal{C}$ is its least *Yoneda* system (with respect to inclusion).

Theorem 7 The collection of all *Yoneda* systems of a small category forms a complete lattice (with respect to inclusion), moreover if the small category is loop-free, then the complete lattice is actually a locale.

[3]Pay attention to the sense of arrows.

The greatest lower bound of a non-empty family $(\Sigma_i)_{i\in I}$ of *Yoneda* systems is its intersection and its least upper bound is the collection of composite

$$\sigma_n \circ \cdots \circ \sigma_0$$

where $n \in \mathbb{N}$ and for each $k \in \{0, \ldots, n\}$ there exists some index i_k in I such that σ_k belong to Σ_{i_k}. The tedious part of the proof consists on cheking that the collection of morphisms we have described is actually a *Yoneda* system, it is not treated here. Let Σ be a *Yoneda* system of a loop-free category and let σ be an element of the intersection of Σ and the least upper bound of a non-empty family $(\Sigma_i)_{i\in I}$ of *Yoneda* systems, then we can write σ as $\sigma_n \circ \cdots \circ \sigma_0$ where σ_k belongs to some Σ_{i_k}. By the Proposition 5 each morphism σ_k actually belongs to $\Sigma \wedge \Sigma_{i_k}$, therefore σ belongs to the least upper bound of the family $(\Sigma \wedge \Sigma_i)_{i\in I}$ hence we have a locale.

4.4 Quotient and Localization

Given a collection Σ of morphisms of a category $\mathcal{C}$, the generalized congruences allow us to define the **quotient** category of $\mathcal{C}$ by Σ, denoted by $\mathcal{C}/\Sigma$, as the unique (up to isomorphism) category such that there exists a functor q_Σ from $\mathcal{C}$ to $\mathcal{C}/\Sigma$ such that

1) for all elements σ of Σ, the morphism $q_\Sigma(\sigma)$ is an identity and

2) for all functors f from $\mathcal{C}$ to $\mathcal{D}$ such that for all elements σ of Σ, the morphism $f(\sigma)$ is an identity, there exists a unique functor g from $\mathcal{C}/\Sigma$ to $\mathcal{D}$ such that $f = g \circ q_\Sigma$

Consider the graph $\mathcal{Q}$ obtained by adding an arrow $\overline{\sigma}$ from $\mathsf{t}(\sigma)$ to $\mathsf{s}(\sigma)$ for each arrow σ of Σ, to the underlying graph of $\mathcal{C}$. Then let $\sim$ be the congruence generated by the following binary relation

$$\Big\{ \big((\overline{\sigma}, \sigma), \mathsf{id}_{\mathsf{s}(\sigma)}\big) \ , \ \big((\sigma, \overline{\sigma}), \mathsf{id}_{\mathsf{t}(\sigma)}\big) \Big| \ \sigma \in \Sigma \Big\}$$

The **localization** of $\mathcal{C}$ by Σ, denoted by $\mathcal{C}[\Sigma^{-1}]$, is the quotient of the free category $F(\mathcal{Q})$ by the congruence $\sim$. We can also define $\mathcal{C}[\Sigma^{-1}]$ as the unique category (up to isomorphism) such that tere exists a functor i_Σ from $\mathcal{C}$ to $\mathcal{C}[\Sigma^{-1}]$ such that

1) for all elements σ of Σ, the morphism $i_\Sigma(\sigma)$ is an isomorphism and

2) for all functors f from $\mathcal{C}$ to $\mathcal{D}$ such that for all elements σ of Σ, the morphism $f(\sigma)$ is an isomorphism, there exists a unique functors g from $\mathcal{C}[\Sigma^{-1}]$ to $\mathcal{D}$ such that $f = g \circ i_\Sigma$

In particular there exists a unique functor p_Σ from $\mathcal{C}[\Sigma^{-1}]$ to $\mathcal{C}/\Sigma$ such that

$$q_\Sigma = p_\Sigma \circ i_\Sigma$$

The next result explains why *Yoneda* systems are specially well fitted to loop-free categories.

Theorem 8 If Σ is a *Yoneda* system of a loop-free category, then the functor p_{Σ} is an equivalence.

We actually prove the existence of a functor j_{Σ} from $\mathcal{C}/\Sigma$ to $\mathcal{C}[\Sigma^{-1}]$ such that

$$p_{\Sigma} \circ j_{\Sigma} = \mathsf{id}_{\mathcal{C}/\Sigma} \quad \text{and} \quad j_{\Sigma} \circ p_{\Sigma} \cong \mathsf{id}_{\mathcal{C}[\Sigma^{-1}]}$$

The **category of components** of a loop-free category $\mathcal{C}$ is the quotient $\mathcal{C}/\overline{\Sigma}$ where $\overline{\Sigma}$ is the greatest *Yoneda* system of $\mathcal{C}$. By extension, we say that the category of components of a pospace $\overrightarrow{X}$ is the category of components of its fundamental category $\overrightarrow{\pi_1}(\overrightarrow{X})$.

4.5 Describing the Category of Components

Let Σ be a *Yoneda* system on a loop-free category $\mathcal{C}$. We denote by $\sim$ for the equivalence relation over the set of objects of $\mathcal{C}$ generated by the following binary relation

$$\big\{(x, x') \bigm| \Sigma \cap \mathcal{C}[x, x'] \neq \emptyset\big\}$$

A **Σ-component** of $\mathcal{C}$ is a full sub-category of $\mathcal{C}$ whose set of objects is a $\sim$-equivalence class.

Theorem 9 Every Σ-component of a loop-free category is a poset[4] in which each pair of elements $\{x, y\}$ admits a greatest lower bound and a least upper bound respectively denoted by $x \wedge y$ and $x \vee y$. Moreover all the morphisms of a Σ-component belong to Σ and the ordered pairs $(x \to x \vee y, y \to x \vee y)$ and $(x \wedge y \to x, x \wedge y \to y)$ are respectively the pushout and the pullback of $(x \wedge y \to x, x \wedge y \to y)$ and $(x \to x \vee y, y \to x \vee y)$.

$$\begin{tikzcd}
& x \vee y & \\
x \arrow[ur] & & y \arrow[ul] \\
& x \wedge y \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

In particular the Theorem 9 implies that if the homset $\mathcal{C}[x, y]$ meets Σ, then $\mathcal{C}[x, y]$ is a singleton, so its unique element belongs to Σ. We digramatically represent this situation by an arrow from x to y carrying the symbol "Σ" over it.

$$x \xrightarrow{\Sigma} y$$

From the Theorem 9 we can also deduce that given two objects x and y we have $x \sim y$ if and only if there exists some object z such that

$$x \xrightarrow{\Sigma} z \xleftarrow{\Sigma} y$$

[4]The order being given by putting $x \sqsubseteq y$ when $\mathcal{C}[x, y] \neq \emptyset$.

which is also equivalent to the existence of an object z' such that

$$x \xleftarrow{\Sigma} z' \xrightarrow{\Sigma} y$$

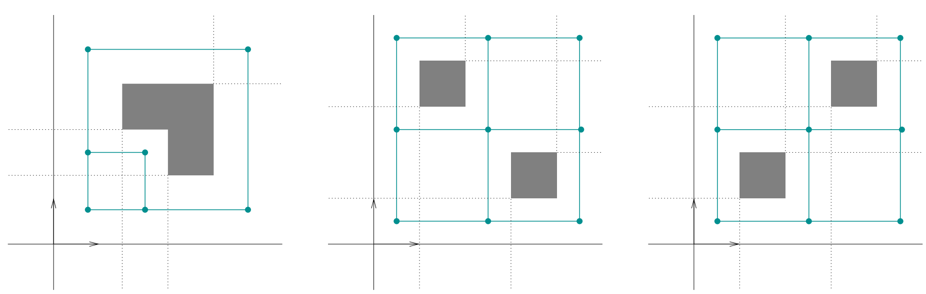
Then we can define a relation $\sim$ over the collection of morphsims of $\mathcal{C}$, writting $\gamma \sim \gamma'$ when

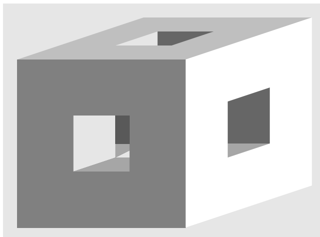
1) $\mathsf{s}(\gamma) \sim \mathsf{s}(\gamma')$ and $\mathsf{t}(\gamma) \sim \mathsf{t}(\gamma')$
2) the following diagram commutes

$$\begin{tikzcd}
 & x \arrow[r, "\gamma"] & y \arrow[dr, "\Sigma"] & \\
x \wedge x' \arrow[ur, "\Sigma"] \arrow[dr, "\Sigma"'] & & & y \vee y' \\
 & x' \arrow[r, "\gamma'"'] & y' \arrow[ur, "\Sigma"'] &
\end{tikzcd}$$

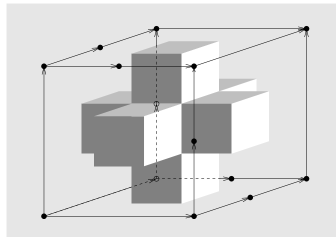
In particular we define the source and the target of the $\sim$-equivalence class of γ as the $\sim$-equivalence classes of $\mathsf{s}(\gamma)$ and $\mathsf{t}(\gamma)$ and we chaeck that if $x \sim y$ then $\mathsf{id}_x \sim \mathsf{id}_y$. We also check that if $\mathsf{s}(\gamma) \sim \mathsf{t}(\delta)$ then there exist two morphisms γ' and δ' such that $\gamma \sim \gamma'$, $\delta \sim \delta'$ and $\mathsf{s}(\gamma') = \mathsf{t}(\delta')$. Then we can define the composition of the $\sim$-equivalence classes γ and δ as the $\sim$-equivalence class of the composite $\gamma' \circ \delta'$ after we have proven that it does not depend on the morphisms γ' and δ'. We have thus defined a category whose identities are the $\sim$-equivalence classes of the identities of $\mathcal{C}$ and we prove that it is isomorphic to $\mathcal{C}/\Sigma$.

We can also describe the quotient $\mathcal{C}/\Sigma$ as a full sub-category of $\mathcal{C}$ provided we suitably pick an object of $\mathcal{C}$ in each of its Σ-components. Denote the set of Σ-components of $\mathcal{C}$ by $\mathcal{K}$ and let ϕ be some function from $\mathcal{K}$ to $\mathsf{Ob}(\mathcal{C})$ such that $\phi(K) \in \mathcal{K}$ for all Σ-components K. If for all Σ-components K and K' we have $\mathcal{C}[\phi(K), \phi(K')] \neq \emptyset$ if and only if there exist some $x \in K$ and $x' \in K'$ such that $\mathcal{C}[x, x'] \neq \emptyset$, then the quotient category $\mathcal{C}/\Sigma$ is isomorphic to the full sub-category of $\mathcal{C}$ whose set of objects if the (direct) image of ϕ. Applying the Thoerem 9 we prove that if the set $\mathcal{K}$ is finite then such a function ϕ exists. The following diagrams exemplify this approach.

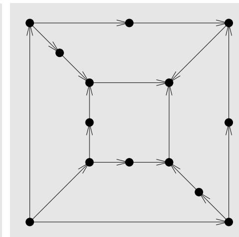




Intrieur de l'espace ordonn



Catgorie des composantes



Aplatie

From the preceding remark it is clear that if Σ is a *Yoneda* system of a loop-free category $\mathcal{C}$, then the quotient $\mathcal{C}/\Sigma$, therefore in particular the category of components of $\mathcal{C}$, is loop-free. Then by the Theorem 8 the quotient $\mathcal{C}/\Sigma$ is the skeleton of the localization $\mathcal{C}[\Sigma^{-1}]$.

Chapter 5

The Algebra of Cubical Areas

The category of components of a a pospace is always well defined, however its construction remains rather abstract and we have not provide any tool allowing actual calculations. In this section we describe a collection of pospaces whose categories of components are finite and can be determine by means of an algorithm, a computer program say. This collection is actually equiped with an additional structure that could be called a "graded boolean algebra" and whose operators suffice to describe the theoretic construction given in the preceding chapters.

5.1 Cubes

A **cube**[1] of dimension $n \in \mathbb{N}$ is a (set theoretic) Cartesian product of n subintervals of $\mathbb{R}_+$. The collection of cubes of dimension n forms a lattice (with respect to inclusion) whose least and greatest elements are the empty set and $\mathbb{R}_+^n$ (the n-fold Cartesian product of $\mathbb{R}_+$). Moreover, the greatest lower bound is given by the set theoretic intersection. In fact the lattice of n-dimensional cubes is the n-fold Cartesian product (in the category of lattices) of the lattice of intervals of $\mathbb{R}_+$: so the calculations are componentwise.

5.2 Cubical Areas

A **cubical algebra** of dimension $n \in \mathbb{N}$ is defined as a subset of $\mathbb{R}_+^n$ that can be written as a finite (set theoretic) union[2] of n-dimensional cubes. One easily sees that the collection of n-dimensional cubical areas forms a sub-structure

[1] Which is often called a **hypercube** when $n > 3$, we will not use this terminology.

[2] The lattice of cubes admits a binary greatest lower bound operator which differs from the set theoretic union.

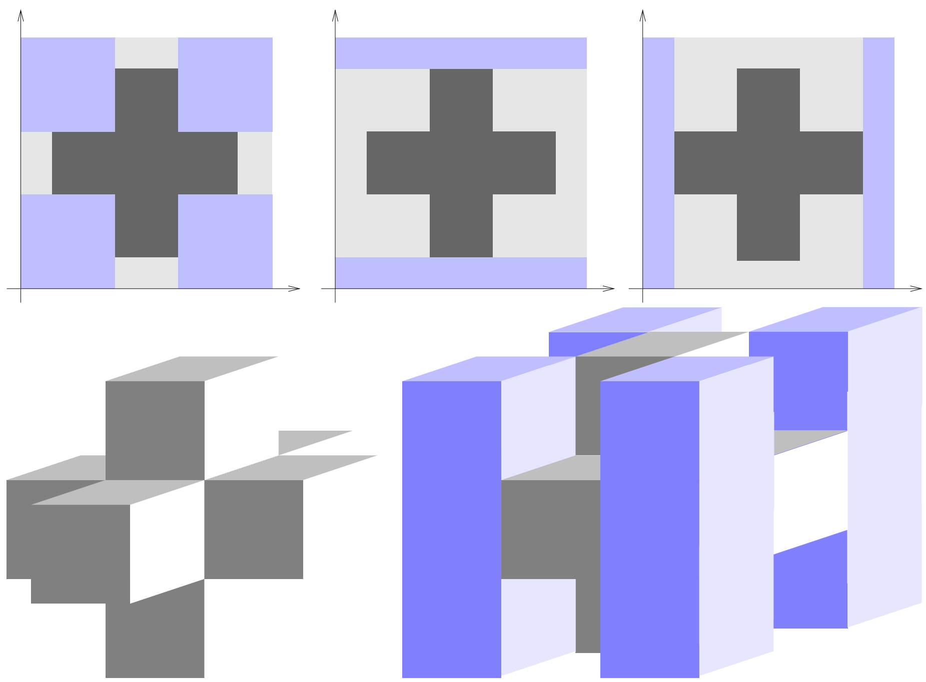
of the Boolean algebra[3] of all subsets of $\mathbb{R}_+^n$. Although the intervals and the cubes are easily represented within a computer's memory, the cubical areas are way harder to deal with : the issue arises from the fact that calculating their "normal forms" involves rather expensive algorithms. Nonetheless we choose to represent cubical areas as finite family of cubes, then one might have several distinct such families representing the same cubical area. From a theoretic point of view, the collection of all finite families of cubes equiped with the following relation

$$(A_i)_{i\in I} \preccurlyeq (B_j)_{j\in J} \quad \text{when} \quad \forall i \in I\ \exists j \in J,\ A_i \subseteq B_j$$

forms a pre-lattice, that is to say a preordered set in which any pair of elements has both a least upper bound and a greatest lower bound. We have an obvious morphism of pre-lattices α which sends a finite family of n-dimensional cubes to its set theoretic union (which is by definition a cubical area)

$$\alpha\big((A_i)_{i\in I}\big) := \bigcup_{i\in I} A_i$$

In addition we have a morphism of pre-lattices γ which takes a cubical area to a finite family of cubes. In order to describe γ, we define a **maximal sub-cube** of a cubical area $\mathcal{A}$ as a cube C such that $C \subseteq \mathcal{A}$ and for all cubes C', if $C \subseteq C' \subseteq \mathcal{A}$ then $C = C'$.



The image by α of the family of maximal sub-cubes of $\mathcal{A}$ is precisely $\mathcal{A}$ and any family of cubes whose image under α is $\mathcal{A}$ is less (with respect to the preorder

[3]Recall that a Boolean algebra is in particular a lattice.

$\preccurlyeq$) than the family of maximal sub-cubes of $\mathcal{A}$. In other words if $(A_i)_{i\in I}$ is a finite family of cubes and $\mathcal{A}$ is a cubical area then we have

$$\alpha\circ\gamma(\mathcal{A}) = \mathcal{A} \;\text{ and }\; (A_i)_{i\in I} \preccurlyeq \gamma\circ\alpha\big((A_i)_{i\in I}\big)$$

In particular the ordered pair (α, γ) forms a *Galois* connection[4].

In order to perform actual calculations in the (abstract) lattice of n-dimensional cubical areas, we are bound to work in the (concrete) pre-lattice of finite families of n-dimensional cubes. When $\alpha\big((A_i)_{i\in I}\big) = \mathcal{A}$ we say that the family $(A_i)_{i\in I}$ is a **cubical covering** of the area $\mathcal{A}$. We say that a finite family of cubes $(A_i)_{i\in I}$ is a **normal form** when $(A_i)_{i\in I} = \gamma\circ\alpha\big((A_i)_{i\in I}\big)$ and a **pre-normal form** when $\gamma\circ\alpha\big((A_i)_{i\in I}\big) \preccurlyeq (A_i)_{i\in I}$. We recall that the relation $\preccurlyeq$ may not be antisymetric, so it is a preorder which therefore induces an equivalence relation $\sim$ over the collection of finite families of cubes : such a family $(A_i)_{i\in I}$ is thus a pre-normal form when it is equivalent to its normal form. In other words for all maximal sub-cubes $C_{\mathcal{A}}$ of $\mathcal{A}$ (the cubical area covered by $(A_i)_{i\in I}$) there exists some index $i\in I$ such that $A_i = C_{\mathcal{A}}$.

So we represent the cubical areas by finite families of cubes whose normal form is provided by the morpism $\gamma\circ\alpha$.

5.3 Some theoretic facts about the calculation of a normal form

We provide here the mathematical facts upon which the normal form algorithm is based on.

Lemma 1 (Greatest lower bound in the lattice of n-cubical areas)
A representative (up to $\sim$-equivalence) of the greatest lower bound of two families of n-cubes $(A_i)_{i\in I}$ and $(B_j)_{j\in J}$ is given by $(A_i\cap B_j)_{(i,j)\in I\times J}$.

Proof : is such that for all indices $k\in K$, there exist $i\in I$ and $j\in J$ such that $C_k\subseteq A_i$ and $C_k\subseteq B_j$ i.e. an ordered pair $(i,j)\in I\times J$ such that $C_k\subseteq A_i\cap B_j$, then we actually have $(C_k)_{k\in K} \preccurlyeq (A_i\cap B_j)_{(i,j)\in I\times J}$.

Lemma 2 (Greatest lower bound preserves the pre-normal forms)
Let $(A_i)_{i\in I}$ and $(B_j)_{j\in J}$ be two finite families of n-cubes, if $(A_i)_{i\in I}$ and $(B_j)_{j\in J}$ are pre-normal forms then so is the family $(A_i\cap B_j)_{(i,j)\in I\times J}$. Besides, if the two first families respectively cover the cubical area $\mathcal{A}$ and $\mathcal{B}$, then the third one covers the cubical area $\mathcal{A}\cap\mathcal{B}$.

[4]In general a *Galois* connection between two preordered sets $(X,\preccurlyeq)$ and $(X',\preccurlyeq')$ is an ordered pair $(\alpha : X\to X', \gamma : X'\to X)$ of morphisms of preordered sets such that for all $x\in X$ and $x'\in X'$ we have $\alpha\circ\gamma(x')\preccurlyeq' x'$ and $x\preccurlyeq\gamma\circ\alpha(x)$. The morphisms α and γ are respectively called the **abstraction** and the **concretisation**.

Preuve :
Since we have normal forms it comes that $(A_i)_{i\in I} \sim \gamma(\mathcal{A})$ and $(B_j)_{j\in J} \sim \gamma(\mathcal{B})$ where $\mathcal{A}$ and $\mathcal{B}$ are the cubical areas which are represented by $(A_i)_{i\in I}$ and $(B_j)_{j\in J}$. Hence $(A_i)_{i\in I} \wedge (B_j)_{j\in J} \sim \gamma(\mathcal{A}) \wedge \gamma(\mathcal{B})$ and since γ is the right side of a *Galois* connection, it preserves (up to equivalence) the greatest lower bounds (i.e. $\gamma(\mathcal{A}\cap\mathcal{B}) \sim \gamma(\mathcal{A}) \wedge \gamma(\mathcal{B})$). We conclude applying the Lemma 1. The second statement is a direct consequence of the following well-known set theoretic formula.

$$\Big(\bigcup_{i\in I} A_i\Big) \cap \Big(\bigcup_{j\in J} B_j\Big) = \bigcup_{(i,j)\in I\times J} (A_i \cap B_j) \quad (1)$$

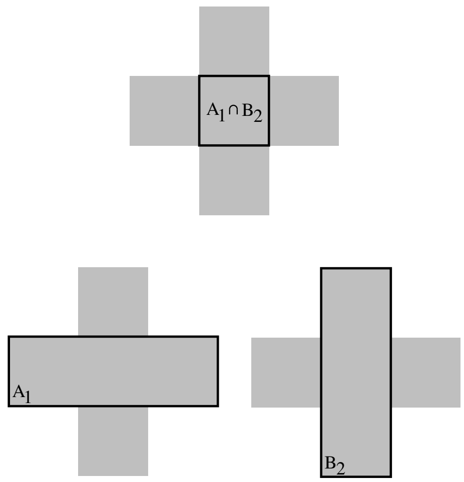
The preceding lemma is false if we substitute “normal” to “pre-normal” : the proof we have given is no more valid since it requires to work up to equivalence. Moreover we can observe a simple counter-example : setting

$$I{=}J{=}\{1,2\},\ A_1{=}B_1{=}[1,4]{\times}[2,3] \text{ and } A_2{=}B_2{=}[2,3]{\times}[1,4],$$

we have $A_1{\cap}B_2{=}[2,3]{\times}[2,3]$ which is not a maximal sub-cube of

$$\mathcal{A}{\cap}\mathcal{B}{=}\big([2,3]{\times}[1,4]\big){\cup}\big([1,4]{\times}[2,3]\big)$$

(see the figure beside).



Anyway, the finiteness of the sets I, J and the elementary formula (1) lead to a (naive) algorithm taking two families of n-cubes as an input and returning

$$\big\{A_i \cap B_j \;\big|\; (i,j) \in I{\times}J\big\}$$

which represents the intersection of the cubical areas represented by the arguments. Ineed by the Lemma 2, if both of them are pre-normal forms, then so is the output of the algorithm.

We now focus on the Boolean algebra structure of the collection of n-dimensional cubical areas which is a sub-structure of the Boolean algebra of all subsets of $\mathbb{R}_+^n$. By a slight abuse of language we identify an n-cube with the area it represents. In addition one easily obtains the normal form of the complement (in $\mathbb{R}_+^n$) of a n-cube : indeed, by definition an n-cube is a Cartesian product of intervals

$$\prod_{k=1}^{n} \mathbb{I}_k$$

therefore a point x of $\mathbb{R}_+^n$ belongs to its complement if and only if there exists some $k \in \{1,\dots,n\}$ such that the k^{th} projection of x lays in the complement (in $\mathbb{R}_+$) of $\mathbb{I}_k$. Yet, the complement of the interval $\mathbb{I}_k$ in $\mathbb{R}_+$ is the disjoint union of an initial segment and a final segment of $\mathbb{R}_+$ (any of which can be empty) :

for each coordinate we thus have something "before" the cube (initial segment) and something "after" it (terminal segment). Finally, the finiteness of the set of indices I, the elementary set theoretic formula

$$^{\mathsf{c}}\big(\bigcup_{i\in I} A_i\big) = \big(\bigcap_{i\in I} {}^{\mathsf{c}}A_i\big)$$

and the Lemma 2 provide a (naive) algorithm taking a family $(A_i)_{i\in I}$ of n-cubes as an input and returning the pre-normal form of the complement of the cubical area represented by the family $(A_i)_{i\in I}$. Up to a slight improvement, this algorithm can be written so as to return a normal form.

5.4 A bit of topology

In general the interior of a union of subsets of a topological space contains, yet might differ from, the union of the interiors of these subsets : the "union" operator does not commute with the "interior" operator. Nevertheless we have

Lemma 3 (Interior of a cubical area)
If a family of n-cubes $(A_i)_{i\in I}$ is a pre-normal form, then the topological interior of the cubical area it covers (seen as a subset of $\mathbb{R}^n_+$) is the union of the interior of the cubes of the family, that is to say

$$\overset{\circ}{\Big(\bigcup_{i\in I} Ai\Big)} = \bigcup_{i\in I} \overset{\circ}{A}i$$

<u>Proof</u> : Let p be a point in the interior of the area covered by the family $(A_i)_{i\in I}$, there exists V a neighbourhood of p contained in this area which can be written as a set theoretic Cartesian product of n open intervals of $(\mathbb{R}^+)^{\mathrm{n}}$ (the open cubes form a basis for the product topology) and since the family$(A_i)_{i\in I}$ is a pre-normal form, there exists some index i such that $V \subseteq A_i$.

The interior of a single n-cube is easily obtained so we can, by means of the preceding lemma, compute the interior of a cubical area from any pre-normal form which covers it. In general the (topological) "closure" operator commutes with the "union" operator. Since the closure of a single cube is as easy to determine as its interior, we obtain the closure of a cubical area from any its cubical coverings. The boundary of a subset of a topological space is defined as the set theoretic difference of its closure and its interior. The topological boundary is directly involved in the detection of "deadlocks" of a PV program.

5.5 Graded Boolean Algebra of Cubical Areas

In the preceding section we have described, for each $n \in \mathbb{N}$, the Boolean algebra of n-dimensional cubical areas and yet it clearly appears that the (set theoretic)

Cartesian product of two cubical areas of dimension n and m, is a cubical area of dimension $n+m$. This remark suggests a graded structure. We get the inspiration from the notion of graded algebra, a structure whose standard example is provided by the (multivariate) polynomials. For each n we denote the Boolean algebra of n-dimensional cubical areas by $\mathbb{B}_n$ and for each oredered pair of natural numbers (n,m) we have a mapping $\mu_{n,m} : \mathbb{B}_n \times \mathbb{B}_m \longrightarrow \mathbb{B}_{n+m}$ which sends any ordered pair of cubical areas $(\mathcal{A},\mathcal{A}')$ to their Cartesian product $\mathcal{A}\times\mathcal{A}'$. The product we have defined is associative in the sense where for all quadruples (n,m,p,q) we have

$$(\mu_{n,m} \circ \mu_{m,p}) \circ \mu_{p,q} = \mu_{n,m} \circ (\mu_{m,p} \circ \mu_{p,q})$$

however it is not commutative since we wish, for example, to distinguish the rectangles $[1,4]\times[2,3]$ and $[2,3]\times[1,4]$. Besides, the analogy with the graded algebras would require that for all cubical areas $\mathcal{A}$, $\mathcal{B}$ of dimension n and m, the partial functions $\mu_{n,m}(\mathcal{A},-)$ and $\mu_{n,m}(-,\mathcal{B})$ be morphisms of Boolean algebra. Each one actually preserves $\wedge$, $\vee$ as well as the null element, which just means that

$$\begin{gathered}
(\mathcal{A}_1\cap\mathcal{A}_2)\times\mathcal{B}=(\mathcal{A}_1\times\mathcal{B})\cap(\mathcal{A}_2\times\mathcal{B}) \\
(\mathcal{A}_1\cup\mathcal{A}_2)\times\mathcal{B}=(\mathcal{A}_1\times\mathcal{B})\cup(\mathcal{A}_2\times\mathcal{B}) \\
\mathcal{A}\times(\mathcal{B}_1\cap\mathcal{B}_2)=(\mathcal{A}\times\mathcal{B}_1)\cap(\mathcal{A}\times\mathcal{B}_2) \\
\mathcal{A}\times(\mathcal{B}_1\cup\mathcal{B}_2)=(\mathcal{A}\times\mathcal{B}_1)\cup(\mathcal{A}\times\mathcal{B}_2) \\
\mathcal{A}\times\emptyset_m=\emptyset_{n+m} \text{ and } \emptyset_n\times\mathcal{B}=\emptyset_{n+m}
\end{gathered}$$

however they do not preserve the unit element since $\mathbb{R}_+^n\times\mathcal{B}\neq\mathbb{R}_+^{n+m}$ as soon as $\mathcal{B}\neq\mathbb{R}_+^m$. Nevertheless we still have the following relation

$${}^{\mathsf{c}}\big(\mu_{n,m}(\mathcal{A},\mathcal{B})\big) = \mu_{n,m}({}^{\mathsf{c}}\!\mathcal{A},\mathcal{B}) \vee \mu_{n,m}(\mathcal{A},{}^{\mathsf{c}}\mathcal{B})$$

or in the case of cubical areas

$${}^{\mathsf{c}}(\mathcal{A}\times\mathcal{B}) = ({}^{\mathsf{c}}\!\mathcal{A}\times\mathcal{B}) \cup (\mathcal{A}\times{}^{\mathsf{c}}\mathcal{B}).$$

This last formula is actually the reason why the (normal form) of the complement of a cube is easily determined : it amounts to a calculation of complement in $\mathbb{B}_1$, that is to say in the Boolean algebra of finite unions of sub-intervals of $\mathbb{R}_+$. It is worth to notice that $\mathbb{B}_0$ only has two elements : the 0-dimensional empty set and the singleton whose only element is the null vector that is $\mathbb{R}_+^0$.

Chapter 6

What has not been told

6.1 Where are the loops ?

In the preceding chapters we have focused on the pospaces for they are sufficient to represent any PV program. Now we would like to go further and enrich the PV language so as it allows loops. The following program is a simple example, it is made of a unique process containing a unique loop.

```
while true
  do
    P(a).V(a)
  done
```

Intuitively, we would like to be able to identify the extremities of the directed segment $\overrightarrow{[0,1]}$ in order to obtain a "directed circle" that is to say for example, the Euclidean circle [1] $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$, together with a structure such that the only directed paths on the "directed circle" are the continuous mappings of the form

$$\begin{array}{ccc} [0,1] & \longrightarrow & S^1 \\ t & \longmapsto & e^{i\theta(t)} \end{array}$$

where θ is an increasing continuous mapping from $[0,1]$ to $\mathbb{R}$. In a broader context, we would like the directed circle to be an object of a category over which we can define a fundamental category functor satisfying the following statements :

1) the category Po can be embedded in this category

2) the fundamental category functor defined over this category is an extension of the functor $\overrightarrow{\pi_1}$ we have defined over Po

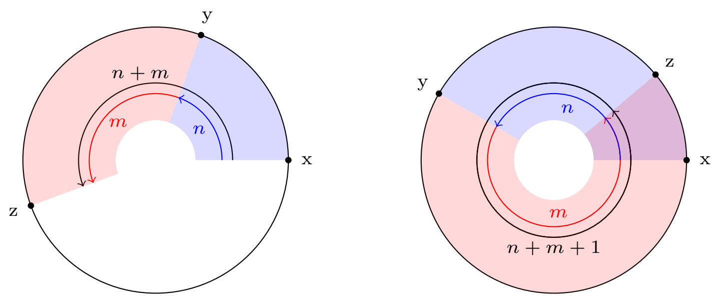
3) the fundamental category of the "directed circle" is isomorphic to the category whose set of objects and set of morphisms are respectively S^1 and

[1] Denoting the field of complex numbers by $\mathbb{C}$ and the norm of z by $|z|$.

$S^1 \times \mathbb{N} \times S^1$, the source and target of (x, n, y) begin given by x and y while the composition is defined as

$$(y, m, z) \circ (x, n, y) = \begin{cases} (x, n+m+1, z) & \text{if } \overset{\curvearrowleft}{xy} \cup \overset{\curvearrowleft}{yz} = S^1 \\ (x, n+m, z) & \text{otherwise} \end{cases}$$

where $\overset{\curvearrowleft}{xy}$ represents the anticlockwise arc from x to y if $x \neq y$ and the singleton $\{x\}$ otherwise.



The identities are the triples $(x, 0, x)$ for $x \in S^1$. We call this category the **directed circle**. One easily checks that the directed circle is isomorphic to its opposite and has no isomorphisms but its identities.

For all such extensions $\overrightarrow{\pi}_1$, we say by a slight abuse of language that an object X picked from the domain of $\overrightarrow{\pi}_1$ is a directed circle when $\overrightarrow{\pi}_1(X)$ is a directed circle. In fact it is not too difficult to find an extension that fulfils the required statements, however we have not find a satisfactory extension of the notion of category of components yet. To be able to treat the categories in which loops may occur, we need to relax the axioms defining the *Yoneda* systems. In the case of the directed circle we would like the morphisms of *Yoneda* to be the triples $(x, 0, y)$ with $x \neq y$, denote by Σ the collection of all such morphisms. Any morphsism of the directed circle can be written as a composition of elements of Σ therefore the quotient of the directed circle by Σ is isomorphic to $\mathbf{1}$. On the contrary, the localization of the directed circle by Σ is isomorphic to the category obtained by replacing the monoid $\mathbb{N}$ by the group $\mathbb{Z}$ and by adapting the composition law in the obvious way that is to say

$$(y, m, z) \circ (x, n, y) = \begin{cases} (x, n+m+1, z) & \text{if } n \geq 0 \ , \ m \geq 0 \text{ and } \overset{\curvearrowleft}{xy} \cup \overset{\curvearrowleft}{yz} = S^1 \\ (x, n+m-1, z) & \text{if } n \leq 0 \ , \ m \leq 0 \text{ and } \overset{\curvearrowright}{xy} \cup \overset{\curvearrowright}{yz} = S^1 \\ (x, n+m, z) & \text{otherwise} \end{cases}$$

where $\overset{\curvearrowright}{xy}$ and $\overset{\curvearrowleft}{xy}$ respectively represent the clockwise and anticlockwise arcs from x to y if $x \neq y$ and the singleton $\{x\}$ otherwise[2]. The quotient and the localization of the directed circle by Σ are therefore not equivalent hence

[2] This category is also isomorphic to the fundamental groupoid of the Euclidean circle.

the Theorem 8 has no straightforward analoguous outside the framework of loop-free categories. However we observe that the directed circle has a unique Σ-component, hence mimicing the description of the category of components given in section 4.5 we define the category of components of the directed circle as the monoid $(\mathbb{N}, +, 0)$. The "right" axioms for the category of components in the context of categories that may contain loops is still an open problem.

6.2 What the notion of "locally loop-free" category may be ?

The collection of pure subcategories of a small category $\mathcal{C}$ forms a locale $\mathcal{X}$ whose greatest element is the category $\mathcal{C}$ and whose smallest one is the empty subcategory. The greatest lower bound is given by the intersection and the least upper bound of the family $(X_i)_{i \in I}$ by the collection of composites

$$\sigma_n \circ \cdots \circ \sigma_0$$

where $n \in \mathbb{N}$ and for each $k \in \{0, \ldots, n\}$ there exists some $i_k \in I$ such that σ_k belongs to X_{i_k}. The pure subcategories of $\mathcal{C}$ can be seen as "open". Then we wonder if the pure loop-free subcategories of $\mathcal{C}$ generates the locale $\mathcal{X}$ and when it is the case, we may say that the category $\mathcal{C}$ is locally loop-free. In fact one has to add some hypothses in order to obtain a relevant notion, our purpose being to have results analoguous to the theorems 5 and 6.

Besides, the functor from $(\mathbb{R}, \leq)$ to the directed circle which takes each object $t \in \mathbb{R}$ to $e^{2i\pi t}$ and each arrow (t, t') to the morphism $(e^{2i\pi t}, q_{t,t'}, e^{2i\pi t'})$ where $q_{t,t'}$ is the smallest integer lower than $t' - t$ suggests that we think of $(\mathbb{R}, \leq)$ as a "directed covering" of the directed circle. The notion of **covering** originally comes from the classical algebraic topology but it has proven to be useful in other branches of mathematics[DD05].

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